# Spectral decomposition of the automorphic spectrum of GSp(4)

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# 1. The Problem

- $G := GSp(4)$, $\nu : G \to \mathbb{G}_m$ ; similitude norm.

$$G = \{g \in GL(4) \,|\, \nu(g) := g\mathrm{Ad}(J)^t g \in \mathbb{G}_m\},$$

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

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**Problem.**

To obtain a spectral ("irreducible") decomposition of the right regular representation of $G(\mathbb{A})$ on

$$\mathcal{L}(G) := L^2(G(F)\mathfrak{A}_G \backslash G(\mathbb{A}))$$

# 2. Cuspidal spectrum

- $B = T_0 U_0 = \left\{ \begin{pmatrix} t_1 & * & * & * \\ 0 & t_2 & * & * \\ 0 & 0 & \nu t_1^{-1} & 0 \\ 0 & 0 & * & \nu t_2^{-1} \end{pmatrix} \in G \right\}$; Borel subgroup.

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- $\mathbf{K} = \prod_v \mathbf{K}_v \subset G(\mathbb{A})$ ; $T_0$-good maximal compact subgroup. We have the Iwasawa decomposition

$$G(\mathbb{A}) = B_0(\mathbb{A})\mathbf{K}.$$

- The constant term along a parabolic subgroup $P = MU \subset G$ is defined by

$$\phi_P(g) := \int_{U(F)\backslash U(\mathbb{A})} \phi(ug)\, du$$

for measurable $\phi : U(F)\backslash G(\mathbb{A}) \to \mathbb{C}$.

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- $\mathcal{H}(G(\mathbb{A}))$ ; Hecke algebra of $G(\mathbb{A})$

  *i.e.* convolution algebra of compactly supported **K**-finite functions on $G(\mathbb{A})$

# 2.1. Proposition

*The convolution operators* $(f \in \mathcal{H}(G(\mathbb{A})))$

$$R(f)\phi(x) = \int_{G(F)\backslash G(\mathbb{A})} f(g)\phi(xg)\,dg$$

*on* $\mathcal{L}_0(G)$ *are compact.*

# 2.2. Theorem (Piatetsky-Shapiro)

$\mathcal{L}_0(G)$ *decomposes into a Hilbert direct sum of irreducible unitary representations of* $G(\mathbb{A})$*, in which each isomorphism class of irreducible unitary representations of* $G(\mathbb{A})$ *occurs with finite multiplicity.*

$$\mathcal{L}_0(G) \simeq \bigoplus_{\pi\in\Pi(G(\mathbb{A}))} \pi^{\oplus m_0(\pi)}.$$

- $\exists m_0(\pi) \in \mathbb{N}$,
- $\Pi(G(\mathbb{A}))$ ; the set of isom. classes of irred. unitary repr.s of $G(\mathbb{A})$.

# 2.3. Remark.

- $\mathcal{L}_0(G)$ is the closure of the space $\mathcal{A}_0(G)$ of cusp forms on $G(\mathbb{A})$ consisting of $\phi : G(F)\mathfrak{A}_G\backslash G(\mathbb{A}) \to \mathbb{C}$ s.t.
  (i) $\phi$ is $\mathfrak{Z}(\mathfrak{g}_\infty)$-finite and **K**-finite;
  (ii) $\phi$ is slowly increasing on $G(\mathbb{A})$;
  (iii) $\phi_P = 0, \forall P \subsetneq G$.
- The description of $m_0(\pi)$ is a difficult problem.
- We concentrate on the spectral decomposition of the orthogonal complement of $\mathcal{L}_0(G)$.

# 3. Parabolic induction

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  ($B_0$-) standard parabolics of $G$.

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- $G$, $P_i = M_i U_i$, $(i = 1, 2)$, $B_0 = T_0 U_0$ ;
  ($B_0$-) standard parabolics of $G$.
- $T_0$ consists of $m_0(t_1, t_2; \nu) = \mathrm{diag}(t_1, t_2, \nu t_2^{-1}, \nu t_1^{-1})$, $t_i, \nu \in \mathbb{G}_m$;
- $M_1$ consists of

$$m_1(t, g) := \begin{pmatrix} t & & & \\ & a & & b \\ & & \nu t^{-1} & \\ & c & & d \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$$

$$t, \nu = \det g \in \mathbb{G}_m;$$

- $M_2$ consists of

$$m_2(g;\nu) := \begin{pmatrix} g & \\ & \nu\,{}^t g^{-1} \end{pmatrix}, \quad \begin{matrix} g \in GL(2) \\ \nu \in \mathbb{G}_m. \end{matrix}$$

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For the above Levi subgroups $M$, the cuspidal multiplicities $m_0(\pi)$, ($\pi \in \Pi(M(\mathbb{A}))$) are well-understood:

- $\mathcal{L}_0(\mathbb{G}_m) = \mathcal{L}(\mathbb{G}_m)$ is described by the abelian classfield theory à la Langlands.
- $m_0(\pi)$, ($\pi \in \Pi(GL(2,\mathbb{A}))$) is at most 1. It is 1 if and only if $\pi$ satisfies the conditions of Jacquet-Langlands' converse theorem.

# 3.1. Induced spaces

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  1. Case $P = B_0$. $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{a}^{G,*}_{0,\mathbb{C}} := \mathbb{C}^2$,

     $$e^\lambda(m_0(t_1, t_2; \nu)) := |t_1|_{\mathbb{A}}^{\lambda_1} |t_2|_{\mathbb{A}}^{\lambda_2} |\nu|_{\mathbb{A}}^{-(\lambda_1+\lambda_2)/2}.$$

  2. Case $P = P_1$. $\lambda \in \mathfrak{a}^{G,*}_{M_1,\mathbb{C}} := \mathbb{C}$,

     $$e^\lambda(m_1(t, g)) := |t|_{\mathbb{A}}^{\lambda} |\det g|_{\mathbb{A}}^{-\lambda/2}.$$

  3. Case $P = P_2$. $\lambda \in \mathfrak{a}^{G,*}_{M_1,\mathbb{C}} := \mathbb{C}$,

     $$e^\lambda(m_2(g; \nu)) := |\det g|_{\mathbb{A}}^{\lambda} |\nu|_{\mathbb{A}}^{-\lambda}.$$

- $\mathcal{A}_0(M)_\pi$ ; $\pi$-isotypic subspace in the $(\mathfrak{m}_\infty, \mathbf{K}_\infty^M) \times M(\mathbb{A}_f)$-module $\mathcal{A}_0(M)$.

