THE RESIDUAL SPECTRUM OF $U(2,2)$

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Dedicated to Professor H. Shimizu on his sixtieth birthday.

ABSTRACT. Let G be the quasi-split unitary group $U(2,2)_{k'/k}$ of four variables attached to a quadratic extension k'/k of number fields. An irreducible decomposition of the non-cuspidal discrete spectrum of L^2-automorphic forms of G is obtained.

CONTENTS

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1. Introduction

Let G be a reductive group defined over a number field k. For simplicity we assume the center of G is anisotropic over k. If we write $\mathbb{A}$ for the ring of adeles of k, then the group of $\mathbb{A}$-points $G(\mathbb{A})$ is a locally compact topological group which contains $G(k)$ as a discrete subgroup with finite covolume. The right regular representation R of $G(\mathbb{A})$ on the space of L^2-*automorphic forms* $L^2(G(k)\backslash G(\mathbb{A}))$;

$$[R(g)\phi](x) := \phi(xg), \quad (\phi \in L^2(G(k)\backslash G(\mathbb{A})),\, g \in G(\mathbb{A}))$$

is a primary object in the modern theory of automorphic forms. In particular we are interested in the "irreducible decomposition" of this R.

We know from the general theory of spectral decomposition that $L^2(G(k)\backslash G(\mathbb{A}))$ decomposes into a direct sum of two $G(\mathbb{A})$-invariant closed subspaces $L^2_{disc}(G)$ and $L^2_{cont}(G)$. $L^2_{disc}(G)$ is a direct sum of irreducible representations of $G(\mathbb{A})$ while $L^2_{cont}(G)$ is a continuous sum of irreducible $G(\mathbb{A})$-modules. Nevertheless, Langlands' spectral theory of Eisenstein series ([La], [MW]) reduces the study of $L^2_{cont}(G)$ to that of $L^2_{disc}(M)$ for Levi subgroups M of G. Thus, the term "irreducible decomposition of $L^2(G(k)\backslash G(\mathbb{A}))$" means that of $L^2_{disc}(G)$.

For $\phi \in L^2(G(k)\backslash G(\mathbb{A}))$, we define its constant term along a k-parabolic subgroup P by

$$\phi_P(g) := \int_{U(k)\backslash U(\mathbb{A})} \phi(ug)\, du.$$

Here U is the unipotent radical of P. Then the space $L^2_0(G(k)\backslash G(\mathbb{A}))$ of L^2-*cusp forms* on $G(\mathbb{A})$ is spanned by those $\phi \in L^2(G(k)\backslash G(\mathbb{A}))$ such that ϕ_P vanishes almost everywhere for each proper k-parabolic subgroup P. This is clearly a $G(\mathbb{A})$-invariant closed subspace in $L^2(G(k)\backslash G(\mathbb{A}))$. Since every $\phi \in L^2_0(G(k)\backslash G(\mathbb{A}))$ is rapidly decreasing, $L^2_0(G(k)\backslash G(\mathbb{A}))$ is contained in $L^2_{disc}(G)$.

Classical Fourier expansions for elliptic modular forms suggest that the Whittaker model for irreducible automorphic representations is a useful tool for the irreducible decomposition of $L^2_0(G(k)\backslash G(\mathbb{A}))$. This idea was carried out by Jacquet and Langlands in the case $G = GL(2)$, and later extended to the case $G = GL(n)$ by Jacquet, Shalika and Piatetskii-Shapiro. Their results are quite satisfactory in these cases. But the Whittaker model cannot capture all the irreducible representations appeared in $L^2_0(G(k)\backslash G(\mathbb{A}))$ if G is not $GL(n)$ or $SL(n)$. Thus the irreducible decomposition of $L^2_0(G(k)\backslash G(\mathbb{A}))$ is open except for the cases $G = GL(n)$, $SL(2)$ and $U(2,1)$. (It seems that the only hope in this direction is the theory of twisted endoscopy.)

There is one problem which is accessible with our present knowledge. It is to determine the irreducible decomposition of the residual discrete spectrum, the orthogonal complement in $L^2_{disc}(G)$ of $L^2_0(G(k)\backslash G(\mathbb{A}))$, for rank 2 classical groups G. In fact

(1) It was shown by Langlands ([La]) that the residual discrete spectrum is spanned by certain residues of Eisenstein series on $G(k)\backslash G(\mathbb{A})$.

(2) Since any Levi factor of k-parabolic subgroups of G is rank one, we know from [Sh] that the poles of these Eisenstein series are given by those of certain automorphic L-functions.
(3) Hence if the poles of these automorphic L-functions can be determined, then the irreducible decomposition of the residual discrete spectrum is reduced to those of $L_0^2(M(k)A_M(\mathbb{A})\backslash M(\mathbb{A}))$ for Levi factors M of k-parabolic subgroups of G.

Here A_M denotes the split component of the center of M.

In the present paper we shall carry out this in the case of rank 2 quasi-split unitary group $G = U(2,2)_{k'/k}$ attached to a quadratic extension k'/k. In this case the automorphic L-functions appeared in (2) above, are products of Hecke L-functions, twisted tensor L-functions ([As], [HLR], [G]) and the product L-functions for $U(1,1) \times \mathrm{Res}_{k'/k}\mathbb{G}_m$. Their poles are either well-known or can be easily determined (cf. Appendices A, B). Also the Levi factors appeared in (3) is one of $M_0 := \mathrm{Res}_{k'/k}\mathbb{G}_m \times \mathrm{Res}_{k'/k}\mathbb{G}_m$, $M_1 := \mathrm{Res}_{k'/k}GL(2)$ and $M_2 := \mathrm{Res}_{k'/k}\mathbb{G}_m \times U(1,1)_{k'/k}$. The irreducible decomposition of $L_0^2(M(k)A_M(\mathbb{A})\backslash M(\mathbb{A}))$ for these M can be deduced from [JL] and [LL]. Now the main result of this paper is stated as follows (Theorem 3.11, Theorem 3.18, Theorem 4.4 and Theorem 5.5).

Theorem 1.1. *The residual spectrum of $G = U(2,2)_{k'/k}$ is a direct sum of the following irreducible representations of $G(\mathbb{A})$. Each occurs with multiplicity one.*

(1) *The one dimensional representations $\chi \circ \det$. Here χ runs over all the characters of $U(1,k)\backslash U(1,\mathbb{A})$.*
(2) *The θ-lifts $R(V_{\mathbb{A}}, \chi)$ of trivial representations of rank 1 unitary groups $U(V,\mathbb{A})$. Here $V_{\mathbb{A}} = V \otimes_k \mathbb{A}$ and V runs over all 1-dimensional Hermitian space over k'. χ runs over all characters of $\mathbb{A}_{k'}^\times/k'^\times$ such that $\chi|_{\mathbb{A}^\times} = \eta_{k'/k}$. $\eta_{k'/k}$ is the quadratic character corresponding to k'/k by the classfield theory.*
(3) *The "θ-lifts" of non-trivial 1-dimensional representations of $U(1,1)_{k'/k}(\mathbb{A})$.*
(4) *The global Langlands' quotients of $\mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_1) \otimes \mathbf{1}_{U_1(\mathbb{A})}]$. Here P_1 is a parabolic subgroup of G whose Levi factor is M_1. $\mathfrak{S}(P_1) = \pi \otimes |\det|_{\mathbb{A}_{k'}}$, where π runs over the irreducible cuspidal automorphic representations of $M_1(\mathbb{A})$ such that*
   (a) *The central character ω_π of π restricted to $\mathbb{A}^\times$ is trivial.*
   (b) *If we write H for the diagonal subgroup $GL(2)_k$ of $M_1 \simeq \mathrm{Res}_{k'/k}GL(2)$, then*
$$\int_{H(k)Z(H,\mathbb{A})\backslash H(\mathbb{A})} f(h)\,dh \neq 0$$
   holds for some f in the automorphic realization of π.
(5) *The global Langlands' quotients of $\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_2,\eta_{k'/k}) \otimes \mathbf{1}_{U_2(\mathbb{A})}]$ and $\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_2,\mathbf{1}) \otimes \mathbf{1}_{U_2(\mathbb{A})}]$, where P_2 is a parabolic subgroup of G whose Levi factor is M_2. $\mathfrak{S}(P_2,\eta_{k'/k})$ runs over all the irreducible cuspidal representations of $M_2(\mathbb{A})$ which satisfies the following conditions. $\mathfrak{S}(P_2,\eta_{k'/k})$ is written as $\chi|\ |_{\mathbb{A}_{k'}} \otimes \tau$ according to $M_2(\mathbb{A}) \simeq \mathbb{A}_{k'}^\times \times U(1,1)_{k'/k}(\mathbb{A})$, where*
   (a) *$\chi|_{\mathbb{A}^\times} = \eta_{k'/k}$.*
   (b) *τ is the θ-lift of its central character ω_τ from $U(1)_{k'/k}(\mathbb{A})$ under the Weil representation $\omega_{\psi,\chi^{-1}}$ (see* B.1*).*

   Also $\mathfrak{S}(P_2,\mathbf{1})$ runs over the irreducible cuspidal representations of $M_2(\mathbb{A})$ of the form $\chi|\ |_{\mathbb{A}_{k'}}^{1/2} \otimes \tau$ such that $\chi|_{\mathbb{A}^\times} = \mathbf{1}$ and $L(s,\tau\times\chi)$ does not vanish at $s = 1/2$.

The residual spectrum of $GL(n)$ was obtained by Moeglin and Waldspurger ([MW2]). Also the same kind of results for $G = Sp(2)$ was obtained independently by H. Kim and the author ([Ki]). Note that the condition (b) in (4) is equivalent to certain functorial properties of π (see [HLR], [Y] and [Fl]).

We now explain the organization of this paper. In section 2, we collect some notations and review the basic tools in Langlands' spectral theory of Eisenstein series from [MW]. In section 3 we determine the contributions of cuspidal Eisenstein series from the Borel subgroup P_0 to the residual spectrum. This will need a detailed calculation of L^2-inner product of these Eisenstein series (see 3.2). Once the inner product is calculated (Theorem 3.4), we proceed to the irreducible decomposition. In this step the results of Kudla, Sweet, S. T. Lee and C. Zhu on the irreducible decomposition of certain degenerate principal series representations of $U(n,n)$ will be used ([KSw], [Le] and [LeZ]). We review these in 3.3.1. Finally we use the arguments of [KRS] §2 to have the statements *(1)* and *(2)* in Theorem 1.1. The statement *(3)* is easily proved by a local Langlands' classification argument.

In section 4 and 5, we determine the contributions of cuspidal Eisenstein series from the Siegel parabolic subgroup P_1 and from the non-Siegel maximal parabolic subgroup P_2 respectively. The main problem in these cases is to determine the poles of certain automorphic L-functions. In the Siegel parabolic case, the twisted tensor L-function appears while the product L-function of $U(1,1) \times \mathrm{Res}_{k'/k}\mathbb{G}_m$ appears in the non-Siegel parabolic case. The poles of these L-functions will be determined in Appendices A, B. Once the poles are determined, the statements *(4)* and *(5)* in Theorem 1.1 can be easily deduced from local Langlands' classification arguments.

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2. Preliminary

2.1. Notations and conventions.

2.1.1. Let k'/k be a quadratic extension of number fields. We write Γ for the Galois group $\mathrm{Gal}(k'/k)$ and σ for its generator. W_k denotes the Weil group of k.

We write $\mathbb{A}$ for the adele ring of k and write $\mathbb{A}_\infty$ and $\mathbb{A}_f$ for its infinite and finite component respectively. $|\ |_{\mathbb{A}}$ denotes the idele norm of $\mathbb{A}^\times$. A place of k is conventionally denoted by v. For each place v of k, we write k_v for the completion of k at v and $|\ |_v$ for the v-adic norm. In particular if v is finite, we let $\mathcal{O}_v$ be the maximal compact subring of k_v, $\mathfrak{p}_v$ be the maximal ideal in $\mathcal{O}_v$ and ϖ_v be a generator of $\mathfrak{p}_v$. Also we write q_v for the cardinality of the residue field of k_v.

We write $\mathbb{A}_{k'}$ for the adele ring of k'. The idele norm of $\mathbb{A}_{k'}^\times$ is written by $|\ |_{\mathbb{A}_{k'}}$. We conventionally write a place of k' by w. The notations $k'_w \supset \mathcal{O}_w \supset \mathfrak{p}_w \ni \varpi_w$ are defined similarly as in the case of k. The following conventions on places of k and k' will be convenient. If a place v is inert in k' then the place of k' lying over v will be denoted by

w. If v splits in k' then the places of k' lying over v will be denoted by w_1 and w_2. We often write k'_v for the semisimple algebra $k' \otimes_k k_v$ over k_v.

2.1.2. We need some notations about the group $\widetilde{G} := \mathrm{Res}_{k'/k}GL(4)$. The k-automorphism on $\widetilde{G}$, which is determined by the action of σ on $GL(4,k')$, is denoted by $\widetilde{\sigma}$. We fix a k'-group isomorphism

$$\widetilde{G}_{k'} \xrightarrow{\sim} GL(4)_{k'} \times GL(4)_{k'},$$

so that the action of σ on $\widetilde{G}$ is transported to

$$\sigma : GL(4)_{k'} \times GL(4)_{k'} \ni (g_1, g_2) \longrightarrow (\sigma(g_2), \sigma(g_1)) \in GL(4)_{k'} \times GL(4)_{k'},$$

and that $\widetilde{\sigma}$ is transported to

$$\widetilde{\sigma} : GL(4)_{k'} \times GL(4)_{k'} \ni (g_1, g_2) \longrightarrow (g_2, g_1) \in GL(4)_{k'} \times GL(4)_{k'}.$$

2.1.3. We define an automorphism θ_2 of $\widetilde{G}_{k'}$ by

$$\theta_2 : GL(4)_{k'} \times GL(4)_{k'} \ni (g_1, g_2) \longrightarrow (\mathrm{Int}J_2({}^t g_1^{-1}), \mathrm{Int}J_2({}^t g_2^{-1})) \in GL(4)_{k'} \times GL(4)_{k'},$$

where

$$J_2 := \left(\begin{array}{c|c} \mathbf{0}_2 & \mathbf{1}_2 \\ \hline -\mathbf{1}_2 & \mathbf{0}_2 \end{array}\right).$$

This clearly commutes with σ and hence is a k-automorphism of $\widetilde{G}$. Then our group $G := U(2,2)_{k'/k}$ is defined by $G := \{g \in \widetilde{G}\,;\, \theta_2 \circ \widetilde{\sigma}(g) = g\}$. Note that $G_{k'} = \{(g, \theta_2(g)) \in GL(4)_{k'} \times GL(4)_{k'}\} \simeq GL(4)_{k'}$. Fix a minimal k-parabolic subgroup P_0 and its Levi factor M_0 of G so that

$$P_{0,k'} := \left\{ \left(\left(\begin{array}{cc|cc} * & * & * & * \\ 0 & * & * & * \\ \hline 0 & & * & 0 \\ & & * & * \end{array}\right), \left(\begin{array}{cc|cc} * & * & * & * \\ 0 & * & * & * \\ \hline 0 & & * & 0 \\ & & * & * \end{array}\right) \right) \in G_{k'} \right\},$$

$$M_{0,k'} := \left\{ d((x_1,y_1),(x_2,y_2)) := \left(\left(\begin{array}{cc|cc} x_1 & & 0 & \\ & x_2 & & \\ \hline 0 & & y_1^{-1} & \\ & & & y_2^{-1} \end{array}\right), \left(\begin{array}{cc|cc} y_1 & & 0 & \\ & y_2 & & \\ \hline 0 & & x_1^{-1} & \\ & & & x_2^{-1} \end{array}\right) \right) \in G_{k'} \right\}.$$

The unipotent radical of P_0 is denoted by U_0. We write an element of M_0 as $d(x_1, x_2) := \mathrm{diag}(x_1, x_2, \widetilde{\sigma}(x_1)^{-1}, \widetilde{\sigma}(x_2)^{-1})$ ($x_i \in \mathrm{Res}_{k'/k}\mathbb{G}_m$, $i = 1, 2$). Then the k-split component A_0 of the center of M_0 is given by $A_0 := \{d(x_1, x_2)\,;\, x_1,\, x_2 \in \mathbb{G}_{m,k}\}$. We write G_v for the algebraic group over k_v given by the scalar extension $G_v := G \otimes_k k_v$. Then $G_v \simeq U(2,2)_{k'_w/k_v}$ if v is inert in k' and $G_v \simeq GL(4)_{k_v}$ if v splits in k'. Also we fix a good maximal compact subgroup $K = \prod_v K_v$ of $G(\mathbb{A})$ as

$$K_v = \begin{cases} G(k_v) \cap GL(4, \mathcal{O}_w), & \text{if } v \nmid \infty \text{ and } v \text{ is inert in } k', \\ GL(4, \mathcal{O}_v), & \text{if } v \nmid \infty \text{ and } v \text{ splits in } k', \\ U(4) \cap G(\mathbb{R}) \simeq U(2) \times U(2), & \text{if } v = \mathbb{R} \text{ and } v \text{ is inert in } k', \\ O(4), & \text{if } v = \mathbb{R} \text{ and } v \text{ splits in } k', \\ U(4) \cap G(\mathbb{C}) = U(4), & \text{if } v = \mathbb{C}. \end{cases}$$

Then we have the Iwasawa decomposition $G(\mathbb{A}) = P_0(\mathbb{A})K$. We write K_∞ for $\prod_{v|\infty} K_v \subset G(\mathbb{A}_\infty)$.

We choose a basis $\{e_1, e_2\}$ of $X^*(A_0)$, the character group of A_0, so that $e_j(d(x_1, x_2)) = x_j$ $(j = 1, 2)$. Let $\{e_1^\vee, e_2^\vee\}$ be its dual basis of $X_*(A_0)$, the group of one-parameter subgroups of A_0: $e_1^\vee(x) = d(x, 1)$, $e_2^\vee(x) = d(1, x)$. Name the P_0-positive roots of A_0 as

$$R^+(P_0, A_0) = \{\alpha_1 := e_1 - e_2,\ \alpha_2 := 2e_2,\ \beta_1 := e_1 + e_2,\ \beta_2 := 2e_1\},$$

then $\Delta(P_0, A_0) := \{\alpha_1, \alpha_2\}$ is the set of simple roots.

2.1.4. Fix a k-splitting of $G_{k'}$ as $(P_{0,k'}, M_{0,k'}, \{X_\alpha\}_\alpha)$, where

$$X_{\alpha_1} := \left(\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)\right),\ X_{\alpha_2} := \left(\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)\right)$$

$X_{\alpha_1'} := \widetilde{\sigma}(X_{\alpha_1})$.

Here

$$\alpha_1(d((x_1, y_1), (x_2, y_2))) = x_1 x_2^{-1}, \quad \alpha_2(d((x_1, y_1), (x_2, y_2))) = x_2 y_2,$$
$$\alpha_1'(d((x_1, y_1), (x_2, y_2))) = y_1 y_2^{-1}$$

are the simple roots of $M_{0,k'}$ in $P_{0,k'}$. The dual group $\widehat{G}$ of $G = U(2,2)_{k'/k}$ equals $GL(4, \mathbb{C})$. We fix a splitting $(\widehat{P}_0, \widehat{M}_0, \{X_{\alpha^\vee}\}_{\alpha^\vee})$ of $\widehat{G}$ to be objects of the form of the first components of $P_{0,k'}$, $M_{0,k'}$, $\{X_\alpha\}_\alpha$. This distinguished splitting is stable under the Γ-action $\sigma(g) := \theta_2(g)$. Now the L-group ${}^LG$ for G is given by ${}^LG := \widehat{G} \rtimes_{\theta_2} W_k$, where W_k acts through Γ by $\sigma(g) := \theta_2(g)$.

2.1.5. The k-parabolic subgroups of G which contain the minimal parabolic subgroup P_0 is called *standard parabolic subgroups*. They contain a complete system of representatives of $G(k)$-conjugacy classes of k-parabolic subgroups. Each standard parabolic subgroup P has a unique Levi component M which contains M_0. Then the set of simple roots $\Delta(M \cap P_0, A_0)$ of A_0 in $M \cap P_0$ is a subset of $\Delta(P_0, A_0)$. This gives a bijection between the standard parabolic subgroups of G and the subsets of $\Delta(P_0, A_0)$. Thus we have two proper standard parabolic subgroups $P_i = M_i U_i$ $(i = 1, 2)$ other than P_0, where P_i corresponds to $\{\alpha_i\} \subset \Delta(P_0, A_0)$ under the above bijection. We also adopt analogous notations for parabolic subgroups of $\widehat{G}$ which contain $\widehat{P}_0$. Then the L-group of $P_i = M_i U_i$ $(i = 1, 2)$ is given by $\widehat{P}_i \rtimes_{\theta_2} W_k$. Here $\widehat{P}_i = \widehat{M}_i \widehat{U}_i$ is such that

$$\Delta(\widehat{P}_0 \cap \widehat{M}_i, \widehat{M}_0) = \begin{cases} \{\alpha_1^\vee, \alpha_1'^\vee\}, & \text{if } i = 1 \\ \{\alpha_2^\vee\}, & \text{if } i = 2. \end{cases}$$

2.1.6. For a standard parabolic subgroup $P = MU$, we write A_M for the k-split component of the center of M. We fix a k-isomorphism $\Psi_M : \mathbb{G}_m^{\dim A_M} \xrightarrow[k]{\sim} A_M$ and write $A_M(\mathbb{R})_+$ for the image of the composit

$$(\mathbb{R}_+^\times)^{\dim A_M} \hookrightarrow \mathbb{G}_m^{\dim A_M}(\mathbb{A}_\infty) \xrightarrow{\Psi_M} A_M(\mathbb{A}_\infty).$$

Here the first injection is the diagonal embedding into $\mathbb{G}_m^{\dim A_M}(\mathbb{A}_\infty) = \prod_{v|\infty} \mathbb{G}_m^{\dim A_M}(k_v)$.

As usual we have real vector spaces $\mathfrak{a}_M := \mathrm{Hom}(X^*(M)_k, \mathbb{R})$ and $\mathfrak{a}_M^* := X^*(M)_k \otimes_{\mathbb{Z}} \mathbb{R}$ dual to each other. Here $X^*(M)_k$ denotes the k-rational character group of M. We write $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ for $\mathfrak{a}_{M_0}$ and $\mathfrak{a}_{M_0}^*$ respectively. $\mathfrak{a}_M$ is always identified with the subspace $\{\lambda \in \mathfrak{a}_0\,;\, \langle \alpha, \lambda \rangle = 0 \text{ for } \alpha \in \Delta(P_0 \cap M, A_0)\}$ of $\mathfrak{a}_0$. On the other hand, the restriction

map $X^*(M)_k \ni \chi \to \chi|_{M_0} \in X^*(M_0)_k$ gives a natural embedding $\mathfrak{a}_M^* \hookrightarrow \mathfrak{a}_0^*$. The Harish-Chandra map $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ is defined by

$$\langle H_M(m), \chi \rangle = \log |\chi(m)|_{\mathbb{A}}, \quad \text{for any } \chi \in X^*(M)_k.$$

We write the kernel of this map as $M(\mathbb{A})^1$. Then we have a direct product decomposition $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M(\mathbb{R})_+$. H_M is often considered as a map from $G(\mathbb{A})$ by the Iwasawa decomposition fixed above:

$$H_M : G(\mathbb{A}) \ni g = umk \longrightarrow H_M(m) \in \mathfrak{a}_M, \quad (u \in U(\mathbb{A}), m \in M(\mathbb{A}), k \in K). \tag{2.1}$$

The Weyl group of A_0 in G is denoted by W. It acts on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$. We write w_i for the simple reflection attached to the simple root α_i $(i = 1, 2)$ and w_- for the longest element $w_1w_2w_1w_2 = w_2w_1w_2w_1$. We also have the Weyl group $W(A_M)$ of A_M in G which acts on $\mathfrak{a}_M$. The inner product on $\mathfrak{a}_0^*$ for which $\{e_1, e_2\}$ is a ortho-normal basis is W-invariant. This determines a $W(A_M)$-invariant inner product on each $\mathfrak{a}_M^*$. We fix these inner products throughout the paper.

Finally we normalize various measures as in [Ar].

2.1.7. *Notations about automorphic forms.* In the rest of this paper, we freely use the notations introduced in [MW] II. We briefly review them.

Fix a standard parabolic subgroup $P = MU$. We write the space of automorphic forms on $U(\mathbb{A})M(k)\backslash G(\mathbb{A})$ as $A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$. $A(M(k)\backslash M(\mathbb{A}))$ denotes the space of automorphic forms on $M(k)\backslash M(\mathbb{A})$. The space of cusp forms on $U(\mathbb{A})M(k)\backslash G(\mathbb{A})$ is written as $A_0(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$. $A_0(M(k)\backslash M(\mathbb{A}))$ is the space of cusp forms on $M(k)\backslash M(\mathbb{A})$.

Next let π be a cuspidal automorphic representation. We write $A(M(k)\backslash M(\mathbb{A}))_\pi$ for the π-isotypic subspace in $A_0(M(k)\backslash M(\mathbb{A}))$, and $A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$ for the space of functions $\phi \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$ such that

$$\phi_k(m) := e^{\langle -\rho_P, H_M(m) \rangle} \phi(mk), \quad (\, m \in M(\mathbb{A})\,)$$

belongs to $A(M(k)\backslash M(\mathbb{A}))_\pi$ for any $k \in K$.

Each $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^* := \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ is identified with the quasi-character

$$M(\mathbb{A}) \ni m \longrightarrow \exp\langle \lambda, H_M(m) \rangle \in \mathbb{C}^\times$$

of $M(\mathbb{A})$ trivial on $M(\mathbb{A})^1$. Recall that two automorphic subrepresentations π and τ of $M(\mathbb{A})$ are *equivalent* if there exists $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ such that $\tau \simeq \pi \otimes \lambda$. We call a pair $(M, \mathfrak{P})$, where $\mathfrak{P}$ is an equivalence class of cuspidal automorphic representations of $M(\mathbb{A})$, a *cuspidal datum*.

Fix a cuspidal datum $(M, \mathfrak{P})$ for the rest of this section. Recall the space of Paley-Wiener sections $P_{(M,\mathfrak{P})}$. For each $\phi \in P_{(M,\mathfrak{P})}$, its Fourier transform $F(\phi)$ was defined.

2.1.8. *Notations about Eisenstein series.* For each $\pi \in \mathfrak{P}$, we write ω_π for its central character and define $\mathrm{Re}\pi \in \mathfrak{a}_M^*$, $\mathrm{Im}\pi \in \mathfrak{P}$ by

$$\mathrm{Re}\pi : M(\mathbb{A}) \ni m \longrightarrow |\omega_\pi(m)|_{\mathbb{A}} \in \mathbb{R}_+^\times, \qquad \mathrm{Im}\pi := (\mathrm{Re}\pi)^{-1} \otimes \pi.$$

We also write $-\pi$ for the contragredient of π and $\overline{\pi}$ for the complex conjugate of π. We have a hermitian pairing between $\phi \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$ and $\phi' \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_{-\overline{\pi}}$ given by

$$\langle \phi,\ \phi' \rangle := \int_K \int_{M(\mathbb{A})^1} \phi(m_1k)\overline{\phi'(m_1k)}\, dm_1 dk.$$

Notice that if $\pi \in \mathfrak{P}$ then $-\overline{\pi} \in \mathfrak{P}$.

For $\phi : \mathfrak{P} \to A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))$ with $\phi(\pi) \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$ ($\pi \in \mathfrak{P}$), the Eisenstein series attached to ϕ is defined by

$$E(\phi, \pi)(g) := \sum_{\gamma \in P(k)\backslash G(k)} \phi(\pi)(\gamma g), \qquad g \in G(\mathbb{A}).$$

This converges absolutely on

$$C_P^+(\mathfrak{P}) := \{\pi \in \mathfrak{P} \ ; \ \langle \mathrm{Re}\pi - \rho_P, \ \alpha^\vee \rangle > 0, \ \text{for all } \alpha \in R^+(A_M, P)\}.$$

Take $w \in W$ such that $w(M)$ is again a Levi factor of a standard parabolic subgroup containing M_0. In $U(2,2)$ case this means $w(M) = M$. We have the intertwining operator

$$(M(w, \pi)\phi_\pi)(g) := \int_{U_w(\mathbb{A})} \phi_\pi(\dot{w}^{-1} u g) \, du,$$

where $\phi_\pi \in A(U(\mathbb{A})M(k)\backslash G(\mathbb{A}))_\pi$, $U_w := U_0 \cap w(U^-)$ and $\dot{w}$ is a representative of w in the normalizer of $A_0(k)$ in $G(k)$. This converges absolutely on

$$C_P^+(\mathfrak{P}, w) := \left\{ \pi \in \mathfrak{P} \ \middle| \ \langle \mathrm{Re}\pi - \rho_P, \ \alpha^\vee \rangle > 0 \ \left(\begin{array}{c} \text{for all } \alpha \in R^+(A_M, P) \\ \text{such that } w\alpha \notin R^+(A_M, P) \end{array} \right) \right\}.$$

2.1.9. *Constant terms of Eisenstein series.* For a standard parabolic subgroup $P = MU$ of G, we have the Weyl group $W^M \subset W$ of M. Let $P' = M'U'$ be another standard parabolic subgroup. Set

$$W_M := \left\{ w \in W \ \middle| \ \begin{array}{l} \text{i) } w \text{ is of minimal length in the coset } wW^M, \\ \text{ii) } w(M) \text{ is again a Levi component of a standard} \\ \quad \text{parabolic containing } M_0. \end{array} \right\},$$

$$W_{M,M'} := \{w \in W_M \mid w(M) \subset M'\}.$$

Now let $E(\phi, \pi)(g)$ be as in 2.1.8. Then the constant term of $E(\phi, \pi)$ along the parabolic subgroup $P' = M'U'$ is given by

$$E_{P'}(\phi, \pi)(m) = \sum_{w \in W_{M,M'}} E^{M'}\Big(M(w, \pi)\phi(\pi), w\pi\Big)(m), \quad (m \in M'(\mathbb{A}))$$

where

$$E^{M'}\Big(M(w, \pi)\phi(\pi), w\pi\Big)(g) := \sum_{\gamma \in (wPw^{-1} \cap M')(k)\backslash M'(k)} M(w, \pi)\phi(\pi)(\gamma g).$$

2.1.10. *Eisenstein pseudo-series.* For each $\phi \in P_{(M,\mathfrak{P})}$, we define

$$\theta_\phi(g) := \sum_{\gamma \in P(k)\backslash G(k)} F(\phi)(\gamma g), \quad (g \in G(\mathbb{A})).$$

Then the sum on the right hand side converges absolutely and uniformly on any compact subsets of $G(\mathbb{A})$. θ_ϕ is in $L^2(G(k)\backslash G(\mathbb{A}))$, and

$$\{\theta_\phi \,|\, \phi \in P_{(M,\mathfrak{P})}, \ (M, \mathfrak{P}) \text{ runs over all cuspidal data.}\}$$

spans a dense subspace of $L^2(G(k)\backslash G(\mathbb{A}))$.

2.1.11. *The inner product formula.* Let $(M,\mathfrak{P})$ and $(M',\mathfrak{P}')$ be two cuspidal data. Set

$$W(\mathfrak{P},\mathfrak{P}') := \{w \in W_{M,M'} \ ; \ w(M,\mathfrak{P}) = (M',\mathfrak{P}')\}.$$

Two data $(M,\mathfrak{P})$ and $(M',\mathfrak{P}')$ are said to be *equivalent* if the set $W(\mathfrak{P},\mathfrak{P}')$ is non-empty. The following theorem is [MW] Théorème II.2.1.

Theorem 2.1. *Let $(M,\mathfrak{P})$ and $(M',\mathfrak{P}')$ be as above. Then for $\phi \in P_{(M,\mathfrak{P})}$ and $\phi' \in P_{(M',\mathfrak{P}')}$, the hermitian inner product of θ_ϕ and $\theta_{\phi'}$ is given by*

$$\begin{aligned}&\langle \theta_\phi,\, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(\mathbb{A}))} \\ &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^{dim\,\mathfrak{a}_M} \int_{\pi\in\mathfrak{P}, Re\pi=\lambda_0} \sum_{w\in W(\mathfrak{P},\mathfrak{P}')} \langle M(w,\pi)\phi(\pi),\, \phi'(-w\overline{\pi})\rangle \, d\pi.\end{aligned} \tag{2.2}$$

Here the hermitian pairing in the integrand is that of 2.1.8 *and $\lambda_0 \in \mathrm{Re}(C_P^+(\mathfrak{P}))$.*

We write $\mathfrak{E}$ for the set of equivalence classes of cuspidal data for G. For each equivalence class $\mathfrak{X} \in \mathfrak{E}$, $L^2(G(k)\backslash G(\mathbb{A}))_{\mathfrak{X}}$ denotes the closed span of θ_ϕ's ($\phi \in P_{(M,\mathfrak{P})}$, $(M,\mathfrak{P}) \in \mathfrak{X}$) in $L^2(G(k)\backslash G(\mathbb{A}))$. Then as a corollary of Theorem 2.1, we have the direct sum decomposition

$$L^2(G(k)\backslash G(\mathbb{A})) = \bigoplus_{\mathfrak{X}\in\mathfrak{E}} L^2(G(k)\backslash G(\mathbb{A}))_{\mathfrak{X}}. \tag{2.3}$$

3. The contributions of cuspidal data attached to P_0

To decompose the non-cuspidal spectrum of G, we first decompose the inner product (2.2) in Theorem 2.1. From (2.3), it is enough to analyze $\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(\mathbb{A}))}$ for ϕ and ϕ' belonging to the same equivalence class of cuspidal data. We say that a cuspidal datum $(M,\mathfrak{P})$ is *attached to* a standard parabolic subgroup P if M is the Levi factor of P. $\mathfrak{E}_P$ denotes the set of equivalence classes of cuspidal data attached to P. In this section, we study $\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(\mathbb{A}))}$ for ϕ, ϕ' belonging to cuspidal data attached to P_0.

As in [MW] V.2.1, we rewrite (2.2) as follows. For $\mathfrak{X} \in \mathfrak{E}_P$, define

$$P_{\mathfrak{X}} := \bigoplus_{(M,\mathfrak{P})\in\mathfrak{X}} P_{(M,\mathfrak{P})}, \quad \theta_{\phi'} := \sum_{(M,\mathfrak{P})\in\mathfrak{X}} \theta_{\phi'_{(M,\mathfrak{P})}},$$

where $\phi' = (\phi'_{(M,\mathfrak{P})})_{(M,\mathfrak{P})\in\mathfrak{X}} \in P_{\mathfrak{X}}$. Take $(M,\mathfrak{P}) \in \mathfrak{X}$ and write

$$A(\phi,\phi')(\pi) := \sum_{(M',\mathfrak{P})\in\mathfrak{X}} \langle \sum_{w\in W(\mathfrak{P},\mathfrak{P}')} M(w,\pi)\phi(\pi),\, \phi'_{(M,\mathfrak{P}')}(-w\overline{\pi})\rangle,$$

where $\phi \in P_{(M,\mathfrak{P})}$ and $\phi' \in P_{\mathfrak{X}}$. From now on we consider

$$\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(\mathbb{A}))} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\dim \mathfrak{a}_M} \int_{\pi\in\mathfrak{P},\, \mathrm{Re}\pi=\lambda_0} A(\phi,\phi')(\pi)\, d\pi \tag{3.1}$$

instead of (2.2).

