CAP forms on $U(2,2)$ II. Cusp forms *

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Abstract

This is a report of our work on non-tempered automorphic representations of $U_{E/F}(2,2)$. Few years ago, we obtained a complete description of the local components of such automorphic forms. This time, we construct all the expected automorphic forms with these components.

Contents

1 Introduction to CAP forms

The term CAP is a short hand for the phrase "Cuspidal but Associated to Parabolic subgroups". This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. An up-to-date definition of CAP forms might be given as follows.

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Let G be a connected reductive group defined over a number field F. We write $\mathbb{A} = \mathbb{A}_F$ for the adele ring of F. By an *automorphic representation* of $G(\mathbb{A})$, we mean an *irreducible subquotient* of the right regular representation

$$R(g)\phi(x) = \phi(xg), \quad g \in G(\mathbb{A})$$

of $G(\mathbb{A})$ on the Hilbert space

$$L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) := \left\{ \begin{array}{c} \phi : G(\mathbb{A}) \to \mathbb{C} \\ \text{measurable} \end{array} \;\middle|\; \begin{array}{l} \text{(i) } \phi(\gamma a g) = \phi(g), \\ \quad \gamma \in G(F),\ a \in \mathfrak{A}_G,\ g \in G(\mathbb{A}) \\ \text{(ii) } \int_{G(F)\mathfrak{A}_G\backslash G(\mathbb{A})} |\phi(g)|^2 \, dg < \infty \end{array} \right\}.$$

Here, $\mathfrak{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in the center $Z(G)(\mathbb{A})$ of $G(\mathbb{A})$ and the measure is taken to be $G(\mathbb{A})$-invariant. The *discrete spectrum* $L^2_{\text{disc}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is the maximum subspace of $L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ which is a direct sum of irreducible subrepresentations. Further this decomposes as

$$L^2_{\text{disc}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) = L^2_0(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) \oplus L^2_{\text{res}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})).$$

Here $L^2_0(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is the completion of the space of cusp forms with respect to the Petersson (*i.e.*, L^2-) norm and called the *cuspidal spectrum*. On the other hand, $L^2_{\text{res}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is spanned by certain iterated residues of Eisenstein series

$$\operatorname{Res}_{\lambda=\mathfrak{s}} E_P^G(\phi), \quad \phi \in \operatorname{Ind}_P^G(\tau_\lambda), \tau \subset L^2_0(M(F)\mathfrak{A}_M\backslash M(\mathbb{A})),$$

where $P = MU \subset G$ is a proper parabolic subgroup. We observe that

- Let us write $t(\tau_v)$ for the *Hecke* (formerly called *Satake*) *matrix* of τ at any unramified place v for M and τ. Then the Hecke matrix for the residue $\operatorname{Res}_{\lambda=\mathfrak{s}} E_P^G(\tau_\lambda)$ is $q_v^{-\mathfrak{s}} t(\tau_v)$. Here q_v is the cardinality of the residue field of F_v.

- According to Langlands' criterion for square integrability, we must have $\Re\varpi^\vee(\mathfrak{s}) > 0$ for any "fundamental coweight" ϖ for P.

In particular, even if τ satisfies the Ramanujan conjecture for M (*i.e.*, $t(\tau_v)^{\mathbb{Z}}$ is bounded), any residue $\operatorname{Res}_{\lambda=\mathfrak{s}} E_P^G(\tau_\lambda)$ in the discrete spectrum cannot satisfy the same conjecture for G.

Now let G^* be the quasisplit inner form of G. At almost all places v of F, $G_v := G \otimes_F F_v$ is isomorphic to G^*_v.

Definition 1.1. *An irreducible cuspidal representation $\pi = \bigotimes_v \pi_v \subset L^2_0(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$ is a* CAP form *if there exists an irreducible residual automorphic representation $\pi^* = \bigotimes_v \pi^*_v \subset L^2_{\text{res}}(G^*(F)\mathfrak{A}_G\backslash G^*(\mathbb{A}))$ of $G^*(\mathbb{A})$ such that the absolute values of the eigenvalues of the Hecke matrices $t(\pi_v)$ and $t(\pi^*_v)$ coincide at almost all v.*

Example 1.2. *(i) Combining the results of Jacquet-Shalika [JS81b], [JS81a] and Moeglin-Waldspurger [MW89], one finds that there are no CAP forms on $G = GL(n)$.*
(ii) If $G = D^\times$, the unit group of a central division algebra over F, the trivial representation $\mathbb{1}_{G(\mathbb{A})}$ is a CAP form.

(iii) The CAP forms on $U_{E/F}(3)$ (any unitary group in 3 variables) are the θ-liftings of automorphic characters on $U_{E/F}(1, \mathbb{A})$ [GR90], [GR91].
(iv) The CAP forms on $Sp(2)$ are either the Saito-Kurokawa liftings (θ-liftings of automorphic representations of the metaplectic cover $\widetilde{SL(2, \mathbb{A})}$) or the θ_{10}-type representations constructed by Howe-Piatetski-Shapiro [PS83] (θ-liftings of automorphic representations of various orthogonal groups in 2-variables). It is expected but I do not know if these two families are disjoint.
(v) Some CAP forms on the split exceptional group of type G_2 are studied by Gan-Gurevich-Jiang [GGJ02].
(vi) The Ikeda lift on $Sp(2n)$ and the Miyawaki lift on $Sp(3)$ [Ike01] are CAP forms.

Besides its importance as counter examples to the Ramanujan conjecture, we propose the following three motivation of studying CAP forms.