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- $\mathcal{A}_0(M)_{\pi_\lambda} := e^\lambda \otimes \mathcal{A}_0(M)_\pi$, $\lambda \in \mathfrak{a}_M$.
- $\mathcal{A}_0(P\backslash G)_{\pi_\lambda} := \mathrm{ind}_{\mathbf{K}\cap P(\mathbb{A})}^{\mathbf{K}} \mathcal{A}_0(M)_{\pi_\lambda}$ ; $\mathbf{K}$-finite induction.

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$\Longrightarrow$

$\mathcal{A}_0(P\backslash G)_{\pi_\lambda} \to \lambda \in \mathfrak{a}_{M,\mathbb{C}}^{G,*}$ ; vector bundle

For any $\mathbf{K}$-type $\kappa$, $\kappa$-isotypic part $\mathcal{A}_0(P\backslash G)_{\pi_\lambda}^\kappa \to \lambda$ is a vector bundle (of finite rank). Above space should be viewed as the union of these bundles.

# 3.2. Poincaré series

- $\mathfrak{A}_M \subset Z_M(\mathbb{A})$ ; $\mathbb{R}$-vector part.

  $M(\mathbb{A})^1 := \bigcap_{\chi \in \mathrm{Hom}(M,\mathbb{G}_m)} \ker |\chi|_{\mathbb{A}}$.

  1. $G(\mathbb{A}) = U(\mathbb{A})\mathfrak{A}_M M(\mathbb{A})^1 \mathbf{K}$.
  2. $\mathrm{Lie}\left(\mathfrak{A}_M/\mathfrak{A}_G\right) \xleftrightarrow{\text{dual}} \mathfrak{a}_M^{G,*}$.

- $P_{(M,\pi)}$ ; space of Paley-Wiener sections of $\mathcal{A}_0(P\backslash G)_{\pi_\lambda} \to \lambda \in \mathfrak{a}_{M,\mathbb{C}}^{G,*}$.

  *i.e.*, the Fourier transform of $\phi \in P_{(M,\pi)}$

$$\widehat{\phi}(g) := \int_{\lambda\in\lambda_0+i\mathfrak{a}_M^{G,*}} \phi_\lambda(g)\, d\lambda$$

$$= \int_{\lambda\in\lambda_0+i\mathfrak{a}_M^{G,*}} \phi_\lambda(mk)e^{\lambda(\log a)}\, d\lambda$$

  $(g = uamk \in U(\mathbb{A})\mathfrak{A}_M M(\mathbb{A})^1\mathbf{K})$ is smooth and compactly supported in $a \in \mathfrak{A}_M$.

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$$\widehat{\phi}(g) := \int_{\lambda\in\lambda_0+i\mathfrak{a}_M^{G,*}} \phi_\lambda(g)\, d\lambda$$

  is smooth and compactly supported on the $\mathfrak{A}_M$-component of $g \in G(\mathbb{A})$.

- $\theta_\phi(g) := \sum_{\gamma\in P(F)\backslash G(F)} \widehat{\phi}(\gamma g), (\phi \in P_{(M,\pi)})$ ;

  Poincaré series

# 3.3. Theorem

*(i)* $\theta_\phi$ *belongs to* $\mathcal{L}(G)$.
*(ii)* $\mathcal{L}(G) = \bigoplus_{[M,\pi]} \mathcal{L}(G)_{[M,\pi]}$, *where*

- $[M, \pi]$ ; $W_0$ *(Weyl group of* $(G, T_0)$*) orbit of* $(M, \pi)$;
- $\mathcal{L}(G)_{[M,\pi]}$
  $:= \mathrm{cl.span}\{\theta_\phi \,|\, \phi \in P_{(M',\pi')},\ (M', \pi') \in [M, \pi]\}$.

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  $:= \mathrm{cl.span}\{\theta_\phi \,|\, \phi \in P_{(M',\pi')},\ (M',\pi') \in [M,\pi]\}$.

This follows from Langlands' lemma:

$$\phi \in \mathcal{L}(G), \phi_P = 0, (B_0 \subset \forall P \subset G) \Longrightarrow \phi = 0.$$

and the orthogonality relation 3.7 below.

# 3.4. Eisenstein series

- $\Delta_0 := \left\{ \begin{array}{l} \alpha_1(m_0(t_1,t_2;\nu)) := t_1 t_2^{-1}, \\ \alpha_2(m_0(t_1,t_2;\nu)) := t_2^2 \nu^{-1} \end{array} \right\}$;
  set of simple roots of $(B_0, T_0)$.
- $\Delta_{M_i} := \{\alpha_{M_i} := \alpha_i|_{Z_{M_i}}\}$ ; set of simple root of $(P_i, Z_{M_i})$, $(i = 1, 2)$

$$\alpha_{M_1}(m_1(t, z\mathbf{1}_2)) = tz^{-1},\ \alpha_{M_2}(m_2(z\mathbf{1}_2;\nu)) = z^2\nu^{-1}.$$

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  $$\alpha_{M_1}(m_1(t, z\mathbf{1}_2)) = tz^{-1}, \ \alpha_{M_2}(m_2(z\mathbf{1}_2; \nu)) = z^2\nu^{-1}.$$

- The corresponding coroots on $\mathfrak{a}_M^{G,*}$ are given by

  $$\alpha_1^\vee(\lambda) = \lambda_1 - \lambda_2, \ \alpha_2^\vee(\lambda) = \lambda_2,$$
  $$\alpha_{M_i}^\vee(\lambda) = \lambda, \ (i = 1, 2).$$

# 3.5. Proposition

*(i) For* $(M, \pi)$ *as above and* $\phi \in P_{(M,\pi)}$, the Eisenstein series

$$E_P(\phi_\lambda, g) := \sum_{\gamma \in P(F)\backslash G(F)} \phi_\lambda(\gamma g)$$

*converges absolutely if* $\alpha^\vee(\Re\lambda - \rho_P) > 0, \forall \alpha \in \Delta_P$.