3.1. Singular hyperplanes. Let the notations be as above. For a subset $\mathfrak{S} \subset \mathfrak{P}$, its vector part $\mathfrak{S}^0$ is defined to be $\{\lambda \in \mathfrak{a}_{M,\mathbb{C}}^* \,|\, \lambda \otimes \mathfrak{S} \subset \mathfrak{S}\}$. A subset $\mathfrak{S} \subset \mathfrak{P}$ is called an *affine subspace with vector part defined over* $\mathbb{R}$ if

(1) $\mathrm{Re}(\mathfrak{S})$ is an affine subspace in $\mathfrak{a}_M^*$ whose vector part is $\mathrm{Re}(\mathfrak{S}^0)$,
(2) $\widetilde{\mathfrak{S}}^0 := \mathrm{Re}(\mathfrak{S}^0) \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{a}_{M,\mathbb{C}}^*$ equals $\mathfrak{S}^0$.

An *affine hyperplane with vector part defined over* $\mathbb{R}$ is an affine subspace $\mathfrak{S}$ with vector part defined over $\mathbb{R}$, such that $\mathrm{codim}_{\mathfrak{a}_M^*}\mathrm{Re}(\mathfrak{S}^0) = 1$. It is known that the singularity set of $A(\phi, \phi')(\pi)$ is a union of affine hyperplanes with vector part defined over $\mathbb{R}$. We write $S^h_{(M,\mathfrak{P})}$ for the set of these singular hyperplanes of $A(\phi, \phi')$ ($\phi \in P_{(M,\mathfrak{P})}$, $\phi' \in P_{\mathfrak{X}}$). In the following, we give a list of elements in $S^h_{(M_0,\mathfrak{P})}$ which intersect the positive chamber.

Each $\pi \in \mathfrak{P}$ is decomposed into a restricted tensor product $\bigotimes_v \pi_v$, where π_v is a smooth irreducible representation of $M(k_v)$ if v is finite and of $(Lie M(k_v) \otimes_{\mathbb{R}} \mathbb{C}, M(k_v) \cap K_v)$ if v is archimedean. We write $\mathfrak{P}_v$ for the set of local representations π_v, which appear as the v-component of some $\pi \in \mathfrak{P}$. By replacing $|\ |_{\mathbb{A}}$ with $|\ |_v$ in 2.1.6, we have the local Harish-Chandra map $H_M : M(k_v) \to \mathfrak{a}_M$. This gives local analogues $\mathrm{Re}\pi_v$, $\mathrm{Im}\pi_v$, $C_P^+(\mathfrak{P}_v)$ and $C_P^+(\mathfrak{P}_v, w)$ of $\mathrm{Re}\pi$, $\mathrm{Im}\pi$, $C_P^+(\mathfrak{P})$ and $C_P^+(\mathfrak{P}, w)$, respectively.

We may assume the Eulerian decomposition

$$M(w,\pi)\phi(\pi) = \bigotimes_v M(w, \pi_v)\phi_v(\pi_v) \quad (\pi \in C_P^+(\mathfrak{P}, \omega)). \tag{3.2}$$

Here $\phi_v(\pi_v)$ is contained in the space $\mathrm{Ind}_{P(k_v)}^{G(k_v)}[\pi_v \otimes \mathbf{1}_{U(k_v)}]$ and

$$\Big(M(w,\pi_v)\phi_v(\pi_v)\Big)(g) := \int_{U_w(k_v)} \phi_v(\pi_v)(\dot{w}^{-1} n g)\, dn \quad (g \in G(k_v))$$

for $\pi_v \in C_P^+(\mathfrak{P}_v, w)$ (cf. [Sh3]).

3.1.1. *Analytic behavior of local intertwining operators.* For the rest of this section, our cuspidal data $(M, \mathfrak{P})$ are always attached to P_0. Since $W_{M_0} = W$, we have to study $M(w, \pi_v)\phi_v(\pi_v)$ for all $w \in W$. We write $\pi \in \mathfrak{P}$ in the form

$$\pi = \mu_1 \otimes \mu_2 : M_0(\mathbb{A}) \ni d(x_1, x_2) \longrightarrow \mu_1(x_1)\mu_2(x_2) \in \mathbb{C}^\times,$$

where $\mu_i = \bigotimes_w \mu_{i,w}$ $(i = 1, 2)$ are quasi-characters of $\mathbb{A}_{k'}^\times / k'^\times$. We fix a non-trivial additive character ψ_v of k_v at each v. In particular we set

$$\psi_v(x) = \begin{cases} \psi_{\mathbb{R}}(x) := \exp(2\pi\sqrt{-1}x) & \text{if } v = \mathbb{R}, \\ \psi_{\mathbb{C}}(x) := \psi_{\mathbb{R}} \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(x) & \text{if } v = \mathbb{C}. \end{cases}$$

Now the normalization factor $r(w, \pi_v)$ for $M(w, \pi_v)$ is given as follows (cf. [KS] §§2–3, [Ar2] §3).

(1) The case of the simple reflection w_1. In this case $(U_0)_{w_1}^\wedge = \widehat{U}_0 \cap \widehat{M}_1$ and $\widetilde{G}_{\alpha_1, D} = \mathrm{Res}_{k'_v/k_v} SL(2)$ in the notation of [KS] §2. Thus

$$r(w_1, \pi_v) := \frac{L(0, \pi_v, r_{w_1})}{L(1, \pi_v, r_{w_1})\varepsilon(0, \pi_v, r_{w_1}, \psi_v)},$$

where

$$L(s,\pi_v,r_{w_1}) := \begin{cases} L_{k'}(s,\mu_{1,w}\mu_{2,w}^{-1}) & \text{if } v \text{ is inert in } k', \\ L_{k'}(s,\mu_{1,w_1}\mu_{2,w_1}^{-1})L_{k'}(s,\mu_{1,w_2}\mu_{2,w_2}^{-1}) & \text{if } v \text{ splits in } k', \end{cases}$$

$$\varepsilon(s,\pi_v,r_{w_1},\psi_v) := \begin{cases} \lambda(k'_w/k_v,\psi_v)\varepsilon_{k'}(s,\mu_{1,w}\mu_{2,w}^{-1},\,\psi_v\circ \mathrm{Tr}_{k'_w/k_v}) & \text{if } v \text{ is inert in } k', \\ \varepsilon_{k'}(s,\mu_{1,w_1}\mu_{2,w_1}^{-1},\psi_v)\varepsilon_{k'}(s,\mu_{1,w_2}\mu_{2,w_2}^{-1},\psi_v) & \text{if } v \text{ splits in } k'. \end{cases}$$

(2) The case of the simple reflection w_2. We have $(U_0)^{\wedge}_{w_2} = \widehat{U}_0 \cap \widehat{M}_2$ and $\widetilde{G}_{\alpha_2,D} = SL(2)_k$. Hence by [KS] §2,

$$r(w_2,\pi_v) := \frac{L(0,\pi_v,r_{w_2})}{L(1,\pi_v,r_{w_2})\varepsilon(0,\pi_v,r_{w_2},\psi_v)},$$

where

$$L(s,\pi_v,r_{w_2}) := \begin{cases} L_k(s,\mu_{2,w}|_{k_v^\times}) & \text{if } v \text{ is inert in } k', \\ L_k(s,\mu_{2,w_1}\mu_{2,w_2}) & \text{if } v \text{ splits in } k', \end{cases}$$

$$\varepsilon(s,\pi_v,r_{w_2},\psi_v) := \begin{cases} \varepsilon_k(s,\mu_{2,w}|_{k_v^\times},\psi_v) & \text{if } v \text{ is inert in } k', \\ \varepsilon_k(s,\mu_{2,w_1}\mu_{2,w_2},\psi_v) & \text{if } v \text{ splits in } k'. \end{cases}$$

(3) The case of general $w \in W$. Each $w \in W$ can be written as a product of simple reflections w_i $(i = 1,2)$. We take such an expression $w = w_{i_1}w_{i_2}\cdots w_{i_k}$ $(0 \le k \le 4)$, which is reduced. Then the normalization factor is given by

$$r(w,\pi_v) = r(w_{i_1},\, w_{i_2}\cdots w_{i_k}\pi_v)r(w_{i_2},\, w_{i_3}\cdots w_{i_k}\pi_v)\cdots r(w_{i_k},\,\pi_v).$$

This does not depend on the choice of the reduced expression of w.

For $w \in W$, write $\mathrm{inv}(w)$ for the set of positive roots $\alpha \in R^+(P_0,A_0)$ such that $w\alpha \notin R^+(P_0,A_0)$. Then the analytic behavior of the local operators $M(w,\pi_v)$ is stated as follows.

Lemma 3.1. *(1) $M(w,\pi_v)\phi_v(\pi_v)$, $r(w,\pi_v)$ are holomorphic and non-zero at $\pi_v \in \mathfrak{P}_v$ such that $\langle \mathrm{Re}\pi_v,\, \alpha^\vee\rangle > 0$ for any $\alpha \in \mathrm{inv}(w)$.*
(2) The normalized intertwining operator $N(w,\pi_v) := r(w,\pi_v)^{-1}M(w,\pi_v)$ is unitary at $\pi_v \in \mathfrak{P}_v$ with $\langle \mathrm{Re}\pi_v,\, \alpha^\vee\rangle = 0$ for any $\alpha \in \mathrm{inv}(w)$.

Proof. (i) The holomorphy of $M(w,\pi_v)\phi_v(\pi_v)$ was proved in [BW] IV 4.3, 4.5 if v is archimedean and in [*loc. cit*] XI 2.6 if v is non-archimedean. The statement for $r(w,\pi_v)$ is clear from its definition.

(ii) This is a special case of [Ar2] Theorem 2.1 when v is archimedean, and of [KS] Theorem 3.1 when v is non-archimedean. □

3.1.2. *Analytic behavior of global intertwining operators.* We go back to the global situation and define

$$r(w,\pi) := \prod_v r(w,\pi_v), \quad N(w,\pi) := r(w,\pi)^{-1}M(w,\pi).$$

For $\pi = \mu_1 \otimes \mu_2 \in \mathfrak{P}$ and $\alpha \in R^+(P_0,A_0)$, define a quasi-character $\alpha^\vee(\pi)$ by

$$\alpha_1^\vee(\pi) := \mu_1\mu_2^{-1}, \quad \alpha_2^\vee(\pi) := \mu_2|_{\mathbb{A}^\times}, \quad \beta_1^\vee(\pi) := \mu_1(\mu_2\circ\widetilde{\sigma}), \quad \beta_2^\vee(\pi) := \mu_1|_{\mathbb{A}^\times}.$$

Lemma 3.2. *(1) $N(w,\pi)\phi(\pi)$ is holomorphic at $\pi \in \mathfrak{P}$ with $\langle \mathrm{Re}\pi, \alpha^\vee \rangle > 0$ for any $\alpha \in \mathrm{inv}(w)$.*
(2) $r(w,\pi)$ has simple poles at $\pi \in \mathfrak{P}$ with

$$\alpha^\vee(\pi) = \begin{cases} |\ |_{\mathbb{A}_{k'}} & \text{if } \alpha = \alpha_1 \text{ or } \beta_1, \\ |\ |_{\mathbb{A}} & \text{if } \alpha = \alpha_2 \text{ or } \beta_2, \end{cases}$$

for some $\alpha \in \mathrm{inv}(w)$, and is holomorphic at other $\pi \in \mathfrak{P}$ with $\langle \mathrm{Re}\pi, \alpha^\vee \rangle > 0$.

Proof. (i) For each $\mathfrak{P}$ and $\phi \in P_{(M_0,\mathfrak{P})}$, we take a finite set S of places of k, so that if $v \notin S$ then $\pi_v \in \mathfrak{P}_v$ is unramified and $\phi_v(\pi_v)$ equals $\phi^0_{\pi_v}$, the K_v-fixed vector in $\mathrm{Ind}_{P_0(k_v)}^{G(k_v)}[\pi_v \otimes \mathbf{1}_{N_0(k_v)}]$ with $\phi^0_{\pi_v}(1) = 1$. Then the Gindikin-Karpelevich formula ([La2] p. 45) gives

$$M(w,\pi_v)\phi_v(\pi_v) = M(w,\pi_v)\phi^0_{\pi_v} = r(w,\pi_v)\phi^0_{\pi_v} \qquad \text{at } v \notin S,$$

and hence

$$\begin{aligned} M(w,\pi)\phi(\pi) &= \bigotimes_{v\in S} r(w,\pi_v)N(w,\pi_v)\phi_v(\pi_v) \otimes \bigotimes_{v\notin S} r(w,\pi_v)\phi^0_{\pi_v} \\ &= r(w,\pi)\bigotimes_{v\in S} N(w,\pi_v)\phi_v(\pi_v) \otimes \bigotimes_{v\notin S} \phi^0_{\pi_v} \\ N(w,\pi)\phi(\pi) &= \bigotimes_{v\in S} N(w,\pi_v)\phi_v(\pi_v) \otimes \bigotimes_{v\notin S} \phi^0_{\pi_v}. \end{aligned}$$

Now the assertion follows from Lemma 3.1.

(ii) This is clear from the definition of $r(w,\pi)$ and fundamental properties of Hecke L-functions. □

3.1.3. *Singular hyperplanes.* We write $S^{h,+}_{(M,\mathfrak{P})}$ for the set of elements in $S^h_{(M,\mathfrak{P})}$, which intersect the closure of the "positive chamber"

$$\{\pi \in \mathfrak{P}\,;\, \langle \mathrm{Re}\pi, \alpha^\vee \rangle > 0, \text{ for any } \alpha \in R^+(P, A_M)\}.$$

Then from Lemma 3.2, we have

$$S^{h,+}_{(M_0,\mathfrak{P})} = \left\{ \begin{array}{l} \mathfrak{S}_1 := \{\, \pi \in \mathfrak{P}\,;\, \alpha_1^\vee(\pi) = |\ |_{\mathbb{A}_{k'}}. \,\} \\ \mathfrak{S}_2 := \{\, \pi \in \mathfrak{P}\,;\, \alpha_2^\vee(\pi) = |\ |_{\mathbb{A}}. \,\} \\ \mathfrak{S}_3 := \{\, \pi \in \mathfrak{P}\,;\, \beta_1^\vee(\pi) = |\ |_{\mathbb{A}_{k'}}. \,\} \\ \mathfrak{S}_4 := \{\, \pi \in \mathfrak{P}\,;\, \beta_2^\vee(\pi) = |\ |_{\mathbb{A}}. \,\} \end{array} \right\}.$$

3.2. Decomposition of the scalar product.

3.2.1. *Notations.* Recall some general notations from [MW] V.2.1. Let $P = MU$ be a standard parabolic subgroup. Take $\mathfrak{X} \in \mathfrak{E}_P$ and $(M,\mathfrak{P}) \in \mathfrak{X}$. $S^+_{(M,\mathfrak{P})}$ denotes the set of intersections of elements in $S^{h,+}_{(M,\mathfrak{P})}$. For $\mathfrak{S} \in S^+_{(M,\mathfrak{P})}$, we define its origin $o(\mathfrak{S}) \in \mathfrak{a}^*_M$ to be $\mathrm{Re}(\mathfrak{S}) \cap (\mathrm{Re}(\mathfrak{S}^0))^\perp$. Here $(\ )^\perp$ denotes the orthogonal complement in $\mathfrak{a}^*_M$ with respect to the fixed $W(A_M)$-invariant inner product. We write $d_{\mathfrak{S}}\pi$ for the Lebesgue measure on $\mathfrak{S}$ defined by the $W(A_M)$-invariant metric on $\mathfrak{a}^*_M$. Recall that $z(\mathfrak{S}) \in \mathrm{Re}(\mathfrak{S})$ is *general but near* $o(\mathfrak{S})$ if for any $\mathfrak{S}' \in S^+_{(M,\mathfrak{P})}$ properly contained in $\mathfrak{S}$,

(1) $z(\mathfrak{S}) \notin \mathrm{Re}(\mathfrak{S}')$
(2) If $\mathrm{Re}(\mathfrak{S}')$ intersects the ball of radius $||z(\mathfrak{S}) - o(\mathfrak{S})||$ with the center $o(\mathfrak{S})$, then $o(\mathfrak{S})$ is contained in $\mathrm{Re}(\mathfrak{S}')$.

3.2.2. *The first step.* Going back to the situation $\mathfrak{X} \in \mathfrak{E}_{P_0}$, $(M_0, \mathfrak{P}) \in \mathfrak{X}$, we have (cf. Figure 1):

$$S^+_{(M_0,\mathfrak{P})} = \left\{ \begin{array}{ll} \mathfrak{P} & (\mathrm{codim}_{\mathfrak{a}_0^*}\mathrm{Re}(\mathfrak{S}) = 0) \\ \mathfrak{S}_1,\ \mathfrak{S}_2,\ \mathfrak{S}_3,\ \mathfrak{S}_4 & (\mathrm{codim}_{\mathfrak{a}_0^*}\mathrm{Re}(\mathfrak{S}) = 1) \\ \mathfrak{S}_{1,2} := \mathfrak{S}_1 \cap \mathfrak{S}_2,\ \mathfrak{S}_{2,4} := \mathfrak{S}_2 \cap \mathfrak{S}_4, & \\ \mathfrak{S}_{2,3} := \mathfrak{S}_2 \cap \mathfrak{S}_3,\ \mathfrak{S}_{1,3} := \mathfrak{S}_1 \cap \mathfrak{S}_3, & \\ \mathfrak{S}_{1,4} := \mathfrak{S}_1 \cap \mathfrak{S}_4 & (\mathrm{codim}_{\mathfrak{a}_0^*}\mathrm{Re}(\mathfrak{S}) = 2) \end{array} \right\}. \tag{3.3}$$

$$\begin{aligned} & o(\mathfrak{P}) = 0 \\ & o(\mathfrak{S}_1) = \alpha_1, \quad o(\mathfrak{S}_2) = \frac{\alpha_2}{2}, \quad o(\mathfrak{S}_3) = \beta_1, \quad o(\mathfrak{S}_4) = \frac{\beta_2}{2} \\ & o(\mathfrak{S}_{i,j}) = \mathrm{Re}(\mathfrak{S}_{i,j}), \quad (1 \le i < j \le 4). \end{aligned} \tag{3.4}$$

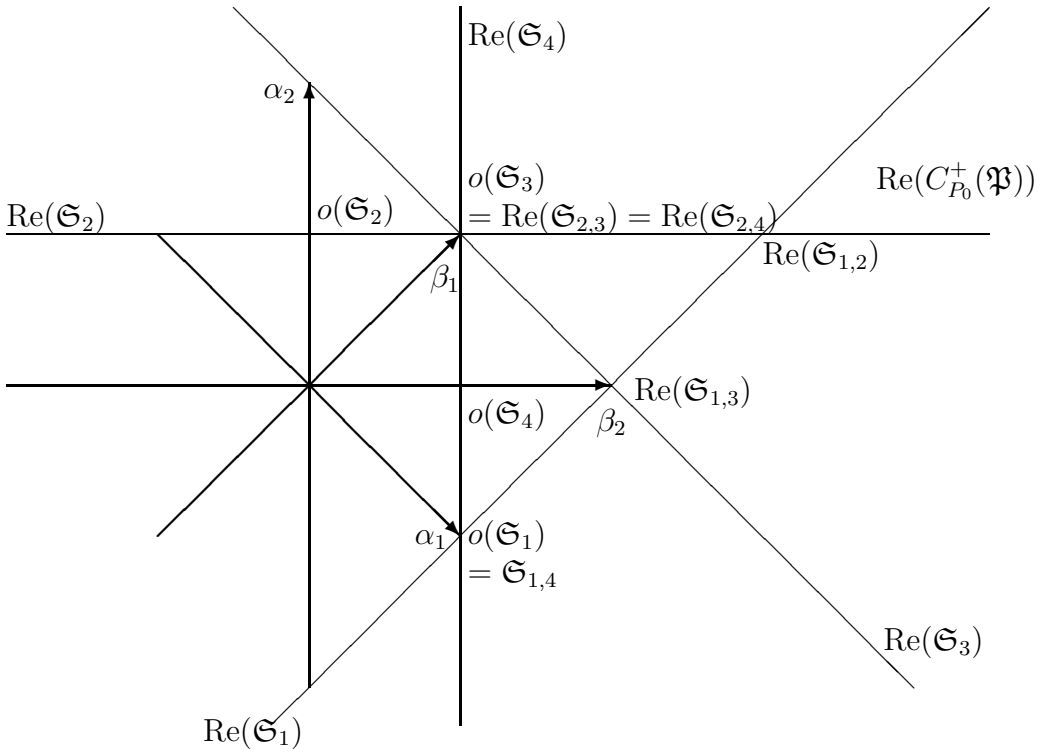

FIGURE 1. $\mathrm{Re}(\mathfrak{S})$ $(\mathfrak{S} \in S^+_{(M_0,\mathfrak{P})})$.

Recall that $\lambda_0 \in \mathfrak{a}_0^*$ in the formula (3.1) is contained in $\mathrm{Re}(C^+_{P_0}(\mathfrak{P}))$. We denote E_0 for the set $\{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \mathfrak{S}_4\}$. Fix a path Γ from λ_0 to $o(\mathfrak{P})$ as in Figure 2. This is *general* in the sense of [MW] V.1.5. We put $y_{\mathfrak{S}} := \mathrm{Re}(\mathfrak{S}) \cap \Gamma$ for $\mathfrak{S} \in E_0$.

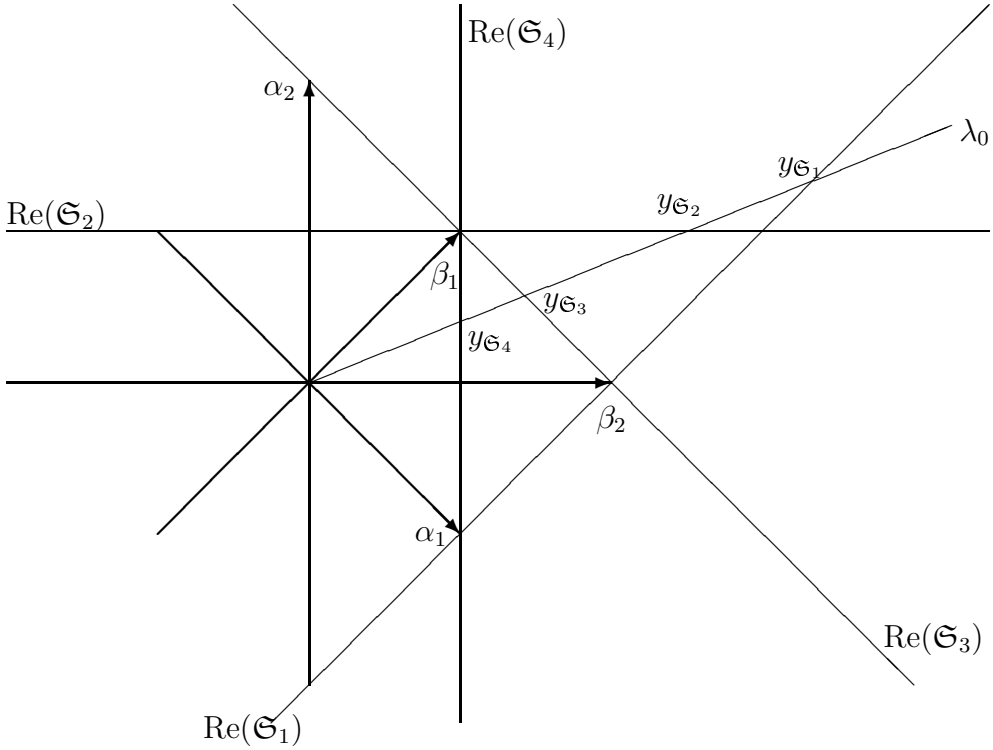

FIGURE 2. The path Γ.

The first step to decompose the scalar product is to move the integration axis from $\mathrm{Re}\pi = \lambda_0$ to $\mathrm{Re}\pi = 0$. But before doing this, we must estimate the integrands to apply the residue theorem. For each $w \in W$ we write

$$A_w(\phi, \phi')(\pi) := \langle M(w,\pi)\phi(\pi), \phi'(-w\overline{\pi})\rangle = r(w,\pi)\langle N(w,\pi)\phi(\pi), \phi'(-w\overline{\pi})\rangle.$$

$r(w,\pi)$ is decomposed into a product of the form

$$r(w,\pi) = (\varepsilon(0,\chi_1)\cdots\varepsilon(0,\chi_p))^{-1}\frac{L(0,\chi_1)\cdots L(0,\chi_p)}{L(1,\chi_1)\cdots L(1,\chi_p)},$$

where $L(s,\chi_i)$ are some Hecke L-functions and $\varepsilon(s,\chi_i)$ are their root numbers. We write $L_\infty(s,\chi)$ and $L_{fin}(s,\chi)$ for the infinite and finite factor of $L(s,\chi)$ respectively. Since $\varepsilon(s,\chi_i)$ are certain exponential functions in s, they are bounded in every vertical strips.

It is well-known that $L_{fin}(s,\chi_i)$ are of the order of a polynomial in $\mathrm{Im}(s)$ in any vertical strip. Also a classical calculation shows that $L_{fin}(s+1,\chi_i)^{-1}$ is of the order of a polynomial in $\log(\mathrm{Im}(s))$ in the region $\mathrm{Re}(s) > -\delta$ for some sufficiently small positive δ. For these facts see [Ay]. Finally we use the Stirling's formula to see that $L_\infty(s,\chi_i)/L_\infty(s+1,\chi_i)$ are slowly increasing as $\mathrm{Im}(s)$ tends to infinity in every vertical strips in $\mathrm{Re}(s) \geq 0$. Now we recall that we have taken ϕ and ϕ' to be of Paley-Wiener type, so that they decrease rapidly as $\mathrm{Im}(s)$ goes to infinity. Summing up, the term $A_w(\phi,\phi')(\pi)$ tends to 0 as $\mathrm{Im}(s)$ goes to infinity, uniformly in each vertical strip. Thus we can apply the usual residue theorem to them.

Applying this to the integrand $A(\phi,\phi')(\pi)$ when it gets across each element of E_0, we obtain

$$\begin{aligned}\langle \theta_\phi,\ \theta_{\phi'}\rangle_{L^2(G(k)\backslash G(\mathbb{A}))} &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{\pi\in\mathfrak{P},\mathrm{Re}\pi=0} A(\phi,\phi')(\pi)\, d\pi \\ &+ \frac{1}{2\pi\sqrt{-1}} \int_{\pi\in\mathfrak{S}_1,\mathrm{Re}\pi=y_{\mathfrak{S}_1}} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1} A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_1}\pi \\ &+ \frac{1}{2\pi\sqrt{-1}} \int_{\pi\in\mathfrak{S}_2,\mathrm{Re}\pi=y_{\mathfrak{S}_2}} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2} A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_2}\pi \\ &+ \frac{1}{2\pi\sqrt{-1}} \int_{\pi\in\mathfrak{S}_3,\mathrm{Re}\pi=y_{\mathfrak{S}_3}} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3} A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_3}\pi \\ &+ \frac{1}{2\pi\sqrt{-1}} \int_{\pi\in\mathfrak{S}_4,\mathrm{Re}\pi=y_{\mathfrak{S}_4}} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_4} A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_4}\pi.\end{aligned} \tag{3.5}$$

Here $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}} A(\phi,\phi')$ denotes the residue of $A(\phi,\phi')$ along $\mathfrak{S}$, which is a meromorphic function on $\mathfrak{S}$.

3.2.3. *Calculation of the residues.* To carry out the second step, we must first calculate the residues $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}} A(\phi,\phi')(\pi)$ appeared in (3.5). We write $L_k(s)$ and $\varepsilon_k(s)$ for the Dedekind zeta function and its root number of k, and $L_{k'}(s)$ and $\varepsilon_{k'}(s)$ for those of k'. c_k and $c_{k'}$ denote the residues

$$c_k := \mathrm{Res}_{s=1}\frac{L_k(s)}{L_k(s+1)\varepsilon_k(s)}, \quad c_{k'} := \mathrm{Res}_{s=1}\frac{L_{k'}(s)}{L_{k'}(s+1)\varepsilon_{k'}(s)}.$$

Then the residues are calculated as:

Proposition 3.3. *The residues* $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}} A(\phi,\phi')(\pi)$ *are calculated as follows* ($\pi \in \mathfrak{S}$).

(i)
$$\begin{aligned}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1} A(\phi,\phi')(\pi) = \sqrt{2}c_{k'}\Big[&\langle N(w_1,\pi)\phi(\pi),\ \phi'(-w_1\overline{\pi})\rangle \\ &+ \langle M(w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\ \phi'(-w_2w_1\overline{\pi})\rangle \\ &+ \langle M(w_1w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\ \phi'(-w_1w_2w_1\overline{\pi})\rangle \\ &+ \langle M(w_2w_1w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\ \phi'(-w_-\overline{\pi})\rangle\Big].\end{aligned}$$

(ii)
$$\begin{aligned}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2} A(\phi,\phi')(\pi) = c_k\Big[&\langle N(w_2,\pi)\phi(\pi),\ \phi'(-w_2\overline{\pi})\rangle \\ &+ \langle M(w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\ \phi'(-w_1w_2\overline{\pi})\rangle \\ &+ \langle M(w_2w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\ \phi'(-w_2w_1w_2\overline{\pi})\rangle \\ &+ \langle M(w_1w_2w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\ \phi'(-w_-\overline{\pi})\rangle\Big].\end{aligned}$$

(iii)
$$\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3} A(\phi,\phi')(\pi)$$

$$
\begin{aligned}
&= \sqrt{2}c_{k'}\Big[\langle N(w_1, w_2\pi)M(w_2,\pi)\phi(\pi),\, \phi'(-w_1w_2\overline{\pi})\rangle \\
&\quad + \langle M(w_2, w_1w_2\pi)N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\, \phi'(-w_2w_1w_2\overline{\pi})\rangle \\
&\quad + \langle N(w_1, w_2w_1\pi)M(w_2w_1,\pi)\phi(\pi),\, \phi'(-w_1w_2w_1\overline{\pi})\rangle \\
&\quad + \langle M(w_1w_2, w_1w_2\pi)N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\, \phi'(-w_-\overline{\pi})\rangle\Big].
\end{aligned}
$$

(iv)
$$
\begin{aligned}
&\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_4}A(\phi,\phi')(\pi) \\
&= c_k\Big[\langle N(w_2, w_1\pi)M(w_1,\pi)\phi(\pi),\, \phi'(-w_2w_1\overline{\pi})\rangle \\
&\quad + \langle N(w_2, w_1w_2\pi)M(w_1w_2,\pi)\phi(\pi),\, \phi'(-w_2w_1w_2\overline{\pi})\rangle \\
&\quad + \langle M(w_1, w_2w_1\pi)N(w_2,w_1\pi)M(w_1,\pi)\phi(\pi),\, \phi'(-w_1w_2w_1\overline{\pi})\rangle \\
&\quad + \langle M(w_2w_1, w_2w_1\pi)N(w_2,w_1\pi)M(w_1,\pi)\phi(\pi),\, \phi'(-w_-\overline{\pi})\rangle\Big].
\end{aligned}
$$

Proof. The proof is straightforward and will be omitted. Note that

$$
\frac{d_{\mathfrak{S}_i}\pi}{d\alpha^\vee(\pi)} = \frac{\|\alpha\|}{\langle \alpha,\, \alpha^\vee\rangle} = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \alpha \text{ is a short root,} \\ 1 & \text{if } \alpha \text{ is a long root.} \end{cases}
$$

□

3.2.4. *The second step.* In the second step, we move the integration axis $\mathrm{Re}\pi = y_{\mathfrak{S}}$ in the right hand side of (3.5) to $\mathrm{Re}\pi = z(\mathfrak{S})$, where $z(\mathfrak{S})$ is general but near $o(\mathfrak{S})$ in the sense of 3.2.1. Then we take the limit as $z(\mathfrak{S})$ tends to $o(\mathfrak{S})$.

(1) Contributions from $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)$. We begin with the second term in the right hand side of (3.5). We take $z(\mathfrak{S}_1)$ to be $o(\mathfrak{S}_1)$ and apply the residue theorem to have

$$
\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_1,\mathrm{Re}\pi=y_{\mathfrak{S}_1}} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_1}\pi \\
&\quad = \frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_1,\mathrm{Re}\pi=o(\mathfrak{S}_1)} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_1}\pi \\
&\qquad + \mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,2}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\mathfrak{S}_{1,2}) + \mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\mathfrak{S}_{1,3}).
\end{aligned} \tag{3.6}
$$

We shall study each terms in the right hand side.

(1-i) The first term: We have to check the well-definedness of

$$
\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_1,\mathrm{Re}\pi=o(\mathfrak{S}_1)} \mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)\, d_{\mathfrak{S}_1}\pi
$$

when $\mathfrak{S}_{1,4} = \mathfrak{S}_1 \cap \mathfrak{S}_4$ is non-empty. In this case $\mathfrak{S}_{1,4} = \mu_1 \otimes \mu_2$ satisfies $\mu_1\mu_2^{-1} = |\ |_{\mathbb{A}_{k'}}$, $\mu_1|_{\mathbb{A}^\times} = |\ |_{\mathbb{A}}$ and in particular $\sigma(\mu_1)\mu_2 = \alpha_1^\vee(w_2w_1\pi) = \mathbf{1}$. We know from Proposition 3.3 (i) that

$$
\begin{aligned}
&\frac{1}{\sqrt{2}c_{k'}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi) \\
&\quad = \langle N(w_1,\pi)\phi(\pi),\, \phi'(-w_1\overline{\pi})\rangle + r(w_2w_1w_2,\, w_1\pi)\langle N(w_-,\pi)\phi(\pi),\, \phi'(-w_-\overline{\pi})\rangle \\
&\qquad + r(w_2,\, w_1\pi)\Big[\langle N(w_2w_1,\,\pi)\phi(\pi),\, \phi'(-w_2w_1\overline{\pi})\rangle \\
&\qquad\qquad + \langle M(w_1,\, w_2w_1\pi)N(w_2w_1,\,\pi)\phi(\pi),\, \phi'(-w_1w_2w_1\overline{\pi})\rangle\Big].
\end{aligned} \tag{3.7}
$$

First consider

$$r(w_2w_1w_2,\, w_1\pi) = \Big(r(w_2,\, w_1w_2w_1\pi)r(w_2,\, w_1\pi)\Big)\cdot r(w_1,\, w_2w_1\pi).$$

The simple zero of the first term kills the simple pole of the second term. Hence the first two terms on the right hand side of (3.7) is well-defined at $\mathfrak{S}_{1,4}$. As for the rest terms, we know from [KS] Proposition 6.3 that $M(w_1,\, w_2w_1\mathfrak{S}_{1,4}) = (-1)\mathrm{Id}$. Thus inside $[\ ]$ has a simple zero at $\pi \in \mathfrak{S}_{1,4}$, which kills the simple pole of $r(w_2,\, w_1\pi)$. The well-definedness is proved.