- Construct and explicitly describe certain mixed motives associated to Shimura varieties. This point of view is discussed in detail in [Har93].
- Capture some periods of automorphic forms. This is related to the Ikeda-Ichino conjecture.
- Construct unipotent and other singular supercuspidal representations of p-adic groups.

In 2003, we have described the expected local components of the CAP forms of the quasisplit unitary group $U_{E/F}(2, 2)$ in 4-variables [KKa]. In this talk, we construct the cusp forms with those local components.

2 A-parameters

In order to put non-tempered automorphic forms into the framework of Langlands' conjecture, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is given through the *A-parameters*. On the other hand, these parameters are not well related to the definition1.1 of CAP forms, because the Ramanujan conjecture is not yet established for any non-abelian reductive group G. In order to obtain a nice framework to study CAP forms, it is best to introduce the following *ad hoc* notion of A-parameters for unitary groups.

Let E/F be a quadratic extension of number fields, and write σ for the generator of $\mathrm{Gal}(E/F)$. We fix an algebraic closure $\bar{F}$ of E (or F) and write W_F (resp. W_E) for the Weil group of $\bar{F}/F$ (resp. $\bar{F}/E$). Recall the (non-split) extension $1 \to W_E \to W_F \to \mathrm{Gal}(E/F) \to 1$. We fix an inverse image $w_\sigma \in W_F$ of σ.

First we consider the group $H_n := \mathrm{Res}_{E/F} GL(n)$. Its *$L$-group* is given by ${}^L H_n = \widehat{H}_n \rtimes_{\rho_{H_n}} W_F$ with $\widehat{H}_n = GL(n, \mathbb{C})^2$ and

$$\rho_{H_n}(w)(h_1, h_2) = \begin{cases} (h_1, h_2) & \text{if } w \in W_E, \\ (h_2, h_1) & \text{otherwise.} \end{cases}$$

We write $\Phi_0(H_n)$ for the set of (isomorphism classes of) irreducible unitary cuspidal representations of $H_n(\mathbb{A})$. Conjecturally, this should be in 1-1 correspondence with the set of isomorphism classes of irreducible n-dimensional representations with bounded image of the hypothetical

Langlands group $\mathcal{L}_E$ of E. We adopt this latter point of view, since it is convenient for some observations. There should be a natural morphism $p_{W_F} : \mathcal{L}_F \to W_F$. As in the Weil group case, $\mathcal{L}_F$ should be an extension $1 \to \mathcal{L}_E \to \mathcal{L}_F \to \mathrm{Gal}(E/F) \to 1$. Again we take an inverse image $w_\sigma \in \mathcal{L}_F$ of the above fixed $w_\sigma \in W_F$. By [Bor79, Prop.8.4], each $\varphi_E \in \Phi_0(H_n)$ is identified with the homomorphism $\varphi : \mathcal{L}_F \to {}^L H_n$ given by

$$\varphi(w) := \begin{cases} (\varphi_E(w), \varphi_E(w_\sigma w w_\sigma^{-1})) \times p_{W_F}(w) & \text{if } w \in \mathcal{L}_E, \\ (\varphi_E(w w_\sigma^{-1}), \varphi_E(w_\sigma w)) \rtimes p_{W_F}(w) & \text{otherwise.} \end{cases} \tag{2.1}$$

Definition 2.1. *An A-*parameter *for H_n is a homomorphism $\phi : \mathcal{L}_F \times SL(2, \mathbb{C}) \to {}^L H_n$ such that*
(i) $\phi|_{SL(2,\mathbb{C})} : SL(2, \mathbb{C}) \to \widehat{H}_n$ is analytic.
(ii) $\mathcal{L}_F \xrightarrow{\phi} {}^L H_n \xrightarrow{\mathrm{proj}} W_F$ coincides with $p_{W_F} : \mathcal{L}_F \to W_F$. Thus ϕ is determined by the representation $\phi_E : \mathcal{L}_E \times SL(2, \mathbb{C}) \xrightarrow{\phi} {}^L H_n \xrightarrow{\text{1st. proj.}} GL(n, \mathbb{C})$ (under (2.1)*).*
(iii) ϕ_E is semisimple, so that we have an irreducible decomposition $\phi_E \simeq \bigoplus_{i=1}^r \varphi_{i,E} \otimes \rho_{d_i}$. Here, $\varphi_{i,E}$ is an m_i-dimensional irreducible representation of $\mathcal{L}_E$ and ρ_d denotes the d-dimensional irreducible representation of $SL(2, \mathbb{C})$. Note $\sum_{i=1}^r d_i m_i = n$.
(iv) $\varphi_{i,E} \in \Phi_0(H_{m_i})$.
A-parameters ϕ, ϕ' for H_n are equivalent *if they are $\widehat{H}_n$-conjugate, or equivalently, if ϕ_E and ϕ'_E are isomorphic. An A-parameter ϕ contributes to the discrete spectrum if and only if it is* elliptic*, i.e., ϕ_E is irreducible.*

Now we turn to the quasisplit unitary group $G = G_n$ in n-variables for E/F. This can be realized in such a way that

$$G_n(R) := \{g \in \mathbb{M}_n(R \otimes_F E)^\times \mid \theta_n(g) = \sigma(g)\},$$

for any abelian F-algebra R. Here $\theta_n(g) := \mathrm{Ad}(I_n)^t g^{-1}$ with

$$I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

The L-group ${}^L G_n = \widehat{G}_n \rtimes_{\rho_{G_n}} W_F$ is given by $\widehat{G}_n = GL(n, \mathbb{C})$ and

$$\rho_{G_n}(w) = \begin{cases} \mathrm{id} & \text{if } w \in W_E, \\ \theta_n & \text{otherwise.} \end{cases}$$

Definition 2.2. *An A-*parameter *for G is a homomorphism $\phi : \mathcal{L}_F \times SL(2, \mathbb{C}) \to {}^L G$ such that*

(BC) $\phi_E : \mathcal{L}_E \times SL(2, \mathbb{C}) \xrightarrow{\phi} {}^L G_n \xrightarrow{\text{1st. proj.}} GL(n, \mathbb{C})$ coincides with ϕ_E^H for some A-parameter ϕ^H for H_n.