- $\rho_P$ ; half of the sum of positive roots of $(P, Z_M)$:

$$2\rho_{B_0}(m_0(t_1, t_2; \nu)) = t_1^4 t_2^2 \nu^{-3},$$

$$\rho_{P_1}(m_1(t, z\mathbf{1}_2)) = t^2 z^{-2},\ 2\rho_{P_1}(m_2(z\mathbf{1}_2; \nu)) = z^6 \nu^{-3}.$$

*(ii) At such “enoughly positive”* $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^{G,*}$, $E_P(\phi_\lambda)$ *is an* *automorphic form* *on* $G(\mathbb{A})$.

*(iii)* $E_P(I_P^G(\pi_\lambda, f)\phi_\lambda) = R(f)E_P(\phi_\lambda)$, $f \in \mathcal{H}(G(\mathbb{A}))$.

($\mathcal{H}(G(\mathbb{A}))$-equivariance.)

Right translation

*(ii) At such "enoughly positive"* $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^{G,*}$, $E_P(\phi_\lambda)$ *is an* *automorphic form* *on* $G(\mathbb{A})$.

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($\mathcal{H}(G(\mathbb{A}))$-equivariance.)

*(iv) For "enoughly positive"* $\lambda_0 \in \mathfrak{a}_M^{G,*}$,

$$\int_{\lambda \in \lambda_0 + i\mathfrak{a}_M^{G,*}} E_P(\phi_\lambda)\, d\lambda = \theta_\phi.$$

# 3.6. Proposition

*(i) For* $w \in W_0/W_0^M$ *s.t.* $w(M) \subset G$ *; standard Levi, the* *intertwining operator*

$$M(w,\pi_\lambda)\phi_\lambda(g) := \int_{(w(U)\cap U_w\backslash U_w)(\mathbb{A})} \phi_\lambda(w^{-1}ug)\,du$$

*converges absolutely if* $\alpha^\vee(\Re\lambda - \rho_P) > 0, \forall\alpha \in \Delta_P$.
*(ii) At such* $\lambda$*,*

$$M(w,\pi_\lambda) : \mathcal{A}_0(P\backslash G)_{\pi_\lambda} \longrightarrow \mathcal{A}_0(P_w\backslash G)_{w(\pi_\lambda)}$$

*is* $\mathcal{H}(G(\mathbb{A}))$*-equivariant.*

- $P_w = M_wU_w := w(M)B_0$ ; standard parabolic.

*(iii) (Functional equation)* $E_P(M(w,\pi_\lambda)\phi_\lambda) = E_P(\phi_\lambda)$.
*(iv) The constant term of* $E_P(\phi_\lambda)$ *along* $P' = M'U'$ *is given by*

$$E_P(\phi_\lambda)_{P'} = \sum_{\substack{w \in W_0/W_0^M \\ w(M)\subset M';\, \textit{standard}}} E_{P_w \cap M'}(M(w,\pi_\lambda)\phi_\lambda)$$

# 3.7. Corollary

*Take* $\phi \in P_{(M,\pi)}$, $\phi' \in P_{(M',\pi')}$.

*(i) If* $[M,\pi] = [M',\pi']$, *the* $L^2$*-inner product of* $\theta_\phi$, $\theta_{\phi'}$ *equals*

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \int_{\lambda_0 + i\mathfrak{a}_M^{G,*}} A(\phi,\phi')(\pi_\lambda)\, d\lambda,$$

$$A(\phi,\phi')(\pi_\lambda) := \sum_{\substack{w \in W_0/W_0^M \\ w(M)=M'}} \underline{\langle M(w,\pi_\lambda)\phi_\lambda, \phi'_{-w(\bar\lambda)} \rangle}$$

$L^2$-inner product of $\mathcal{L}(M')$

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$$A(\phi,\phi')(\pi_\lambda) := \sum_{\substack{w \in W_0/W_0^M \\ w(M)=M'}} \langle M(w,\pi_\lambda)\phi_\lambda, \phi'_{-w(\bar{\lambda})} \rangle$$

*(ii) Otherwise,* $\langle \theta_\phi, \theta_{\phi'} \rangle = 0$.

# 4. Analytic behavior of $M(w, \pi_\lambda)$

- ${}^L G = \widehat{G} \times W_F$ ; $L$-group of $G$:

  $\widehat{G} = GSp(4, \mathbb{C})$, $W_F$ ; Weil group of $F$.

# 4. Analytic behavior of $M(w, \pi_\lambda)$

- ${}^LG = \widehat{G} \times W_F$ ; $L$-group of $G$:

  $\widehat{G} = GSp(4, \mathbb{C})$, $W_F$ ; Weil group of $F$.

- ${}^LB_0 = {}^LT_0 \ltimes \widehat{U}_0 = B_0(\mathbb{C}) \times W_F$,

  ${}^LP_1 = {}^LM_1 \ltimes \widehat{U}_1 = P_2(\mathbb{C}) \times W_F$,

  ${}^LP_2 = {}^LM_2 \ltimes \widehat{U}_2 = P_1(\mathbb{C}) \times W_F$.