(1-ii) The second term: We calculate the residue $\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,2}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)$. The terms

$$\langle N(w_1,\pi)\phi(\pi),\, \phi'(-w_1\overline{\pi})\rangle, \quad \langle M(w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\, \phi'(-w_2w_1\overline{\pi})\rangle$$
$$\langle M(w_1w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\, \phi(-w_1w_2w_1\overline{\pi})\rangle$$

are holomorphic at $\mathfrak{S}_{1,2}$, while

$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,2}}\langle M(w_2w_1w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\, \phi'(-w_-\overline{\pi})\rangle\\
&= \frac{d_{\mathfrak{S}_1}\pi}{d\alpha_1^\vee(\pi)}c_k\langle N(w_2,w_1w_2w_1\mathfrak{S}_{1,2})M(w_1w_2,w_1\mathfrak{S}_{1,2})N(w_1,\,\mathfrak{S}_{1,2})\phi(\mathfrak{S}_{1,2}),\, \phi'(-w_-\overline{\mathfrak{S}}_{1,2})\rangle\\
&= \sqrt{2}c_k\langle N(w_2,w_1w_2w_1\mathfrak{S}_{1,2})M(w_1w_2,w_1\mathfrak{S}_{1,2})N(w_1,\,\mathfrak{S}_{1,2})\phi(\mathfrak{S}_{1,2}),\, \phi'(-w_-\overline{\mathfrak{S}}_{1,2})\rangle.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,2}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\mathfrak{S}_{1,2})\\
&= 2c_kc_{k'}\langle N(w_2,w_1w_2w_1\mathfrak{S}_{1,2})M(w_1w_2,w_1\mathfrak{S}_{1,2})N(w_1,\mathfrak{S}_{1,2})\phi(\mathfrak{S}_{1,2}),\, \phi'(-w_-\overline{\mathfrak{S}}_{1,2})\rangle.
\end{aligned} \tag{3.8}$$

(1-iii) The third term: We calculate the residue $\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)$. Among the terms on the right hand side of Proposition 3.3 (i),

$$\langle N(w_1,\pi)\phi(\pi),\, \phi'(-w_1\overline{\pi})\rangle, \quad \langle M(w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\, \phi'(-w_2w_1\overline{\pi})\rangle$$

are holomorphic. As for the rest two terms, we have the following two possibilities. Note that writing $\mathfrak{S}_{1,3} = \mu_1\otimes\mu_2$, we have $\mu_2\sigma(\mu_2) = \mu_2\circ \mathrm{N}_{k'/k} = \mathbf{1}$.

Case 1. $\mu_2|_{\mathbb{A}^\times} = \eta_{k'/k}$.
In this case, noting $M(w_2, w_1w_2w_1\mathfrak{S}_{1,3}) = \mathrm{Id}$ by [KS] Proposition 6.3, we have

$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,3}}\langle M(w_1w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\, \phi'(-w_1w_2w_1\overline{\pi})\rangle\\
&\qquad = \sqrt{2}c_{k'}\langle N(w_1,w_2w_1\mathfrak{S}_{1,3})M(w_2,w_1\mathfrak{S}_{1,3})N(w_1,\,\mathfrak{S}_{1,3})\phi(\mathfrak{S}_{1,3}),\, \phi'(-w_-\overline{\mathfrak{S}}_{1,3})\rangle,
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,3}}\langle M(w_2w_1w_2,w_1\pi)N(w_1,\pi)\phi(\pi),\, \phi'(-w_-\overline{\pi})\rangle\\
&= \sqrt{2}c_{k'}\langle N(w_1,w_2w_1\mathfrak{S}_{1,3})M(w_2,w_1\mathfrak{S}_{1,3})N(w_1,\,\mathfrak{S}_{1,3})\phi(\mathfrak{S}_{1,3}),\, \phi'(-w_-\overline{\mathfrak{S}}_{1,3})\rangle
\end{aligned} \tag{3.10}$$

Case 2. $\mu_2|_{\mathbb{A}^\times}$ is trivial.
The formula (3.9) still holds. But in this case, $M(w_2, w_1w_2w_1\mathfrak{S}_{1,3}) = (-1)\mathrm{Id}$ by [KS] Proposition 6.3. Thus the formula (3.10) must be multiplied by -1.

In both cases, we take summations of the two terms and have

(3.11)
$$\begin{aligned}&\mathrm{Res}^{\mathfrak{S}_1}_{\mathfrak{S}_{1,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)\\&=\begin{cases}4c_{k'}^2\langle N(w_1,w_2w_1\mathfrak{S}_{1,3})M(w_2,w_1\mathfrak{S}_{1,3})N(w_1,\,\mathfrak{S}_{1,3})\phi(\mathfrak{S}_{1,3}),\,\phi'(-w_-\overline{\mathfrak{S}}_{1,3})\rangle\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } \mu_2|_{\mathbb{A}^\times}=\eta_{k'/k},\\ 0\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } \mu_2|_{\mathbb{A}^\times}\text{ is trivial.}\end{cases}\end{aligned}$$

(2) Contributions from $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)$. Next comes the third term in the right hand side of (3.5). We take $z(\mathfrak{S}_2)$ to be $o(\mathfrak{S}_2)$ and apply the residue theorem to see

$$\begin{aligned}(3.12)\quad&\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_2,\mathrm{Re}\pi=y_{\mathfrak{S}_2}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_2}\pi\\&=\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_2,\mathrm{Re}\pi=o(\mathfrak{S}_2)}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_2}\pi+\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\\&\qquad\qquad+\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\end{aligned}$$

We study each terms on the right hand side in turn. The first term has no problem.

(2-i) The second term: We calculate the residue $\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)$. First note that $\mathfrak{S}_{2,3}=\mu_1\otimes\mu_2$ satisfies $\mu_2|_{\mathbb{A}^\times}=|\ |_{\mathbb{A}}$, $\mu_1\sigma(\mu_2)=|\ |_{\mathbb{A}^\times}$ and hence $\sigma(\mu_1\mu_2^{-1})=\alpha_1^\vee(w_2w_1w_2\pi)$ is trivial. Among the terms in the right hand side of Proposition 3.3 (ii), $\langle N(w_2,\pi)\phi(\pi),\,\phi'(-w_2\overline{\pi})\rangle$ is holomorphic at $\mathfrak{S}_{2,3}$, and

$$\begin{aligned}&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,3}}\langle M(w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\,\phi'(-w_1w_2\overline{\pi})\rangle\\&\qquad=2c_{k'}\frac{d_{\mathfrak{S}_2}\pi}{d\beta_1^\vee(\pi)}\langle N(w_1w_2,\,\mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\,\phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle\\&\qquad\qquad=2c_{k'}\langle N(w_1w_2,\,\mathfrak{S}_{2,3}),\,\phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle.\end{aligned}$$

As for the rest two terms, we have

$$\begin{aligned}&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,3}}\Big[\langle M(w_2w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\,\phi'(-w_2w_1w_2\overline{\pi})\rangle\\&\qquad\qquad+\langle M(w_1w_2w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\,\phi'(-w_-\overline{\pi})\rangle\Big]\\&=2c_{k'}\lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_2}}\Big[r(w_2,w_1w_2\pi)\Big(M(w_1,w_2w_1w_2\pi)+\mathrm{Id}\Big)\Big]\\&\qquad\qquad\times\langle N(w_2w_1w_2,\,\mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\,\phi'(-w_-\overline{\mathfrak{S}}_{2,3})\rangle.\end{aligned}$$

From [KS] Proposition 6.3, $M(w_1,w_2w_1w_2\pi)+\mathrm{Id}$ has a simple zero at $\mathfrak{S}_{2,3}$, which kills the simple pole of $r(w_2,\,w_1w_2\pi)$. Thus the limit exists. Summing up, we have

(3.13)
$$\begin{aligned}&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\\&=2c_kc_{k'}\Big[\langle N(w_1w_2,\,\mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\,\phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle\\&\qquad+\lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_2}}[r(w_2,\,w_1w_2\pi)\Big(M(w_1,w_2w_1w_2\pi)+\mathrm{Id}\Big)]\\&\qquad\qquad\times\langle N(w_2w_1w_2,\,\mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\,\phi'(-w_-\overline{\mathfrak{S}}_{2,3})\rangle\Big]\end{aligned}$$

(2-ii) The third term: Calculate the residue $\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)$. $\mathfrak{S}_{2,4}=\mu_1\otimes\mu_2$ satisfies $\mu_1|_{\mathbb{A}^\times}=\mu_2|_{\mathbb{A}^\times}=|\ |_{\mathbb{A}}$. Since the case $\mu_1=\mu_2$ has already been treated in (2-i), we assume $\mu_1\neq\mu_2$. Then

$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\langle N(w_2,\pi)\phi(\pi),\ \phi'(-w_2\overline{\pi})\rangle=0\\
&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\langle M(w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\ \phi'(-w_1w_2\overline{\pi})\rangle=0\\
&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\langle M(w_2w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\ \phi'(-w_2w_1w_2\overline{\pi})\rangle\\
&=c_k\langle N(w_2,w_1w_2\mathfrak{S}_{2,4})M(w_1,w_2\mathfrak{S}_{2,4})N(w_2,\ \mathfrak{S}_{2,4})\phi(\mathfrak{S}_{2,4}),\ \phi'(-w_2w_1w_2\overline{\mathfrak{S}}_{2,4})\rangle\\
&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\langle M(w_1w_2w_1,w_2\pi)N(w_2,\pi)\phi(\pi),\ \phi'(-w_-\overline{\pi})\rangle\\
&=c_k\langle M(w_1,w_2w_1w_2\mathfrak{S}_{2,4})N(w_2,w_1w_2\mathfrak{S}_{2,4})M(w_1,w_2\mathfrak{S}_{2,4})N(w_2,\ \mathfrak{S}_{2,4})\phi(\mathfrak{S}_{2,4}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\phi'(-w_-\overline{\mathfrak{S}}_{2,4})\rangle.
\end{aligned}$$

Hence

(3.14)
$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{S}_2}_{\mathfrak{S}_{2,4}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\\
&=c_k^2\langle N(w_2,w_1w_2\mathfrak{S}_{2,4})M(w_1,w_2\mathfrak{S}_{2,4})N(w_2,\ \mathfrak{S}_{2,4})\phi(\mathfrak{S}_{2,4}),\ \phi'(-w_2w_1w_2\overline{\mathfrak{S}}_{2,4})\rangle\\
&\quad+c_k^2\langle M(w_1,w_2w_1w_2\mathfrak{S}_{2,4})N(w_2,w_1w_2\mathfrak{S}_{2,4})M(w_1,w_2\mathfrak{S}_{2,4})N(w_2,\ \mathfrak{S}_{2,4})\phi(\mathfrak{S}_{2,4}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\phi'(-w_-\overline{\mathfrak{S}}_{2,4})\rangle.
\end{aligned}$$

(3) Contributions from $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi)$**.** We choose $z(\mathfrak{S}_3)$ to be any point on the segment tying $y_{\mathfrak{S}_3}$ and $o(\mathfrak{S}_3)$ but not to be $o(\mathfrak{S}_3)$. We use the principal value integration theorem, at first formally, to have

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_3,\mathrm{Re}\pi=y_{\mathfrak{S}_3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_3}\pi\\
=&\frac{1}{4\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_3,\mathrm{Re}\pi=z(\mathfrak{S}_3)}\Big[\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi)+\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(w_1\pi)\Big]\,d_{\mathfrak{S}_3}\pi\\
&+\frac{1}{2}\mathrm{Res}^{\mathfrak{S}_3}_{\mathfrak{S}_{2,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi).
\end{aligned}\tag{3.15}$$

(3-i) The first term: First we need to check the limit

$$\lim_{z(\mathfrak{S}_3)\to o(\mathfrak{S}_3)}\int_{\pi\in\mathfrak{S}_3,\mathrm{Re}\pi=z(\mathfrak{S}_3)}\frac{1}{2}\Big[\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi)+\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(w_1\pi)\Big]\,d_{\mathfrak{S}_3}\pi\tag{3.16}$$

exists. From Proposition 3.3 (iii), we know that

$$\begin{aligned}
&\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi)+\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(w_1\pi)\\
\text{(I)}\quad&=\sqrt{2}c_{k'}\Big[\langle N(w_1,w_2\pi)\Big(M(w_2,\pi)\phi(\pi)+M(w_2w_1,w_1\pi)\phi(w_1\pi)\Big),\ \phi'(-w_1w_2\overline{\pi})\rangle\\
\text{(II)}\quad&\quad+\langle N(w_1,w_2w_1\pi)\Big(M(w_2,w_1\pi)\phi(w_1\pi)+M(w_2w_1,\pi)\phi(\pi)\Big),\ \phi'(-w_1w_2w_1\overline{\pi})\rangle\\
\text{(III)}\quad&\quad+\Big(\langle M(w_2,w_1w_2\pi)N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\ \phi'(-w_2w_1w_2\overline{\pi})\rangle\\
&\qquad+\langle M(w_1w_2,w_1w_2w_1\pi)N(w_1,w_2w_1\pi)M(w_2,w_1\pi)\phi(w_1\pi),\ \phi'(-w_2w_1w_2\overline{\pi})\rangle\Big)
\end{aligned}$$

(IV)

$$+\Big(\langle M(w_1w_2,w_1w_2\pi)N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\ \phi'(-w_-\overline{\pi})\rangle + \langle M(w_2,w_1w_2w_1\pi)N(w_1,w_2w_1\pi)M(w_2,w_1\pi)\phi(w_1\pi),\ \phi'(-w_-\overline{\pi})\rangle\Big)\Big]$$

As for (I), we have

$$M(w_2,\pi)\phi(\pi)+M(w_2w_1,w_1\pi)\phi(w_1\pi) = r(w_2,\pi)N(w_2,\pi)\Big[\phi(\pi)+M(w_1,w_1\pi)\phi(w_1\pi)\Big].$$

Since $\alpha_1(\mathfrak{S}_{2,3})=\mathbf{1}$, we see from [KS] Proposition 6.3 that $\phi(\mathfrak{S}_{2,3})+M(w_1,w_1\mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3})=0$. This kills the simple pole of $r(w_2,\pi)$ and (I) is well-defined at $\mathfrak{S}_{2,3}$. (II) can be treated in the same manner. The terms (III) and (IV) must be handled together. (III) becomes

$$\Big\langle\Big(r(w_2,w_1w_2\pi)r(w_2,\pi)N(w_2w_1w_2,\pi)\phi(\pi)+r(w_1w_2,w_1w_2w_1\pi)r(w_2,w_1\pi) N(w_-,w_1\pi)\phi(w_1\pi)\Big),\ \phi'(-w_2w_1w_2\overline{\pi})\Big\rangle$$
$$=\Big\langle r(w_2,w_1w_2\pi)r(w_2,\pi)N(w_2w_1w_2,\pi)\Big(\phi(\pi)+M(w_1,w_1\pi)\phi(w_1\pi)\Big), \phi'(-w_2w_1w_2\overline{\pi})\Big\rangle,$$

while (IV) becomes

$$\Big\langle r(w_1w_2,w_1w_2\pi)r(w_2,\pi)N(w_-,\pi)\phi(\pi) + r(w_2,w_1w_2w_1\pi)r(w_2,w_1\pi)N(w_2w_1w_2,w_1\pi)\phi(w_1\pi),\ \phi'(-w_-\overline{\pi})\Big\rangle$$
$$\Big\langle M(w_1,w_2w_1w_2\pi)r(w_2,w_1w_2\pi)r(w_2,\pi)N(w_2w_1w_2,\pi) \times\Big(\phi(\pi)+M(w_1,w_1\pi)\phi(w_1\pi)\Big),\ \phi'(-w_-\overline{\pi})\Big\rangle$$

We now apply Proposition 6.3 of [KS] to $M(w_1,w_1\pi)$ and to $M(w_1,w_2w_1w_2\pi)$ to see that (III)+(IV) is well-defined at $\mathfrak{S}_{2,3}$. Thus the limit (3.16) exists.

(3-ii) The second term: We calculate the half of the residue $\mathrm{Res}^{\mathfrak{S}_3}_{\mathfrak{S}_{2,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi)$. Among those terms appeared in the right hand side of Proposition 3.3 (iii), the first two have the residue

$$\mathrm{Res}^{\mathfrak{S}_3}_{\mathfrak{S}_{2,3}}\Big[\langle N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\ \phi'(-w_1w_2\overline{\pi})\rangle + \langle N(w_1,w_2w_1\pi)M(w_2w_1,\pi)\phi(\pi),\ \phi'(-w_1w_2w_1\overline{\pi})\rangle\Big]$$
$$=\frac{d_{\mathfrak{S}_3}\pi}{d\alpha_2^\vee(\pi)}c_k\Big[\langle N(w_1w_2,\ \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\ \phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle - \langle N(w_1w_2,w_1\mathfrak{S}_{2,3})M(w_1,\ \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\ \phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle\Big]$$
$$=-2\sqrt{2}c_k\langle N(w_1w_2,\ \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\ \phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle$$

Here we have used $w_1\mathfrak{S}_{2,3}=\mathfrak{S}_{2,3}$ and $M(w_1,\ \mathfrak{S}_{2,3})=(-1)\mathrm{Id}$ (Proposition 6.3 in [KS]). As for the rest two terms, we have

$$\mathrm{Res}^{\mathfrak{S}_3}_{\mathfrak{S}_{2,3}}\Big[\langle M(w_2,w_1w_2\pi)N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\ \phi'(-w_2w_1w_2\overline{\pi})\rangle + \langle M(w_1w_2,w_1w_2\pi)N(w_1,w_2\pi)M(w_2,\pi)\phi(\pi),\ \phi'(-w_-\overline{\pi})\rangle\Big]$$

$$
\begin{aligned}
=&\mathrm{Res}^{\mathfrak{S}_3}_{\mathfrak{S}_{2,3}} r(w_2, w_1w_2\pi)r(w_2,\pi)\Big[\langle N(w_2w_1w_2,\pi)\phi(\pi),\, \phi'(-w_2w_1w_2\overline{\pi})\rangle \\
&\qquad + \langle M(w_1, w_2w_1w_2\pi)N(w_2w_1w_2,\pi)\phi(\pi),\, \phi'(-w_-\overline{\pi})\rangle\Big] \\
=&\frac{d_{\mathfrak{S}_3}\pi}{d\alpha_2^\vee(\pi)}c_k \lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_3}}\Big[r(w_2, w_1w_2\pi)\Big(M(w_1, w_2w_1w_2\pi)+\mathrm{Id}\Big)\Big] \\
&\qquad \times \langle N(w_2w_1w_2,\, \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\, \phi'(-w_-\overline{\mathfrak{S}}_{2,3})\rangle \\
=&2\sqrt{2}c_k \lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_2}}\Big[r(w_2, w_1w_2\pi)\Big(M(w_1, w_2w_1w_2\pi)+\mathrm{Id}\Big)\Big] \\
&\qquad \times \langle N(w_2w_1w_2,\, \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\, \phi'(-w_-\overline{\mathfrak{S}}_{2,3})\rangle.
\end{aligned}
$$

Note that $M(w_1, w_2w_1w_2\mathfrak{S}_{2,3}) = (-1)\mathrm{Id}$ by [KS] Proposition 6.3, hence the limit in the above formula exists. Also we have used

$$\lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_3}}(\quad) = 2\lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_2}}(\quad).$$

Now we conclude

$$
\begin{aligned}
&\frac{1}{2}\mathrm{Res}^{\mathfrak{S}_3}_{\mathfrak{S}_{2,3}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(\pi) \\
&= -2c_kc_{k'}\Big[\lim_{\substack{\pi\to\mathfrak{S}_{2,3}\\ \pi\in\mathfrak{S}_2}}\Big[r(w_2, w_1w_2\pi)\Big(M(w_1, w_2w_1w_2\pi)+\mathrm{Id}\Big)\Big] \\
&\qquad \times \langle N(w_2w_1w_2,\, \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\, \phi'(-w_-\overline{\mathfrak{S}}_{2,3})\rangle \\
&\qquad + \langle N(w_1w_2,\, \mathfrak{S}_{2,3})\phi(\mathfrak{S}_{2,3}),\, \phi'(-w_1w_2\overline{\mathfrak{S}}_{2,3})\rangle\Big]
\end{aligned}
\tag{3.17}
$$

(4) Contributions from $\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_4}A(\phi,\phi')(\pi)$**.** $z(\mathfrak{S}_4)$ is chosen to be $o(\mathfrak{S}_4)$. From Cauchy's integration theorem, we have

$$
\begin{aligned}
&\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_4,\mathrm{Re}\pi=y_{\mathfrak{S}_4}}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_4}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_4}\pi \\
&\qquad = \frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_4,\mathrm{Re}\pi=o(\mathfrak{S}_4)}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_4}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_4}\pi.
\end{aligned}
\tag{3.18}
$$

3.2.5. *The final form of the inner product formula.* Now we compile the formulae (3.6), (3.19), (3.20), (3.12), (3.13), (3.21), (3.15), (3.17) and (3.18) and apply them to (3.5).

Theorem 3.4. *Take* $\mathfrak{X}\in\mathfrak{E}_{P_0}$ *and* $(M_0,\mathfrak{P})\in\mathfrak{X}$*. Then the* L^2*-inner product of* θ_ϕ ($\phi\in P_{(M_0,\mathfrak{P})}$) *and* $\theta_{\phi'}$ ($\phi'\in P_{\mathfrak{X}}$) *is given by*

$$
\begin{aligned}
\langle\theta_\phi,\theta_{\phi'}\rangle_{L^2(G(k)\backslash G(\mathbb{A}))} &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2\int_{\pi\in\mathfrak{P},\mathrm{Re}\pi=0}A(\phi,\phi')(\pi)\,d\pi \\
&+\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_1,\mathrm{Re}\pi=o(\mathfrak{S}_1)}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_1}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_1}\pi
\end{aligned}
$$

$$+\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_2,\mathrm{Re}\pi=o(\mathfrak{S}_2)}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_2}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_2}\pi$$

$$+\lim_{z(\mathfrak{S}_3)\to o(\mathfrak{S}_3)}\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_3,\mathrm{Re}\pi=z(\mathfrak{S}_3)}\frac{1}{2}\sum_{w=1,w_1}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_3}A(\phi,\phi')(w\pi)\,d_{\mathfrak{S}_3}\pi$$

$$+\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{S}_4,\mathrm{Re}\pi=o(\mathfrak{S}_4)}\mathrm{Res}^{\mathfrak{P}}_{\mathfrak{S}_4}A(\phi,\phi')(\pi)\,d_{\mathfrak{S}_4}\pi$$

(3.19)
$$+2c_kc_{k'}\langle N(w_2,w_1w_2w_1\mathfrak{S}_{1,2})M(w_1w_2,\,w_1\mathfrak{S}_{1,2})N(w_1,\,\mathfrak{S}_{1,2})\phi(\mathfrak{S}_{1,2}),\,\phi'(-w_-\overline{\mathfrak{S}}_{1,2})\rangle$$

(3.20)
$$+4c_{k'}^2\langle N(w_1,w_2w_1\mathfrak{S}_{1,3})M(w_2,w_1\mathfrak{S}_{1,3})N(w_1,\,\mathfrak{S}_{1,3})\phi(\mathfrak{S}_{1,3}),\,\phi'(-w_-\overline{\mathfrak{S}}_{1,3})\rangle$$

(3.21)
$$+4c_{k'}^2\Big[\langle N(w_2,w_1w_2\mathfrak{S}_{2,4})M(w_1,w_2\mathfrak{S}_{2,4})N(w_2,\,\mathfrak{S}_{2,4})\phi(\mathfrak{S}_{2,4}),\,\phi'(-w_2w_1w_2\overline{\mathfrak{S}}_{2,4})\rangle$$
$$+\langle M(w_1,w_2w_1w_2\mathfrak{S}_{2,4})N(w_2,w_1w_2\mathfrak{S}_{2,4})M(w_1,w_2\mathfrak{S}_{2,4})N(w_2,\,\mathfrak{S}_{2,4})\phi(\mathfrak{S}_{2,4}),$$
$$\phi'(-w_-\overline{\mathfrak{S}}_{2,4})\rangle\Big].$$

Here in the term (3.21), $\mathfrak{S}_{2,4}\neq\mathfrak{S}_{2,3}$.

3.3. The residual spectrum from P_0. In this subsection, we describe the contributions of cuspidal data attached to P_0 to the residual spectrum. Only the terms (3.19), (3.20) and (3.21) contribute to the residual discrete spectrum. To determine these contributions, we shall study the images of intertwining operators appearing in these terms. We need some results from [KSw] and [LeZ].

3.3.1. *Review of the results of Kudla, Sweet, Lee and Zhu.* Their results describe the complete irreducible decomposition of certain degenerate principal series representations of $U(n,n)$ (inert case) and $GL(2n)$ (split case) over a local field. Though our review concentrates on the case $n=2$, their results cover the general rank case.

(1) The non-archimedean inert case. Take a non-archimedean place v of k which is inert in k', and let w be the place of k' lying over v. For each character $\chi_v: {k'}_w^\times\to\mathbb{C}^1$ and $s\in\mathbb{C}$, we write $I(s,\chi_v)$ for the degenerate principal series representation;

$$I(s,\chi_v):=\mathrm{Ind}_{P_1(k_v)}^{G(k_v)}[(\chi_v\circ\det)|\det|_w^s\otimes\mathbf{1}_{U_1(k_v)}].$$

Proposition 3.5 (Theorem 1.1 in [KSw]**).** *(1) $I(s,\chi_v)$ is reducible if and only if $\chi_v\circ\mathrm{N}_{k'_w/k_v}$ is trivial.*
(2) If $\chi_v|_{k_v^\times}=\mathbf{1}$, then the points of reducibility are $s_0=-1,\ 0,\ 1$.
(3) If $\chi_v|_{k_v^\times}=\eta_{k'_w/k_v}$, then the points of reducibility are $s_0=-1/2,\ 1/2$.
Here $\eta_{k'_w/k_v}$ is the quadratic character of $k_v^\times$ corresponding to k'_w/k_v by the classfield theory.

To describe the irreducible constituents of $I(s,\chi_v)$ at its reducible points, we need oscillator representations for unitary dual pairs. We write a reducible point $s_0=(m-2)/2$ where m equals one of $m=0,1,2,3,4$. We first consider the case $m\neq0$. The m-dimensional non-degenerate Hermitian spaces $(V,\langle\ ,\ \rangle_V)$ over k'_w are classified as follows.

(1) When $m=1$, there are, up to equivalence, two possibilities

$$V_{1,i}:=k'_w,\quad \langle x,y\rangle_{V_{1,i}}:=xQ_i\tilde{\sigma}(y),\qquad (i=1,2),$$

where $Q_1=1$ and $Q_2=\gamma\in k_v^\times-\mathrm{N}_{k'_w/k_v}({k'}_w^\times)$.

(2) When $m=2$, there are two possibilities up to equivalence

$$V_{2,i} := k_w'^{\oplus 2}, \quad \langle x,y\rangle_{V_{2,i}} := xQ_i\widetilde{\sigma}({}^t y), \qquad (i=1,2),$$

where

$$Q_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 := \begin{pmatrix} \gamma & 0 \\ 0 & -1 \end{pmatrix}.$$

γ is same as in (1).

(3) If $3 \le m \le 4$, then there are two possibilities $V_{m,1}$ and $V_{m,2}$, where

$$V_{m,i} := \begin{cases} V_{1,i} \oplus V_{2,1} & \text{if } m=3, \\ V_{2,i} \oplus V_{2,1} & \text{if } m=4. \end{cases}, \qquad (i=1,2).$$

We write $U(V_{m,i})$ for the unitary group over k_v attached to $(V_{m,i}, \langle\ ,\ \rangle_{V_{m,i}})$. $(W, (\ ,\ )_W)$ denotes the anti-Hermitian space attached to G_v;

$$W := k_w'^{\oplus 4}, \quad (x,y)_W := xJ_2\widetilde{\sigma}({}^t y).$$

Then we have the symplectic space

$$\Big(\mathbf{W}_{m,i} := W \otimes_{k_w'} V_{m,i},\ \langle\ ,\ \rangle_{\mathbf{W}} := \mathrm{Tr}_{k_w'/k_v}\big((\ ,\ )_W \otimes \widetilde{\sigma}(\langle\ ,\ \rangle_{V_{m,i}})\big)\Big),$$

and write $Sp(\mathbf{W}_{m,i})$ for its symplectic group. $G_v \times U(V_{m,i})$ is a dual reductive pair in $Sp(\mathbf{W}_{m,i})$.

Fix a non-trivial character $\psi_v : k_v \to \mathbb{C}^1$. Then we have the metaplectic group

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp(\mathbf{W}_{m,i})_v \longrightarrow Sp(\mathbf{W}_{m,i}, k_v) \longrightarrow 1,$$

and its oscillator representation ω_{ψ_v}. We identify $Mp(\mathbf{W}_{m,i})_v$ with $Sp(\mathbf{W}_{m,i}, k_v) \times \mathbb{C}^1$ as a set. The multiplication law is given by

$$(g_1,\epsilon_1)(g_2,\epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2 c_v(g_1,g_2)), \quad (g_i \in Sp(\mathbf{W}_{m,i}, k_v),\ \epsilon_i \in \mathbb{C}^1)$$

where $c_v(g_1,g_2)$ is the metaplectic two cocycle calculated in [P].

Note that our χ_v satisfies $\chi_v|_{k_v^\times} = \eta_{k_w'/k_v}^m$. Then from [Ku] Theorem 3.1, we have the splitting

$$\iota_{\chi_v} : G(k_v) \times U(V_{m,i}, k_v) \longrightarrow Mp(\mathbf{W}_{m,i})_v$$

attached to χ_v. Hence we can define the auxiliary oscillator representation ω'_{ψ_v,χ_v} of $G(k_v) \times U(V_{m,i}, k_v)$ by $\omega'_{\psi_v,\chi_v} := \omega_{\psi_v} \circ \iota_{\chi_v}$. For simplicity, we twist the action of $U(V_{m,i}, k_v)$ by the character $(\chi_v \circ \det)^{-1}$ and obtain the oscillator representation ω_{ψ_v,χ_v}. Some explicit formulae for its Schrödinger model $\mathcal{S}(V_{m,i}^{\oplus 2})$ are given by

$$\omega_{\psi_v,\chi_v}\Big(\begin{pmatrix} A & \mathbf{0}_2 \\ \mathbf{0}_2 & \widetilde{\sigma}({}^t A^{-1}) \end{pmatrix}\Big)\Phi(x) = \chi_v(\det A)|\det A|_w^{m/2}\Phi(xA), \quad (A \in GL(2,k_w')) \tag{3.22}$$

$$\omega_{\psi_v,\chi_v}\Big(\begin{pmatrix} \mathbf{1}_2 & B \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}\Big)\Phi(x) = \psi_v(\mathrm{tr}(\langle x,x\rangle_{V_{m,i}} B))\Phi(x), \quad (B \in \mathrm{Her}_2(k_w'/k_v)) \tag{3.23}$$

$$\omega_{\psi_v,\chi_v}\Big(\begin{pmatrix} \mathbf{0}_2 & -\mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix}\Big)\Phi(x) = \frac{1}{\gamma_{m,i}^2}\int_{V_{m,i}^{\oplus 2}} \Phi(y)\psi_v(\mathrm{Tr}_{k_w'/k_v}(\mathrm{tr}\langle x,y\rangle_{V_{m,i}}))\,dy. \tag{3.24}$$

$$\omega_{\psi_v,\chi_v}(h)\Phi(x) = \Phi(h^{-1}x), \quad (h \in H(k_v)) \tag{3.25}$$

Here, $\langle x, y\rangle_{V_{m,i}}$ denotes the matrix

$$\begin{pmatrix} \langle x_1, y_1\rangle_{V_{m,i}} & \langle x_1, y_2\rangle_{V_{m,i}} \\ \langle x_2, y_1\rangle_{V_{m,i}} & \langle x_2, y_2\rangle_{V_{m,i}} \end{pmatrix}, \quad \text{for } x = (x_1, x_2),\ y = (y_1, y_2) \in V_{m,i}^{\oplus 2},$$

and the measure in (3.24) is chosen to be self-dual with respect to the pairing $\mathrm{tr}\langle\ ,\ \rangle_{V_{m,i}}$. $\mathrm{Her}_2(k'_w/k_v)$ denotes the space of 2×2 Hermitian matrices over k'_w, and $\gamma^2_{m,i}$ is the usual Weil constant for $\mathrm{Tr}_{k'_w/k_v}(\langle\ ,\ \rangle_{V_{m,i}})$ with respect to ψ_v (cf. [P]).

Now we write $R(V_{m,i}, \chi_v)$ for the image of the map

$$\mathcal{S}(V_{m,i}^{\oplus 2}) \ni \Phi \longrightarrow \Big(g \to \omega_{\psi_v,\chi_v}(g)\Phi(0)\Big) \in I(s_0, \chi_v)$$

(recall that $s_0 = (m-2)/2$). Then Rallis's coinvariant theorem ([Ra] Theorem II.1.1) extended to the general case in [MVW] Chapter 3-IV Théorème 7 asserts that

$$R(V_{m,i}, \chi_v) \simeq (\omega_{\psi_v,\chi_v})_{U(V_{m,i},k_v)},\ \text{(the } U(V_{m,i}, k_v)\text{-coinvariant space)}.$$

Next we consider the case $m = 0$. $U(1)_{k'/k}$ denotes the unitary group

$$U(1)_{k'/k} := \{g \in \mathrm{Res}_{k'/k}\mathbb{G}_m\,;\, \tilde{\sigma}(g)^{-1} = g\}.$$

Then we have the k-homomorphism $\det : G \to U(1)_{k'/k}$. When $m = 0$, χ_v satisfies $\chi_v|_{k_v^\times} = \mathbf{1}$. This allows us to define a character χ'_v of $U(1, k_v)_{k'/k}$ by $\chi'_v(x\tilde{\sigma}(x)^{-1}) := \chi_v(x)$. (Note that every $u \in U(1, k_v)_{k'/k}$ can be written in the form $x\tilde{\sigma}(x)^{-1}$ for some $x \in {k'}_w^\times$ by Hilbert 90.) We write χ_v^G for the 1-dimensional representation $\chi'_v \circ \det$ of $G(k_v)$. We define $R(0, \chi_v) := \chi_v^G$.