Two A-parameters are equivalent *if they are $\widehat{G}$-conjugate. Let $\Psi(G)$ be the set of equivalence classes of A-parameters for G. For an A-parameter ϕ, we write $S_\phi(G)$ for the centralizer of $\phi(\mathcal{L}_F \times SL(2,\mathbb{C}))$ in $\widehat{G}$, and $\mathcal{S}_\phi(G)$ for the group of connected components of $S_\phi(G)/Z(\widehat{G})^{\mathrm{Gal}(\bar{F}/F)}$. $\phi \in \Psi(G)$ is called* elliptic *if the identity component $S_\phi(G)^0$ of $S_\phi(G)$ is contained in $Z(\widehat{G})^{\mathrm{Gal}(\bar{F}/F)}$. We write $\Psi_0(G)$ for the subset elliptic classes in $\Psi(G)$. An elliptic ϕ is of* CAP-type *if $\phi|_{SL(2,\mathbb{C})}$ is non-trivial. We write $\Psi_{\mathrm{CAP}}(G)$ for the set of classes of CAP-type in $\Psi_0(G)$.*

An elementary exercise in representation theory shows that each $\phi \in \Phi_0(G_n)$ can be written as

$$\phi_E \simeq \bigoplus_{i=1}^{r} \xi_i \cdot \varphi_{i,E} \otimes \rho_{d_i} \tag{2.2}$$

where,

- $\varphi_i \in \Psi(G_{m_i})$ is such that $\varphi_{i,E}|_{\mathcal{L}_E}$ is irreducible;
- ξ_i is an idele class character of E such that $\xi_i|_{\mathbb{A}^\times} = \omega_{E/F}^{n-d_i-m_i+1}$. Here $\omega_{E/F}$ is the quadratic character of $\mathbb{A}^\times/F^\times$ associated to E/F by the classfield theory.
- $\xi_i \cdot \varphi_{i,E} \not\simeq \xi_j \cdot \varphi_{j,E}$, $(1 \le i \ne j \le r)$.

Thus it suffices to describe the set

$$\Phi_{\mathrm{st}}(G_m) := \{\varphi \in \Psi_0(G_m) \,|\, \varphi_E|_{\mathcal{L}_E} \text{ is irreducible}\}.$$

For $\varphi \in \Phi_{\mathrm{st}}(G_m)$, φ_E viewed as a parameter for H_m corresponds to a cuspidal automorphic representation π_E of $H_m(\mathbb{A})$. According to Langlands' functoriality conjecture, the map $\varphi \mapsto \varphi_E$ corresponds to the *standard base change* lifting from $G_m(\mathbb{A})$ to $H_m(\mathbb{A})$ [Rog90]. Hence the description of $\Phi_0(G_m)$ amounts to that of the image of the standard base change. As for this question, the following expectation is well-known.

Conjecture 2.3. *Let π_E be an irreducible cuspidal representation of $H_m(\mathbb{A})$ and $\varphi^H : \mathcal{L}_F \to {}^L H_m$ be its Langlands parameter. Take an idele class character μ of E such that $\mu|_{\mathbb{A}^\times} = \omega_{E/F}$. Then $\varphi_E^H = \varphi_E$ for some $\varphi \in \Phi_{\mathrm{st}}(G_m)$ (i.e., π_E is the standard base change lift of some stable L-packet of $G_m(\mathbb{A})$) if and only if*

(i) $\sigma(\pi_E) := \pi_E \circ \sigma \simeq \pi_E^\vee$ (the contragredient);

(ii) the twisted tensor L-function $L_{\mathrm{Asai}}(s, \mu^{n+1}(\det)\pi_E)$ [Gol94] has a pole at $s = 1$.

Using the base change for $GU_{E/F}(2)$, we deduced the case $m = 2$ of the conjecture from [HLR86, Th.3.12] ([KKa, Cor.3.3]). This avails us to deduce the following description of $\Psi_{\mathrm{CAP}}(G_4)$ from (2.2). Note that this does not involve the hypothetical Langlands group $\mathcal{L}_F$ anymore.