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- ${}^L G = \widehat{G} \times W_F$ ; $L$-group of $G$:

  $\widehat{G} = GSp(4, \mathbb{C})$, $W_F$ ; Weil group of $F$.

- ${}^L B_0 = {}^L T_0 \ltimes \widehat{U}_0 = B_0(\mathbb{C}) \times W_F$,

  ${}^L P_1 = {}^L M_1 \ltimes \widehat{U}_1 = P_2(\mathbb{C}) \times W_F$,

  ${}^L P_2 = {}^L M_2 \ltimes \widehat{U}_2 = P_1(\mathbb{C}) \times W_F$.

- $\varphi_\tau$: $\mathcal{L}_F \to GL(2, \mathbb{C})$ ; (conjectural) global Langlands parameter of a cuspidal autom. repr. $\tau$ of $GL(2, \mathbb{A})$).

- Identify $\mathrm{Hom}_{\mathrm{cont}}(\mathbb{A}^\times / F^\times, \mathbb{C}^\times) = \mathrm{Hom}_{\mathrm{cont}}(W_F, \mathbb{C}^\times)$

  (abelian CFT à la Langlands)

# 4.1. Langlands parameters

The Langlands parameter $\varphi_\pi$ of a cuspidal repr. $\pi$ of $M(\mathbb{A})$ is given by:

1. Case $P = B_0$, $\pi(m_0(t_1, t_2; \nu)) = \omega_1(t_1)\omega_2(t_2)\omega(\nu)$.

$$\varphi_\pi = \mathrm{diag}(\omega_1\omega_2\omega, \omega_1\omega, \omega, \omega_2\omega).$$

2. Case $P = P_1$, $\pi(m_1(t, g)) = \omega(t)\tau(g)$.

$$\varphi_\pi = \begin{pmatrix} \omega\varphi_\tau & \mathbf{0}_2 \\ \mathbf{0}_2 & \omega_\tau {}^t\varphi_\tau^{-1} \end{pmatrix}.$$

$\omega_\tau$ ; central character of $\tau$.

3. Case $P = P_2$, $\pi(m_2(g;\nu)) = \omega(\nu)\tau(g)$.

$$\varphi_\pi = \begin{pmatrix} \omega\omega_\tau & & & \\ & \omega a_\tau & & \omega b_\tau \\ & & \omega & \\ & \omega c_\tau & & \omega d_\tau \end{pmatrix}$$

We have written $\varphi_\tau = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}$.

# 4.2. Normalization factor

- $\widehat{\mathfrak{u}}$ ; Lie algebra of $\widehat{U}$.
- $w \in W_0/W_0^M$ with $w(M)$ standard Levi, consider the adjoint repr.

$$r_w : {}^L M \rightarrow GL(\widehat{\mathfrak{u}}/w^{-1}(\widehat{\mathfrak{u}}_w) \cap \widehat{\mathfrak{u}}))$$

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$L(s, \pi, r_w) = L(s, r_w \circ \varphi_\pi)$,
$\varepsilon(s, \pi, r_w) = \varepsilon(s, r_w \circ \varphi_\pi)$
associated automorphic $L$, $\varepsilon$-functions.

- $r(w, \pi_\lambda) := \dfrac{L(0, \pi_\lambda, r_w)}{L(1, \pi_\lambda, r_w)\varepsilon(0, \pi_\lambda, r_w)}$
Langlands' normalization factor for $M(w, \pi_\lambda)$

# 4.3. Example

1. The case $P = B_0$, $\pi = \omega_1 \otimes \omega_2 \otimes \omega$. For simple reflections $s_i$ attached to $\alpha_i$,

$$L(0, \pi_\lambda, r_{s_1}) = L(\lambda_1 - \lambda_2, \omega_1 \omega_2^{-1}),\ L(0, \pi_\lambda, r_{s_2}) = L(\lambda_2, \omega_2)$$

The other factors are product of these ones for various $w(\pi)$, $(w \in W_0)$.

2. The case $P = P_1$, $\pi = \omega \otimes \tau$. (unique non-trivial $w =w_1$)

$$L(0, \pi_\lambda, r_{M_1}) = L(\lambda, \mathrm{Ad}^2(\tau) \times \omega).$$

3. The case $P = P_2$, $\pi = \tau \otimes \omega$. (unique non-trivial $w =w_2$)

$$L(0, \pi_\lambda, r_{M_2}) = L(\lambda, \tau) L(2\lambda, \omega_\tau).$$

# 4.4. Theorem. (Langlands, Shahidi)

*(i)* $E_P(\phi_\lambda)$, $M(w,\pi_\lambda)\phi_\lambda$ *have meromorphic continuation to* $\mathfrak{a}^{G,*}_{M,\mathbb{C}}$.

*(ii) The properties Prop.3.5 (ii), (iii), Prop.3.6 (ii), (iii), (iv) holds for these meromorphic functions.*

*(iii)* $N(w,\pi_\lambda) := r(w,\pi_\lambda)^{-1}M(w,\pi_\lambda)$ *is holomorphic in the region* $\alpha^\vee(\Re\lambda) \geq 0$, $\forall\alpha > 0$, $w(\alpha) < 0$.

- (i), (ii) is due to the general theory of Langlands. (iii) is a part of the Langlands-Shahidi theory and is valid only for generic cuspidal $\pi$.
- In general the singularities of $M(w, \pi_\lambda)\phi_\lambda$ are known to be locally finite for each $\phi$. The above (iii) shows, for $\Re\lambda$ positive, they are finite and depend only on $\pi$ !

# 4.5. Singularities of Eisenstein series

Theorem 4.4 combined with Example 4.3 gives the following list of singularities of $A(\phi, \phi')(\pi_\lambda)$ intersecting the positive chamber.