Proposition 3.6 (Theorem 1.2 and Proposition 5.8 in [KSw]**).** *(1) The representation $R(V_{4,1}, \chi_v)$ equals $I(1, \chi_v)$ and $R(V_{4,2}, \chi_v)$ is the unique irreducible submodule of $I(1, \chi_v)$. The image of the normalized intertwining operator $N(w_2w_1w_2,\ (\chi_v \circ \det)|\det|_w)$ equals $R(0, \chi_v)$ and its kernel equals $R(V_{4,2}, \chi_v)$.*
(2) $R(V_{1,1}, \chi_v)$ and $R(V_{1,2}, \chi_v)$ are two inequivalent irreducible submodules of $I(-1/2, \chi_v)$. $R(V_{3,1}, \chi_v)$ and $R(V_{3,2}, \chi_v)$ are two distinct maximal submodules of $I(1/2, \chi_v)$. The normalized operator $N(w_2w_1w_2,\ (\chi_v \circ \det)|\det|_w^{1/2})$ induces isomorphisms

$$R(V_{3,i}, \chi_v)\Big/(R(V_{3,1}, \chi_v) \cap R(V_{3,2}, \chi_v)) \xrightarrow{\sim} R(V_{1,i}, \chi_v), \quad (i = 1, 2).$$

(2) The non-archimedean split case. Let v be a finite place of k which splits in k', and let w_1, w_2 be the places of k' lying over v. Then $G_v = GL(4)_{k_v}$ and

$$M_{1,v} = \left\{ \begin{pmatrix} A & \mathbf{0}_2 \\ \mathbf{0}_2 & D \end{pmatrix} \middle|\ A,\ D \in GL(2)_{k_v} \right\}, \quad U_{1,v} = \left\{ \begin{pmatrix} \mathbf{1}_2 & B \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix} \middle|\ B \in \mathbb{M}_2 \right\}.$$

For each character $\chi_v = \chi_{w_1} \otimes \chi_{w_2}$ of k'_v and $s \in \mathbb{C}$, define

$$I(s, \chi_v) := \mathrm{Ind}_{P_1(k_v)}^{G(k_v)}[\Big((\chi_{w_1} \circ \det)|\det|_v^s \otimes (\chi_{w_2}^{-1} \circ \det)|\det|_v^{-s}\Big) \otimes \mathbf{1}_{U_1(k_v)}].$$

Then we have

Proposition 3.7 (Theorem 1.3 in [KSw]**).** *(1) $I(s, \chi_v)$ is reducible if and only if $\chi_v \circ \mathrm{N}_{k'_v/k_v} = \chi_{w_1}\chi_{w_2} = \mathbf{1}$ and $s = s_0$ with $s_0 \in \{-1, -1/2, 0, 1/2, 1\}$.*
(2) $I(s, \chi_v)$ in (1) at its reducible point $s = s_0$ has a unique irreducible submodule A, and the quotient $I(s_0, \chi_v)\big/A$ is also irreducible. Moreover the irreducible quotient of

$I(1/2,\chi_v)$ ($I(1,\chi_v)$ resp.) and the irreducible submodule of $I(-1/2,\chi_v)$ ($I(-1,\chi_v)$ resp.) are both isomorphic to

$$\mathrm{Ind}_{P_{(3,1)}(k_v)}^{G(k_v)}[\Big((\chi_{w_1}\circ\det)\otimes\chi_{w_2}^{-1}\Big)\otimes\mathbf{1}_{U_{(3,1)}(k_v)}],\quad (\chi_{w_1}\circ\det\ \textit{resp.}).$$

Here $P_{(3,1)}=M_{(3,1)}U_{(3,1)}$ is the standard parabolic subgroup of G_v such that $M_{(3,1)}\simeq GL(3)\times\mathbb{G}_m$.
(3) Above irreducible representations are related to the Weil representation as follows. Write $s_0=(m-2)/2$ with $m\in\{0,1,\dots,4\}$, and consider the dual reductive pair $G_v\times H_v$ with $H_v:=GL(m)_{k_v}$. We can construct an oscillator representation ω_{ψ_v,χ_v} of $G(k_v)\times H(k_v)$, for which explicit formulae for its Schrödinger model $\mathcal{S}(k_v^{\oplus 4m})$ are given by

(3.26)

$$\omega_{\psi_v,\chi_v}\Big(\begin{pmatrix}A&\mathbf{0}_2\\ \mathbf{0}_2&D\end{pmatrix}\Big)\Phi(x,y)=\chi_{w_1}(\det(AD))|\det(AD)|_v^{m/2}\Phi(xA,y^tD^{-1}),\ (A,D\in GL(2,k_v))$$

$$\omega_{\psi_v,\chi_v}\Big(\begin{pmatrix}\mathbf{1}_2&B\\ \mathbf{0}_2&\mathbf{1}_2\end{pmatrix}\Big)\Phi(x,y)=\psi_v(xB^ty)\Phi(x,y),\quad (B\in\mathbb{M}_2(k_v)) \tag{3.27}$$

$$\omega_{\psi_v,\chi_v}\Big(\begin{pmatrix}\mathbf{0}_2&\mathbf{1}_2\\ -\mathbf{1}_2&\mathbf{0}_2\end{pmatrix}\Big)\Phi(x,y)=\int_{k_v^{\oplus 2m}}\Phi(u,v)\psi_v(u^tx+v^ty)\,dudv \tag{3.28}$$

$$\omega_{\psi_v,\chi_v}(h)\Phi(x,y)=\Phi(h^{-1}(x,y)),\quad (h\in H(k_v)). \tag{3.29}$$

We write $R(V_m,\chi_v)$ for the image of the map

$$\mathcal{S}(k_v^{\oplus 4m})\ni\Phi\longrightarrow\Big(g\to\omega_{\psi_v,\chi_v}(g)\Phi(0)\Big)\in I(\tfrac{m-2}{2},\chi_v).$$

Then

1. *$R(V_3,\chi_v)=I(1/2,\chi_v)$ and $R(V_4,\chi_v)=I(1,\chi_v)$.*
2. *$R(V_1,\chi_v)$ is the unique irreducible submodule of $I(-1/2,\chi_v)$.*
3. *$R(0,\chi_v)$ defined to be $\chi_{w_1}\circ\det$ is the unique irreducible submodule of $I(-1,\chi_v)$.*

(5) These irreducible constituents are related by the normalized intertwining operators as follows.

$$R(V_1,\chi_v)=\mathrm{Im}N\Big(w_2w_1w_2,\ (\chi_{w_1}\circ\det)|\det|_v^{1/2})\otimes(\chi_{w_2}^{-1}\circ\det)|\det|_v^{-1/2})\Big),$$
$$R(0,\chi_v)=\mathrm{Im}N\Big(w_2w_1w_2,\ (\chi_{w_1}\circ\det)|\det|_v)\otimes(\chi_{w_2}^{-1}\circ\det)|\det|_v^{-1})\Big).$$

(3) The archimedean inert case. The situation is $k_w'/k_v=\mathbb{C}/\mathbb{R}$. For each character χ_v of $\mathbb{C}^\times$ and $s\in\mathbb{C}$, we construct $I(s,\chi_v)$ as in the non-archimedean case. We may assume that χ_v is of the form $\chi_v(z)=(z/\overline{z})^{\nu/2}$ ($\nu\in\mathbb{Z}$).

Proposition 3.8 (Theorem 6.2 in [Le]**).** *The reducible points of $I(s,\chi_v)$ are $s_0\in\nu/2+\mathbb{Z}$.*

Next we describe the irreducible constituents of $I(s,\chi_v)$ at its reducible points s_0. For our purpose, it is sufficient to do this at

$$s_0=\begin{cases}-1/2,1/2 & \text{if } \nu\notin 2\mathbb{Z}\\ -1,0,1 & \text{if } \nu\in 2\mathbb{Z}.\end{cases}$$

We again write these s_0 as $(m-2)/2$ ($m=0,1,2,3,4$). For $m\neq 0$ we classify the m-dimensional Hermitian spaces $V_{m,i}$ ($i=1,2$) just as in the non-archimedean inert case (by

replacing γ with -1). For each $V_{m,i}$ we have the oscillator representation $(\omega_{\psi_v,\chi_v}, \mathcal{S}(V_{m,i}^{\oplus 2}))$ of the dual pair $(G(k_v), U(V_{m,i}, k_v))$ as in the non-archimedean case.

Let $V_{m,i} = V_{m,i}^+ \oplus V_{m,i}^-$ be the direct sum decomposition such that $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\langle\ ,\ \rangle_{m,i})$ is positive (negative resp.) definite (as a 2×2 symmetric matrix !) on $V_{m,i}^+$ ($V_{m,i}^-$ resp.). We define an inner product $\langle\ ,\ \rangle_{m,i}^+$ on $V_{m,i}$ by

$$\langle x, y\rangle_{m,i}^+ := \begin{cases} \langle x, y\rangle_{m,i} & \text{if } x,\ y \in V_{m,i}^+, \\ -\langle x, y\rangle_{m,i} & \text{if } x,\ y \in V_{m,i}^-, \\ 0 & \text{if } x \in V_{m,i}^+ \text{ and } y \in V_{m,i}^-. \end{cases}$$

Using this we define the Gaussian function on $V_{m,i}^{\oplus 2}$ by

$$\Phi^0_{V_{m,i}}(x) := \exp(-\pi \mathrm{tr}\langle x, x\rangle_{m,i}^+) \in \mathcal{S}(V_{m,i}^{\oplus 2}).$$

Then the resulting space

$$\mathcal{S}^0(V_{m,i}^{\oplus 2}) := \{\Phi^0_{V_{m,i}}(x)P(x) \,|\, P(x) \text{ is a polynomial.}\}$$

of "standard functions" with respect to ψ_v is the underlying $(\mathfrak{g}_v, K_v)\times(\mathfrak{u}(V_{m,i})_{\mathbb{C}}, K_{m,i})$-bimodule of ω_{ψ_v,χ_v}. Here $\mathfrak{u}(V_{m,i})_{\mathbb{C}}$ is the complexified Lie algebra of $U(V_{m,i},\mathbb{R})$ and $K_{m,i}$ is a certain maximal compact subgroup of $U(V_{m,i},\mathbb{R})$.

Proposition 3.9 (Theorems 4.2 and 5.9 in [LeZ]**).** *Assume that the character χ_v is such that $\chi_v|_{\mathbb{R}^\times} = \mathrm{sgn}^m$ $(m = 0,\dots,4)$.*
(1) If we write $R(V_{m,i},\chi_v)$ for the image of the map

$$\mathcal{S}^0(V_{m,i}^{\oplus 2}) \ni \Phi \longrightarrow (g \to \omega_{\psi_v,\chi_v}(g)\Phi(0)) \in I(\tfrac{m-2}{2}, \chi_v),$$

then this map induces an isomorphism $\mathcal{S}^0(V_{m,i}^{\oplus 2})_{(\mathfrak{u}(V_{m,i})_{\mathbb{C}}, K_{m,i})} \xrightarrow{\sim} R(V_{m,i},\chi_v)$.
(2) $R(V_{4,1},\chi_v)$ equals $I(1,\chi_v)$ and $R(V_{4,2},\chi_v)$ is an irreducible submodule. The kernel of the normalized intertwining operator

$$N(w_2w_1w_2,\ (\chi_v\circ\det)|\det|_{\mathbb{C}}) : I(1,\chi_v) \longrightarrow I(-1,\chi_v)$$

equals $R(V_{4,2},\chi_v)$ and its image is χ_v^G. Here χ_v^G is defined similarly as in the non-archimedean case.
(3) $R(V_{3,i},\chi_v)$ $(i=1,2)$ are two distinct maximal submodules of $I(1/2,\chi_v)$. $R(V_{1,i},\chi_v)$ $(i=1,2)$ are two inequivalent irreducible submodules of $I(-1/2,\chi_v)$. The normalized intertwining operator $N(w_2w_1w_2,\ (\chi_v\circ\det)|\det|_{\mathbb{C}}^{1/2})$ induces isomorphisms

$$R(V_{3,i},\chi_v)\Big/(R(V_{3,1},\chi_v)\cap R(V_{3,1},\chi_v)) \xrightarrow{\sim} R(V_{1,i},\chi_v), \quad (i=1,2).$$

(2) Archimedean split case. In this case we have $G(k_v) = GL(4,\mathbb{R})$ or $GL(4,\mathbb{C})$. For each character $\chi_v = \chi_{w_1}\otimes\chi_{w_2}$ of $(k_v')^\times$ and $s\in\mathbb{C}$, we define $I(s,\chi_v)$ as in the non-archimedean case. Then the following is essentially a classical result of E. M. Stein.

Proposition 3.10 (Lemma 2.4 in [V]**).** *(1) $I(s,\chi_v)$ is irreducible unless $\chi_{w_1}\chi_{w_2} = \mathbf{1}$.*
(2) If $\chi_{w_1}\chi_{w_2} = \mathbf{1}$, then it is reducible at $s = s_0$ with $s_0 \in \{-1,-1/2,1/2,1\}$. The unique irreducible submodule of $I(-1/2,\chi_v)$ and the unique irreducible quotient of $I(1/2,\chi_v)$ are both isomorphic to $\mathrm{Ind}_{P_{(3,1)}(k_v)}^{G(k_v)}[\left((\chi_{w_1}\circ\det)\otimes\chi_{w_2}^{-1}\right)\otimes\mathbf{1}_{U_{(3,1)}(k_v)}]$. Here $P_{(3,1)}$ is as in Proposition 3.7. An isomorphism between them is given by $N(w_2w_1w_2,\ (\chi_{w_1}\circ\det)|\det|_v^{1/2}\otimes(\chi_{w_2}^{-1}\circ\det)|\det|_v^{-1/2})$.

(3) We adopt those notations of Proposition 3.7 *on Weil representations. Then* $R(V_3,\chi_v) = I(1/2,\chi_v)$ *and* $R(1,\chi_v) = \mathrm{Ind}_{P_{(3,1)}(k_v)}^{G(k_v)}[\big((\chi_{w_1}\circ\det)\otimes\chi_{w_2}^{-1}\big)\otimes\mathbf{1}_{U_{(3,1)}(k_v)}]$.

3.3.2. *The contributions of* (3.19) *and* (3.20). Let $V_{\mathbb{A}} = (V_v)_v$ be a collection of 1-dimensional Hermitian spaces V_v over k'_v at each v. We fix a non-trivial character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/k$. For each character $\chi = \bigotimes_v \chi_v$ of $\mathbb{A}_{k'}^\times/k'^\times$ such that $\chi|_{\mathbb{A}^\times} = \eta_{k'/k}$, we can construct a collection of irreducible representations $R(V_v,\chi_v)$ of $G(k_v)$ using ψ. Moreover $R(V_v,\chi_v)$ is unramified at almost all v and hence we have a smooth irreducible representation $R(V_{\mathbb{A}},\chi) := \bigotimes_v R(V_v,\chi_v)$ of $G(\mathbb{A})$.

Theorem 3.11. *(1) The contribution of* (3.19) *in Theorem* 3.4 *to the residual spectrum is multiplicity free and consists of one dimensional representations* $\chi^G = \chi'\circ\det$. *Here* $\chi' : U(1,\mathbb{A})_{k'/k} \to \mathbb{C}^1$ *is defined by* $\chi'(x\tilde{\sigma}(x)^{-1}) := \chi(x)$, *where* χ *runs over characters of* $\mathbb{A}_{k'}^\times/k'^\times$ *such that* $\chi|_{\mathbb{A}^\times} = \mathbf{1}$.
(2) The contribution of (3.20) *in Theorem* 3.4 *is multiplicity free and consists of* $R(V_{\mathbb{A}},\chi)$, *where* $V_{\mathbb{A}} = \bigotimes_v V_v$ *is obtained from one dimensional Hermitian space* V *over* k' *by* $V_v = V\otimes_k k_v$, *and* χ *is a character of* $\mathbb{A}_{k'}^\times/k'^\times$ *such that* $\chi|_{\mathbb{A}^\times} = \eta_{k'/k}$.

The proof of this will occupy the following three subsubsections. We begin with some review from [KRS].

3.3.3. *Lemmas on coinvariant.* We still fix the nontrivial character ψ of 3.3.2. For brevity we write

$$m(A) := \begin{pmatrix} A & \mathbf{0}_2 \\ \mathbf{0}_2 & \sigma({}^tA^{-1}) \end{pmatrix}\ (a\in \mathrm{Res}_{k'/k}GL(2)),\quad u(B) := \begin{pmatrix} \mathbf{1}_2 & B \\ \mathbf{0}_2 & \mathbf{1}_2 \end{pmatrix}\ (B\in \mathrm{Her}_2(k'/k)),$$

and

$$w := \begin{pmatrix} \mathbf{0}_2 & \mathbf{1}_2 \\ -\mathbf{1}_2 & \mathbf{0}_2 \end{pmatrix}.$$

(1) The non-archimedean case. First take a non-archimedean v which is inert in k'. For each $\beta\in \mathrm{Her}_2(k'_w/k_v)$ we have a character of $U_1(k_v)$ defined by

$$\psi_\beta : U_1(k_v)\ni u(B)\longrightarrow \psi_v(\mathrm{tr}(B\beta))\in\mathbb{C}^1.$$

For each smooth representation V of $G(k_v)$, we define its twisted coinvariant space by

$$V_{U_1,\beta} := V\Big/\mathrm{Span}\{u.\xi-\psi_\beta(u)\xi\,|\,\xi\in V,\ u\in U_1(k_v)\}.$$

Now recall the Hermitian space $V_v = V_{1,i}$ $(i=1,2)$ and the oscillator representation $(\omega_{\psi_v,\chi_v},\mathcal{S}(V_v^{\oplus 2}))$ of $G(k_v)\times U(V_v,k_v)$. For β as above, we set

$$\Omega_\beta(V_v) := \{x\in V_v^{\oplus 2}\,;\,\langle x,x\rangle_{V_v} = \beta\}.$$

Lemma 3.12 (Lemma 1.3 in [KRS]**).** *(1) The canonical projection* $\mathcal{S}(V_v^{\oplus 2})\twoheadrightarrow \mathcal{S}(V_v^{\oplus 2})_{U_1,\beta}$ *factors through the "restriction to* $\Omega_\beta(V_v)$*" map* $\mathcal{S}(V_v^{\oplus 2})\twoheadrightarrow\mathcal{S}(\Omega_\beta(V_v))$.
(2) In particular if $\det\beta\neq 0$, *then* $\Omega_\beta(V_v)$ *consists of a single* $U(V_v,k_v)$*-orbit and the projection* $\mathcal{S}(V_v^{\oplus 2})\twoheadrightarrow(\mathcal{S}(V_v^{\oplus 2})_{U_1,\beta})_{U(V_v,k_v)}$ *is given by*

$$\Phi\longrightarrow\int_{\Omega_\beta(V_v)}\Phi(x)\,d_\beta x.$$

Here $d_\beta x$ *is the unique (up to a factor)* $U(V_v,k_v)$*-invariant measure.*

Proof. (1) is merely a combination of the explicit formula (3.23) and [BZ] Lemma 2.33. As for *(2)*, the first assertion is clear and the second follows from the first and *(1)*. □

The corresponding result at split v is almost trivial and will be omitted.

(2) The archimedean case. Again we treat the inert case only. Thus $k'_w/k_v = \mathbb{C}/\mathbb{R}$. As in the non-archimedean case, we have a character ψ_β of $U_1(\mathbb{R})$ for each $\beta \in \mathrm{Her}_2(\mathbb{C}/\mathbb{R})$. We write $d\psi_\beta : \mathrm{Lie}(U_1(\mathbb{R}))_{\mathbb{C}} \to \mathbb{C}$ for its differential. $\Omega_\beta(V_v)$ is defined similarly as in the non-archimedean case. Again recall the Hermitian space $V_v = V_{1,i}$ $(i = 1, 2)$ and the $(\mathfrak{g}_v, K_v) \times (\mathfrak{u}(V_v)_{\mathbb{C}}, K_{V_v})$-module $(\omega_{\psi_v,\chi_v}, \mathcal{S}^0(V_v^{\oplus 2}))$ and $R(V_v, \chi_v)$. We write $\mathcal{S}^0(V_v^{\oplus 2})^*_\beta$ for the space of $U(V_v, \mathbb{R})$-invariant tempered distributions T satisfying

$$T(d\omega_{\psi_v,\chi_v}(X)\Phi) = d\psi_\beta(X)T(\Phi), \quad \forall X \in \mathrm{Lie}(U_1(\mathbb{R})).$$

Also the space of linear functionals $\mathfrak{L}$ on $R(V_V, \chi_v)$ which satisfies

$$\mathfrak{L}(X.\Phi) = d\psi_\beta(X) \cdot \mathfrak{L}(\Phi), \quad \forall X \in \mathrm{Lie}(U_1(\mathbb{R}))$$

is denoted by $R(V_v, \chi_v)^*_\beta$. Then

Lemma 3.13 (Proposition 1.7 and Corollary 1.8 in [KRS]**).** *(1) If $\Omega_\beta(V_v)$ is empty, then both $(\mathcal{S}^0(V_v^{\oplus 2})^*_\beta)^{U(V_v,\mathbb{R})}$ and $R(V_v, \chi_v)^*_\beta$ are $\{0\}$.*
(2) If $\Omega_\beta(V_v)$ is not empty, then $(\mathcal{S}^0(V_v^{\oplus 2})_\beta)^{U(V_v,\mathbb{R})}$ is spanned by the orbital integral

$$\Phi \longrightarrow \int_{\Omega_\beta(V_v)} \Phi(x)\, d_\beta x.$$

Consequently $\dim R(V_v, \chi_v)^*_\beta = 1$.

Proof. The proof in the symplectic case was given in [Ra2] Lemma 4.2. The unitary case can be treated similarly using [LeZ]. □

3.3.4. *Generalized Whittaker models for $R(V_{\mathbb{A}}, \chi)$.* Here we review some facts about the generalized Whittaker models from [KRS] §2.

Every character of $U_1(\mathbb{A})/U_1(k)$ is of the form

$$\psi_\beta : U_1(\mathbb{A}) \ni u(B) \longrightarrow \psi(\mathrm{tr}(B\beta)) \in \mathbb{C}^1,$$

for some $\beta \in \mathrm{Her}_2(k'/k)$. We restrict ourselves to those ψ_β with $\det \beta \neq 0$. Then the space of generalized Whittaker functionals for an irreducible smooth representation (π, V_π) of $G(\mathbb{A})$ is defined by

$$\mathcal{W}_\beta(\pi) := \left\{ \begin{array}{l|l} \mathfrak{L} : V_\pi \to \mathbb{C} & \text{(i) } \mathfrak{L}(\pi(u)f) = \psi_\beta(u)\mathfrak{L}(f),\ \forall u \in U_1(\mathbb{A}_f) \\ \text{linear map} & \text{(ii) } \mathfrak{L}(d\pi(X)f) = d\psi_\beta(X)\mathfrak{L}(f),\ \forall X \in \mathrm{Lie}U_1(\mathbb{A}_\infty) \end{array} \right\}$$

Now we suppose that (π, V_π) is an automorphic subrepresentation. For each $f \in V_\pi$, we define its β-th Fourier coefficient as

$$W_\beta(f)(g) := \int_{U_1(k)\backslash U_1(\mathbb{A})} f(ug)\overline{\psi_\beta(u)}\, du,$$

and a Whittaker functional $W_\beta \in \mathcal{W}_\beta(\pi)$ by

$$V_\pi \ni f \longrightarrow W_\beta(f)(1) \in \mathbb{C}.$$

Let $V_{\mathbb{A}} = \bigotimes_v V_v$ and $R(V_{\mathbb{A}}, \chi) = \bigotimes_v R(V_v, \chi_v)$ be as in 3.3.2.

Lemma 3.14 (Lemma 2.5 in [KRS]**).** *If an intertwining map D from $R(V_{\mathbb{A}}, \chi)$ to the space of L^2-automorphic forms $L^2(G(k)\backslash G(\mathbb{A}))$ satisfies*

$$W_\beta \circ D = 0, \quad \forall \beta \in \mathrm{Her}_2(k'/k) \text{ with } \det \beta \neq 0,$$

then D is zero.

Proof. We write $\pi = \bigotimes_v \pi_v$ for the representation of $G(\mathbb{A})$ on $R(V_{\mathbb{A}}, \chi)$. Take a finite place v which is unramified or split in k'. We write $\mathcal{O}_{k'_v}$ for the integral closure of $\mathcal{O}_v$ in k'_v.

Let $\mathfrak{O}(V_v) := \{\beta \in \mathrm{Her}_2(k'_v/k_v) \mid \det \beta \neq 0,\ \beta = \langle x, x\rangle_{V_v}\ (\exists x \in V_v^{\oplus 2})\}$. Take a Schwartz function h on $\mathrm{Her}_2(k'_v/k_v)$ such that its Fourier transform

$$\widehat{h}(b) := \int_{\mathrm{Her}_2(k'_v/k_v)} h(B)\psi_v(\mathrm{tr}(Bb))\, dB$$

is the characteristic function of $\mathfrak{O}(V_v) \cap \mathbb{M}_2(\mathcal{O}_{k'_v})$. We write f_h for the function on $U_1(k_v)$ given by $f_h(u(b)) := h(b)$. Then from the definition of $R(V_v, \chi_v)$, we have

$$\pi_v(f_h)\varphi_v(w) = \int_{\mathrm{Her}_2(k'_v/k_v)} \varphi_v(wu(b))h(b)\, db$$

choosing $\Phi_v \in \mathcal{S}(V_v^{\oplus 2})$ which projects to φ_v,

$$= \int_{\mathrm{Her}_2(k'_v/k_v)} \omega_{\psi_v,\chi_v}(wu(b))\Phi_v(0)h(b)\, db$$

from (3.24) or (3.28),

$$= \int_{\mathrm{Her}_2(k'_v/k_v)} \frac{1}{\gamma(\mathrm{Tr}_{k'_v/k_v}(\langle\ ,\ \rangle_{V_v}))^2} \int_{V_v^{\oplus 2}} h(b)\psi_v(\mathrm{tr}(b\langle x, x\rangle_{V_v}))\, db\, \Phi_v(x)\, dx$$
$$= \frac{1}{\gamma(\mathrm{Tr}_{k'_v/k_v}(\langle\ ,\ \rangle_{V_v}))^2} \int_{\substack{x \in V_v^{\oplus 2} \\ \langle x,x\rangle_{V_v} \in \mathfrak{O}(V_v) \cap \mathbb{M}_2(\mathcal{O}_{k'_v})}} \Phi_v(x)\, dx.$$

Thus we can choose Φ_v so that this does not vanish.

Now using this specific Φ_v, we construct $\Phi = \bigotimes_{v' \neq v} \Phi_{v'} \otimes \Phi_v \in \mathcal{S}(V_{\mathbb{A}}^{\oplus 2})$ and write $\varphi = \bigotimes_{v' \neq v} \varphi_{v'} \otimes \varphi_v$ for its image in $R(V_{\mathbb{A}}, \chi)$. Then using the Fourier expansion on $U_1(k)\backslash U_1(\mathbb{A})$, one has

$$R(f_h)D(\varphi)(g) = \int_{U_1(k_v)} D(\varphi)(gu)f_h(u)\, du$$
$$= \int_{U_1(k_v)} \sum_{\beta \in \mathrm{Her}_2(k'/k)} (W_\beta \circ D)(\varphi)(gu)f_h(u)\, du$$

If we take $g_v = 1$, then this equals

$$= \int_{\mathrm{Her}_2(k'_w/k_v)} \sum_{\beta \in \mathrm{Her}_2(k'/k)} \psi_v(\mathrm{tr}(b\beta))h(b)\, db\, (W_\beta \circ D)(\varphi)(g)$$
$$= \sum_{\beta \in \mathrm{Her}_2(k'/k)} \widehat{h}(\beta)(W_\beta \circ D)(\varphi)(g).$$

This is zero from our hypothesis. Since this means $D(\pi(f_h)\varphi)$ restricted to the dense subset $G(k)\prod_{v' \neq v} G(k_{v'})$ of $G(\mathbb{A})$ identically vanishes, $D(\pi(f_h)\varphi)$ is forced to be identically zero. But our choice of φ asserts $\pi(f_h)\varphi \neq 0$. That is, D has non-trivial kernel. Hence $D = 0$ because $R(V_{\mathbb{A}}, \chi)$ is irreducible. □

$\mathrm{Hom}_{G(\mathbb{A})}(R(V_{\mathbb{A}}, \chi), L^2(G(k)\backslash G(\mathbb{A})))$ denotes the space of $G(\mathbb{A}_f) \times (\mathfrak{g}_\infty, K_\infty)$-equivariant maps from $R(V_{\mathbb{A}}, \chi)$ to the space of smooth K_∞-finite vectors in $L^2(G(k)\backslash G(\mathbb{A}))$.

Proposition 3.15 (Theorem 2.2 in [KRS]**).** *Assume $V_{\mathbb{A}}$ comes from a Hermitian space V over k', i.e. $V_{\mathbb{A}} = V \otimes_k \mathbb{A}$. Then* $\dim \mathrm{Hom}_{G(\mathbb{A})}(R(V_{\mathbb{A}}, \chi), L^2(G(k)\backslash G(\mathbb{A}))) \leq 1$.

Proof. Take two elements $A,\ B \in \mathrm{Hom}_{G(\mathbb{A})}(R(V_{\mathbb{A}},\chi), L^2(G(k)\backslash G(\mathbb{A})))$ and set $A_\beta := W_\beta \circ A$ ($B_\beta := W_\beta \circ B$ resp.) for $\beta \in \mathrm{Her}_2(k'/k)$. Also set $\mathfrak{O}(V) := \{\beta \in \mathrm{Her}_2(k'/k)\,;\, \beta = \langle \xi, \xi\rangle_V,\ \exists \xi \in V\}$. If A_β is not zero, then by applying Lemma 3.12 and Lemma 3.13 to $A_\beta|_{R(V_v,\chi_v)}$ and to $B_\beta|_{R(V_v,\chi_v)}$, we see that $\beta \in \mathfrak{O}(V)$ and there exists a constant c_β such that $B_\beta = c_\beta A_\beta$. Moreover since

$$\begin{aligned} A_\beta(\pi(m(a))\Phi) &= W_\beta(R(m(a))A(\Phi)) \\ &= W_{{}^t a\beta\widetilde{\sigma}(a)}(A(\Phi)) = A_{{}^t a\beta\widetilde{\sigma}(a)}(\Phi), \end{aligned}$$

for $a \in GL(2,k')$ and $\mathfrak{O}(V)$ is a single $GL(2,k')$ orbit, we know that $c = c_\beta$ is independent of β. Thus if we set $D := B - cA$, we have $W_\beta \circ D = 0$ for any $\beta \in \mathrm{Her}_2(k'/k)$ with $\det\beta \neq 0$. Now we apply Lemma 3.14 to finish the proof. □

Lemma 3.16 (Proposition 2.6 in [KRS]**).** *Suppose that there is no Hermitian space V over k' such that $V_{\mathbb{A}} = V \otimes_k \mathbb{A}$, then* $\mathrm{Hom}_{G(\mathbb{A})}(R(V_{\mathbb{A}},\chi), L^2(G(k)\backslash G(\mathbb{A}))) = 0$.

Proof. Take a intertwining map $D : R(V_{\mathbb{A}},\chi) \to L^2(G(k)\backslash G(\mathbb{A}))$ and consider $W_\beta \circ D$ for $\beta \in \mathrm{Her}_2(k'/k)$. Then by Lemma 3.12 and Lemma 3.13, at every v it is necessary that $\Omega_\beta(V_v)$ is not empty for $W_\beta \circ D$ restricted to $R(V_v,\chi_v)$ to be non-zero. But this means that the Hermitian space ($V = k'$, $\langle x, y\rangle_V := x\beta\widetilde{\sigma}(y)$) over k' satisfies $V_{\mathbb{A}} = V \otimes_k \mathbb{A}$, which contradicts our assumption. Thus $W_\beta \circ D = 0$ for any $\beta \in \mathrm{Her}_2(k'/k)$ must holds. Now the lemma follows from Lemma 3.14. □

3.3.5. *The proof of Theorem* 3.11. First note that we can choose a character χ of $\mathbb{A}_{k'}^\times/k'^\times$ such that

$$\mathfrak{S}_{1,2} = \chi|\ |_{\mathbb{A}_{k'}}^{3/2} \otimes \chi|\ |_{\mathbb{A}_{k'}}^{1/2},\ \chi|_{\mathbb{A}^\times} = \mathbf{1}, \quad (\mathfrak{S}_{1,3} = \chi|\ |_{\mathbb{A}_{k'}} \otimes \chi,\ \chi|_{\mathbb{A}^\times} = \eta_{k'/k} \text{ resp.})$$

The contribution to be calculated is the image $\mathrm{Im}N(w_-, \mathfrak{S}_{1,2})$ ($\mathrm{Im}N(w_-, \mathfrak{S}_{1,3})$ resp.). But since $w_- = (w_2w_1w_2)w_1$ and

$$\begin{aligned} \mathrm{Im}N(w_1, \mathfrak{S}_{1,2}) &= \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}[(\chi\circ\det)|\det|_{\mathbb{A}_{k'}} \otimes \mathbf{1}_{U_1(\mathbb{A})}], \\ (\mathrm{Im}N(w_1, \mathfrak{S}_{1,3}) &= \mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}[(\chi\circ\det)|\det|_{\mathbb{A}_{k'}}^{1/2} \otimes \mathbf{1}_{U_1(\mathbb{A})}] \text{ resp.}) \end{aligned}$$

the statement *(1)* follows immediately from Proposition 3.5 *(2)*, Proposition 3.6 *(2)*, Proposition 3.7, Proposition 3.9, Proposition 3.10.

To prove *(2)*, we have to determine the irreducible constituents of

$$N(w_2w_1w_2,\ \chi\otimes\chi|\ |_{\mathbb{A}_{k'}})\Big(\mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}[(\chi\circ\det)|\det|_{A_{k'}}^{1/2} \otimes \mathbf{1}_{U_1(\mathbb{A})}]\Big).$$

A local component of this consists of at most two irreducible representations $R(V_{1,i},\chi_v)$ ($i = 1, 2$) by Proposition 3.5 *(2) (3)*, Proposition 3.6 *(2)*, Proposition 3.7, Proposition 3.9 and Proposition 3.10. Thus the irreducible constituents are contained in

$$\left\{ R(V_{\mathbb{A}},\chi) \;\middle|\; \begin{array}{l} \text{(i) } V_{\mathbb{A}} \text{ is a collection of one-dimensional Hermitian} \\ \quad \text{space } V_v \text{ over } k_v \text{ at each } v, \\ \text{(ii) } \chi \text{ is such that } \chi|_{\mathbb{A}^\times} = \eta_{k'/k}. \end{array} \right\}$$

But we know from Lemma 3.16 that $R(V_{\mathbb{A}},\chi)$ with $V_{\mathbb{A}}$ does not come from a global V must be excluded. Also Proposition 3.15 assures that the multiplicity of $R(V_{\mathbb{A}},\chi)$ is at most 1.

Finally we need to check that $R(V_{\mathbb{A}},\chi)$ with $V_{\mathbb{A}} = V \otimes_k \mathbb{A}$ for some V over k' really contributes to the residual spectrum (cf. [KRS] Proposition 3.3). For such a Hermitian

space V, we write its unitary group $U(V)$. We can construct the global oscillator representation $(\omega_{\psi,\chi}, \mathcal{S}^0(V_{\mathbb{A}}^{\oplus 2}))$ of $G(\mathbb{A}) \times U(V,\mathbb{A})$ (see 3.3.7 below). Using this we have the usual theta kernel

$$\theta_\chi(g,h;\Phi) := \sum_{\xi \in V} \omega_{\psi,\chi}(g,h)\Phi(\xi), \quad (g \in G(\mathbb{A}),\ h \in U(V,\mathbb{A})).$$

Since V is always anisotropic, the theta integral

$$I_\chi(g,\Phi) := \int_{U(V,k)\backslash U(V,\mathbb{A})} \theta_\chi(g,h;\Phi)\, dh$$

is well-defined. Moreover the map $I_\chi : \mathcal{S}^0(V_{\mathbb{A}}^{\oplus 2}) \to L^2(G(k)\backslash G(\mathbb{A}))$ is non-zero (calculate the ψ_β-Fourier coefficients) and factors $R(V_{\mathbb{A}},\chi)$. This completes the proof of Theorem 3.11. (q.e.d.)