Proposition 2.4. *The set $\Psi_{\mathrm{CAP}}(G_4)$ consists of the following classes. We write η, μ for typical idele class characters of E such that $\eta|_{\mathbb{A}^\times} = \mathbb{1}$, $\mu|_{\mathbb{A}^\times} = \omega_{E/F}$, respectively.*

Name	ϕ_E	$\{(d_i, m_i)\}$	$\mathcal{S}_\phi(G)$
(1.a) ϕ_η	$\eta \otimes \rho_4$	$\{(4,1)\}$	$\{1\}$
(1.b) $\phi_{\pi_E,\mu}$	$\mu\varphi_{\pi_E} \otimes \rho_2$	$\{(2,2)\}$	$\{1\}$
(2.a) $\phi_{\underline{\mu}}$	$(\mu \otimes \rho_3) \oplus \mu'$	$\{(3,1),(1,1)\}$	$\mathbb{Z}/2\mathbb{Z}$
(2.b) $\phi_{\pi_E,\eta}$	$(\eta \otimes \rho_2) \oplus \varphi_{\pi_E}$	$\{(2,1),(1,2)\}$	$\mathbb{Z}/2\mathbb{Z}$
(2.c) $\phi_{\underline{\eta}}$	$(\eta \otimes \rho_2) \oplus (\eta' \otimes \rho_2)$	$\{(2,1),(2,1)\}$	$\mathbb{Z}/2\mathbb{Z}$
(2.d) $\phi_{\eta,\underline{\mu}}$	$(\eta \otimes \rho_2) \oplus \mu \oplus \mu'$	$\{(2,1),(1,1),(1,1)\}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Here, in (1.b), (2.b), π_E runs over the set of irreducible cuspidal automorphic representation of $H_2(\mathbb{A})$ such that $\sigma(\pi_E) \simeq \pi_E^\vee$ and $L_{\mathrm{Asai}}(s, \pi_E)$ is holomorphic at $s = 1$. In (2.a) $\underline{\mu} = (\mu, \mu')$ where μ' can be μ. In (2.c) $\underline{\eta} = (\eta, \eta')$ modulo permutation, with $\eta \neq \eta'$. Finally, in (2.d) $\underline{\mu} = (\mu, \mu')$ modulo permutation and $\mu \neq \mu'$.

3 Review of the local theory

Let ϕ be an A-parameter for $G = G_4$. By restriction, we obtain the local component

$$\phi_v : \mathcal{L}_{F_v} \times SL(2, \mathbb{C}) \to {}^L G_v$$

of ϕ at each place v of F. Here the local Langlands group $\mathcal{L}_{F_v}$ is given by $W_{F_v} \times SU(2, \mathbb{R})$ if v is non-archimedean and W_{F_v} otherwise [Kot84, §12]. ${}^L G_v$ is the L-group of the scalar extension $G_v = G \otimes_F F_v$. Arthur's local conjecture, among other things, associates to each ϕ_v a finite set $\Pi_{\phi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an *A-packet*. At all but finite number of v, $\Pi_{\phi_v}(G_v)$ is expected to contain a unique unramified element π_v^1. Using such elements, we can form the global A-packet associated to ϕ:

$$\Pi_\phi(G) := \left\{ \bigotimes_v \pi_v \;\middle|\; \begin{array}{ll} \text{(i)} & \pi_v \in \Pi_{\phi_v}(G_v),\ \forall v; \\ \text{(ii)} & \pi_v = \pi_v^1,\ \forall' v \end{array} \right\}.$$

It is conjectured that any CAP-form on G is contained in $\Pi_\phi(G)$ for some $\phi \in \Psi_{\mathrm{CAP}}(G)$. Thus our problem can be stated as follows.

Problem 3.1. *(i) Describe $\Pi_\phi(G)$ (or equivalently, its local components $\Pi_{\phi_v}(G_v)$.*
(ii) Describe the multiplicity of each $\pi \in \Pi_\phi(G)$ in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$. (Note $\mathfrak{A}_G = \{1\}$ for the unitary group G.)

Example 3.2. *The A-packets associated to some of the parameters listed in Prop.2.4 can be easily described.*

(1.a) For ϕ_η, we have $\Pi_\phi(G) = \{\eta_G := \eta_u(\det)\}$, where $\eta_u : U_{E/F}(1, \mathbb{A}) \ni z/\sigma(z) \mapsto \eta(z) \in \mathbb{C}^\times$.

(1.b) For $\phi_{\pi_E,\mu}$, $\Pi_\phi(G)$ consists of the unique irreducible quotient $J_P^G(\mu(\det)\pi_E|\det|_{\mathbb{A}_E}^{1/2})$, of the global parabolically induced representation from the Siegel parabolic subgroup $P = MU$.

(2.a) For $\phi_{\underline{\mu}}$, $\Pi_\phi(G)$ consists of the θ-lifting $\theta_\mu((\mu/\mu')_u, W)$ of the automorphic character $(\mu/\mu')_u$ of $U_{E/F}(1, \mathbb{A})$.

In particular, no CAP forms occur in these cases. All of these representations are known to occur in the residual discrete spectrum [Kon98]. Hence from now on, we concentrate on the rest cases (2.b–d).

Local A-packets Now let E/F be a quadratic extension of non-archimedean local fields of characteristic zero. We also have corresponding results in the archimedean case, but we need some extra notation to state them. Let ϕ be (local analogue of) an A-parameter of type (2.b–2.d). In [KKa], we have constructed $\Pi_\phi(G)$ by the local θ-correspondence. Let us briefly recall the construction. First note that ϕ can be written in the form