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1. Case $P = B_0$, $\pi = \omega \otimes \omega_1 \otimes \omega_2$.

$$\mathfrak{S}_1 = (\omega_1 = \omega_2, \lambda_1 = 1 + \lambda_2),\ \mathfrak{S}_2 = (\omega_2 = \mathbf{1}, \lambda_2 = 1),$$

$$\mathfrak{S}_3 = (\omega_1 = \omega_2^{-1}, \lambda_1 = 1 - \lambda_2),\ \mathfrak{S}_4 = (\omega_1 = \mathbf{1}, \lambda_1 = 1).$$

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Theorem 4.4 combined with Example 4.3 gives the following list of singularities of $A(\phi, \phi')(\pi_\lambda)$ intersecting the positive chamber.

1. Case $P = B_0$, $\pi = \omega \otimes \omega_1 \otimes \omega_2$.

$$\mathfrak{S}_1 = (\omega_1 = \omega_2, \lambda_1 = 1 + \lambda_2),\ \mathfrak{S}_2 = (\omega_2 = \mathbf{1}, \lambda_2 = 1),$$

$$\mathfrak{S}_3 = (\omega_1 = \omega_2^{-1}, \lambda_1 = 1 - \lambda_2),\ \mathfrak{S}_4 = (\omega_1 = \mathbf{1}, \lambda_1 = 1).$$

2. Case $P = P_1$, $\pi = \omega \otimes \tau$.

$$\mathfrak{S}_{E,\theta} = \omega_{E/F} |\cdot|_{\mathbb{A}} \otimes \pi(\theta) |\cdot|_{\mathbb{A}}^{-1/2}, \quad \left( \begin{array}{c} E/F \text{ ; quad. ext.} \\ \theta : \mathbb{A}_E^\times / E^\times \rightarrow \mathbb{C}^1 \end{array} \right)$$

quad. char. ass. to $E/F$

$\varphi_{\pi(\theta)} = \mathrm{ind}_{W_E}^{W_F} \theta$

# 4.5. Singularities of Eisenstein series

Theorem 4.4 combined with Example 4.3 gives the following list of singularities of $A(\phi, \phi')(\pi_\lambda)$ intersecting the positive chamber.

1. Case $P = B_0$, $\pi = \omega \otimes \omega_1 \otimes \omega_2$.

$$\mathfrak{S}_1 = (\omega_1 = \omega_2, \lambda_1 = 1 + \lambda_2),\ \mathfrak{S}_2 = (\omega_2 = \mathbf{1}, \lambda_2 = 1),$$

$$\mathfrak{S}_3 = (\omega_1 = \omega_2^{-1}, \lambda_1 = 1 - \lambda_2),\ \mathfrak{S}_4 = (\omega_1 = \mathbf{1}, \lambda_1 = 1).$$

2. Case $P = P_1$, $\pi = \omega \otimes \tau$.

$$\mathfrak{S}_{E,\theta} = \omega_{E/F}|\cdot|_{\mathbb{A}} \otimes \pi(\theta)|\cdot|_{\mathbb{A}}^{-1/2}, \quad \begin{pmatrix} E/F \text{ ; quad. ext.} \\ \theta : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^1 \end{pmatrix}$$

3. Case $P = P_2$, $\pi = \tau \otimes \omega$.

$$\mathfrak{S}_{\tau,\omega} = \tau|\det|_{\mathbb{A}} \otimes \omega|\cdot|_{\mathbb{A}}^{-1}, \quad (\omega_\tau = \mathbf{1}, L(1/2, \tau) \neq 0)$$

# 5. Inner product formula

In order to obtain $G(\mathbb{A})$-equivariant inner product formula from Cor.3.7, we need to move the integration axis to the unitary axis $\Re\lambda = 0$.

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In order to obtain $G(\mathbb{A})$-equivariant inner product formula from Cor.3.7, we need to move the integration axis to the unitary axis $\Re\lambda = 0$.

- $\phi, \phi' \in P_{(M,\pi)}$.
- $c_F := \mathrm{Res}_{s=1}\, \zeta_F(s)/\zeta_F(2)$.

In the cases $P = P_1, P_2$, usual residue theorem yields the following.

# 5.1. Proposition

*(i) Case* $P = P_1$.

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \int_{i\mathbb{R}} A(\phi, \phi')(\pi_\lambda)\, d\lambda$$

$$+ \frac{c_F L_E(1, \theta\sigma(\theta)^{-1})}{L_E(2, \theta\sigma(\theta)^{-1})\varepsilon_E(2, \theta\sigma(\theta)^{-1})} \langle N(w_1, \mathfrak{S}_{E,\theta})\phi_1, \phi_1' \rangle.$$