3.3.6. *The contributions of* (3.21)*—the result.* The contribution of the term (3.21) is relatively easy to describe. First we prove the following.

Lemma 3.17. *Decompose $\mathfrak{S}_{2,4}$ into a restricted tensor product $\bigotimes_v (\mathfrak{S}_{2,4})_v$. Then the images*

$$\mathrm{Im}[N(w_2,\ w_1w_2(\mathfrak{S}_{2,4})_v)M(w_1,\ w_2(\mathfrak{S}_{2,4})_v)N(w_2,\ (\mathfrak{S}_{2,4})_v)]$$
$$\mathrm{Im}[M(w_1,\ w_2w_1w_2(\mathfrak{S}_{2,4})_v)N(w_2,\ w_1w_2(\mathfrak{S}_{2,4})_v)M(w_1,\ w_2(\mathfrak{S}_{2,4})_v)N(w_2,\ (\mathfrak{S}_{2,4})_v)]$$

coincide to each other and are irreducible.

Proof. We write $\mathfrak{S}_{2,4}$ as $\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2} \otimes \chi_2|\ |_{\mathbb{A}_{k'}}^{1/2}$ where χ_i $(i=1,2)$ are distinct characters of $\mathbb{A}_{k'}^\times / k'^\times$ such that $\chi_i|_{\mathbb{A}^\times} = \mathbf{1}$.

The case of inert v.

We take a inert place v of k. Then $(\mathfrak{S}_{2,4})_v := \chi_{1,v}|\ |_w^{1/2} \otimes \chi_{2,v}|\ |_w^{1/2}$ and hence

$$r(w_2,\ (\mathfrak{S}_{2,4})_v) = \frac{L_k(1,\chi_{v,2}|_{k_v^\times})}{L_k(2,\chi_{v,2}|_{k_v^\times})\varepsilon_k(1,\chi_{v,2}|_{k_v^\times},\psi_v)}$$
$$r(w_2,\ w_1w_2(\mathfrak{S}_{2,4})_v) = \frac{L_k(1,\chi_{v,1}|_{k_v^\times})}{L_k(2,\chi_{v,1}|_{k_v^\times})\varepsilon_k(1,\chi_{v,1}|_{k_v^\times},\psi_v)}$$

are both defined and non-zero. Thus we may replace $N(w,\bullet)$ by $M(w,\bullet)$ and study the images

$$\mathrm{Im}M(w_2w_1w_2,\ (\mathfrak{S}_{2,4})_v), \quad \mathrm{Im}M(w_-,\ (\mathfrak{S}_{2,4})_v).$$

But since $(\pi_v, V_{\pi_v}) := \mathrm{Ind}_{(P_0\cap M_1)(k_v)}^{M_1(k_v)}[(\mathfrak{S}_{2,4})_v \otimes \mathbf{1}_{(U_0\cap M_1)(k_v)}]$ is essentially tempered, irreducible and $M(w_1,\ (\mathfrak{S}_{2,4})_v) : V_{\pi_v} \to V_{\pi_v}$ is an isomorphism, we know that the above images are both isomorphic to the Langlands' quotient of $\mathrm{Ind}_{P_1(k_v)}^{G(k_v)}[\pi_v \otimes \mathbf{1}_{U_1(k_v)}]$.

The case of split v.

At a split v, $(\mathfrak{S}_{2,4})_v$ is of the form

$$\chi_{1,w_1}|\ |_v^{1/2} \otimes \chi_{2,w_1}|\ |_v^{1/2} \otimes \chi_{1,w_2}^{-1}|\ |_v^{-1/2} \otimes \chi_{2,w_2}^{-1}|\ |_v^{-1/2},$$

where $\chi_{1,v} = \chi_{1,w_1} \otimes \chi_{1,w_2},\ \chi_{2,v} = \chi_{2,w_1} \otimes \chi_{2,w_2}$, and hence

$$r(w_2,\ (\mathfrak{S}_{2,4})_v) = \frac{L_k(1,\chi_{2,w_1}\chi_{2,w_2})}{L_k(2,\chi_{2,w_1}\chi_{2,w_2})\varepsilon_k(1,\chi_{2,w_1}\chi_{2,w_2},\psi_v)}$$
$$r(w_2,\ w_1w_2(\mathfrak{S}_{2,4})_v) = \frac{L_k((1,\chi_{1,w_1}\chi_{1,w_2}))}{L_k((2,\chi_{1,w_1}\chi_{1,w_2}))\varepsilon_k(1,\chi_{1,w_1}\chi_{1,w_2}),\psi_v)}.$$

These again allow us to replace the normalized intertwining operators by the unnormalized ones. Thus we study

$$\mathrm{Im}M(w_2w_1w_1'w_2,\,(\mathfrak{S}_{2,4})_v),\quad \mathrm{Im}M(w_-,\,(\mathfrak{S}_{2,4})_v),$$

where w_1' denotes the simple reflection of $G_v \simeq GL(4)_{k_v}$ attached to the simple root α_1' (see 2.1.4). Again notice that $M_{1,v} \simeq GL(2)_{k_v} \times GL(2)_{k_v}$, we see both of the images are the Langlands' quotient of $\mathrm{Ind}_{P_1(k_v)}^{G(k_v)}[\pi_v \otimes \mathbf{1}_{U_1(k_v)}]$ where $\pi_v := \mathrm{Ind}_{(P_0\cap M_1)(k_v)}^{M_1(k_v)}[(\mathfrak{S}_{2,4})_v \otimes \mathbf{1}_{(U_0\cap M_1)(k_v)}]$. □

We may call the resulting irreducible representation

$$\mathrm{Im}N(w_2w_1w_2,\,\mathfrak{S}_{2,4}) = \mathrm{Im}N(w_-,\,\mathfrak{S}_{2,4})$$

the *global Langlands' quotient* of $\mathrm{Ind}_{P_0(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}_{2,4} \otimes \mathbf{1}_{U_0(\mathbb{A})}]$.

Theorem 3.18. *The contribution of the term* (3.21) *in Theorem* 3.4 *to the residual spectrum is multiplicity free and consists of irreducible representations corresponding to non-trivial one-dimensional representations of* $U(1,1)_{k'/k}$ *by the theta correspondence.*

Write H for the group $U(1,1)_{k'/k}$ defined by

$$H := \left\{ h \in \mathrm{Res}_{k'/k}GL(2) \;\middle|\; h\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\tilde{\sigma}({}^th) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \right\}.$$

All we have to show is that the irreducible representation

$$\mathrm{Im}N(w_2w_1w_2,\,\mathfrak{S}_{2,4}) = \mathrm{Im}N(w_-,\,\mathfrak{S}_{2,4})$$

of $G(\mathbb{A})$ corresponding to the one dimensional representation $(\chi_1\chi_2^{-1})^H := (\chi_1\chi_2^{-1})' \circ \det$, where $(\chi_1\chi_2)'$ is defined similarly as in Theorem 3.11. To do this we calculate the constant terms of theta series on $G(\mathbb{A}) \times H(\mathbb{A})$ (cf. [Ra]).

3.3.7. *Constant terms of theta series.* **(1) Global oscillator representations for $G(\mathbb{A})\times H(\mathbb{A})$.** We begin with a short review of theta series on $G(\mathbb{A}) \times H(\mathbb{A})$. As in the local case (3.3.1), we consider $G \times H$ as a dual reductive pair in $Sp(\mathbf{W})$. Here the Hermitian space $V_{m,i}$ is replaced by

$$\left(V := k'^{\oplus 2},\, \langle\,,\,\rangle_V := \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right).$$

As in [I] §1, we have the global metaplectic group $Mp(\mathbf{W}_{\mathbb{A}})$ of $Sp(\mathbf{W}_{\mathbb{A}})$. This is a restricted direct product of $Mp(\mathbf{W}_v)$ with respect to their appropriate compact subgroups $\mathbf{K}_v$ divided by the central subgroup $\{(\epsilon_v)_v \in \bigoplus_v \mathbb{C}^1\,;\, \prod_v \epsilon_v = 1\}$. The group $Sp(\mathbf{W}, k)$ is naturally considered as a closed subgroup of $Mp(\mathbf{W})$. Also for a character ψ of $\mathbb{A}/k$, we have the oscillator representation ω_ψ of $Mp(\mathbf{W}_{\mathbb{A}})$ associated to it.

Recall that our $\mathfrak{S}_{2,4}$ was written in the form $\chi_1|\;|_{\mathbb{A}_{k'}}^{1/2} \otimes \chi_2|\;|_{\mathbb{A}_{k'}}^{1/2}$. χ_i $(i = 1, 2)$ are two distinct characters of $\mathbb{A}_{k'}^\times/k'^\times$ whose restriction to $\mathbb{A}^\times$ are trivial. Using this χ_1 we can construct a system of local splittings (cf. [Ku] Theorem 3.1);

$$\iota_{\chi_{1,v}} : G(k_v) \times H(k_v) \longrightarrow Mp(\mathbf{W})_v.$$

Then from explicit formulae (3.22), (3.23), (3.24), (3.25) and (3.26), (3.27), (3.28), (3.29) we see that $\iota_{\chi_{1,v}}(K_v) \subset \mathbf{K}_v$ at almost all v. This allows us to construct a global splitting

$$\iota_{\chi_1} : G(\mathbb{A}) \times H(\mathbb{A}) \ni (g, h) \to (\iota_{\chi_{1,v}}(g_v, h_v))_v \in Mp(\mathbf{W})_{\mathbb{A}}.$$

It follows from the product formula for the Weil constant that $\iota_{\chi_1}(G(k)\times H(k))\subset Sp(\mathbf{W},k)$. As in the local case we have the auxiliary oscillator representation $\omega'_{\psi,\chi_1}:=\omega_\psi\circ\iota_{\chi_1}$ and twist the $H(\mathbb{A})$-action by the character $\chi_1\circ\det$ to obtain the final oscillator representation $(\omega_{\psi,\chi_1},\mathcal{S}(V_{\mathbb{A}}^{\oplus 2}))$.

Explicit formulae for $(\omega_{\psi,\chi_1},\mathcal{S}(V_{\mathbb{A}}^{\oplus 2}))$ are given by

$$\omega_{\psi,\chi_1}(m(A))\Phi(x)=\chi_1(\det A)|\det A|_{\mathbb{A}_{k'}}\Phi(xA),\ (A\in GL(2,\mathbb{A}_{k'})) \tag{3.30}$$

$$\omega_{\psi,\chi_1}(u(B))\Phi(x)=\psi(\mathrm{tr}(\langle x,x\rangle_V B))\Phi(x),\ (B\in \mathrm{Her}_2(\mathbb{A}_{k'}/\mathbb{A})) \tag{3.31}$$

$$\omega_{\psi,\chi_1}(w)=\int_{V_{\mathbb{A}}^{\oplus 2}}\Phi(y)\psi(\mathrm{Tr}_{k'/k}(\mathrm{tr}\langle x,y\rangle_V))\,dy \tag{3.32}$$

$$\omega_{\psi,\chi_1}(h)\Phi(x)=\Phi(h^{-1}x),\ (h\in H(\mathbb{A})). \tag{3.33}$$

If we replace $\mathcal{S}(V_v^{\oplus 2})$ with $\mathcal{S}^0(V_v^{\oplus 2})$ defined in 3.3.1 at each archimedean v, then we have the underlying $[G(\mathbb{A}_f)\times(\mathfrak{g}(\mathbb{A}_\infty)_{\mathbb{C}},K_\infty)]\times[H(\mathbb{A}_f)\times(\mathfrak{h}(\mathbb{A}_\infty)_{\mathbb{C}},K_\infty^H)]$-bimodule $\mathcal{S}^0(V_{\mathbb{A}}^{\oplus 2})$. Now for $\Phi\in\mathcal{S}^0(V_{\mathbb{A}}^{\oplus 2})$, define the theta series attached to it by

$$\theta_\Phi(g,h):=\sum_{\xi\in V_k^{\oplus 2}}\omega_{\psi,\chi_1}(g,h)\Phi(\xi).$$

This is a slowly increasing smooth $K_\infty\times K_\infty^H$-finite function on $G(k)\backslash G(\mathbb{A})\times H(k)\backslash H(\mathbb{A})$.

(3) Calculation of the constant term. At this point we identify $V^{\oplus 2}$ with $\mathbb{M}_2(k')$, on which G (H resp.) acts by right (left resp.) translation. We fix a point $x_1\in V^{\oplus 2}$ as

$$x_1:=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

Lemma 3.19. *If we set $F_\Phi(h)(g):=\sum_{\zeta\in Z(M_1,k)}\omega_{\psi,\chi_1}(\zeta g,h)\Phi(x_1)$ for $\Phi\in\mathcal{S}(V_{\mathbb{A}}^{\oplus 2})$, then the constant term of θ_Φ along $P_0\subset G$ is given by*

$$\begin{aligned}\theta_\Phi(g,h)_{P_0}=\omega_{\psi,\chi_1}(g,h)\Phi(0)&+\sum_{\gamma\in B_H^-(k)\backslash H(k)}F_\Phi(\gamma h)(g)\\&+\sum_{\gamma\in B_H^-(k)\backslash H(k)}\int_{U_0^{M_1}(\mathbb{A})}F_\Phi(\gamma h)(w_1^{-1}ug)\,du\end{aligned} \tag{3.34}$$

Here B_H^- denotes the Borel subgroup of H consisting of lower triangular elements.

Proof. We follow Rallis's argument appeared in the proof of Theorem I.1.1 in [Ra]. First we calculate the constant term along P_1:

$$\begin{aligned}\theta_\Phi(g,h)_{P_1}&=\int_{U_1(k)\backslash U_1(\mathbb{A})}\sum_{\xi\in V_k^{\oplus 2}}\omega_{\psi,\chi_1}(ug,h)\Phi(\xi)\,du\\&=\int_{\mathrm{Her}_2(k'/k)\backslash\mathrm{Her}_2(\mathbb{A}_{k'}/\mathbb{A})}\sum_{\xi\in V_k^{\oplus 2}}\omega_{\psi,\chi_1}(u(B))\Phi(\xi)\,dB\\&=\sum_{\xi\in V_k^{\oplus 2}}\omega_{\psi,\chi_1}(g,h)\Phi(\xi)\int_{\mathrm{Her}_2(k'/k)\backslash\mathrm{Her}_2(\mathbb{A}_{k'}/\mathbb{A})}\psi(\mathrm{tr}(\langle\xi,\xi\rangle_V B))\,dB\\&=\sum_{\substack{\xi\in V_k^{\oplus 2}\\ \langle\xi,\xi\rangle_V=0}}\omega_{\psi,\chi_1}(g,h)\Phi(\xi).\end{aligned}$$

Next set $X_0 := \{\xi \in V_k^{\oplus 2} \mid \langle \xi, \xi \rangle_V = 0\}$. Then there are two $M_1(k) \times H(k)$-orbits in X_0;

$$X_{0,0} := \{0\}, \quad X_{0,1} := \{(\xi, \eta) \in X_0 \mid \dim \mathrm{Span}\{\xi, \eta\} = 1\}.$$

The element x_1 above is a representative of $X_{0,1}$. If we write S_1 for the stabilizer of x_1 in $M_1(k) \times H(k)$, then

$$\theta_\Phi(g,h)_{P_1} = \omega_{\psi,\chi_1}(g,h)\Phi(0) + \sum_{(\delta,\gamma) \in S_1 \backslash H(k) \times M_1(k)} \omega_{\psi,\chi_1}(\delta g, \gamma h)\Phi(x_1).$$

Next we proceed to

$$\theta_\Phi(g,h)_{P_0} := \int_{U_0^{M_1}(k) \backslash U_0^{M_1}(\mathbb{A})} \theta_\Phi(ug,h)_{P_1}\, du,$$

where $P_0^{M_1} = M_0 U_0^{M_1}$ is the intersection of P_0 and M_1. Note that

$$S_1 = \left\{ \left(m\left(\begin{pmatrix} x & y \\ 0 & \widetilde{\sigma}(a)^{-1} \end{pmatrix} \right), \begin{pmatrix} a & 0 \\ b & \widetilde{\sigma}(a)^{-1} \end{pmatrix} \right) \in M_1(k) \times H(k) \right\}.$$

This combined with the Bruhat decomposition yields

$$\theta_\Phi(g,h)_{P_1} = \omega_{\psi,\chi_1}(g,h)\Phi(0) + \sum_{\gamma \in B_H^-(k) \backslash H(k)} \sum_{\delta \in P(M_1)(k) \backslash M_1(k)} \omega_{\psi,\chi_1}(\delta g, \gamma h)\Phi(x_1)$$

where $P(M_1) := \left\{ m\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \right) \in M_1 \right\}$,

$$= \sum_{\gamma \in B_H^-(k) \backslash H(k)} \Big[\omega_{\psi,\chi_1}(g,h)\Phi(0) + \sum_{\zeta \in Z(M_1,k)} \omega_{\psi,\chi_1}(\zeta g, \gamma h)\Phi(x_1) + \sum_{\zeta \in Z(M_1,k)} \sum_{\epsilon \in U_0^{M_1}(k)} \omega_{\psi,\chi_1}(w_1^{-1}\zeta\epsilon g, \gamma h)\Phi(x_1) \Big].$$

On the other hand,

$$\int_{U_0^{M_1}(k) \backslash U_0^{M_1}(\mathbb{A})} \Big(\omega_{\psi,\chi_1}(ug,h)\Phi(0) + \sum_{\zeta \in Z(M_1,k)} \omega_{\psi,\chi_1}(\zeta ug, \gamma h)\Phi(x_1) \Big)\, du$$
$$= \omega_{\psi,\chi_1}(g,h)\Phi(0) + \sum_{\zeta \in Z(M_1,k)} \omega_{\psi,\chi_1}(\zeta g, \gamma h)\Phi(x_1),$$
$$\int_{U_0^{M_1}(k) \backslash U_0^{M_1}(\mathbb{A})} \sum_{\zeta \in Z(M_1,k)} \sum_{\epsilon \in U_0^{M_1}(k)} \omega_{\psi,\chi_1}(w_1^{-1}\zeta\epsilon ug, \gamma h)\Phi(x_1)\, du$$
$$= \int_{U_0^{M_1}(\mathbb{A})} \sum_{\zeta \in Z(M_1,k)} \omega_{\psi,\chi_1}(\zeta w_1^{-1} ug, \gamma h)\Phi(x_1)\, du.$$

Hence the lemma follows. □

3.3.8. *Proof of Theorem* 3.18. For $s \in \mathbb{C}$, define

$$F_\Phi(h,s)(g) := \int_{\mathbb{A}_{k'}^\times} \omega_{\psi,\chi_1}(g,h)\Phi(ax_1)\chi_1\chi_2^{-1}(a)|a|_{\mathbb{A}_{k'}}^s\, da^\times, \quad \Phi \in \mathcal{S}^0(V_{\mathbb{A}}^{\oplus 2}).$$

Since the main part of the integral is $L_{k'}(s,\chi_1\chi_2^{-1})$ and $\chi_1 \neq \chi_2$, this converges absolutely for $\mathrm{Re}(s) >> 0$ and extended to an entire function on s. Moreover it satisfies

$$F_\Phi\Big(\begin{pmatrix} t & 0 \\ * & \widetilde{\sigma}(t)^{-1}\end{pmatrix} h, s\Big)(g) = \widetilde{\sigma}(\chi_1^{-1}\chi_2)(t)|t|_{\mathbb{A}_{k'}}^{-s} F_\Phi(h,s)(g),$$

$$F_\Phi(h,s)\Big(m\Big(\begin{pmatrix} x & * \\ 0 & y\end{pmatrix}\Big)g\Big) = \chi_1(x)\chi_2(y)|x|_{\mathbb{A}_{k'}}|y|_{\mathbb{A}_{k'}}^{1-s} F_\Phi(h,s)(g).$$

That is, $F_\Phi(h,s)(g)$ is a holomorphic section of $A(B_H^-(\mathbb{A})\backslash H(\mathbb{A}))_{\widetilde{\sigma}(\chi_1^{-1}\chi_2)|\ |_{\mathbb{A}_{k'}}^{-s-1/2}}$ and of $A(P_0(\mathbb{A})\backslash M_1(\mathbb{A}))_{(\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2}\otimes\chi_2|\ |_{\mathbb{A}_{k'}}^{3/2-s})}$. Recall that

(1) The Eisenstein series

$$E(F_\Phi,h;s)(g) := \sum_{\gamma\in B_H^-(k)\backslash H(k)} F_\Phi(\gamma h,s)(g)$$

on $H(k)\backslash H(\mathbb{A})$ converges absolutely for $\mathrm{Re}(s) > 1$ and meromorphically continued to the whole plane.

(2) It has a simple pole at $s = 1$ and its residue there spans the one dimensional representation $(\chi_1\chi_2^{-1})^H$.

Write Θ_{χ_1,P_0} for the space of functions on $M_0(k)\backslash M_0(\mathbb{A}) \times H(k)\backslash H(\mathbb{A})$ spanned by $\theta_\Phi(g,h)_{P_0}$ ($\Phi \in \mathcal{S}^0(V_{\mathbb{A}}^{\oplus 2})$). We consider the integral

$$\begin{aligned}\int_{Z(M_1,k)\backslash Z(M_1,\mathbb{A})} \Big[&\sum_{\gamma\in B_H^-(k)\backslash H(k)} F_\Phi(\gamma h)(zg) \\ &+ \sum_{\gamma\in B_H^-(k)\backslash H(k)} \int_{U_0^{M_1}(\mathbb{A})} F_\Phi(\gamma h)(w_1^{-1}uzg)\, du\Big]\chi_1\chi_2^{-1}(z)|z|_{\mathbb{A}_{k'}}^s\, dz.\end{aligned} \tag{3.35}$$

If $\mathrm{Re}(s) > 1$, then

$$\begin{aligned}&\sum_{\gamma\in B_H^-(k)\backslash H(k)} \int_{Z(M_1,k)\backslash Z(M_1,\mathbb{A})} F_\Phi(\gamma h)(zg)\chi_1\chi_2^{-1}(z)|z|_{\mathbb{A}_{k'}}^s\, dz \\ &\quad= \sum_{\gamma\in B_H^-(k)\backslash H(k)} \int_{\mathbb{A}_{k'}^\times/k'^\times} \sum_{\zeta\in Z(M_1,k)} \omega_{\psi,\chi_1}(\zeta g,h)\Phi(ax_1)\chi_1\chi_2^{-1}(a)|a|_{\mathbb{A}_{k'}}^s\, da^\times \\ &\quad= \sum_{\gamma\in B_H^-(k)\backslash H(k)} F_\Phi(\gamma h,s)(g) = E(F_\Phi,h;s)(g) \\ &\sum_{\gamma\in B_H^-(k)\backslash H(k)} \int_{U_0^{M_1}(\mathbb{A})} \int_{Z(M_1,k)\backslash Z(M_1,\mathbb{A})} F_\Phi(\gamma h)(w_1^{-1}uzg)\, dz\, du \\ &\quad= \sum_{\gamma\in B_H^-(k)\backslash H(k)} \int_{U_0^{M_1}(\mathbb{A})} F_\Phi(\gamma h,s)(w_1^{-1}ug)\, du \\ &\quad= \sum_{\gamma\in B_H^-(k)\backslash H(k)} M^{M_1}(w_1,\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2}\otimes\chi_2|\ |_{\mathbb{A}_{k'}}^{3/2-s})F_\Phi(\gamma h,s)(g)\end{aligned}$$

$$=E(M^{M_1}(w_1,\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2}\otimes\chi_2|\ |_{\mathbb{A}_{k'}}^{3/2-s})F_\Phi,h;s)(g)$$

are absolutely convergent and are equal to

$$\int_{Z(M_1,k)\backslash Z(M_1,\mathbb{A})}\sum_{\gamma\in B_H^-(k)\backslash H(k)}F_\Phi(\gamma h)(zg)\chi_1\chi_2^{-1}(z)|z|_{\mathbb{A}_{k'}}^s\,dz,$$

$$\int_{Z(M_1,k)\backslash Z(M_1,\mathbb{A})}\sum_{\gamma\in B_H^-(k)\backslash H(k)}\int_{U_0^{M_1}(\mathbb{A})}F_\Phi(\gamma h)(w_1^{-1}uzg)\,du\,\chi_1\chi_2^{-1}(z)|z|_{\mathbb{A}_{k'}}^s\,dz$$

respectively. Thus (3.35) is well-defined for $\mathrm{Re}(s)>1$ and meromorphically continued to define an intertwining operator

$$A(s):\Theta_{\chi_1,P_0}\ni\theta_\Phi(g,h)_{P_0}$$
$$\longrightarrow E(F_\Phi,h;s)(g)+E(M^{M_1}(w_1,\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2}\otimes\chi_2|\ |_{\mathbb{A}_{k'}}^{3/2-s})F_\Phi,h;s)(g)\in A(H(k)\backslash H(\mathbb{A})),$$

for $s\in\mathbb{C}$.

From (2) above, we know that $A(s)$ has a simple pole at $s=1$, and the image of the residue $\mathrm{Res}_{s=1}A(s)$ is the constant term of the automorphic representation corresponding to $(\chi_1\chi_2^{-1})^H$ by the Howe duality. But as a function on $G(k)\backslash G(\mathbb{A})$, $\mathrm{ImRes}_{s=1}A(s)$ is still of the form

$$\phi(g)+M(w_1,\,\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2}\otimes\chi_2|\ |_{\mathbb{A}_{k'}}^{1/2})\phi(g)$$

for some $\phi\in A(P_0(\mathbb{A})\backslash M_1(\mathbb{A}))_{(\chi_1|\ |_{\mathbb{A}_{k'}}^{1/2}\otimes\chi_2|\ |_{\mathbb{A}_{k'}}^{1/2})}$. This is precisely the constant term of the global Langlands' quotient of $\mathrm{Ind}_{P_0(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}_{2,4}\otimes\mathbf{1}_{U_0(\mathbb{A})}]$ and we have done. (q.e.d.)

4. The contributions of cuspidal data attached to P_1

In this section we describe the contribution of cuspidal data attached to the Siegel parabolic subgroup $P_1=M_1U_1$ to the residual spectrum.

4.1. Singular hyperplanes. We retain the notations defined at the beginning of the previous section. In particular we take $\mathfrak{X}\in\mathfrak{E}_{P_1}$ and $(M_1,\mathfrak{P})\in\mathfrak{X}$. For $\phi\in P_{(M_1,\mathfrak{P})}$ and $\phi'\in P_{\mathfrak{X}}$ we consider

$$\langle\theta_\phi,\theta_{\phi'}\rangle_{L^2(G(k)\backslash G(\mathbb{A}))}=\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{P},\mathrm{Re}\pi=\lambda_0}A(\phi,\phi')(\pi)\,d\pi,$$

where

$$A(\phi,\phi')=\sum_{(M_1,\mathfrak{P}')\in\mathfrak{X}}\Big\langle\sum_{w\in W(\mathfrak{P},\mathfrak{P}')}M(w,\,\pi)\phi(\pi),\,\phi'(-w\overline{\pi})\Big\rangle.$$

In the Siegel parabolic case W_{M_1} consists of 1 and $w(P_1):=w_2w_1w_2$, hence the singular hyperplanes for $A(\phi,\phi')(\pi)$ are those of $M(w(P_1),\,\pi)\phi(\pi)$.

4.1.1. *Analytic behavior of the local intertwining operator.* Take $\pi=\bigotimes_v\pi_v\in\mathfrak{P}$. We begin with the review of the normalization factor for $M(w(P_1),\,\pi_v)$ ([Sh]). Recall that our $M_{1,v}$ is isomorphic to $\mathrm{Res}_{k'_v/k_v}GL(2)$. We have the following two cases.

(1) At inert v. Let $w|v$ be as before and write $\Gamma_v:=\mathrm{Gal}(k'_w/k_v)$. We identify the generator of Γ_v with σ. Just as in the global case, one has ${}^LG_v=\widehat{G}\rtimes_{\theta_2}W_{k_v}$, where the Weil group W_{k_v} acts on $\widehat{G}$ through Γ_v by $\sigma=\theta_2$. Hence we have the L-isomorphism

$$(4.1)\qquad {}^LM_1\ni\begin{pmatrix}A&\mathbf{0}_2\\ \mathbf{0}_2&D\end{pmatrix}\rtimes w\xrightarrow{\sim}(A,{}^tD^{-1})\rtimes w\in{}^L(\mathrm{Res}_{k'_w/k_v}GL(2)),$$

and identify ${}^LM_1$ with ${}^L(\mathrm{Res}_{k'_w/k_v}GL(2))$ by this.

In the notation of [Sh], we have $(U_1)_{w(P_1)} = U_1$. The adjoint representation $r_{w(P_1)}$ of ${}^LM_1$ on $Lie((U_1)^\wedge_{w(P_1)}) \simeq \mathbb{M}_2(\mathbb{C})$ is given by

$$r_{w(P_1)}|_{\widehat{M}_1} = \mathrm{St}_2 \otimes \mathrm{St}_2, \quad r_{w(P_1)}(1 \rtimes \sigma)(v_1 \otimes v_2) = v_2 \otimes v_1, \tag{4.2}$$

where St_2 denotes the standard representation of $GL(2,\mathbb{C})$. The L-function attached to π_v and $r_{w(P_1)}$ is the twisted tensor L-function $L_{Asai}(s,\pi_v)$ ([As], [HLR]). The normalization factor $r(w(P_1),\,\pi_v)$ for $M(w(P_1),\,\pi_v)$ is defined by

$$r(w(P_1),\,\pi_v) := \frac{L_{Asai}(0,\pi_v)}{L_{Asai}(1,\pi_v)\varepsilon_{Asai}(0,\pi_v,\psi_v)}.$$

(2) At split v. Let $w_1,\,w_2|v$ be as before and write

$$\pi_v = \pi_{w_1} \otimes \pi_{w_2} : M_1(k_v) \ni \begin{pmatrix} A & \mathbf{0}_2 \\ \mathbf{0}_2 & D \end{pmatrix} \longrightarrow \pi_{w_1}(A) \otimes \pi_{w_2}({}^tD^{-1}) \in GL(V_{\pi_v}).$$

The normalization factor in this case is well-known and given by

$$r(w(P_1),\,\pi_v) := \frac{L(0,\pi_{w_1}\times\pi_{w_2})}{L(1,\pi_{w_1}\times\pi_{w_2})\varepsilon(0,\pi_{w_1}\times\pi_{w_2},\psi_v)},$$

where $L(s,\pi_{w_1}\times\pi_{w_2})$ and $\varepsilon(s,\pi_{w_1}\times\pi_{w_2},\psi_v)$ are the Rankin product L-factor and its root number for π_{w_1} and π_{w_2} defined in [Ja].

Lemma 4.1. *The local normalized intertwining operator $N(w(P_1),\,\pi_v) := r(w(P_1),\,\pi_v)^{-1}$ $M(w(P_1),\,\pi_v)$ is holomorphic on the closed positive chamber $\{\pi_v \in \mathfrak{P}_v\,;\,\langle \mathrm{Re}\pi_v, \beta_1^\vee\rangle \geq 0\}$. (Note that β_1 is the only positive root of A_{M_1} in P_1.)*

Proof. At splitting v, this was proved in [MW2]. Hence we assume v is inert in k'. When π_v is essentially tempered, this follows from [Ar2] Theorem 2.1 if v is archimedean and from Proposition 7.2 in [Sh] if v is finite. Thus we may assume that π_v is in the complementary series. Then π_v is of the form

$$\pi_v = \mathrm{Ind}^{M_1(k_v)}_{(P_0\cap M_1)(k_v)}[(\mu_w|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{-\lambda/2}) \otimes \mathbf{1}_{(U_0\cap M_1)(k_v)}],$$

where μ_w is a quasi-character of $k'^\times_w$ and $0 < \lambda < 1$. From 3.1.1, $M(w(P_1),\,\pi_v)$ and its normalization factor can be written as

$$\begin{aligned} M(w(P_1),\,\pi_v) = M(w_2,\,\widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{\lambda/2}) M(w_1,\,\mu_w|\ |_w^{\lambda/2} \otimes \widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2}) \\ M(w_2,\,\mu_w|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{-\lambda/2}), \\ r(w(P_1),\,\pi_v) = r(w_2,\,\widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{\lambda/2}) r(w_1,\,\mu_w|\ |_w^{\lambda/2} \otimes \widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2}) \\ r(w_2,\,\mu_w|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{-\lambda/2}). \end{aligned}$$

Since $\langle \mathrm{Re}\pi_v, \beta_1^\vee\rangle \geq 0$ is equivalent to $\mathrm{Re}\mu_w \geq 0$, both of

$$\begin{aligned} r(w_2,\,\widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{\lambda/2})^{-1} M(w_2,\,\widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{\lambda/2}) \\ r(w_1,\,\mu_w|\ |_w^{\lambda/2} \otimes \widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2})^{-1} M(w_1,\,\mu_w|\ |_w^{\lambda/2} \otimes \widetilde{\sigma}(\mu_w^{-1})|\ |_w^{\lambda/2}) \end{aligned}$$

are holomorphic on the region $\langle \mathrm{Re}\pi_v, \beta_1^\vee\rangle \geq 0$. On the other hand, it follows from [Sh] Corollary 7.6 that both $r(w_2,\,\mu_w|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{-\lambda/2})$ and $M(w_2,\,\mu_w|\ |_w^{\lambda/2} \otimes \mu_w|\ |_w^{-\lambda/2})$ have a simple pole at $\mu_w = |\ |_w^{\lambda/2}$, and are holomorphic and non-zero at other π_v in our positive cone. □

4.1.2. *Analytic behavior of the global intertwining operator.* As in §3, we set

$$r(w(P_1),\,\pi) := \prod_v r(w(P_1),\,\pi_v), \quad N(w(P_1),\,\pi) := r(w(P_1),\,\pi)^{-1} M(w(P_1),\,\pi).$$

Then the following proposition will be proved in Appendix A.

Proposition 4.2. *For an irreducible representation π of $GL(2,\mathbb{A}_{k'})$, we write its central character ω_π. We write H for the diagonal subgroup $GL(2)_k$ of $\mathrm{Res}_{k'/k}GL(2)$. Then the only pole of $r(w(P_1),\,\pi)$ in the region $\langle \mathrm{Re}\pi, \beta_1^\vee\rangle \geq 0$ occurs at*

$$\mathfrak{S}(P_1) := \left\{ \pi \in \mathfrak{P} \;\middle|\; \begin{array}{l} \text{(i) } \omega_\pi|_{\mathbb{A}^\times} = |\;|_{\mathbb{A}}^2, \\ \text{(ii) } \int_{H(k)Z(H,\mathbb{A})\backslash H(\mathbb{A})} f(h)|\det(h)|_{\mathbb{A}}^{-1}\,dh \neq 0 \end{array} \right\},$$

and it is simple.

4.1.3. *Singular hyperplanes.* From Proposition 4.2, we have $S^{h,+}_{(M_1,\mathfrak{P})} = \{\mathfrak{S}(P_1)\}$.