$$\phi_E = \varphi_{\pi_E} \oplus (\eta \otimes \rho_2). \tag{3.1}$$

Here $\varphi_{\pi_E} : \mathcal{L}_E \to GL(2, \mathbb{C})$ corresponds to an irreducible admissible representation π_E of $H_2(F) = GL(2, E)$ under the local Langlands correspondence [HT01], [Kut80]. Also notice that $\mathcal{S}_\phi(G) = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For a 2-dimensional hermitian space $(V, (\cdot,\cdot))$, we write G_V for its unitary group. $(W, \langle\cdot,\cdot\rangle) = (W_n, \langle\cdot,\cdot\rangle_n)$ denotes the hyperbolic skew-hermitian space of dimension $2n$, so that $G = G_4$ is the unitary group G_{W_2} of W_2. Fix a character pair $\underline{\xi} = (\mathbb{1}, \eta)$ of $E^\times$ such that $\eta|_{F^\times} = \mathbb{1}$, and a non-trivial character $\psi_F : F \to \mathbb{C}^\times$. These specify the Weil representation $\omega_{V,W,\underline{\xi}} = \omega_{W,\mathbb{1}} \times \omega_{V,\eta}$ of $G_V(F) \times G_W(F)$. As usual, this determines the *local θ-correspondence*

$$\mathscr{R}(G_V, \omega_{W,\mathbb{1}}) \ni \begin{matrix} \pi_V & \longmapsto & \theta_{\underline{\xi}}(\pi_V, W) \\ \theta_{\underline{\xi}}(\pi_W, V) & \longleftarrow & \pi_W \end{matrix} \in \mathscr{R}(G_W, \omega_{V,\eta})$$

between certain subsets $\mathscr{R}(G_V, \omega_{W,\mathbb{1}}) \subset \Pi(G_V(F))$, $\mathscr{R}(G_W, \omega_{V,\eta}) \subset \Pi(G_W(F))$. Here $\Pi(G_V(F))$ denotes the set of isomorphism classes of irreducible admissible representations of $G_V(F)$.

Definition 3.3. *In the notation of 3.1, let $\Pi_{\eta\pi_E^\vee}(G_V)$ be the L-packet of $G_V(F)$ whose standard base change to $H_2(F)$ is $\eta(\det)\pi_E^\vee$. (Empty if V is anisotropic and π_E is in the principal series.) We define*

$$\Pi_\phi(G) := \coprod_V \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_V), W),$$

where V runs over the set of isometry classes of 2-dimensional hermitian space over E.

4 Presentation of the problem

We now go back to the global setting. Let ϕ be an A-packet of type (2.b–d) in Prop.2.4. Having defined the local A-packets, we have the global packet $\Pi_\phi(G) = \bigotimes_v \Pi_{\phi_v}(G_v)$. In the present case, the multiplicity formula in Arthur's conjecture is stated as follows.

Conjecture 4.1. *There exists a pairing $\langle\cdot,\cdot\rangle : \mathcal{S}_{\phi_v}(G_v) \times \Pi_{\phi_v}(G_v) \to \{\pm 1\}$ such that the multiplicity of $\pi = \bigotimes_v \pi_v \in \Pi_\phi(G)$ in $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$ is given by*

$$m(\pi) := \frac{1}{|\mathcal{S}_\phi(G)|} \sum_{\boldsymbol{s} \in \mathcal{S}_\phi(G)} \epsilon_\phi(\boldsymbol{s}) \prod_v \langle \boldsymbol{s}, \pi_v \rangle.$$

Here, ϵ_ϕ is the sign character of the first $\mathbb{Z}/2\mathbb{Z}$ of $\mathcal{S}_\phi(G)$ if $\varepsilon(1/2, \pi_E \times \eta^{-1}) = -1$, and is the trivial character otherwise.

Our main result states that $m(\pi)$ is equal to or larger than the right hand side of the conjectural formula. But this makes sense only after the pairing $\langle\cdot,\cdot\rangle : \mathcal{S}_{\phi_v}(G_v)\times\Pi_{\phi_v}(G_v)\to\{\pm 1\}$ is described.

Pairing in the stable case The pairing $\langle\cdot,\cdot\rangle : \mathcal{S}_{\phi_v}(G_v)\times\Pi_{\phi_v}(G_v)\to\{\pm 1\}$ is given locally as the notation indicates. Thus we may go back to the local non-archimedean situation of §3. First we recall some basic requirements on $\Pi_\phi(G)$ from [Art89].

(i) For $\phi\in\Psi(G)$, we have a Langlands' parameter

$$\varphi_\phi : \mathcal{L}_F \ni w \longmapsto \phi\Big(w, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2}\end{pmatrix}\Big) \rtimes p_{W_F}(w) \in {}^L G,$$

where $|\cdot|_F$ is the transport of the module of F by the reciprocity isomorphism $F^\times \xrightarrow{\sim} W_{F,\mathrm{ab}}$ (or its composite with $\mathcal{L}_F \overset{p_{W_F}}{\twoheadrightarrow} W_F \twoheadrightarrow W_{F,\mathrm{ab}}$). Then the associated L-packet $\Pi_{\varphi_\phi}(G)$ should be contained in $\Pi_\phi(G)$.

(ii) More precisely, there exists a parabolic subgroup $P_\phi = M_\phi U_\phi$ such that $\phi(\mathcal{L}_F)\subset {}^L M_\phi$ and

$$\mu_\phi : W_F \ni w \longmapsto \phi\Big(1, \begin{pmatrix} |w|_F^{1/2} & \\ & |w|_F^{-1/2}\end{pmatrix}\Big) \in {}^L G$$