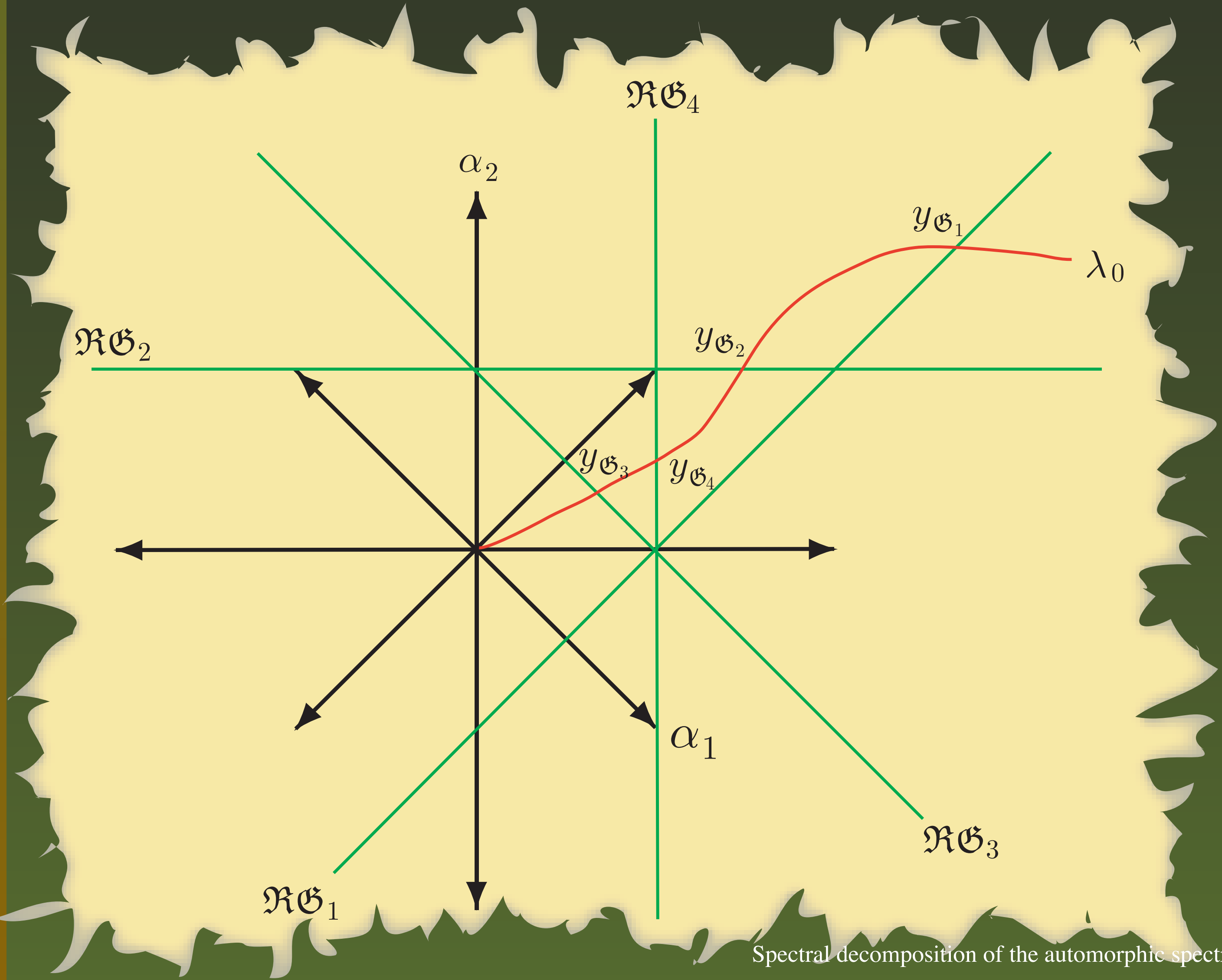
*(ii) Case* $P = P_2$.

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \int_{i\mathbb{R}} A(\phi, \phi')(\pi_\lambda)\, d\lambda$$

$$+ \frac{c_F L(1/2, \tau)}{\sqrt{2} L(3/2, \tau)\varepsilon(1/2, \tau)} \langle N(w_2, \mathfrak{S}_{\tau,\omega})\phi_1, \phi_1' \rangle.$$

# 5.2. Inner product in the case $P = B_0$

In the case $P = B_0$, we move the axis following the map:

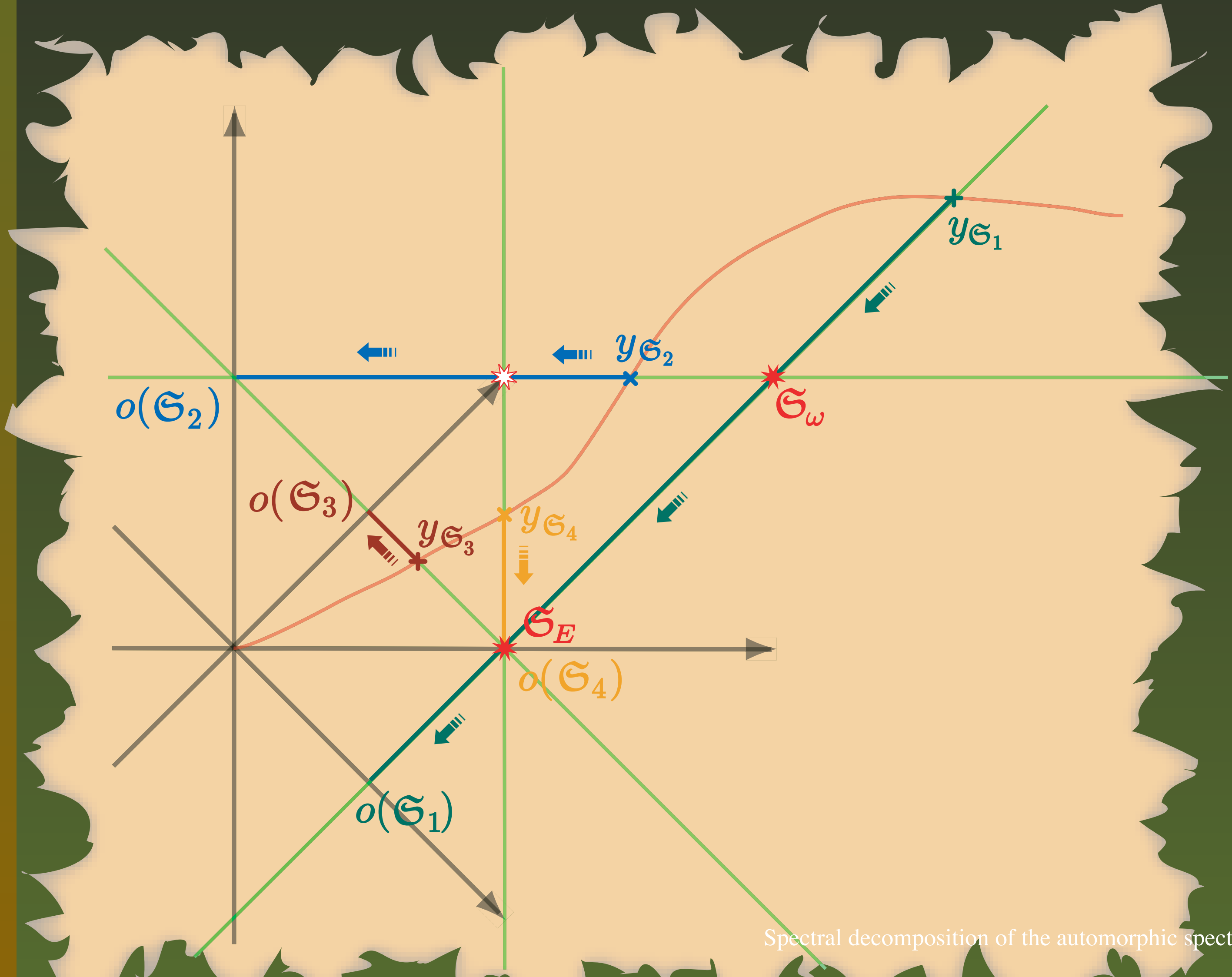


The result is

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \int_{i\mathfrak{a}_0^{G,*}} A(\phi,\phi')(\pi_\lambda)\, d\lambda + \sum_{j=1}^{4} \int_{y_{\mathfrak{S}_j}+i\mathfrak{a}_j^*} \mathrm{Res}_{\mathfrak{S}_j} A(\phi,\phi')(\pi_\lambda)\, d\lambda$$

- $\mathrm{Res}_{\mathfrak{S}_j}$ ; residue along $\mathfrak{S}_j$,
- $\mathfrak{a}_1 := \{(\lambda,\lambda) \in \mathfrak{a}_0^{G,*}\}$, $\mathfrak{a}_2 := \{(\lambda,0) \in \mathfrak{a}_0^{G,*}\}$, $\mathfrak{a}_3 := \{(\lambda,-\lambda) \in \mathfrak{a}_0^{G,*}\}$, $\mathfrak{a}_4 := \{(0,\lambda) \in \mathfrak{a}_0^{G,*}\}$.

Then move the axis $y_{\mathfrak{S}_j} + i\mathfrak{a}_j^*$ to $o(\mathfrak{S}_j) + i\mathfrak{a}_j^*$:



# 5.3. Proposition

The $L^2$-inner product in the case $P = B_0$ is given by

$$\langle \theta_\phi, \theta_{\phi'} \rangle = \int_{i\mathfrak{a}_0^{G,*}} A(\phi,\phi')(\pi_\lambda)\, d\lambda + \sum_{j=1}^{3} \int_{o(\mathfrak{S}_j)+i\mathfrak{a}_j^*} \mathrm{Res}_{\mathfrak{S}_j} A(\phi,\phi')(\pi_\lambda)\, d\lambda$$

$$+ \lim_{t\to 0} \int_{o(\mathfrak{S}_4)+i\mathfrak{a}_j^*} \frac{1}{2}\Big( \mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_{\lambda+t}) + \mathrm{Res}_{\mathfrak{S}_4} A(\phi,\phi')(\pi_{\lambda-t}) \Big)\, d\lambda$$

$$+ c_F^2 \langle N(s_2, w_1\mathfrak{S}_\omega) M(s_1s_2, s_1\mathfrak{S}_\omega) N(s_1, \mathfrak{S}_\omega)\phi_{\rho_{B_0}}, \phi'_{\rho_{B_0}} \rangle$$

$$+ c_F^2 \langle M(s_2, w_1\mathfrak{S}_E) N(s_1, s_2s_1\mathfrak{S}_E) M(s_2, s_1\mathfrak{S}_E) N(s_1, \mathfrak{S}_E)\phi_{(1,0)}, \phi'_{(1,0)} \rangle$$

- $\mathfrak{S}_\omega := |\cdot|_{\mathbb{A}}^2 \otimes |\cdot|_{\mathbb{A}} \otimes \omega|\cdot|_{\mathbb{A}}^{-3/2}$,
- $\mathfrak{S}_E := \omega_{E/F}|\cdot|_{\mathbb{A}} \otimes \omega_{E/F} \otimes \omega|\cdot|_{\mathbb{A}}^{-1/2}$.

# 6. Spectral decomposition

The corresponding residues of Eisenstein series are given by

- On $o(\mathfrak{S}_1)$, $o(\mathfrak{S}_3)$, the Siegel Eisenstein series

$$\operatorname{Res}_{\mathfrak{S}_j} E_{B_0}(\pi_{(1/2,-1/2)}) = E_{P_2}(I_{P_2}^G(\omega_{1,it}(\det))).$$

- On $o(\mathfrak{S}_2)$, $o(\mathfrak{S}_4)$, the Klingen Eisenstein series

$$\operatorname{Res}_{\mathfrak{S}_j} E_{B_0}(\pi_{(0,1)}) = E_{P_2}(I_{P_1}^G(\omega_{1,it} \otimes \omega_{-it/2}(\det))).$$

- At $\mathfrak{S}_\omega$, $\omega \circ \nu$ (automorphic character).
- At $\mathfrak{S}_E$, $J_{P_1}^G(\omega_{E/F,1} \otimes I_B^{GL_2}(\omega\omega_{E/F} \otimes \omega)_{-1/2})$.
  (unique irred. quotient of $I_{P_1}^G(-)$.)

# Case $P = P_1, P_2$

- At $\mathfrak{S}_{E,\theta}$, $J_{P_1}^G(\omega_{E/F,1} \otimes \pi(\theta)_{-1/2})$.
- At $\mathfrak{S}_{\tau,\omega}$, $J_{P_2}^G(\tau_1 \otimes \omega_{-1})$.