4.2. Decomposition of the scalar product. In this case $S^{+}_{(M_1,\mathfrak{P})} = \{\mathfrak{P},\, \mathfrak{S}(P_1)\}$ and we set $o(\mathfrak{P}) := 0$.

Theorem 4.3. *For $\phi \in P_{(M_1,\mathfrak{P})}$ and $\phi' \in P_{\mathfrak{X}}$, we have*

$$\begin{aligned} \langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(\mathbb{A}))} =& \frac{1}{2\pi\sqrt{-1}} \int_{\pi\in\mathfrak{P},\mathrm{Re}\pi=0} A(\phi,\phi')(\pi)\,d\pi \\ &+ c_1 \langle N(w(P_1),\,\mathfrak{S}(P_1))\phi(\mathfrak{S}(P_1)),\, \phi'(-w(P_1)\overline{\mathfrak{S}(P_1)})\rangle \end{aligned} \tag{4.3}$$

for some non-zero constant c_1.

Proof. Since our P_1 is a maximal parabolic subgroup, we can apply Lemma 101 in [HC]. This allows us to apply the usual residue theorem to $A(\phi,\phi')$, and the assertion is obvious. □

4.3. The residual spectrum from P_1. By Theorem 4.3, it is enough to determine the image

$$\mathrm{Im}N(w(P_1),\,\mathfrak{S}(P_1)) = \bigotimes_v \mathrm{Im}N(w(P_1),\,\mathfrak{S}(P_1)_v).$$

If $\mathfrak{S}(P_1)_v$ is essentially tempered, then $\mathrm{Im}N(w(P_1),\,\mathfrak{S}(P_1)_v)$ is irreducible by the Langlands' classification. Next we handle $\mathfrak{S}(P_1)_v$ in the complementary series. Then its exponent is of the form

$$\begin{cases} |\;|_w^{(1+\lambda)/2} \otimes |\;|_w^{(1-\lambda)/2},\ (0<\lambda<1) & \text{if } v \text{ is inert,} \\ |\;|_v^{(1+\lambda)/2} \otimes |\;|_v^{(1-\lambda)/2} \otimes |\;|_v^{-(1+\mu)/2} \otimes |\;|_v^{-(1-\mu)/2},\ (0<\lambda,\mu<1) & \text{if } v \text{ splits.} \end{cases}$$

In both cases this is regular and

$$\mathrm{Im}N(w(P_1),\,\mathfrak{S}(P_1)_v) = \mathrm{Ind}_{P_1(k_v)}^{G(k_v)}[\mathfrak{S}(P_1)_v \otimes \mathbf{1}_{U_1(k_v)}]$$

is irreducible. We call the irreducible representation $\mathrm{Im}N(w(P_1),\,\mathfrak{S}(P_1))$ the *global Langlands' quotient* of $\mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_1)\otimes\mathbf{1}_{U_1(\mathbb{A})}]$. Then our conclusion is

Theorem 4.4. *The contribution of the cuspidal data attached to P_1 to the residual discrete spectrum is multiplicity free and consists of the global Langlands' quotients of $\mathrm{Ind}_{P_1(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_1)\otimes\mathbf{1}_{U_1(\mathbb{A})}]$, where $\mathfrak{S}(P_1)$ were defined in Proposition 4.2.*

5. The contributions of cuspidal data attached to P_2

In this section we study the contributions of cuspidal data attached to the non-Siegel maximal parabolic subgroup P_2 to the residual spectrum. For this purpose, it is more convenient to redefine our group $G = U(2,2)_{k'/k}$ by replacing J_2 in 2.1.3 with

$$J_2 := \left(\begin{array}{cc|cc} & & 0 & 1 \\ & 0 & 1 & 0 \\ \hline 0 & -1 & & \\ -1 & 0 & & 0 \end{array}\right).$$

Then the Borel subgroup P_0 consists of upper triangular elements in G and the Cartan subgroup M_0 consists of elements of the form $d(x_1,x_2) := \mathrm{diag}(x_1, x_2, \widetilde{\sigma}(x_2^{-1}), \widetilde{\sigma}(x_1^{-1}))$. Also $P_2 = M_2U_2$ and $\widehat{P}_2 = \widehat{M}_2\widehat{U}_2$ becomes

$$M_2 = \left\{ \left(\begin{array}{c|c|c} x & & \\ \hline & A & \\ \hline & & \widetilde{\sigma}(x^{-1}) \end{array}\right) \;\middle|\; \begin{array}{l} x \in \mathrm{Res}_{k'/k}\mathbb{G}_m, \\ A \in U(1,1)_{k'/k}. \end{array} \right\},\ U_2 = \left\{ \left(\begin{array}{c|c|c} 1 & * & * \\ \hline & \mathbf{1}_2 & * \\ \hline & & 1 \end{array}\right) \in G \right\}$$

$$\widehat{M}_2 = \left\{ \left(\begin{array}{c|c|c} x & & \\ \hline & A & \\ \hline & & y \end{array}\right) \;\middle|\; \begin{array}{l} x,\, y \in \mathbb{G}_m(\mathbb{C}), \\ A \in GL(2,\mathbb{C}). \end{array} \right\},\ \widehat{U}_2 = \left\{ \left(\begin{array}{c|c|c} 1 & * & * \\ \hline & \mathbf{1}_2 & * \\ \hline & & 1 \end{array}\right) \in \widehat{G} \right\}.$$

Here the group $U(1,1)_{k'/k}$ is defined by

$$U(1,1)_{k'/k} := \{ g \in \mathrm{Res}_{k'/k}GL(2)\,;\, gJ_1\widetilde{\sigma}({}^t g) = J_1 \},\ \text{where } J_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We conventionally write G_2 for this group.

5.1. Singular hyperplanes. We take $\mathfrak{X} \in \mathfrak{E}_{P_2}$, $(M_2, \mathfrak{P}) \in \mathfrak{X}$ and consider

$$\langle \theta_\phi, \theta_{\phi'} \rangle_{L^2(G(k)\backslash G(\mathbb{A}))} = \frac{1}{2\pi\sqrt{-1}} \int_{\pi \in \mathfrak{P}, \mathrm{Re}\pi = \lambda_0} A(\phi,\phi')(\pi)\, d\pi,$$

where

$$A(\phi,\phi')(\pi) := \sum_{(M_2,\mathfrak{P}') \in \mathfrak{X}} \Big\langle \sum_{w \in W(\mathfrak{P},\mathfrak{P}')} M(w,\pi)\phi(\pi), \phi'(-w\overline{\pi}) \Big\rangle$$

for $\phi \in P_{(M_2,\mathfrak{P})}$, $\phi' \in P_{\mathfrak{X}}$. Since W_{M_2} consists of 1 and $w(P_2) := w_1w_2w_1$, the analytic behavior of $A(\phi,\phi')(\pi)$ is determined by that of $M(w(P_2),\,\pi)$.

5.1.1. *Analytic behavior of the local intertwining operator.* Take $\pi = \bigotimes_v \pi_v \in \mathfrak{P}$ and consider $M(w(P_2),\pi)\phi(\pi) = \bigotimes_v M(w(P_2),\pi_v)\phi_v(\pi_v)$. We first recall the normalization factor $r(w(P_2),\pi_v)$ defined in [Sh]. We write $\pi \in \mathfrak{P}$ as $\chi \otimes \tau$, where χ is a quasi-character of $\mathbb{A}_{k'}^\times / k'^\times$ and τ is an irreducible cuspidal automorphic representation of $G_2(\mathbb{A})$.

(1) At inert v. Let $w|v$ be as before and Γ_v be as in 4.1.1. We use the isomorphism

$${}^L M_2 \ni \left(\begin{array}{c|c|c} x & & \\ \hline & A & \\ \hline & & y \end{array}\right) \rtimes w \xrightarrow{\sim} [(x, y^{-1}) \times A] \rtimes w \in {}^L(\mathrm{Res}_{k'/k}\mathbb{G}_m \times G_2)$$

to identify these L-groups. Then the adjoint representation $r_{w(P_2)}$ of ${}^LM_2$ on $\mathrm{Lie}(U_2)^\wedge_{w(P_2)} = \mathrm{Lie}\widehat{U}_2$ decomposes into the direct sum $r_{w(P_2),1} \oplus r_{w(P_2),2}$, where

$$r_{w(P_2),1}|_{\widehat{M}_2} = [(\mathrm{St}_1 \otimes \mathbf{1}) \otimes \widetilde{\mathrm{St}}_2] \oplus [(\mathbf{1} \otimes \mathrm{St}_1) \otimes \mathrm{St}_2], \quad r_{w(P_2),1}(\sigma)(v_1 \oplus v_2) = v_2 \oplus v_1,$$
$$r_{w(P_2),2}|_{\widehat{M}_2} = (\mathrm{St}_1 \otimes \mathrm{St}_1) \otimes \mathbf{1}, \quad r_{w(P_2),2}(\sigma)((v_1 \otimes v_2) \otimes u) = (v_2 \otimes v_1) \otimes u$$

Here St_1 denotes the standard representation of $\mathbb{G}_m(\mathbb{C})$ and $\widetilde{\mathrm{St}}_2$ is the contragredient of St_2. The L-factor attached to $\pi_v = \chi_v \otimes \tau_v$ and $r_{w(P_2),1}$ is the product L-factor of $\tau_v \times \chi_v$;

$$L(s, \pi_v, r_{w(P_2),1}) = L(s, \tau_v \times \chi_v).$$

The L-factor attached to π_v and $r_{w(P_2),2}$ is simply $L_k(s, \chi_v|_{k_v^\times})$. Thus from [Sh] §7, we have

$$r(w(P_2), \pi_v) = \frac{L(0, \tau_v \times \chi_v)}{L(1, \tau_v \times \chi_v)\varepsilon(0, \tau_v \times \chi_v, \psi_v)} \frac{L_k(0, \chi_v|_{k_v^\times})}{L_k(1, \chi_v|_{k_v^\times})\varepsilon_k(0, \chi_v|_{k_v^\times}, \psi_v)}.$$

(2) At splitting v. Again let $w_1,\ w_2|v$ be as before. We identify $M_{2,v}$ with $(\mathbb{G}_m \times \mathbb{G}_m) \times GL(2)$ by

$$M_2 \ni \begin{pmatrix} x & & \\ & A & \\ & & y \end{pmatrix} \xrightarrow{\sim} (x, y^{-1}) \times A \in (\mathbb{G}_m \times \mathbb{G}_m) \times GL(2).$$

Accordingly $\pi_v = \chi_v \otimes \tau_v$ is written as $(\chi_{w_1} \otimes \chi_{w_2}) \otimes \tau_v$. The normalization factor in this case is well-known and is given by

$$r(w(P_2), \pi_v) = \frac{L(0, \pi_v, r_{w(P_2),1})}{L(1, \pi_v, r_{w(P_2),1})\varepsilon(0, \pi_v, r_{w(P_2),1}, \psi_v)} \frac{L(0, \pi_v, r_{w(P_2),2})}{L(1, \pi_v, r_{w(P_2),2})\varepsilon(0, \pi_v, r_{w(P_2),2}, \psi_v)},$$

where, writing $\widetilde{\tau}_v$ for the contragredient of τ_v,

$$L(s, \pi_v, r_{w(P_2),1}) = L(s, \widetilde{\tau}_v \otimes \chi_{w_1})L(s, \tau_v \otimes \chi_{w_2})$$
$$\varepsilon(s, \pi_v, r_{w(P_2),1}, \psi_v) = \varepsilon(s, \widetilde{\tau}_v \otimes \chi_{w_1}, \psi_v)\varepsilon(s, \tau_v \otimes \chi_{w_2}, \psi_v)$$
$$L(s, \pi_v, r_{w(P_2),2}) = L_k(s, \chi_{w_1}\chi_{w_2}), \quad \varepsilon(s, \pi_v, r_{w(P_2),2}, \psi_v) = \varepsilon_k(s, \chi_{w_1}\chi_{w_2}, \psi_v).$$

Lemma 5.1. *The normalized operator $N(w(P_2), \pi_v) := r(w(P_2), \pi_v)^{-1}M(w(P_2), \pi_v)$ is holomorphic on the region $\langle \mathrm{Re}\pi, \beta_2^\vee \rangle \geq 0$. (Note that β_2 is the only positive root of A_{M_2} in P_2.)*

Proof. If v splits in k', then this was proved in [MW2]. Hence we assume v is inert in k' and let $w|v$. First assume π_v is essentially tempered. Archimedean case is a special case of Theorem 2.1 in [Ar2]. Non-archimedean case follows immediately from [Sh] Proposition 7.2 (b). Next comes π_v with τ_v in the complementary series. Then τ_v is of the form

$$\tau_v = \mathrm{Ind}_{B_2(k_v)}^{G_2(k_v)}[\nu_v \otimes |\ |_w^{\lambda/2} \otimes \mathbf{1}_{N_2(k_v)}] \quad (0 < \lambda < 1),$$

where ν_v is a character of $k_w'^\times$ such that $\nu_v|_{k_v^\times} = \mathbf{1}$. $B_2 = T_2N_2$ is the Borel subgroup consisting of upper triangular elements in G_2. It follows from

$$L(s, \pi_v, r_{w(P_2),1}) = L(s - \lambda/2, \chi_v\nu_v^{-1})L(s + \lambda/2, \widetilde{\sigma}(\chi_v)\nu_v)$$

that $r(w(P_2), \pi_v)$ has its only pole in the region $\langle \mathrm{Re}\pi_v, \beta_2^\vee \rangle \geq 0$ at $\chi_v = \nu_v|\ |_w^{\lambda/2}$ and it is simple. On the other hand, we apply Corollary 7.6 of [Sh] to $M(w_1, w_2w_1\pi_v)$ and note

$$M(w(P_2), \pi_v) = M(w_1, w_2w_1\pi_v)M(w_2, w_1\pi_v)M(w_1, \pi_v)$$

to see that $M(w(P_2),\pi_v)$ has its only pole in the positive region at $\chi_v=\nu_v|\ |_w^{\lambda/2}$ and it is simple. The lemma is proved. □

5.1.2. *Analytic behavior of the global intertwining operator.* For $\pi\in\mathfrak{P}$, we define

$$r(w(P_2),\pi):=\prod_v r(w(P_2),\pi_v),\quad N(w(P_2),\pi):=r(w(P_2),\pi)^{-1}M(w(P_2),\pi).$$

Then the following proposition will be proved in Appendix B.

Proposition 5.2. *Here again, we write $\pi\in\mathfrak{P}$ as $\chi\otimes\tau$. The central character of τ is denoted by ω_τ. For each character μ of $U(1,\mathbb{A})_{k'/k}$, we write $\Theta(\xi,\psi)_\mu$ for the theta-lifts of μ to $G_2(\mathbb{A})$ under the Weil representation $\omega_{\psi,\xi}$ of $G_2(\mathbb{A})\times U(1,\mathbb{A})_{k'/k}$. (See Appendix B for more details.) Then the only poles of $M(w(P_2),\pi)$ in the region $\langle \mathrm{Re}\pi,\beta_2^\vee\rangle\geq 0$ locate at*

$$\begin{aligned}&\mathfrak{S}(P_2,\eta_{k'/k}):=\{\pi=\chi\otimes\tau\in\mathfrak{P}\,;\,\chi|_{\mathbb{A}^\times}=|\ |_{\mathbb{A}}^2\eta_{k'/k},\,\tau=\Theta(\chi^{-1}|\ |_{\mathbb{A}_{k'}},\psi)_{\omega_\tau}\}\\&\mathfrak{S}(P_2,\mathbf{1}):=\{\pi=\chi\otimes\tau\in\mathfrak{P}\,;\,\chi|_{\mathbb{A}^\times}=|\ |_{\mathbb{A}},\ L(0,\tau\times\chi)\neq 0\},\end{aligned}$$

and they are simple. Here in the definition of $\mathfrak{S}(P_2,\eta_{k'/k})$, ψ has been chosen so that τ_v is generic with respect to it.

Remark 5.3. *If we exchange ψ in the definition of $\mathfrak{S}(P_2,\eta_{k'/k})$, then τ changes to another irreducible cuspidal representation in the same global L-packet (cf.* [LL]*).*

5.1.3. *Singular hyperplanes.* From Proposition 5.2, we have

$$S^{h,+}_{(M_2,\mathfrak{P})}:=\{\mathfrak{S}(P_2,\eta_{k'/k}),\ \mathfrak{S}(P_2,\mathbf{1})\}.$$

5.2. Decomposition of the scalar product. We have $S^+_{\mathfrak{X}}=\{\mathfrak{P},\,\mathfrak{S}(P_2,\eta_{k'/k}),\,\mathfrak{S}(P_2,\mathbf{1})\}$ and set $o(\mathfrak{P}):=0$. Then the following is proved in the same manner as Theorem 4.3.

Theorem 5.4. *For $\phi\in P_{(M_2,\mathfrak{P})}$ and $\phi'\in P_{\mathfrak{X}}$, we have*

$$\begin{aligned}\langle\theta_\phi,\theta_{\phi'}\rangle_{L^2(G(k)\backslash G(\mathbb{A}))}&=\frac{1}{2\pi\sqrt{-1}}\int_{\pi\in\mathfrak{P},\mathrm{Re}\pi=0}A(\phi,\phi')(\pi)\,d\pi\\&+c_2\langle N(w(P_2),\mathfrak{S}(P_2,\eta_{k'/k}))\phi(\mathfrak{S}(P_2,\eta_{k'/k})),\,\phi'(-w(P_2)\overline{\mathfrak{S}(P_2,\eta_{k'/k})})\rangle\\&+c_2'\langle N(w(P_2),\mathfrak{S}(P_2,\mathbf{1}))\phi(\mathfrak{S}(P_2,\mathbf{1})),\,\phi'(-w(P_2)\overline{\mathfrak{S}(P_2,\mathbf{1})})\rangle\end{aligned}\tag{5.1}$$

for some non-zero constants c_2 and c_2'.

5.3. The residual spectrum from P_2. By Theorem 5.4, it is enough to determine the images

$$\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\eta_{k'/k}))=\bigotimes_v\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\eta_{k'/k})_v)$$
$$\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\mathbf{1}))=\bigotimes_v\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\mathbf{1})_v).$$

(1) $\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\eta_{k'/k})_v)$. If $\mathfrak{S}(P_2,\eta_{k'/k})_v$ is essentially tempered, then $\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\eta_{k'/k})_v)$ is irreducible by the Langlands' classification. Next we handle $\mathfrak{S}(P_2,\eta_{k'/k})_v=\chi_v\otimes\tau_v$ with τ_v in the complementary series. Then its exponent is of the form

$$\begin{cases}|\ |_w\otimes|\ |_w^{\lambda/2},\ (0<\lambda<1) & \text{if } v \text{ is inert},\\ |\ |_v\otimes|\ |_v^{\mu/2}\otimes|\ |_v^{-\mu/2}\otimes|\ |_v^{-1},\ (0<\mu<1) & \text{if } v \text{ splits}.\end{cases}$$

In both cases this is regular and hence $\mathrm{Im}N(w(P_2),\,\mathfrak{S}(P_2,\eta_{k'/k})_v)$ is irreducible.

(2) $\mathrm{Im}N(w(P_2), \mathfrak{S}(P_2, \mathbf{1})_v)$. Again the tempered case is cared by the Langlands' classification. In the non-tempered case, the exponent of $\mathfrak{S}(P_2, \mathbf{1})_v$ is of the form

$$\begin{cases} |\ |_w^{1/2} \otimes |\ |_w^{\lambda/2},\ (0 < \lambda < 1) & \text{if } v \text{ is inert,} \\ |\ |_v^{1/2} \otimes |\ |_v^{\mu/2} \otimes |\ |_v^{-\mu/2} \otimes |\ |_v^{-1/2},\ (0 < \mu < 1) & \text{if } v \text{ splits.} \end{cases}$$

Again this is regular and hence $\mathrm{Im}N(w(P_2), \mathfrak{S}(P_2, \mathbf{1})_v) = \mathrm{Ind}_{P_2(k_v)}^{G(k_v)}[\mathfrak{S}(P_2, \mathbf{1})_v \otimes \mathbf{1}_{U_2(k_v)}]$ is irreducible.

The resulting irreducible representations $\mathrm{Im}N(w(P_2), \mathfrak{S}(P_2, \eta_{k'/k}))$ and $\mathrm{Im}N(w(P_2), \mathfrak{S}(P_2, \mathbf{1}))$ will be called the *global Langlands' quotients* of $\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_2, \eta_{k'/k}) \otimes \mathbf{1}_{U_2(\mathbb{A})}]$ and $\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_2, \mathbf{1}) \otimes \mathbf{1}_{U_2(\mathbb{A})}]$ respectively.

Theorem 5.5. *The contribution of cuspidal data attached to P_2 to the residual discrete spectrum is multiplicity free and consists of the global Langlands' quotients defined above of $\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_2, \eta_{k'/k}) \otimes \mathbf{1}_{U_2(\mathbb{A})}]$ and $\mathrm{Ind}_{P_2(\mathbb{A})}^{G(\mathbb{A})}[\mathfrak{S}(P_2, \mathbf{1}) \otimes \mathbf{1}_{U_2(\mathbb{A})}]$, where $\mathfrak{S}(P_2, \eta_{k'/k})$ and $\mathfrak{S}(P_2, \mathbf{1})$ are as in Proposition* 5.2.

Appendix A. Poles of the twisted tensor L-function

In this appendix we determine the poles of the twisted tensor L-function in the half plane $\mathrm{Re}(s) > 0$ and prove Proposition 4.2 as a corollary. In fact this is little more than Satz 3.13 in [HLR]. We begin with the integral representation of $L_{Asai}(s, \pi)$.

A.1. The zeta integral.

A.1.1. *The zeta integral $Z(f, \Phi, s)$ and its unfolding.* Let k'/k and Γ be as before. We write G_1 for $\mathrm{Res}_{k'/k}GL(2)$ and H for the k-subgroup $GL(2)_k$ of G_1. We fix a Borel subgroup $B = TN$ consisting of upper triangular elements, together with the Levi factor T consisting of diagonal elements in G_1. Set $B_H := B \cap H$, $T_H = T \cap H$ and $N_H := N \cap H$.

For $\Phi \in \mathcal{S}(\mathbb{A}^{\oplus 2})$ and a quasi-character ω of $\mathbb{A}^\times/k^\times$ we define a meromorphic section $f_\Phi(\omega, s; h)$ for $\mathrm{Ind}_{B_H(\mathbb{A})}^{H(\mathbb{A})}[(|\ |_{\mathbb{A}}^{(s-1)/2} \otimes |\ |_{\mathbb{A}}^{(1-s)/2}\omega^{-1}) \otimes \mathbf{1}_{N_H(\mathbb{A})}]$ as

$$f_\Phi(\omega, s; h) := |\det(h)|_{\mathbb{A}}^{s/2} \int_{\mathbb{A}^\times} \Phi([0, t]h)\omega(t)|t|_{\mathbb{A}}^s\, dt^\times, \quad (h \in H(\mathbb{A})).$$

Using this we define the Eisenstein series

$$E_\Phi(\omega, s; h) := \sum_{\gamma \in B_H(k)\backslash H(k)} f_\Phi(\omega, s; h), \quad (h \in H(\mathbb{A})).$$

This converges absolutely for $\mathrm{Re}\omega + \mathrm{Re}(s) \gg 0$ and is meromorphically continued to the whole s-plane.

Now let π be an irreducible cuspidal automorphic representation of $G_1(\mathbb{A})$ and write ω_π for its central character. We define the Zeta integral $Z(f, \Phi, s)$, where f is a cusp form in the space of π and $\Phi \in \mathcal{S}(\mathbb{A}^{\oplus 2})$, by

$$Z(f, \Phi, s) := \int_{H(k)Z(H,\mathbb{A})\backslash H(\mathbb{A})} f(h)E_\Phi(\omega_\pi, 2s; h)\, dh. \tag{A.1}$$

Here $Z(H)$ is the center of H. Since f is rapidly decreasing and $E_\Phi(\omega_\pi, 2s; h)$ is slowly increasing, this converges absolutely for $\mathrm{Re}\omega + \mathrm{Re}(s) \gg 0$ and meromorphically continued to the whole plane. Also note that the functional equation for $Z(f, \Phi, s)$ can be deduced from that of $E_\Phi(\omega_\pi, 2s; h)$ as was done in [As].

The zeta integral (A.1) is decomposed into the direct product of local zeta integrals as follows (cf. [HLR] pp. 76–77). We fix a non-trivial character $\psi_{k'}$ of $\mathbb{A}_{k'}/k'$ such that $\psi_{k'}|_{\mathbb{A}^\times} = \mathbf{1}$ and take the Whittaker model $\mathcal{W}(\pi, \psi_{k'})$ for π. Then we have the Fourier expansion on $N_H(k)\backslash N_H(\mathbb{A})$;

$$f(h) = \sum_{\alpha \in k'^\times} W_f\Big(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h\Big), \quad (\exists W_f \in \mathcal{W}(\pi, \psi_{k'})).$$

Using this and the definition of $E_\Phi(\omega_\pi, 2s; h)$, we have

$$Z(f, \Phi, s) = \int_{N_H(\mathbb{A})Z(H,\mathbb{A})\backslash H(\mathbb{A})} \sum_{\alpha \in k^\times} W_f\Big(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h\Big) f_\Phi(\omega_\pi, 2s; h)\, dh.$$

We note that $W(h) := \sum_{\alpha \in k^\times} W_f(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} h)$ is contained in $\mathcal{W}(\pi, \psi_{k'})$. Further we may assume that $W(h)$ and Φ are of the form

$$W(h) = \bigotimes_v W_v(h_v),\ W_v \in \mathcal{W}(\pi_v, \psi_{k',v}), \quad \Phi(x) = \bigotimes_v \Phi_v(x_v),\ \Phi_v \in \mathcal{S}(k_v^{\oplus 2})$$

(restricted tensor products). Now we have

$$Z(f, \Phi, s) = \prod_v Z_v(W_v, \Phi_v, s), \tag{A.2}$$

where

$$\begin{aligned} Z_v(W_v, \Phi_v, s) &:= \int_{N_H(k_v)Z(H,k_v)\backslash H(k_v)} W_v(h) f_{\Phi_v}(\omega_{\pi_v}, 2s; h)\, dh, \\ f_{\Phi_v}(\omega_v, s; h) &:= |\det(h)|_v^{s/2} \int_{k_v^\times} \Phi_v([0,t]h)\omega_v(t)|t|_v^s\, dt^\times \\ &\in \mathrm{Ind}_{B_H(k_v)}^{H(k_v)}[(|\ |_v^{(s-1)/2} \otimes \omega_v^{-1}|\ |_v^{(1-s)/2}) \otimes \mathbf{1}_{N_H(k_v)}]. \end{aligned}$$

These local zeta integrals converge absolutely for $\mathrm{Re}(\omega_{\pi_v}) + \mathrm{Re}(s) \gg 0$ and are meromorphically continued to the whole plane. (A.2) can be considered as a equality of these meromorphic functions.

A.2. Comparisons of local factors. Here we shall check that $Z_v(W_v, \Phi_v, s)$ has the same poles as those of the L-factor $L_{Asai}(s, \pi)$ at least in the region $\langle \mathrm{Re}\pi, \beta_1^\vee \rangle > 0$. We treat various types of π_v separately.

A.2.1. *The case of splitting v.* Let w_1 and w_2 be the places of k' lying over v. From our choice of $\psi_{k'}$ we can take a non-trivial character ψ_v of k_v such that $\psi_{k',w_1} = \psi_v$ and $\psi_{k',w_2} = \overline{\psi}_v$. We identify $G(k_v)$ with $GL(2,k_v) \times GL(2,k_v)$ and write π_v as $\pi_v = \pi_{w_1} \otimes \pi_{w_2}$ accordingly. Then we may assume $W_v = W_{w_1} \otimes W_{w_2}$ where $W_{w_1} \in \mathcal{W}(\pi_{w_1}, \psi_v)$ and $W_{w_2} \in \mathcal{W}(\pi_{w_2}, \overline{\psi}_v)$. Noting that $W(h) \xrightarrow{\sim} W(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h)$ gives an isomorphism from $\mathcal{W}(\pi_{w_2}, \psi_v)$ to $\mathcal{W}(\pi_{w_2}, \overline{\psi}_v)$, we have

$$Z_v(W_v, \Phi_v, s) = \int_{N_H(k_v)Z(H,k_v)\backslash H(k_v)} W_{w_1}(h) W'_{w_2}\Big(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h\Big) z(|\ |_v^{2s}\omega_\pi, h.\Phi) |\det(h)|^s\, dh.$$

Here $W_{w_1} \in \mathcal{W}(\pi_{w_1}, \psi_v)$, $W'_{w_2} \in \mathcal{W}(\pi_{w_2}, \psi_v)$, and $z(\omega, \Phi)$ was defined in the introduction of [Ja]. Hence from [Ja], we know that

(1) The local integrals $Z_v(W_v, \Phi_v, s)$ have $L_{Asai}(s, \pi_v) = L(s, \pi_{w_1} \times \pi_{w_2})$ as their G.C.D.

(2) (Unramified situation.) We assume that $\pi_v = \pi_{w_1} \otimes \pi_{w_2}$ is unramified, Φ_v is Φ_v^0, the characteristic function of $\mathcal{O}_v \oplus \mathcal{O}_v$, and ψ_v is of order 0. Then for the class-1 Whittaker function $W_v^0 \in \mathcal{W}(\pi_v, \psi_{k',v})$ with $W_v^0(1) = 1$, we have

$$Z_v(W_v, \Phi_v, s) = L(s, \pi_{w_1} \times \pi_{w_2}).$$

A.2.2. *The case of inert unramified v.* Let w be the place of k' lying over v. We assume that

(1) k'_w/k_v is an unramified quadratic extension,
(2) π_v is of the form $\mathrm{Ind}_{B(k_v)}^{G_1(k_v)}[(\mu \otimes \nu) \otimes \mathbf{1}_{N(k_v)}]$, where μ and ν are unramified quasi-characters of ${k'}_w^\times$,
(3) Φ_v equals Φ_v^0 (see A.2.1). $W_v^0 \in \mathcal{W}(\pi_v, \psi_{k',v})$ is the class-1 Whittaker function for π_v;

$$W_v^0(1) = 1, \qquad W_v^0(gk) = W_v^0(g) \quad \text{for } \forall k \in GL(2, \mathcal{O}_w).$$

Lemma A.1 (Lemma 3.14 in [HLR]**).** *Under these assumptions we have*

$$Z_v(W_v^0, \Phi_v^0, s) = L_{Asai}(s, \pi_v).$$

A.2.3. *The case of non-archimedean inert v.* Let $w|v$ be as before. It follows from the well-known integration formula that

$$Z_v(W_v, \Phi_v, s) = \int_{k_v^\times} \int_{GL(2,\mathcal{O}_v)} W_v\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\Big) f_{\Phi_v}(\omega_{\pi_v}, 2s; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k) |a|_v^{-1} \, dk \, da^\times.$$

But since

$$f_{\Phi_v}(\omega_{\pi_v}, 2s; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k) = |a|_v^s L(2s, \omega_\pi|_{k_v^\times}) P(q_v^s, q_v^{-s})$$

with $P(X, Y)$ a polynomial, and $W_v(h)$ is $GL(2, \mathcal{O}_v)$-finite, it is enough to consider

$$Z_v(\varphi_v, \Phi_v, s) := \int_{k_v^\times} \varphi_v(a) |a|_v^{s-1} \, da^\times L(2s, \omega_\pi|_{k_v^\times}) P(q_v^s, q_v^{-s}).$$

Here φ_v is in the Kirillov model $\mathcal{K}(\pi_v, \psi_{k',v})$ of π_v with respect to $\psi_{k',v}$.

Recall that the space $\mathcal{K}(\pi_v, \psi_{k',v})$ was calculated by Godement as follows.

(1) If $\pi_v = \mathrm{Ind}_{B(k_v)}^{G_1(k_v)}[(\mu \otimes \nu) \otimes \mathbf{1}_{N(k_v)}]$, then

$$\mathcal{K}(\pi_v, \psi_{k',v}) = \begin{cases} \{f(a)\mu(a)|a|_w^{1/2} + g(a)\nu(a)|a|_w^{1/2} \mid f,\, g \in \mathcal{S}(k'_w)\} & \text{if } \mu \neq \nu, \\ \{f(a)\mu(a)|a|_w^{1/2} + g(a)\mu(a)\mathrm{ord}_{k'_w}(a)|a|_w^{1/2} \mid f,\, g \in \mathcal{S}(k'_w)\} & \text{if } \mu = \nu. \end{cases}$$

(2) If π_v is a special representation embedded in $\mathrm{Ind}_{B(k_v)}^{G_1(k_v)}[(\mu \otimes \mu|\ |_w^{-1}) \otimes \mathbf{1}_{N(k_v)}]$ as a submodule, then

$$\mathcal{K}(\pi_v, \psi_{k',v}) = \{f(a)\mu(a)|a|_w^{1/2} \mid f \in \mathcal{S}(k'_w)\}.$$

(3) If π_v is supercuspidal, then $\mathcal{K}(\pi_v, \psi_{k',v}) = \mathcal{S}({k'}_w^\times)$.

In the case (1), we see

$$Z_v(\varphi_v, \Phi_v, s) = L_k(s, \mu|_{k_v^\times}) L_k(s, \nu_v|_{k_v^\times}) L_k(2s, \omega_{\pi_v}|_{k_v^\times}) P'(q_v^s, q_v^{-s})$$

for some polynomial $P'(X, Y)$, while we have from 3.1.1

$$L_{Asai}(s, \pi_v) = L_k(s, \mu|_{k_v^\times}) L_k(s, \nu|_{k_v^\times}) L_{k'}(s, \omega_{\pi_v}).$$

But both $L_k(2s, \omega_{\pi_v}|_{k_v^\times})$ and $L_{k'}(s, \omega_{\pi_v})$ are holomorphic and non-zero on the region $\langle \mathrm{Re}\pi_v, \beta_1 \rangle > 0$ and $\mathrm{Re}(s) > 0$. Thus $Z_v(\varphi_v, \Phi_v, s)$ has the same poles as $L_{Asai}(s, \pi_v)$ has in that positive region.

In the case (2), we have

$$Z_v(\varphi_v, \Phi_v, s) = L_k(s, \mu|_{k_v^\times}) L_k(2s, \omega_{\pi_v}|_{k_v^\times}) P'(q_v^s, q_v^{-s}).$$

On the other hand the twisted tensor L-factor in this case was calculated in [G] Theorem 5.6;

$$L_{Asai}(s, \pi_v) = L_k(s, \mu|_{k_v^\times}) L_k(s-1, (\mu|_{k_v^\times}) \otimes \eta_{k'_w/k_v}).$$

Here again, both $L_k(2s, \omega_{\pi_v}|_{k_v^\times})$ and $L_k(s-1, (\mu|_{k_v^\times}) \otimes \eta_{k'_w/k_v})$ are holomorphic and non-zero on the region $\langle \mathrm{Re}\pi, \beta_1^\vee \rangle = \mathrm{Re}(\mu^2|\ |_w^{-1}) > 0$ and $\mathrm{Re}(s) > 0$. Hence $Z_v(\varphi_v, \Phi_v, s)$ and $L_{Asai}(s, \pi_v)$ have the same poles in this region.