is a P_ϕ-dominant element of $\mathfrak{a}^*_{M_\phi} = (\mathrm{Lie}\mathfrak{A}_{M_\phi})^*$. Then $\Pi_{\varphi_\phi}(G) = \{J^G_{P_\phi}(\pi\otimes e^{\mu_\phi})\,|\,\pi\in\Pi_{\phi|_{\mathcal{L}_F}}(M_\phi)\}$, where $J^G_{P_\phi}(\pi\otimes e^{\mu_\phi})$ is the "Langlands' quotient[1]" of the standard parabolically induced representation $I^G_{P_\phi}(\pi\otimes e^{\mu_\phi})$. Now let us fix a Borel subgroup $\boldsymbol{B} = \boldsymbol{TU}$ and a non-degenerate character $\psi_{\boldsymbol{U}}$ of $\boldsymbol{U}(F)$. According to the *generic packet conjecture*, $\Pi_{\phi|_{\mathcal{L}_F}}(M_\phi)$ contains a unique generic representation π_1 with respect to $\psi_{\boldsymbol{U}}|_{(\boldsymbol{U}\cap M_\phi)(F)}$. Then, the pairing between $\Pi_\phi(G)$ and $\Pi(\mathcal{S}_\phi(G))$ should be chosen in such a way that $\langle J^G_{P_\phi}(\pi_1\otimes e^{\mu_\phi}),\cdot\rangle$ is the trivial character of $\mathcal{S}_\phi(G)$.

(iii) The following diagram should commute.

$$\begin{array}{ccc} \Pi_{\varphi_\phi}(G)\ni J^G_{P_\phi}(\tau\otimes e^{\mu_\phi}) & \longrightarrow & \langle\cdot,\tau\rangle\in\Pi(\mathcal{S}_{\phi|_{\mathcal{L}_F}}(M_\phi)) \\ \text{inclusion}\big\downarrow & & \big\downarrow\text{inclusion} \\ \Pi_\phi(G)\ni\pi & \longrightarrow & \langle\cdot,\pi\rangle\in\Pi(\mathcal{S}_\phi(G)). \end{array}$$

Going back to ϕ of type (2.b–d), the construction of the local packet $\Pi_\phi(G)$ involved the following, so-called ε*-dichotomy* property of the local θ-correspondence. Recall that there are only two isometry classes of 2-dimensional hermitian space V over E. They are classified by the signature $\omega_{E/F}(-\det V)$.

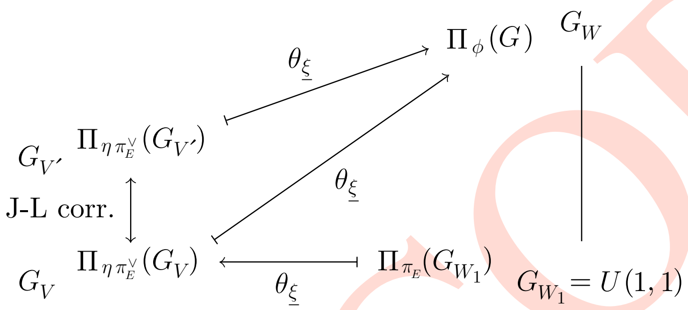
Theorem 4.2 ([KKa] Th.6.4)**.** *We adopt the notation of Def.3.3. The local* θ*-correspondent* $\theta_{\underline{\xi}}(\Pi_{\pi_E}(G_2),V)$ *of the L-packet* $\Pi_{\pi_E}(G_2)$ *to* $G_V(F)$ *is the L-packet* $\Pi_{\eta\pi_E^\vee}(G_V)$ *if*

$$\varepsilon(1/2,\pi_E\times\eta^{-1},\psi_E)\omega_{\Pi_{\pi_E}(G_2)}(-1) = \omega_{E/F}(-\det V),$$

[1] Again not precisely, because π is not always tempered in our definition of A-parameters.

and is zero otherwise. Here $\psi_E := \psi_F \circ \mathrm{Tr}_{E/F}$ *and* $\varepsilon(s, \pi_E \times \eta^{-1}, \psi_E)$ *is the Jacquet-Langlands local constant of* $\pi_E \times \eta^{-1}$. *Also* $\omega_{\Pi_{\pi_E}(G_2)}$ *denotes the common central character of the members of* $\Pi_{\pi_E}(G_2)$.

If we write V for the (isometry class of the) 2-dimensional hermitian space over E satisfying the condition of Th.4.2 and V' for the other one, the construction of $\Pi_\phi(G)$ is summerized in the following diagram.



Moreover, the *induction principle* of the local θ-correspondence [Kud86], [MVW87, Ch.3] shows that $\Pi_{\varphi_\phi}(G) = \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_V), W_2)$. This together with the requirement (iii) above yield the following.

Theorem 4.3. *Suppose* $\Pi_{\pi_E}(G_2)$ *is* stable*, i.e., consists of a single element, so that* $\mathcal{S}_\phi(G) \simeq \mathbb{Z}/2\mathbb{Z}$. *Then we have*

$$\langle \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_V), W), \cdot \rangle = \mathrm{sgn}, \quad \langle \theta_{\underline{\xi}}(\Pi_{\eta\pi_E^\vee}(G_{V'}), W), \cdot \rangle = \mathbb{1},$$

where V *and* V' *are labeled as above.*

5 Endoscopy for $U_{E/F}(2)$

It remains to consider the case where $\Pi_{\pi_E}(G_2)$ is *endoscopic*. This is the case (2.d) in Prop.2.4 (see [KKb, 4.3]):

$$\varphi_E = \mu \oplus \mu', \quad \pi_E = I(\mu \otimes \mu').$$

We write $\Pi_{\underline{\mu}}(G_V) := \Pi_{\pi_E}(G_V) = \{\pi_V(\underline{\mu})^\pm\}$ with $\underline{\mu} = (\mu, \mu')$.