# Case $P = P_1, P_2$

- At $\mathfrak{S}_{E,\theta}$, $J_{P_1}^G(\omega_{E/F,1} \otimes \pi(\theta)_{-1/2})$.
- At $\mathfrak{S}_{\tau,\omega}$, $J_{P_2}^G(\tau_1 \otimes \omega_{-1})$.

Combining these with Prop.5.1, Prop.5.3, we obtain the final theorem.

# 6.1. Theorem

*We have a direct sum decomposition*
$\mathcal{L}(G) = \mathcal{L}_0(G) \oplus \mathcal{L}_{\text{res}}(G) \oplus \mathcal{L}_{\text{cont}}(G)$ *such that:*
*(i)* $\mathcal{L}_{\text{res}}(G)$ *is a direct sum of*

*(a)* 1-*dim. representations* $\omega \circ \nu$;

*(b)* $J_{P_1}^{G}(\omega_{E/F,1} \otimes I_B^{GL_2}(\omega\omega_{E/F} \otimes \omega)_{-1/2})$
$\hookrightarrow I_{P_2}^{G}(\omega_{E/F,-1/2}(\det) \otimes \omega_{1/2})$;

*(c)* $J_{P_1}^{G}(\omega_{E/F,1} \otimes \pi(\theta)_{-1/2})$;

*(d)* $J_{P_2}^{G}(\tau_1 \otimes \omega_{-1})$.

$E/F$ *; quad. ext.,* $\theta$, $\omega$ *; idele class char.s of* $E$, $F$, *resp.*

$\tau$ *; cusp. rep. of* $GL(2, \mathbb{A})$ *s.t.* $\omega_\tau = \mathbf{1}$, $L(1/2, \tau) \neq 0$.

*(ii)* $\mathcal{L}_{\mathrm{cont}}(G)$ *is a direct sum of*

*(a)* $\displaystyle\int_{i\mathfrak{a}_M^{G,*}} I_P^G(\pi_\lambda)\,d\lambda$, *for* $[M,\pi]$ *as before;*

*(b)* $\displaystyle\int_{i\mathbb{R}} I_{P_1}^G(\omega_{1,it}(\det)\otimes\omega_{-it})\,dt;$

*(c)* $\displaystyle\int_{i\mathbb{R}} I_{P_2}^G(\omega_{1,it}\otimes\omega_{-it/2}(\det))\,dt.$

# 6.2. $A$-parameters

We briefly discuss the $A$-parameters $\phi : L_F \times SL(2,\mathbb{C}) \rightarrow {}^L G$ associated to the repr.s in Th.6.1 (i).

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We briefly discuss the $A$-parameters $\phi : L_F \times SL(2, \mathbb{C}) \rightarrow {}^L G$ associated to the repr.s in Th.6.1 (i).

- $\phi$ can be written as $\phi = \bigoplus_{i=1}^{r} \varphi_{m_i} \otimes \rho_{d_i}$,

  $\varphi_{m_i} : L_F \longrightarrow GL(m_i, \mathbb{C})$ ; irred. repr.,
  $\rho_d$ ; irred. $d$-dim. repr. of $SL(2, \mathbb{C})$.
- $\{(m_i, d_i)\}_i$ ; Jordan block of $\phi$.

# 6.2. $A$-parameters

We briefly discuss the $A$-parameters $\phi : L_F \times SL(2,\mathbb{C}) \to {}^LG$ associated to the repr.s in Th.6.1 (i).

- $\phi$ can be written as $\phi = \bigoplus_{i=1}^{r} \varphi_{m_i} \otimes \rho_{d_i}$,

  $\varphi_{m_i} : L_F \longrightarrow GL(m_i, \mathbb{C})$ ; irred. repr.,
  $\rho_d$ ; irred. $d$-dim. repr. of $SL(2,\mathbb{C})$.
- $\{(m_i, d_i)\}_i$ ; Jordan block of $\phi$.
- $S_\phi(G)$:= $\mathrm{Cent}(\phi, {}^LG)$, $\mathcal{S}_\phi(G)$:= $\pi_0(S_\phi(G))$ control the endoscopy for $\Pi_\phi(G)$, the $A$-packet of $\phi$.

# Table

(i.a) Jordan block $\{(1,4)\}$: $\phi = \omega \otimes \rho_4$.

$$\mathcal{S}_\phi(G) = \pi_0(Z(\widehat{G})) = \{1\},\ \Pi_\phi(G) = \{\omega \circ \nu\}.$$

(i.c) $\{(2,2)\}$: $\phi = \omega_{E/F}\mathrm{ind}_{W_E}^{W_F}\theta \otimes \rho_2$.

$$\mathcal{S}_\phi(G) = \pi_0(Z(GO(2))) = \{1\},$$
$$\Pi_\phi(G) = \{J_{P_1}^G(\omega_{E/F,1} \otimes \pi(\theta)_{-1/2})\}.$$

(i.d) $\{(2,1),(1,2)\}$: $\phi = \omega(\det)\varphi_\tau \oplus (\omega \otimes \rho_2)$, $(\omega_\tau = \mathbf{1})$.

$$\mathcal{S}_\phi(G) = \pi_0(\{\mathrm{diag}(z, \pm z, z, \pm z) \in \widehat{G}\}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

$\Pi_\phi(G) \ni J_{P_2}^G(\tau_1 \otimes \omega_{-1})$ iff $L(1/2,\tau) \neq 0$.

This is the Saito-Kurokawa packet.

(i.b) $\{(1,2),(1,2)\}$: $\phi = (\omega \otimes \rho_2) \oplus (\omega\omega_{E/F} \otimes \rho_2)$.

$\mathcal{S}_\phi(G) \simeq \mathbb{Z}/2\mathbb{Z}$ as in (i.d).

$\Pi_\phi(G) \ni J_{P_1}^{G}(\omega_{E/F,1} \otimes I_B^{GL_2}(\omega\omega_{E/F} \otimes \omega)_{-1/2})$.

This is the Howe-PS (or $\theta_{10}$-type) packet.