In case (3), the G.C.D. of $Z_v(\varphi_v, \Phi_v, s)$ is $L_k(2s, \omega_{\pi_v}|_{k_v^\times})$. This is holomorphic on our positive region. On the other hand [Sh] Proposition 7.2 (a) asserts that $L_{Asai}(s, \pi_v)$ is also holomorphic in that region.

A.2.4. *The case of archimedean ramified v.* Since $k'_w/k_v = \mathbb{C}/\mathbb{R}$, we may assume $\pi_v = \mathrm{Ind}_{B(k_v)}^{G_1(k_v)}[(\mu \otimes \nu) \otimes \mathbf{1}_{N(k_v)}]$. As usual we write $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $L(s, \chi) := \Gamma_{\mathbb{R}}(s + r + m)$ for $\chi = |\ |_{\mathbb{R}}^r \mathrm{sgn}^m$ ($r \in \mathbb{C}$, $m \in \{0, 1\}$). We may assume

$$f_{\Phi_v}(\omega_{\pi_v}, 2s; \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k(\theta)) = e^{\sqrt{-1}n\theta} |a|_{\mathbb{R}}^s L(2s, \omega_{\pi_v}|_{\mathbb{R}^\times}) P(s),$$

$$W_v(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k(\theta)) = e^{\sqrt{-1}m\theta} W_v(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}), \quad k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $P(s)$ is a holomorphic function. Then it is enough to consider

$$Z(\Phi_w, \Phi_v, s) := \int_{\mathbb{R}^\times} \varphi_{\Phi_w}(a) |a|_{\mathbb{R}}^{s-1}\, da^\times\, L(2s, \omega_{\pi_v}|_{k_v^\times}),$$

with

$$\varphi_{\Phi_w}(a) = W_{\Phi_w}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big), \quad \Phi_w \in \mathcal{S}(\mathbb{C}^{\oplus 2}),$$

$$W_{\Phi_w}(g) := \mu(\det(g)) |\det(g)|_{\mathbb{C}}^{1/2} \int_{\mathbb{C}^\times} [r(g)\Phi_w](z, z^{-1}) \mu\nu^{-1}(z)\, dz^\times.$$

Here $r(g)$ denotes the Weil representation of $GL(2, \mathbb{C})$ on the linear model $\mathcal{S}(\mathbb{C}^2)$ given in [JL] Lemma 1.6.

Now one has

$$\begin{aligned} &\int_{\mathbb{R}^\times} \varphi_{\Phi_w}(a) |a|_{\mathbb{R}}^{s-1}\, da^\times \\ &\quad = \int_{\mathbb{R}^\times} \mu(a) |a|_{\mathbb{R}}^s \int_{\mathbb{C}^\times} \Phi_w\Big([z, z^{-1}] \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big) \mu\nu^{-1}(z)\, dz^\times\, da^\times \\ &\quad = \int_{\mathbb{R}^\times} \int_{\mathbb{C}^\times} \mu(a) |a|_{\mathbb{R}}^s \Phi_w(az^{-1}, z) \mu^{-1}\nu(z)\, dz^\times\, da^\times \end{aligned}$$

decomposing $\mathbb{C}^\times$ into $\mathbb{R}^\times \times (\mathbb{C}^1/\pm 1)$,

$$= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \int_{\mathbb{C}^1/\pm 1} \mu(ax^{-1}u^{-1}) |a|_{\mathbb{R}}^s \Phi_w(ax^{-1}u^{-1}, xu) \nu(xu)\, du\, dx^\times\, da^\times$$

$$= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \left(\int_{\mathbb{C}^1/\pm 1} \mu^{-1}\nu(u)\Phi_w(au^{-1}, xu)\, du \right) \mu(a)|a|_{\mathbb{R}}^s \nu(x)|x|_{\mathbb{R}}^s \, da^\times \, dx^\times$$

writing $\Phi'_w(a,x)$ for the inside of ( ),

$$= \int_{\mathbb{R}^\times} \int_{\mathbb{R}^\times} \Phi'_w(a,x)\mu(a)|a|_{\mathbb{R}}^s \nu(x)|x|_{\mathbb{R}}^s \, da^\times \, dx^\times$$

Since $\Phi'_w(a,x)$'s span a dense subspace of $\mathcal{S}(\mathbb{R}^{\oplus 2})$, the G.C.D. of these integrals is $L(s, \mu|_{k_v^\times})L(s, \nu|_{k_v^\times})$. Again noting $L_k(2s, \omega_{\pi_v}|_{\mathbb{R}^\times})$ and $L_{k'}(s, \omega_{\pi_v})$ are holomorphic on the region $\langle \mathrm{Re}\pi, \beta_1^\vee \rangle > 0$ and $\mathrm{Re}(s) > 0$, we conclude that $Z(\Phi_w, \Phi_v, s)$ has the same pole as $L_{Asai}(s, \pi_v)$ has.

A.3. Poles of the twisted tensor L-function. We now state the main result of this appendix.

Proposition A.2. *Let π be an irreducible unitary cuspidal automorphic representation of $G_1(\mathbb{A})$ and write ω_π for its central character. Then the only possible pole of $L_{Asai}(s,\pi)$ in the region*

$$\langle \mathrm{Re}\pi, \beta_1^\vee \rangle > 0, \quad \mathrm{Re}(s) > 0$$

locates at $s = 1$ with $\omega_\pi|_{\mathbb{A}^\times} = \mathbf{1}$. It is a pole if and only if

$$\int_{H(k)Z(H,\mathbb{A})\backslash H(\mathbb{A})} f(h)\, dh \neq 0, \quad \exists f \in V_\pi \tag{A.3}$$

holds.

Proof. From A.2, it is sufficient to determine the poles of $Z(f, \Phi, s)$ in the positive domain of the proposition. Then these poles were calculated in [HLR] Satz 3.13. □

Corollary A.3 (Proposition 4.2). *Let the notations be as above. Then the only pole of $r(w(P_1), \pi)$ in the region $\langle \mathrm{Re}\pi, \beta_1^\vee \rangle \geq 0$ occurs at*

$$\mathfrak{S}(P_1) := \left\{ \pi \in A_0(G_1(k)\backslash G_1(\mathbb{A})) \;\middle|\; \begin{array}{l} (i)\ \omega_\pi|_{\mathbb{A}^\times} = |\ |_{\mathbb{A}}^2, \\ (ii)\ \int_{H(k)Z(H,\mathbb{A})\backslash H(\mathbb{A})} f(h)|\det(h)|_{\mathbb{A}}^{-1}\, dh \neq 0 \end{array} \right\},$$

and it is simple.

Proof. Recall that

$$r(w(P_1), \pi) = \frac{L_{Asai}(0,\pi)}{L_{Asai}(1,\pi)\varepsilon_{Asai}(0,\pi)}.$$

The infinite product $\varepsilon_{Asai}(s,\pi) = \prod_v \varepsilon_{Asai}(s, \pi_v, \psi_v)$ is in fact a finite product ([Sh] Theorem 3.5 (1)) and each local factor is an exponential function. Thus $\varepsilon_{Asai}(0,\pi)$ contributes nothing to the analytic property of $r(w(P_1), \pi)$. On the other hand $L_{Asai}(1,\pi)$ does not vanish in our positive region by Shahidi's non-vanishing theorem ([Sh3] Theorem 5.1). Thus the poles of $r(w(P_1), \pi)$ are the same as those of $L_{Asai}(0,\pi)$, and the corollary follows from Proposition A.2. □

Appendix B. Poles of the product L-function of $U(1,1) \times \mathrm{Res}_{k'/k}\mathbb{G}_m$

Here we shall determine the poles of the product L-function of $G_2 \times \mathrm{Res}_{k'/k}\mathbb{G}_m$ and deduce Proposition 5.2 as a corollary. For this we use the Shimura type integral representation (cf. [GJ]), and hence we begin with some review of theta series on $G_2(\mathbb{A})$.

B.1. Oscillator representations for $U(1,1) \times U(1)$.

B.1.1. *Preliminary for the local theory.* Let F be a local field of characteristic 0. E denotes a 2-dimensional abelian semisimple algebra over F and let $\{1, \epsilon\}$ be a basis of E over F. We may assume that $\epsilon^2 = -\delta \in F^\times$. We write σ for the non-trivial involution of E trivial on F. We write $|\ |_F$ for the modulus of F. If F is non-archimedean we write $\mathcal{O}_F$, $\mathfrak{p}_F$, ϖ_F and q_F for the maximal compact subring of F, the maximal ideal in $\mathcal{O}_F$, a generator of $\mathfrak{p}_F$ and the cardinality of the residue field of F, respectively. $\eta_{E/F}$ denotes the quadratic character of $F^\times$ which corresponds to E/F by the classfield theory.

The group G_2 to be considered is defined by

$$G_2 = U(1,1)_{E/F} := \{g \in \mathrm{Res}_{E/F} GL(2)\,;\, gJ_1\widetilde{\sigma}({}^t g) = J_1\},$$

where J_1 is as in the beginning of §5. This is attached to the skew-Hermitian space $(V, \Phi) = (E^{\oplus 2}, J_1)$. We define the symplectic space (W, ϕ) over F by $W := \mathrm{Res}_{E/F} V$ and $\phi(\ ,\ ) := \mathrm{Tr}_{E/F}\Phi(\ ,\ )$ and write $Sp(W)$ for its symplectic group. Of course we have the natural embedding $\iota : G_2 \hookrightarrow Sp(W)$. We fix a Borel pair (B_2, T_2) of G_2 such that B_2 consists of upper triangular elements and T_2 consists of $d(a) := \mathrm{diag}(a, \widetilde{\sigma}(a)^{-1})$, $(a \in \mathrm{Res}_{E/F}\mathbb{G}_m)$. Then G_2 is generated by $H := SL(2)_F$ and T_2. We take the Borel pair (B_H, T_H) to be the intersection of (B_2, T_2) with H. We choose a suitable symplectic basis of W and identify $Sp(W)$ with $Sp(2)$, so that the above embedding ι is given by

$$\iota\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) = \left(\begin{array}{cc|cc} a & 0 & 2b & 0 \\ 0 & a & 0 & 2b\delta \\ \hline c/2 & 0 & d & 0 \\ 0 & c/2\delta & 0 & d \end{array}\right), \quad \Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H\Big)$$

$$\iota(d(x)) = \left(\begin{array}{cc|cc} \xi_1 & \xi_2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -\xi_2\delta & \xi_1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \eta_1 & \eta_2\delta \\ & & -\eta_2 & \eta_1 \end{array}\right), \quad \left(\begin{array}{c} x = \xi_1 + \xi_2\epsilon \in \mathrm{Res}_{E/F}\mathbb{G}_m \text{ with} \\ \widetilde{\sigma}(x)^{-1} = \eta_1 + \eta_2\epsilon \end{array}\right)$$

B.1.2. *Splitting of the metaplectic 2-cocycle and local oscillator representations.* It is known that the metaplectic 2-cocycle on $Sp(W) = Sp(2)$ splits over $\iota(G_2(F))$. The explicit splitting is calculated in [MVW] Chapter 3.I.4 (see also [Ku] Theorem 3.1). To state the result, we fix a character ξ of $E^\times$ such that $\xi|_{F^\times} = \eta_{E/F}$. We define three 1-cocycles as follows.

(1) Define a 1-cocycle λ_1 on $H(F)$ by

$$\lambda_1\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big) := \begin{cases} \langle c/2, -\delta\rangle_F & \text{if } c \neq 0, \\ \langle d, -\delta\rangle_F & \text{otherwise,} \end{cases}$$

where $\langle\ ,\ \rangle_F$ denotes the Hilbert symbol for F.

(2) Define a 1-cocycle λ_2 on $T_2(F)$ by

$$\lambda_2(d(x)) := \frac{\gamma(\mathrm{N}_{E/F}(x))}{\gamma(1)}.$$

Here γ is the Weil constant (cf. [P]).

(3) Define a 1-cocycle λ_3 on $G_2(F)$ for $h \in H(F)$ and $d(x) \in T_2(F)$,

$$\lambda_3(h \cdot d(x)) = \lambda_3(d(x) \cdot h) := \begin{cases} \langle \mathrm{N}_{E/F}(x), \delta\rangle_F & \text{if } c \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then we define a 1-cocycle λ_ξ on $G_2(F)$ by

$$\lambda_\xi(g = h \cdot d(x)) = \lambda_\xi(d(x) \cdot h) := \lambda_1(h)\lambda_2(d(x))\xi(x)\lambda_3(g), \quad (h \in H(F),\, d(x) \in T_2(F)).$$

This is independent of the decomposition $g = h \cdot d(x)$ and the metaplectic 2-cocycle $c(g_1, g_2)$ ([P]) restricted to $\iota(G_2(F))$ becomes

$$c(\iota(g_1), \iota(g_2)) = \frac{\lambda_\xi(g_1 g_2)}{\lambda_\xi(g_1)\lambda_\xi(g_2)}.$$

Thus we have the splitting

$$\iota_\xi : G_2(F) \ni g \longrightarrow (\iota(g), \lambda_\xi(g)) \in Mp(W).$$

We fix a non-trivial character ψ of F and write ω_ψ for the oscillator representation of $Mp(W)$ associated with ψ. The above splitting gives the oscillator representation $\omega_{\psi,\xi} := \omega_\psi \circ \iota_\xi$ on $\mathcal{S}(E)$. We have explicit formulae:

$$\omega_{\psi,\xi}(d(a))\Phi(x) = \xi(a)|a|_E^{1/2}\Phi(ax), \quad (a \in E^\times) \tag{B.1}$$

$$\omega_{\psi,\xi}(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\Phi(x) = \psi(b\mathrm{N}_{E/F}(x))\Phi(x) \tag{B.2}$$

$$\omega_{\psi,\xi}(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})\Phi(x) = \frac{1}{\gamma(2\mathrm{N}_{E/F})}\int_E \Phi(y)\psi(\mathrm{Tr}_{E/F}(y\sigma(x)))\, dy \tag{B.3}$$

B.1.3. *Local Howe duality for $U(1,1) \times U(1)$.* Here we assume that E/F is a quadratic extension of fields. Note that the group $U(1)_{E/F}$ is contained in G_2 as its center Z_2. Since $Z_2(F)$ is compact, we can define the local theta integral

$$\mathrm{Pr}_\chi\Phi(x) := \int_{Z_2(F)} \overline{\chi}\xi(z)\Phi(xz)\, dz. \tag{B.4}$$

This defines a projection on $\mathcal{S}(E)$ whose image $(\omega_{\psi,\xi}(\chi), \mathcal{S}(E)_\chi)$ is the χ-isotypic subspace of $(\omega_{\psi,\xi}, \mathcal{S}(E))$. Then the following lemma was essentially proved in [ST] 1.1.

Lemma B.1. *(i)* Pr_χ *commutes with* $\omega_{\psi,\xi}(g)$ *and hence*

$$(\omega_{\psi,\xi}, \mathcal{S}(E)) = \bigoplus_{\chi \in \mathrm{Hom}(Z_2(F), \mathbb{C}^1)} (\omega_{\psi,\xi}(\chi), \mathcal{S}(E)_\chi).$$

(ii) Each $(\omega_{\psi,\xi}(\chi), \mathcal{S}(E)_\chi)$ *is non-zero and irreducible.*
(iii) Suppose F is non-archimedean. If $\chi \neq \xi$, then $\omega_{\psi,\xi}(\chi)$ is supercuspidal. If $\chi = \xi$ then $\omega_{\psi,\xi}(\chi)$ is a limit of discrete series representation. Another limit of discrete series representation in the same L-packet will be obtained by replacing ψ.
(iv) When $F = \mathbb{R}$, (iii) still holds if the term supercuspidal is replaced by discrete series.

Proof. (i) is clear since $Z_2(F)$ is compact. For *(ii)* we use a classical argument in [ST]. We first consider the non-archimedean case. Take $A \in \mathrm{Hom}_{G_2(F)}(\omega_{\psi,\xi}(\chi), \omega_{\psi,\xi}(\chi))$. Then from (B.2),

$$\begin{aligned}\psi(b\mathrm{N}_{E/F}(x))A\Phi(x) =& \omega_{\psi,\xi}\Big(\chi; \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\Big)A\Phi(x) = A\omega_{\psi,\xi}\Big(\chi; \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\Big)\Phi(x) \\ =& A\psi(b\mathrm{N}_{E/F}(x))\Phi(x).\end{aligned}$$

But since $(\omega_{\psi,\xi}(\chi), \mathcal{S}(E))$ extends to a continuous representation on $L^2(E)$, A has a kernel function $K(x,y)$;

$$A\Phi(x) = \int_E K(x,y)\Phi(y)\,dy.$$

Then we must have

$$\int_E [\psi(b\mathrm{N}_{E/F}(x)) - \psi(b\mathrm{N}_{E/F}(y))]K(x,y)\Phi(y)\,dy = 0, \quad \forall \Phi \in \mathcal{S}(E).$$

It follows that $K(x,\bullet) = k(x)\delta_x$, where δ_x is the Dirac distribution supported at x and $k(x)$ is a bounded measurable function; $A\Phi(x) = k(x)\Phi(x)$. Moreover it follows from (B.1) that $k(xa) = k(x)$ for any $a \in E^\times$. Now we have two $E^\times$-orbit in E, the open orbit $E^\times$ and the closed one $\{0\}$. If $\chi \neq \xi$, then

$$\mathrm{Pr}_\chi \Phi(0) = \int_{Z_2(F)} \overline{\chi}\xi(z)\,dz\,\Phi(0) = 0.$$

Hence $\dim \mathrm{Hom}_{G_2(F)}(\omega_{\psi,\xi}(\chi), \omega_{\psi,\xi}(\chi)) = 1$ and $\omega_{\psi,\xi}(\chi)$ is irreducible. Next we assume $\chi = \xi$. Then we apply Rallis's invariant distribution theorem to have

$$\mathrm{Pr}_\xi \mathcal{S}(E) \simeq \mathrm{Span}\{\omega_{\psi,\xi}(g)\delta_0\}.$$

Again this asserts that $\dim \mathrm{Hom}_{G_2(F)}(\omega_{\psi,\xi}(\xi), \omega_{\psi,\xi}(\xi)) = 1$ and $\omega_{\psi,\xi}(\xi)$ is irreducible. This proof is also valid in the archimedean case, but notations will become more complicated.

(iii) and (iv) were proved in [JL]. □

B.1.4. *Local Howe duality for $GL(2) \times \mathbb{G}_m$.* Next we assume $E = F \oplus F$. Then $G_2 = GL(2)_F$ and ξ is identified with a character of $F^\times$ by $\xi(xy^{-1}) := \xi(x,y)$. Then the explicit formulae for $(\omega_{\psi,\xi}, \mathcal{S}(F^{\oplus 2}))$ become

$$\omega_{\psi,\xi}\Big(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\Big)\Phi(x,y) = \xi(a_1a_2)|a_1a_2|_F^{1/2}\Phi(a_1x, a_2^{-1}y) \tag{B.5}$$

$$\omega_{\psi,\xi}\Big(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\Big)\Phi(x,y) = \psi(bxy)\Phi(x,y) \tag{B.6}$$

$$\omega_{\psi,\xi}\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Big)\Phi(x,y) = \int_E \Phi(u,v)\psi(uy+vx)\,du\,dv. \tag{B.7}$$

Now the following results were proved in [JL].

Lemma B.2. *(i) The partial Fourier transform;*

$$\mathcal{F}_2\Phi(x,y) := \int_F \Phi(x,u)\psi(yu)\,du, \quad (\Phi \in \mathcal{S}(E))$$

gives a $G_2(F)$-equivariant isomorphism

$$(\omega_{\psi,\xi}, \mathcal{S}(E)) \xrightarrow{\sim} ((\xi \circ \det)|\det|_F^{1/2} \otimes R, \mathcal{S}(F^{\oplus 2})).$$

Here R denotes the right translation action.
(ii) The map

$$\mathcal{S}(F^{\oplus 2}) \ni \Phi \longrightarrow \xi(\det(g))|\det(g)|_F^{1/2}\Phi([0,1]g) \in \mathrm{ind}_{B_2^0(F)}^{G_2(F)}[\xi|\ |_F^{1/2} \otimes \mathbf{1}_{N_2(F)}]$$

is a $G_2(F)$-equivariant isomorphism. Here B_2^0 denotes the subgroup of B_2 consisting of elements of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$. $\mathrm{ind}_{B_2^0(F)}^{G_2(F)}$ assigns the unnormalized induction.

(iii) We define a projector Pr'_χ on $\mathrm{ind}_{B_2^0(F)}^{G_2(F)}[\xi|\ |_F^{1/2} \otimes \mathbf{1}_{N_2(F)}]$ by

$$\mathrm{Pr}'_\chi \phi(g) := \int_{Z_2(F)} \overline{\chi}(z)\phi(zg)\, dz.$$

This is a surjection to $\mathrm{Ind}_{B_2(F)}^{G_2(F)}[(\xi \otimes \chi\xi^{-1}) \otimes \mathbf{1}_{N_2(F)}]$. Hence the χ-isotypic quotient $\omega_{\psi,\xi}(\chi)$ of $\omega_{\psi,\xi}$ is isomorphic to $\mathrm{Ind}_{B_2(F)}^{G_2(F)}[(\xi \otimes \chi\xi^{-1}) \otimes \mathbf{1}_{N_2(F)}]$.

B.1.5. *The global theory.* Now let k'/k be a quadratic extension of number fields. We use notations defined in 2.1 and in §5. As in the local case we have the skew-Hermitian space (V, Φ) over k' and obtain the symplectic space (W, ϕ) over k by the restriction of scalars. We again write $Sp(W)$ for the symplectic group of (W, ϕ). Fix a non-trivial character $\psi = \bigotimes_v \psi_v$. As in [I] §1, we can construct the global metaplectic group $Mp(W)_{\mathbb{A}}$ and its oscillator representation $(\omega_\psi, \mathcal{S}(\mathbb{A}_{k'}))$.

We also fix a character $\xi = \bigotimes_v \xi_v$ of $\mathbb{A}_{k'}^\times / k'^\times$ whose restriction to $\mathbb{A}^\times$ equals $\eta_{k'/k}$. Then using the arguments of 3.3.7 (1), we have the global splitting $\iota_\xi : G_2(\mathbb{A}) \hookrightarrow Mp(W)_{\mathbb{A}}$. This time we define the oscillator representation $\omega_{\psi,\xi}$ of $G_2(\mathbb{A})$ by $\omega_{\psi,\xi} := \omega_\psi \circ \iota_\xi$. The explicit formulae are given as follows:

$$\omega_{\psi,\xi}(d(a))\Phi(x) = \xi(a)|a|_{\mathbb{A}_{k'}}^{1/2}\Phi(ax), \quad (a \in \mathbb{A}_{k'}^\times) \tag{B.8}$$

$$\omega_{\psi,\xi}(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\Phi(x) = \psi(b\mathrm{N}_{k'/k}(x))\Phi(x) \tag{B.9}$$

$$\omega_{\psi,\xi}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\Phi(x) = \int_{\mathbb{A}_{k'}} \Phi(y)\psi(\mathrm{Tr}_{k'/k}(y\sigma(x)))\, dy \tag{B.10}$$

Now we define a tempered distribution, called θ-distribution, on $\mathcal{S}(\mathbb{A}_{k'})$ by

$$\mathcal{S}(\mathbb{A}_{k'}) \ni \Phi \longrightarrow \theta(\Phi) := \sum_{\delta \in k'} \Phi(\delta) \in \mathbb{C}.$$

The theta series $\theta_\Phi(g)$ attached to $\Phi \in \mathcal{S}(\mathbb{A}_{k'})$ is defined by $\theta_\Phi(g) := \theta(\omega_{\psi,\xi}(g)\Phi)$ ($g \in G_2(\mathbb{A})$). We write the space of automorphic forms spanned by θ_Φ's with $\Phi \in \mathcal{S}^0(\mathbb{A}_{k'})$ as $\Theta(\xi, \psi)$. Here $\mathcal{S}^0(\mathbb{A}_{k'})$ is defined similarly as in 3.3.7 (1).

Next we take a character $\chi = \bigotimes_v \chi_v$ of $Z_2(k)\backslash Z_2(\mathbb{A})$. At inert v we have the projection Pr_{χ_v}. At split v Pr_{χ_v} can still be defined by (B.4) if $x \in k'_v$ satisfies $\mathrm{N}_{k'_v/k_v}(x) \neq 0$. This defines $\mathrm{Pr}_{\chi_v}\Phi_v(x)$ ($\Phi_v \in \mathcal{S}(k'_v)$) at such x, and we define $\mathrm{Pr}_{\chi_v}\Phi_v \in \mathcal{S}(k'_v)$ to be its extension to k'_v by zero. We now define the global projector Pr_χ on $\mathcal{S}(\mathbb{A}_{k'})$ by

$$\mathrm{Pr}_\chi : \mathcal{S}(\mathbb{A}_{k'}) \ni \bigotimes_v \Phi_v \longrightarrow \bigotimes_v \mathrm{Pr}_{\chi_v}\Phi_v \in \mathcal{S}(\mathbb{A}_{k'}). \tag{B.11}$$

This is clearly well-defined and $G_2(\mathbb{A})$-equivariant. We write $\mathcal{S}(\mathbb{A}_{k'})_\chi$ for its image and $\omega_{\psi,\xi}(\chi)$ for the restriction of $\omega_{\psi,\xi}$ to $\mathcal{S}(\mathbb{A}_{k'})_\chi$. On the other hand since $Z_2(k)\backslash Z_2(\mathbb{A})$ is compact, the distribution

$$\Phi \longrightarrow \theta_\chi(\Phi) := \int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\chi}\xi(z)\theta(\Phi(\bullet z))\, dz$$

is also tempered. We define $\theta_{\Phi,\chi}(g) := \theta_\chi(\omega_{\psi,\xi}(g)\Phi)$ ($g \in G_2(\mathbb{A})$). The space of automorphic forms spanned by $\theta_{\Phi,\chi}$ ($\Phi \in \mathcal{S}^0(\mathbb{A}_{k'})$) is denoted by $\Theta(\xi, \psi)_\chi$.

Proposition B.3. *(i) We have the direct sum decomposition*

$$\Theta(\xi,\psi) = \bigoplus_{\substack{\chi\in A(Z_2(k)\backslash Z_2(\mathbb{A}))\\ \chi \text{ is unitary}}} \Theta(\xi,\psi)_\chi.$$

(ii) Each $\Theta(\xi,\psi)_\chi$ is an irreducible automorphic representation. Moreover if $\chi\neq\xi$, then $\Theta(\xi,\psi)_\chi$ is cuspidal, while $\Theta(\xi,\psi)_\xi$ is not cuspidal.

Proof. (i) is clear. The first statement and the cuspidality in *(ii)* were proved in [ST] Proposition 4.9. The last statement follows from the fact that $\theta_{\Phi,\xi}$ has the non-trivial constant term $\omega_{\psi,\xi}(g)\Phi(0)$ along B_2. □

B.2. The zeta integral—definition and unfolding. Here we fix a irreducible cuspidal automorphic representation $\tau=\otimes_v\tau_v$ of $G_2(\mathbb{A})$ and give an integral representation for $L(s,\tau\times\chi)$.

B.2.1. *Classification of τ_v at inert v.* Let $w|v$ be as before. The results of [JL] combined with [Si] Theorem 2.5.9 and [Sh] Corollary 7.6 gives:

Lemma B.4 (Non-archimedean case). *(i) If τ_v is not supercuspidal, then we can choose a character ν_0 of $k_w'^\times$ and $s\in\mathbb{C}$ such that τ_v is a submodule of $I(\nu_0,s):=\mathrm{Ind}_{B_2(k_v)}^{G_2(k_v)}[(\nu_0|\ |_w^{s/2})\otimes\mathbf{1}_{N_2(k_v)}]$.*
(ii) $I(\nu_0,s)$ is irreducible unless $\nu_0|_{k_v^\times}=\eta_{k_w'/k_v}$ and $s=0$ or $\nu_0|_{k_v^\times}=\mathbf{1}$, and $s\in\{-1,1\}$.
(iii) If $\nu_0|_{k_v^\times}=\eta_{k_w'/k_v}$, then $I(\nu_0,0)$ is a direct sum of two distinct limit of discrete series representations.
(iv) If $\nu_0|_{k_v^\times}=\mathbf{1}$, then $I(\nu_0,1)$ has a unique irreducible submodule, the Steinberg representation $St(\nu_0)$ with the central character ν_0. The quotient of $I(\nu_0,1)$ by $St(\nu_0)$ is the one-dimensional representation $\nu_0^{G_2}$ defined similarly as in 3.3.1. *$I(\nu_0,-1)$ has $\nu_0^{G_2}$ as its unique submodule and the quotient of $I(\nu_0,-1)$ by it is $St(\nu_0)$.*

The corresponding archimedean results are very well-known and will be omitted (cf. [JL] Lemma 5.7).

B.2.2. *Whittaker models for τ_v at inert v.* Note that in the classification above, finite dimensional irreducible representations can never be a τ_v. We begin with the non-archimedean case. Take a non-trivial character ψ_v and consider it as a character of $N_2(k_v)$. The space of Whittaker functional on τ_v is defined by

$$\mathcal{W}_{\psi_v}(\tau_v):=\mathrm{Hom}_{N_2(k_v)}(\tau_v|_{N_2(k_v)},\psi_v)=(\tau_v)^*_{N_2(k_v),\psi_v^{-1}}.$$

From [LL], we know that $\mathcal{W}_{\psi_v}(\tau_v)\neq 0$ with some suitable ψ_v. For this suitable ψ_v, we choose a non-zero $\mathfrak{L}_v\in\mathcal{W}_{\psi_v}(\tau_v)$ and define the Whittaker model of τ_v with respect to ψ_v by

$$\mathcal{W}(\tau_v,\psi_v):=\{W_f^{\psi_v}(g):=\mathfrak{L}_v(\tau_v(g)f)\mid f\in V_{\tau_v}\}.$$

Finally we remark that if τ_v is in the principal series, $\mathcal{W}_{\psi_v}(\tau_v)\neq 0$ for any ψ_v.

Next assume $k_w'/k_v=\mathbb{C}/\mathbb{R}$. The non-trivial characters $\psi_v=\psi_\mathbb{R}$ or $\psi_\mathbb{R}^{-1}$ are considered as a character of $N_2(\mathbb{R})$. Its differential is denoted by $d\psi_v$. Define

$$\mathcal{W}_{\psi_v}(\tau_v):=\left\{\begin{array}{c|l}\begin{array}{c}\mathfrak{L}_v:V_{\tau_v}\to\mathbb{C}\\ \text{linear functional}\end{array} & \begin{array}{l}\text{(i) } \mathfrak{L}_v \text{ is continuous with respect to the Schwartz}\\ \quad\text{topology,}\\ \text{(ii) } \mathfrak{L}_v(d\tau_v(X)f)=d\psi_v(X)\mathfrak{L}_v(f),\ (X\in Lie(N_2(\mathbb{R})))\end{array}\end{array}\right\}.$$

Then again $\mathcal{W}_{\psi_v}(\tau_v)\neq 0$ for $\psi_v=\psi_\mathbb{R}$ or $\psi_\mathbb{R}^{-1}$. With this suitable ψ_v, we choose a non-zero $\mathfrak{L}_v\in\mathcal{W}_{\psi_v}(\tau_v)$. Extend $\mathfrak{L}_v$ to a map on a unitary completion $(\tau_v,\overline{V}_{\tau_v})$. Then using this

extended $\mathfrak{L}_v$, we define the Whittaker model $\mathcal{W}(\tau_v, \psi_v)$ as in the non-archimedean case. By the uniqueness of the Whittaker model, this does not depend on the choice of the unitary completion.

B.2.3. *Whittaker models for τ.* From above, we can choose a non-trivial character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}/k$ so that $\mathcal{W}_{\psi_v}(\tau_v) \neq 0$ at any v. Moreover at almost all v, τ_v is unramified and ψ_v is of order 0. At such v we write W_v^0 for the element of $\mathcal{W}(\tau_v, \psi_v)$ such that W_v^0 is right $K_{2,v}$-invariant and $W_v^0(1) = 1$. Here $K_{2,v}$ is a suitable maximal compact subgroup of $G_2(k_v)$. The global Whittaker model $\mathcal{W}(\tau, \psi)$ of τ is given by the restricted tensor product of $\mathcal{W}(\tau_v, \psi_v)$'s with respect to W_v^0. It was shown by Shalika that $\mathcal{W}(\tau, \psi)$ is unique for a fixed ψ.