We briefly recall the endoscopic lifting for G_V from [KKb]. The unique non-trivial *elliptic endoscopic data* for G_V is $(H, {}^LH, s, \xi)$, where $H = U_{E/F}(1)^2$, $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\xi : {}^LH \hookrightarrow {}^LG_2$ is the L-embedding given by

$$\xi : \begin{array}{rcl} \widehat{H} \ni (z_1, z_2) & \longmapsto & \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \times 1 \\ W_E \ni w & \longmapsto & \begin{pmatrix} \mu_0(w) & 0 \\ 0 & \mu_0'(w) \end{pmatrix} \times w \\ w_\sigma & \longmapsto & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rtimes w_\sigma \end{array} \in {}^LG_2.$$

Here $\underline{\mu}_0 = (\mu_0, \mu_0')$ are characters of $E^\times$ such that $\mu_0|_{F^\times} = \mu_0'|_{F^\times} = \omega_{E/F}$. The isomorphism class of the data is independent of $\underline{\mu}_0$.

We fix a generator δ of E over F such that $\mathrm{Tr}_{E/F}(\delta)=0$, and take $\varepsilon\in F^\times\smallsetminus \mathrm{N}_{E/F}(E^\times)$. We may realize $(V,(\cdot,\cdot))$ as $V=E^2$ and

$$(v,v')=\begin{cases} {}^t\sigma(v)\begin{pmatrix}0 & (2\delta)^{-1}\\ -(2\delta)^{-1} & 0\end{pmatrix}v' & \text{if } V \text{ is hyperbolic,}\\ {}^t\sigma(v)\begin{pmatrix}-\varepsilon & 0\\ 0 & 1\end{pmatrix}v' & \text{if } V \text{ is anisotropic.}\end{cases}$$

Then we fix an embedding

$$\eta_V: H\ni\gamma_H=(zz',z\sigma(z'))\longmapsto\begin{cases} z\begin{pmatrix}x & -y\varDelta\\ y & x\end{pmatrix}\in G_V & \text{if } V \text{ is hyperbolic,}\\ \begin{pmatrix}zz' & 0\\ 0 & z\sigma(z')\end{pmatrix}\in G_V & \text{if } V \text{ is anisotropic.}\end{cases}$$

Here, each element $\gamma_H\in H$ is written as $(zz',z\sigma(z'))$ for some $z,z'\in\mathrm{Res}_{E/F}\mathbb{G}_m$ with $\mathrm{N}_{E/F}(z)=\mathrm{N}_{E/F}(z')^{-1}$ and $\varDelta:=-\delta^2$. These data together with the non-trivial character ψ_F in §3 determines the *Langlands-Shelstad transfer factor* $\Delta_V: H(F)_{G\text{-reg}}\times G_V(F)_{\mathrm{reg}}\to\mathbb{C}$. This is characterized by the formula

$$\Delta_V(\gamma_H,\eta_V(\gamma_H))=\lambda(E/F,\psi_F)\omega_{E/F}\Big(\frac{z'-\sigma(z')}{-2\delta}\Big)\mu_0(x_1)\mu_0'(x_2)\frac{|z'-\sigma(z')|_E^{1/2}}{|z'|_E^{1/2}}. \tag{5.1}$$

Here $\lambda(E/F,\psi_F)$ is *Langlands' λ-factor* for E/F with respect to ψ_F, and we have written $zz'=x_1/\sigma(x_1)$, $z\sigma(z')=x_2/\sigma(x_2)$ for some $x_1,x_2\in E^\times$.

Fact 5.1 (Labesse-Langlands, [KKb] Ch.3)**.** *For any $f\in C_c^\infty(G_V(F))$,*

$$f^H: H(F)_{G\text{-}reg}\ni\gamma_H\longmapsto\sum_{\substack{\gamma\in\mathrm{Ad}(G_V(\bar F))\eta_V(\gamma_H)\cap G_V(F)\\ mod.\ G_V(F)\text{-}conj.}}\Delta_V(\gamma_H,\gamma)O_\gamma(f)\in\mathbb{C}$$

extends to an element of $C_c^\infty(H(F))$. Here $O_\gamma(f)$ denotes the orbital integral *of f at γ.*

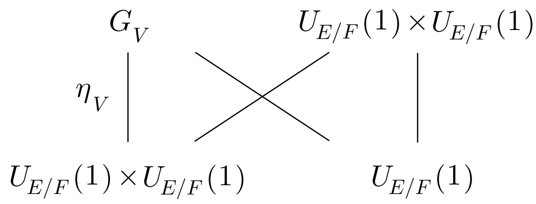
The *endoscopic lifting* which we need is the adjoint map of $f\mapsto f^H$ from the space of invariant distributions on $G(F)$ to that on $H(F)$. In particular, the L-packet $\Pi_{\underline{\mu}}(G_V)=\{\pi_V(\underline{\mu})^\pm\}$ is labeled in such a way that

$$\mathrm{tr}\pi_V(\underline{\mu})^+(f)-\mathrm{tr}\pi_V(\underline{\mu})^-(f)=\big((\mu/\mu_0)_u\otimes(\mu'/\mu_0')_u\big)(f^H)$$

holds. If V is hyperbolic in the realization (5.1), then $\pi_V(\underline{\mu})^+$ is the unique generic member in $\Pi_{\underline{\mu}}(G_V)$ with respect to the character [KKb, Prop.4.8]

$$\psi_{\boldsymbol{U}_2}:\boldsymbol{U}_2(F)\ni\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\longmapsto\psi_F(b)\in\mathbb{C}^\times.$$

(This is a consequence of the *Whittaker normalization* of the transfer factor (5.1).) Combining these with the *seesaw duality* [Kud84]



we obtain a *Saito-Tunnell type* character formula for $\pi_V(\underline{\mu})^{\pm}$.

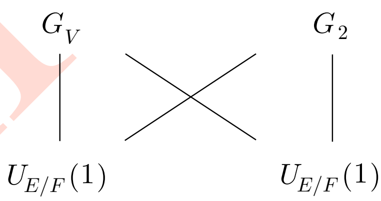
Theorem 5.2. *For a character μ such that $\mu|_{F^\times} = \omega_{E/F}$, we introduce a sign $\varepsilon_{\psi_F}(\mu) := \varepsilon(1/2, \mu, \psi_E)\mu(-\delta)$.*
(i) If V is hyperbolic, the character (function) $\Theta_{\pi_V(\underline{\mu})^{\pm}}$ of $\pi_V(\underline{\mu})^{\pm}$ is given by (respecting signs)

$$\Theta_{\pi_V(\underline{\mu})^{\pm}} \circ \eta_V = \sum_{\eta|_{F^\times} = \mathbb{1}} \frac{(1 \pm \varepsilon_{\psi_F}(\eta\mu^{-1}))(1 \pm \varepsilon_{\psi_F}(\eta\mu'^{-1}))}{4} (\mu\mu'\eta)_u \otimes \eta_u.$$

(ii) If V is anisotropic, we have (respecting signs)

$$\Theta_{\pi_V(\underline{\mu})^{\pm}} \circ \eta_V = \sum_{\eta|_{F^\times} = \mathbb{1}} \frac{(1 \mp \varepsilon_{\psi_F}(\eta\mu^{-1}))(1 \pm \varepsilon_{\psi_F}(\eta\mu'^{-1}))}{4} (\mu\mu'\eta)_u \otimes \eta_u.$$

Of course, these formulae indicates various interesting speculations. But this is not a place to discuss them. We only remark that the same formulae are also valid in the archimedean case. Now we combine the theorem with the seesaw duality



to obtain the following.

Theorem 5.3 (Howe duality for $\Pi_{\underline{\mu}}(G_V)$)**.** *We write $\Pi_{\underline{\mu}}(G_2) = \{\pi(\underline{\mu})^{\pm}\}$ as above. Suppse $(V, (\cdot,\cdot))$ satisfies the condition of Th.4.2. Then we have $\theta_{\underline{\xi}}(\pi(\underline{\mu})^{\pm}, V) = \pi_V(\eta\underline{\mu}^{-1})^{\pm\varepsilon_{\psi_F}(\mu)}$, where $\eta\underline{\mu}^{-1} := (\eta\mu^{-1}, \eta\mu'^{-1})$.*

Pairing in the endoscopic case We now define the pairing $\langle\cdot,\cdot\rangle : \Pi_\phi(G) \times \mathcal{S}_\phi(G) \to \{\pm 1\}$ for ϕ in Prop.2.4 (2.d). We retain the notation of the above discussion.

Definition 5.4. *Recall that $\mathcal{S}_\phi(G)$ for $\phi_E \simeq (\eta \otimes \rho_2) \oplus \mu \oplus \mu'$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The pairing is defined by*

$$\langle \cdot, \theta_{\underline{\xi}}(\pi_V(\eta\underline{\mu}^{-1})^{\pm}, W_2)\rangle := \mathrm{sgn}^{(1-\varepsilon_{V,\eta}(\underline{\mu}))/2} \otimes \mathrm{sgn}^{(1 \mp \varepsilon_{\psi_F}(\mu))/2},$$

where $\varepsilon_{V,\eta}(\underline{\mu}) := \varepsilon_{\psi_F}(\eta\mu^{-1})\varepsilon_{\psi_F}(\eta\mu'^{-1})\omega_{E/F}(-\det V)$.

6 Automorphic forms

We now go back to the global situation, and consider the A-parameters ϕ of type (2.b)–(2.d) in Prop.2.4. As is announced in §4, our principal result is the following.

Theorem 6.1. *Each $\pi = \bigotimes_v \pi_v \in \Pi_\phi(G)$ occurs in the discrete spectrum $L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A}))$ with the multiplicity at least:*

$$\frac{1}{|\mathcal{S}_\phi(G)|} \sum_{\boldsymbol{s} \in \mathcal{S}_\phi(G)} \epsilon_\phi(\boldsymbol{s}) \prod_v \langle \pi_v, \boldsymbol{s} \rangle. \tag{6.1}$$

Here, ϵ_ϕ is the sign character of the first $\mathbb{Z}/2\mathbb{Z}$ of $\mathcal{S}_\phi(G)$ if $\varepsilon(1/2, \pi_E \times \eta^{-1}) = -1$, and is the trivial character otherwise.

The proof involves the global θ-correspondence between $G_V(\mathbb{A})$ and $G(\mathbb{A})$ and the description of the discrete spectrum of $G_V(\mathbb{A})$ [KKb].

Remark 6.2. *Those $\pi \in \Pi_\phi(G)$ such that $\varepsilon(1/2, \pi_E \times \eta^{-1}) = 1$ and $\langle \pi_v, \cdot \rangle$ are trivial on the first $\mathbb{Z}/2\mathbb{Z} \subset \mathcal{S}_{\phi_v}(G_v)$ at all v are the* residual discrete *automorphic representations of $G(\mathbb{A})$ [Kon98]. All the other π with non-zero* (6.1) *are CAP autmorphic forms.*

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