Our final remark is about the Fourier expansion. For a cusp form $\varphi \in V_\tau$ we can define an element of $\mathcal{W}(\tau, \psi)$ by

$$W_\varphi^\psi(g) := \int_{N_2(k)\backslash N_2(\mathbb{A})} \varphi(ng)\overline{\psi(n)}\, dn, \quad (g \in G_2(\mathbb{A})).$$

If we write $\psi^\delta(x) := \psi(\delta x)$ for $\delta \in k$, then the Fourier expansion on $N_2(k)\backslash N_2(\mathbb{A})$ reads

$$\varphi(g) = \sum_{\delta \in k^\times} W_\varphi^{\psi^\delta}(g).$$

B.2.4. *The zeta integral and its unfolding.* We retain the notations defined above. In particular fix a character ξ of $\mathbb{A}_{k'}^\times / k'^\times$ such that $\xi|_{\mathbb{A}^\times} = \eta_{k'/k}$. Additionally for each character χ of $\mathbb{A}_{k'}^\times / k'^\times$, we choose a holomorphic section $F_s(\chi\xi)$ of $A(N_2(\mathbb{A})T_2(k)\backslash G_2(\mathbb{A}))_{\chi\xi|\ |_{\mathbb{A}_{k'}}^s}$ $(s \in \mathbb{C})$. Then we have the Eisenstein series

$$E(F(\chi\xi), s)(g) := \sum_{\gamma \in B_2(k)\backslash G_2(k)} F_s(\chi\xi; \gamma g)$$

on $G_2(\mathbb{A})$. The properties of $E(F(\chi\xi), s)$ were given in 3.3.8. We choose a non-trivial character ψ of $\mathbb{A}/k$ so that τ has a global Whittaker model with respect to it. Then we construct the oscillator representation $\omega_{\psi,\xi}$ and θ-series using this ψ. Now for fixed data $\varphi \in V_\tau$, $\Phi \in \mathcal{S}(\mathbb{A}_{k'})$ and $F_s(\chi\xi)$, we define the zeta integral by

$$Z(\varphi, \theta_\Phi, \chi; s) := \int_{G_2(k)\backslash G_2(\mathbb{A})} \varphi(g)\overline{\theta_\Phi(g)} E(F(\chi\xi), s - 1/2)(g)\, dg. \tag{B.12}$$

Proposition B.5. *Suppose that the data appeared in the definition of the zeta integral are chosen to be of the form*

$$\Phi = \bigotimes_v \Phi_v \in \mathcal{S}(\mathbb{A}_{k'}), \quad F_s(\chi\xi)(g) = \bigotimes_v F_{s,v}(\chi_v\xi_v)(g_v), \quad W_\varphi^\psi(g) = \bigotimes_v W_v(g_v).$$

If we define the local zeta integral $Z_v(W_v, \Phi_v, \chi_v; s)$ by

$$\int_{K_{2,v}} \int_{k_v'^\times} F_{s-1/2,v}(\chi_v\xi_v)(k)\overline{\omega_{\psi_v,\xi_v}(k)\Phi_v(a)} W_v(d(a)k)\chi_v(a)|a|_{k_v'}^{s-1/2}\, da^\times\, dk, \tag{B.13}$$

then

$$Z(\varphi, \theta_\Phi, \chi; s) = \prod_v Z_v(W_v, \Phi_v, \chi_v; s).$$

Proof. Since the analogous argument in the case of $\widetilde{SL(2,\mathbb{A})}$ was given in [GJ], we only give the formal outline and the convergence argument will be omitted. From the definition of $E(F(\chi\xi),s)$, we have

$$\begin{aligned}
Z(\varphi,\theta_\Phi,\chi;s) &= \int_{B_2(k)\backslash G_2(\mathbb{A})} \varphi(g)\overline{\theta_\Phi(g)}F_{s-1/2}(\chi\xi;g)\,dg \\
&= \int_{K_2}\int_{k'^\times\backslash\mathbb{A}_{k'}^\times}\int_{k\backslash\mathbb{A}} \varphi\overline{\theta_\Phi}F_{s-1/2}(\chi\xi)\Big(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}d(a)k\Big)\,dx\,\frac{da^\times}{|a|_{\mathbb{A}_{k'}}}\,dk \\
&= \int_{K_2}\int_{k'^\times\backslash\mathbb{A}_{k'}^\times} F_{s-1/2}(\chi\xi;k)\chi\xi(a)|a|_{\mathbb{A}_{k'}}^s \int_{k\backslash\mathbb{A}} \varphi\overline{\theta_\Phi}\Big(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}d(a)k\Big)\,dx\,\frac{da^\times}{|a|_{\mathbb{A}_{k'}}}\,dk.
\end{aligned}$$

Here K_2 is a certain maximal compact subgroup of $G_2(\mathbb{A})$. From (B.9), our inner integral becomes

$$\begin{aligned}
&\int_{k\backslash\mathbb{A}} \varphi\overline{\theta_\Phi}\Big(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\Big)\,dx \\
&\quad= \int_{k\backslash\mathbb{A}} \varphi\Big(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\Big)\sum_{\alpha\in k'^\times}\overline{\psi(x\mathrm{N}_{k'/k}(\alpha))\omega_{\psi,\xi}(g)\Phi(\alpha)}\,dx \\
&\quad= \sum_{\alpha\in k'^\times}\overline{\omega_{\psi,\xi}(g)\Phi(\alpha)}\int_{k\backslash\mathbb{A}} \varphi\Big(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\Big)\overline{\psi(x\mathrm{N}_{k'/k}(\alpha))}\,dx.
\end{aligned}$$

But notice that

$$\begin{aligned}
\int_{k\backslash\mathbb{A}} \varphi\Big(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\Big)\overline{\psi(x\mathrm{N}_{k'/k}(\alpha))}\,dx &= \int_{k\backslash\mathbb{A}} \varphi\Big(\begin{pmatrix}1 & \mathrm{N}_{k'/k}(\alpha)^{-1}x\\ 0 & 1\end{pmatrix}g\Big)\overline{\psi(x)}\,dx \\
&= \int_{k\backslash\mathbb{A}} \varphi\Big(\begin{pmatrix}\alpha^{-1} & 0\\ 0 & \sigma(\alpha)\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \sigma(\alpha)^{-1}\end{pmatrix}g\Big)\overline{\psi(x)}\,dx \\
&= W_\varphi^\psi(d(\alpha)g).
\end{aligned}$$

Thus we conclude

$$\begin{aligned}
Z(\varphi,\theta_\Phi,\chi;s) = \int_{K_2}\int_{k'^\times\backslash\mathbb{A}_{k'}^\times} &F_{s-1/2}(\chi\xi;k)\chi\xi(a)|a|_{\mathbb{A}_{k'}}^s \\
&\times\sum_{\alpha\in k'^\times}\overline{\omega_{\psi,\xi}(d(a)k)\Phi(\alpha)}W_\varphi^\psi(d(\alpha a)k)\,\frac{da^\times}{|a|_{\mathbb{A}_{k'}}}\,dk
\end{aligned}$$

putting $a := \alpha a$ and using (B.8),

$$\begin{aligned}
&= \int_{K_2}\int_{\mathbb{A}_{k'}^\times} F_{s-1/2}(\chi\xi;k)\overline{\omega_{\psi,\xi}(k)\Phi(a)}W_\varphi^\psi(d(a)k)\chi(a)|a|_{\mathbb{A}_{k'}}^{s-1/2}\,da^\times\,dk \\
&= \prod_v Z_v(W_v,\Phi_v,\chi_v;s).
\end{aligned}$$

□

B.3. Comparisons of local factors. Here again we shall see that $Z_v(W_v,\Phi_v,\chi_v;s)$ equals certain quotient of $L(s,\tau_v\times\chi_v)$ at almost all v. Then we shall prove that, at every v in the region $\mathrm{Re}(s)>0$, the G.C.D. has the same set of poles as $L(s,\tau_v\times\chi_v)$ has.

B.3.1. *Unramified calculations.* We begin with the inert case, hence k'_w/k_v be an unramified quadratic extension. Further assume that χ_v and ξ_v are unramified, ψ_v is of order 0, $\Phi_v = \Phi_v^0$ (the characteristic function of $\mathcal{O}_w$) and $F_{s,v}(\chi\xi)$ is the spherical vector such that $F_{s,v}(\chi\xi;k) = 1$ for $(k \in K_{2,v})$. The spherical Whittaker function W_v^0 was defined in B.2.3.

Lemma B.6. *Under these assumptions we have*

$$Z_v(W_v^0, \Phi_v^0, \chi_v; s) = \frac{L_{k'}(s, \chi_v\nu_v)L_{k'}(s, \chi_v\nu_v^{-1})}{L_k(2s, (\chi_v\xi_v)|_{k_v^\times})} = \frac{L(s, \tau_v \times \chi_v)}{L_k(2s, (\chi_v\xi_v)|_{k_v^\times})}.$$

Proof.

$$\begin{aligned} Z_v(W_v^0,\Phi_v^0, \chi_v; s) &= \int_{{k'_w}^\times} \Phi_v^0(a)W_v^0(d(a))\chi_v(a)|a|_w^{s-1/2}\, da^\times \\ &= \int_{\mathcal{O}_w \cap {k'}_w^\times} W_v^0(d(a))\chi_v(a)|a|_w^{s-1/2}\, da^\times \\ &= \sum_{n=0}^{\infty} W_v^0(d(\varpi_v^n))\chi_v(\varpi_v)^n q_w^{-n(s-1/2)}. \end{aligned}$$

We need to calculate $W_v^0(d(\varpi_v^n))$. $\tau_v = \mathrm{Ind}_{B_2(k_v)}^{G_2(k_v)}[\nu_v \otimes \mathbf{1}_{N_2(k_v)}]$ is restricted to $\tau_v|_{H(k_v)} = \mathrm{Ind}_{B_H(k_v)}^{H(k_v)}[\nu_v|_{k_v^\times} \otimes \mathbf{1}_{N_H(k_v)}]$, then extends to a representation

$$\widehat{\tau}_v = \mathrm{Ind}_{B(k_v)}^{GL(2,k_v)}[(\nu_v|_{k_v^\times} \otimes \mathbf{1}) \otimes \mathbf{1}_{N(k_v)}]$$

of $GL(2, k_v)$. Here $B = TN$ denotes the upper triangular Borel subgroup of $GL(2)_{k_v}$. Accordingly W_v^0 is restricted to $H(k_v)$ and then extended to the spherical Whittaker function for $\widehat{\tau}_v$. For this an explicit formula is available (cf. [JL] p. 123);

$$\begin{aligned} W_v^0(d(\varpi_v^n)) =& W_v^0\Big(\begin{pmatrix} \varpi_v^{-n} & 0 \\ 0 & \varpi_v^{-n} \end{pmatrix}\begin{pmatrix} \varpi_v^{2n} & 0 \\ 0 & 1 \end{pmatrix}\Big) \\ =& \nu_v(\varpi_v)^n|\varpi_v|_v^n \int_{k_v^\times} \mathrm{Ch}_{\mathcal{O}_v\oplus\mathcal{O}_v}(\varpi_v^{2n}t, t^{-1})\nu_v(t)\, dt^\times \\ =& \nu_v(\varpi_v)^n q_v^{-n} \sum_{j=0}^{2n} \nu_v(\varpi_v)^{-j} = \frac{\nu_v(\varpi_v)^{n+1} - \nu_v(\varpi_v)^{-n}}{\nu_v(\varpi_v) - 1} q_w^{-n/2}. \end{aligned}$$

Here $\mathrm{Ch}_{\mathcal{O}_v\oplus\mathcal{O}_v}$ denotes the characteristic function of $\mathcal{O}_v \oplus \mathcal{O}_v$. Hence we have

$$\begin{aligned} Z_v(W_v^0,\Phi_v^0, \chi_v; s) =& \frac{1}{\nu_v(\varpi_v) - 1} \sum_{n=0}^{\infty} \Big(\chi_v\nu_v(\varpi_v^n)\nu_v(\varpi_v) - \chi_v\nu_v^{-1}(\varpi_v^n)\Big) q_w^{-ns} \\ =& \frac{1}{\nu_v(\varpi_v) - 1}\left(\frac{\nu_v(\varpi_v)}{1 - \chi_v\nu_v(\varpi_v)q_w^{-s}} - \frac{1}{1 - \chi_v\nu_v^{-1}(\varpi_v)q_w^{-s}}\right) \\ =& \frac{1 + \chi_v(\varpi_v)q_w^{-s}}{(1 - \chi_v\nu_v(\varpi_v)q_w^{-s})(1 - \chi_v\nu_v^{-1}(\varpi_v)q_w^{-s})} \end{aligned}$$

noting $\xi_v(\varpi_v) = \eta_{k'_w/k_v}(\varpi_v) = -1$,

$$= \frac{L_{k'}(s, \chi_v\nu_v)L_{k'}(s, \chi_v\nu_v^{-1})}{L_k(2s, (\chi_v\xi_v)|_{k_v^\times})}.$$

□

Next comes a split v. Let $w_1,\ w_2|v$ be as before. Assume that $\chi_v = \chi_{w_1}\otimes\chi_{w_2}$ and $\xi_v(x,y) = \xi_v(xy^{-1})$ are unramified, ψ_v is of order 0, $\Phi_v = \Phi_v^0$ (the characteristic function of $\mathcal{O}_{w_1}\oplus\mathcal{O}_{w_2}$) and $F_{s,v}(\chi_v\xi_v)$ is the spherical vector such that $F_{s,v}(\chi_v\xi_v;k)=1$ for $k\in K_{2,v}$. Further assume that τ_v is of the form $\mathrm{Ind}_{B_2(k_v)}^{G_2(k_v)}[(\nu_{w_1}\otimes\nu_{w_2})\otimes\mathbf{1}_{N_2(k_v)}]$, where ν_{w_i} are unramified characters of $k_v^\times$. The spherical Whittaker function W_v^0 is as in B.2.3.

Lemma B.7. *Under these assumptions we have*

$$Z_v(W_v^0,\Phi_v^0,\chi_v;s) = \frac{L(s,\tau_v\otimes\chi_{w_1})L(s,\widetilde{\tau}_v\otimes\chi_{w_2})}{L(2s,(\chi_v\xi_v)|_{k_v^\times})} = \frac{L(s,\tau_v\times\chi_v)}{L_k(2s,(\chi_v\xi_v)|_{k_v^\times})}.$$

Proof. In this case $G_2(k_v)=GL(2,k_v)$ and we have from [JL] p. 123 that

$$\begin{aligned}
W_v^0(d(\varpi_v^n,\varpi_v^m)) &= \nu_{w_1}\nu_{w_2}^{-1}(\varpi_v^{-m})\nu_{w_1}(\varpi_v^{n+m})|\varpi_v^{n+m}|_v^{1/2}\\
&\qquad\times\int_{k_v^\times}\mathrm{Ch}_{\mathcal{O}_v\oplus\mathcal{O}_v}(\varpi_v^{n+m}t,t^{-1})\nu_{w_1}\nu_{w_2}(t)\,dt^\times\\
&= \nu_{w_1}(\varpi_v^n)\nu_{w_2}(\varpi_v^m)q_v^{-(n+m)/2}\sum_{j=0}^{n+m}\nu_{w_1}\nu_{w_2}(\varpi_v)^{-j}\\
&= \begin{cases} q_v^{-(n+m)/2}\sum_{j=-m}^{n}\nu_{w_1}(\varpi_v)^j\nu_{w_2}(\varpi_v)^{j+m-n} & \text{if } m+n\geq 0,\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

Thus one has

$$\begin{aligned}
Z_v(W_v^0,\Phi_v^0,\chi_v;s) &= \int_{k_v^\times}\int_{k_v^\times}\Phi_v^0(a,b)W_v^0(d(a,b))\chi_{w_1}(a)\chi_{w_2}(b)|ab|_v^{s-1/2}\,da^\times\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}W_v^0(d(\varpi_v^n,\varpi_v^m))\chi_{w_1}(\varpi_v)^n\chi_{w_2}(\varpi_v)^m q_v^{-(n+m)(s-1/2)}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=-m}^{n}\nu_{w_1}(\varpi_v)^j\nu_{w_2}(\varpi_v)^{j-n+m}\chi_{w_1}(\varpi_v)^n\chi_{w_2}(\varpi_v)^m q_v^{-(n+m)s}.
\end{aligned}$$

We now use abbreviations

$$\alpha := \nu_{w_1}(\varpi_v),\quad \beta := \nu_{w_2}(\varpi_v),\quad A := \chi_{w_1}(\varpi_v)q_v^{-s},\quad B := \chi_{w_2}(\varpi_v)q_v^{-s}.$$

Then

$$\begin{aligned}
Z_v(W_v^0,\Phi_v^0,\chi_v;s) &= \sum_{n,m\geq 0}A^nB^m(\alpha^n\beta^m+\alpha^{n-1}\beta^{m-1}+\cdots+\alpha^{-m}\beta^{-n})\\
&= \sum_{n,m\geq 0}A^nB^m\frac{\alpha^{n+1}\beta^{m+1}-\alpha^{-m}\beta^{-n}}{\alpha\beta-1}\\
&= \frac{\alpha\beta}{\alpha\beta-1}\sum_{n,m\geq 0}(A\alpha)^n(B\beta)^m - \frac{1}{\alpha\beta-1}\sum_{n,m\geq 0}(A\beta^{-1})^n(B\alpha^{-1})^m\\
&= \frac{\alpha\beta}{(\alpha\beta-1)(1-A\alpha)(1-B\beta)} - \frac{1}{(\alpha\beta-1)(1-A\beta^{-1})(1-B\alpha^{-1})}\\
&= \frac{\alpha\beta(1-A\beta^{-1}-B\alpha^{-1}+AB\alpha^{-1}\beta^{-1})-(1-A\alpha-B\beta+AB\alpha\beta)}{(1-A\alpha)(1-A\beta^{-1})(1-B\alpha^{-1})(1-B\beta)(\alpha\beta-1)}\\
&= \frac{1-AB}{(1-A\alpha)(1-A\beta^{-1})(1-B\alpha^{-1})(1-B\beta)}
\end{aligned}$$

$$= \frac{1-\chi_{w_1}\chi_{w_1}(\varpi_v)q_v^{-2s}}{(1-\chi_{w_1}\nu_{w_1}(\varpi_v)q_v^{-s})(1-\chi_{w_1}\nu_{w_2}^{-1}(\varpi_v)q_v^{-s})(1-\chi_{w_2}\nu_{w_1}^{-1}(\varpi_v)q_v^{-s})(1-\chi_{w_2}\nu_{w_2}(\varpi_v)q_v^{-s})}$$
$$= \frac{L(s,\tau_v\otimes\chi_{w_1})L(s,\widetilde{\tau}_v\otimes\chi_{w_2})}{L(2s,\chi_v\xi_v|_{k_v^\times})}$$

□

B.3.2. *Ramified split case.* We want to show that the G.C.D. of $Z_v(W_v,\Phi_v,\chi_v;s)$ has the same analytic behavior as that of $L(s,\tau_v\times\chi_v)$ in the region $\mathrm{Re}(s)>0$. At a split v this is reduced to the corresponding results in [Ja]. The key point is the following lemma.

Lemma B.8. *If we set*

$$\tau_2 := \mathrm{Ind}_{B_2(k_v)}^{G_2(k_v)}[(\chi_{w_1}\otimes\omega_{\tau_v}\chi_{w_2}^{-1})\otimes\mathbf{1}_{N_2(k_v)}] = \mathrm{Ind}_{B(k_v)}^{GL(2,k_v)}[(\chi_{w_1}\otimes\omega_{\tau_v}^{-1}\chi_{w_2})\otimes\mathbf{1}_{N(k_v)}],$$

then we can choose $W_2\in\mathcal{W}(\tau_2,\psi_v)$ *and* $\Phi_2\in\mathcal{S}(k_v^{\oplus 2})$ *(*$\mathcal{S}^0(k_v^{\oplus 2})$ *if* v *is archimedean) so that*

$$Z_v(W_v,\Phi_v,\chi_v;s) = \Psi(s,W_v,W_2,\Phi_2).$$

Here the right hand side is the Rankin product local zeta integral defined in [Ja].

Proof. We may assume $\mathrm{Re}(s)\gg 0$ throughout the proof. From definition, we have

$$Z_v(W_v,\Phi_v,\chi_v;s) = \int_{K_{2,v}}\int_{k_v^\times}\int_{k_v^\times} F_{s-1/2}(\chi_v\xi_v;k)\overline{\omega_{\psi_v,\xi_v}(k)\Phi_v(a,b)}W_v\Big(\begin{pmatrix}a&0\\0&b^{-1}\end{pmatrix}k\Big)$$
$$\times\chi_{w_1}(a)\chi_{w_2}(b)|ab|_v^{s-1/2}\,da^\times\,db^\times\,dk$$

putting $x:=ab$, $y=b^{-1}$,

$$= \int_{K_{2,v}}\int_{k_v^\times}\int_{k_v^\times} F_{s-1/2,v}(\chi_v\xi_v;k)\overline{\omega_{\psi_v,\xi_v}(k)\Phi_v(xy,y^{-1})}W_v\Big(\begin{pmatrix}xy&0\\0&y\end{pmatrix}k\Big)\chi_{w_1}(xy)$$
$$\chi_{w_2}^{-1}(y)|x|_v^{s-1/2}\,dx^\times\,dy^\times\,dk$$
$$= \int_{K_{2,v}}\int_{k_v^\times}\overline{\int_{k_v^\times}[\omega_{\psi_v,\xi_v}(k)\Phi_v](xy,y^{-1})\omega_{\tau_v}^{-1}\chi_{w_1}^{-1}\chi_{w_2}(y)\,dy^\times\,F_{s-1/2,v}(\chi_v\xi_v;k)}$$
$$W_v\Big(\begin{pmatrix}x&0\\0&1\end{pmatrix}k\Big)\chi_{w_1}(x)|x|_v^{s-1/2}\,dx^\times\,dk.$$

Here we note

$$\int_{k_v^\times}[\omega_{\psi_v,\xi_v}(k)\Phi_v](xy,y^{-1})\omega_{\tau_v}^{-1}\chi_{w_1}^{-1}\chi_{w_2}(y)\,dy^\times$$
$$= \int_{k_v^\times}\overline{\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1}(y)}[\omega_{\psi_v,\xi_v}(k)\Phi_v](xy,y^{-1})\,dy^\times = \mathrm{Pr}_{(\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})}[\omega_{\psi_v,\xi_v}(k)\Phi_v](x,1)$$
$$= \omega_{\psi_v,\xi_v}(k)(\mathrm{Pr}_{(\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})}\Phi_v)(x,1),$$

and hence

$$Z_v(W_v,\Phi_v,\chi_v;s)$$
$$= \int_{K_{2,v}}\int_{k_v^\times}\overline{\omega_{\psi_v,\xi_v}(k)(\mathrm{Pr}_{(\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})}\Phi_v)(x,1)}F_{s-1/2,v}(\chi_v\xi_v;k)W_v\Big(\begin{pmatrix}x&0\\0&1\end{pmatrix}k\Big)$$

$$\chi_{w_1}(x)|x|_v^{s-1/2}\,dx^\times\,dk$$

$$= \int_{N_2(k_v)Z_2(k_v)\backslash G_2(k_v)} F_{s-1/2,v}(\chi_v\xi_v;g)\overline{\omega_{\psi_v,\xi_v}(g)\mathrm{Pr}_{(\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})}\Phi_v(1,1)}W_v(g)\,dg.$$

Next we specify the integrand. We may assume $F_{s-1/2,v}(\chi_v\xi_v;g)$ is of the form

$$\begin{aligned} f_{\Phi_2}(\chi_v\xi_v;g) &= \chi_{w_1}\xi_v(\det(g))|\det(g)|_v^s\int_{k_v^\times}\Phi_2([0,t]g)\chi_{w_1}\chi_{w_2}(t)|t|_v^{2s}\,dt^\times \\ &= \chi_{w_1}\xi_v(\det(g))|\det(g)|_v^s z(|\ |_v^{2s}\chi_{w_1}\chi_{w_2},g.\Phi_2), \end{aligned}$$

where $\Phi_2\in\mathcal{S}(k_v^{\oplus 2})$ ($\in\mathcal{S}^0(k_v^{\oplus 2})$ if $v|\infty$) and $z(\omega,\Phi)$ was defined in the introduction of [Ja]. Moreover if we set

$$\tau_2' := \mathrm{Ind}_{B(k_v)}^{GL(2,k_v)}[(\xi_v\otimes\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})\otimes\mathbf{1}_{N(k_v)}],$$

then from Lemma B.2 we know $\omega_{\psi_v,\xi_v}(\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})\simeq\tau_2'$ and

$$W^*(g) := \omega_{\psi_v,\xi_v}(g)\mathrm{Pr}_{(\omega_{\tau_v}\chi_{w_1}\chi_{w_2}^{-1})}\Phi_v(1,1)$$

is in the Whittaker model $\mathcal{W}(\tau_2',\psi_v)$ (cf. [JL]). Note that $W^*(g)$ spans a dense subspace of $\mathcal{W}(\tau_2',\psi_v)$ as Φ_v changes. Now the zeta integral becomes

$$\begin{aligned} &Z_v(W_v,\Phi_v,\chi_v;s) \\ &\quad= \int_{N_2(k_v)Z_2(k_v)\backslash G_2(k_v)} W_v(g)\overline{W^*(g)}\chi_{w_1}\xi_v(\det(g))z(|\ |_v^{2s}\chi_{w_1}\chi_{w_2},g.\Phi_2)|\det(g)|_v^s\,dg. \end{aligned}$$

On the other hand, since $\overline{W^*(g)}$ runs over the Whittaker functions $W_2'(\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)g)$ ($W_2'\in\mathcal{W}(\overline{\tau}_2',\psi_v)$), $W_2(g):=\chi_{w_1}\xi_v(\det(g))W_2'(g)$ runs over Whittaker functions for

$$\tau_2 = (\chi_{w_1}\xi_v\circ\det)\otimes\overline{\tau}_2' = \mathrm{Ind}_{B(k_v)}^{GL(2,k_v)}[(\chi_{w_1}\otimes\omega_{\tau_v}^{-1}\chi_{w_2})\otimes\mathbf{1}_{N(k_v)}]$$

with respect to ψ_v. Hence we conclude

$$\begin{aligned} &Z_v(W_v,\Phi_v,\chi_v;s) \\ &\quad= \int_{N_2(k_v)Z_2(k_v)\backslash G_2(k_v)} W_v(g)W_2\Big(\begin{pmatrix}1&0\\0&-1\end{pmatrix}g\Big)z(|\ |_v^{2s}\chi_{w_1}\chi_{w_2},g.\Phi_2)|\det(g)|_v^s\,dg \\ &\quad= \Psi(s,W_v,W_2,\Phi_2). \end{aligned}$$

□

Corollary B.9. *The G.C.D. of $Z_v(W_v,\Phi_v,\chi_v;s)$ equals*

$$L(s,\tau_v\otimes\chi_{w_1})L(s,\widetilde{\tau}_v\otimes\chi_{w_2}) = L(s,\tau_v\times\chi_v).$$

B.3.3. *Ramified inert non-archimedean case.* Let $w|v$ be as before. We first state the result.

Proposition B.10. *$Z_v(W_v,\Phi_v,\chi_v;s)$ has the same poles in the region $\mathrm{Re}(s)>0$ as*

$$\frac{L(s,\tau_v\times\chi_v)}{L(2s,(\chi_v\xi_v)|_{k_v^\times})}$$

has.

(1) A key lemma. The proof of the proposition is a case-by-case argument and quite lengthy. We start with a key lemma which will be commonly used in all cases.

Lemma B.11. *$Z_v(W_v, \Phi_v, \chi_v; s)$ has the same set of poles as*

$$H(k,s) := \int_{k'^{\times}_w} \overline{\omega_{\psi,\xi}(k)\Phi(a)} W_v(d(a)k)\chi_v(a)|a|_w^{s-1/2}\, da^{\times}, \quad (k \in K_{2,v})$$

has.

Proof. Since

$$Z_v(W_v, \Phi_v, \chi_v; s) = \int_{B_2(k_v)\cap K_{2,v}\backslash K_{2,v}} F_{s-1/2,v}(\chi_v\xi_v; k)H(k,s)\, dk,$$

all poles of $Z_v(W_v, \Phi_v, \chi_v; s)$ come from those of $H(k,s)$.

Conversely if we set

$$K_{2,v}(\mathfrak{p}_w^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{2,v} \;\middle|\; \begin{array}{l} a,\ d \in 1+\mathfrak{p}_w^n \\ b,\ c \in \mathfrak{p}_w^n \end{array} \right\}, \quad \widetilde{K_{2,v}}(\mathfrak{p}_w^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{2,v} \;\middle|\; c \in \mathfrak{p}_w^n \right\},$$

then we have for sufficiently large n,

$$\begin{array}{l} \text{(i) } \widetilde{K_{2,v}}(\mathfrak{p}_w^n) = (B_2(k_v)\cap K_{2,v})K_{2,v}(\mathfrak{p}_w^n) \\ \text{(ii) } H(kl,s) = H(k,s), \quad (\forall l \in K_{2,v}(\mathfrak{p}_w^n),\ k \in K_{2,v}). \end{array}$$

Thus if we choose

$$F_{s-1/2,v}(\chi_v\xi_v; k) = \begin{cases} \chi_v\xi_v(a) & \text{if } k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \widetilde{K_{2,v}}(\mathfrak{p}_w^n), \\ 0 & \text{otherwise,} \end{cases}$$

then $Z_v(W_v, \Phi_v, \chi_v; s)$ becomes

$$\int_{B_2(k_v)\cap\widetilde{K_{2,v}}(\mathfrak{p}_w^n)\backslash\widetilde{K_{2,v}}(\mathfrak{p}_w^n)} F_{s-1/2,v}(\chi_v\xi_v; k)H(k,s)\, dk = c\cdot H(1,s).$$

Hence every pole of $H(1,s)$ contributes to that of $Z_v(W_v, \Phi_v, \chi_v; s)$. □

(2) Whittaker functions for generalized principal series. To prove the proposition for non-supercuspidal τ_v, we need explicit formulae for the Whittaker functions for them. For this we calculate the Whittaker functions for the induced module $I(\nu_v) = \mathrm{Ind}_{B_2(k_v)}^{G_2(k_v)}[\nu_v \otimes \mathbf{1}_{N_2(k_v)}]$, where ν_v is a quasi-character of $k'^{\times}_w$.

We define the Whittaker functional on $I(\nu_v)$ by the principal value integral

$$W_{\psi_v}(f)(g) := \sum_{n\in\mathbb{Z}} \int_{\mathfrak{p}_v^n - \mathfrak{p}_v^{n+1}} f\Big(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\Big)\overline{\psi_v(x)}\, dx,$$

where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If this is well-defined, then it gives a Whittaker functions at least for an irreducible quotient of $I(\nu_v)$. We formally have

$$\begin{aligned} W_{\psi_v}(f)(d(a)) &= \sum_{n\in\mathbb{Z}} \int_{\mathfrak{p}_v^n - \mathfrak{p}_v^{n+1}} f\Big(\begin{pmatrix} \sigma(a)^{-1} & 0 \\ 0 & a \end{pmatrix} w^{-1}\begin{pmatrix} 1 & \mathrm{N}_{k'_w/k_v}(a)^{-1}x \\ 0 & 1 \end{pmatrix}\Big)\overline{\psi_v(x)}\, dx \\ &= \nu_v(\sigma(a))^{-1}|a|_w^{1/2} \sum_{n\in\mathbb{Z}} \int_{\mathfrak{p}_v^n - \mathfrak{p}_v^{n+1}} f\Big(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\Big)\overline{\psi_v(\mathrm{N}_{k'_w/k_v}(a)x)}\, dx. \end{aligned}$$

But since

$$f\Big(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\Big) = f\Big(\begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}\Big) = \nu_v(x)^{-1}|x|_v^{-1}f(1)$$

for $|x|_v \gg 1$, the principal value integral converges if $\mathrm{Re}\nu_v > -1$ (cf. [JL]). Moreover using the explicit formulae for the Kirillov models on $GL(2,k_v)$ (see A.2.3), we see that the Kirillov functions

$$\sum_{n\in\mathbb{Z}} \int_{\mathfrak{p}_v^n - \mathfrak{p}_v^{n+1}} f\Big(w^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\Big)\overline{\psi_v(\mathrm{N}_{k'_w/k_v}(a)x)}\,dx$$

are contained in

$$\begin{cases} \{f_1(\mathrm{N}_{k'_w/k_v}(a))\nu_v(\mathrm{N}_{k'_w/k_v}(a)) + f_2(\mathrm{N}_{k'_w/k_v}(a)) \mid f_1,\ f_2 \in \mathcal{S}(k_v)\} & \text{if } \nu_v|_{k_v^\times} \neq |\ |_v^{-1}, \mathbf{1}, \\ \{f_1(\mathrm{N}_{k'_w/k_v}(a))\mathrm{ord}_v(\mathrm{N}_{k'_w/k_v}(a)) + f_2(\mathrm{N}_{k'_w/k_v}(a)) \mid f_1,\ f_2 \in \mathcal{S}(k_v)\} & \text{if } \nu_v|_{k_v^\times} = \mathbf{1}, \\ \{f_1(\mathrm{N}_{k'_w/k_v}(a)) \mid f_1 \in \mathcal{S}(k_v)\} & \text{if } \nu_v|_{k_v^\times} = |\ |_v^{-1}. \end{cases}$$

Hence $W_{\psi_v}(f)(d(a))$ is of the form

(B.14)

$$\begin{cases} (f_1(\mathrm{N}_{k'_w/k_v}(a))\nu_v(a) + f_2(\mathrm{N}_{k'_w/k_v}(a))\nu_v^{-1}(\sigma(a)))|a|_w^{1/2} & \text{if } \nu_v|_{k_v^\times} \neq |\ |_v^{-1}, \mathbf{1}, \\ (f_1(\mathrm{N}_{k'_w/k_v}(a))\nu_v(a)\mathrm{ord}_v(\mathrm{N}_{k'_w/k_v}(a)) + f_2(\mathrm{N}_{k'_w/k_v}(a))\nu_v^{-1}(\sigma(a)))|a|_w^{1/2} & \text{if } \nu_v|_{k_v^\times} = \mathbf{1}, \\ f_1(\mathrm{N}_{k'_w/k_v}(a))\nu_v^{-1}(\sigma(a))|a|_w^{1/2} & \text{if } \nu_v|_{k_v^\times} = |\ |_v^{-1}, \end{cases}$$

where f_1 and f_2 are in $\mathcal{S}(k_v)$.

(3) Proof of Proposition B.10 for non-supercuspidal τ_v. First consider τ_v in the principal or complementary series. It equals some $I(\nu_v)$ with (i) ν_v is unitary and $\nu_v|_{k_v^\times} \neq \eta_{k'_w/k_v}$ or (ii) $\nu_v|_{k_v^\times} = |\ |_v^\lambda$ with $0 < \lambda < 1$. Hence from (B.14), $H(1,s)$ is of the form

$$\int_{{k'_w}^\times} \Phi_1(a)\chi_v\nu_v(a)|a|_w^s\,da^\times + \int_{{k'_w}^\times} \Phi_2(a)\chi_v(\nu_v\circ\sigma)^{-1}(a)|a|_w^s\,da^\times$$

if $\nu_v|_{k_v^\times} \neq \mathbf{1}$ and

$$\int_{{k'_w}^\times} \Phi_1(a)\chi_v\nu_v(a)\mathrm{ord}_v(\mathrm{N}_{k'_w/k_v}(a))|a|_w^s\,da^\times + \int_{{k'_w}^\times} \Phi_2(a)\chi_v(\nu_v\circ\sigma)^{-1}(a)|a|_w^s\,da^\times$$

if $\nu_v|_{k_v^\times} = \mathbf{1}$, with $\Phi_i \in \mathcal{S}(k_v)$ $(i = 1,2)$. In both cases the G.C.D. of $H(1,s)$ is $L_{k'}(s,\chi_v\nu_v)L_{k'}(s,\chi_v(\nu_v\circ\sigma)^{-1})$ which equals $L(s,\tau_v\times\chi_v)$.

Next we treat τ_v in the limit of discrete series. This τ_v is a direct summand of $I(\nu_v)$ with $\nu_v|_{k_v^\times} = \eta_{k'_w/k_v}$. As in the principal case, we have from (B.14) that the G.C.D. of $H(1,s)$ equals

$$L_{k'}(s,\chi_v\nu_v)L_{k'}(s,\chi_v(\nu_v\circ\sigma)^{-1}).$$

This is holomorphic for $\mathrm{Re}(s) > 0$, while $L(s,\tau_v\times\chi_v)$ is also holomorphic in that region by [Sh] Proposition 7.2 (b).

Finally we care τ_v in the discrete series. It is a unique irreducible quotient of $I(\nu_v)$ with $\nu_v|_{k_v^\times} = |\ |_v^{-1}$. Hence from (B.14),

$$H(1,s) = \int_{{k'}_w^\times} \Phi(a)\chi_v(\nu_v\circ\sigma)^{-1}(a)|a|_w^s\,da^\times$$

and their G.C.D. is $L_{k'}(s,\chi_v(\nu_v\circ\sigma)^{-1})$. This is holomorphic for $\mathrm{Re}(s) > 0$ and again we use [Sh] Proposition 7.2 (b) to finish the proof. □

(4) Whittaker functions for supercuspidal τ_v. To obtain an explicit formula for the Kirillov space of such τ_v, we use the following variant of the Gel'fand-Kazhdan theory (cf. [BZ] Chapter III). We freely use the notations and terminologies of [BZ]. In particular

Ind_H^G denotes the induction functor from $Alg(H)$ to $Alg(G)$ and ind_H^G denotes the finite induction functor between these categories.

We write the space of characters of $N_2(k_v)$ as $\widehat{N_2(k_v)}$. Then our ψ_v allows us to identify $\widehat{N_2(k_v)}$ with the ℓ-space k_v and $C_c^\infty(N_2(k_v))$ with $\mathcal{S}(\widehat{N_2(k_v)})$ (by the Fourier transform). This gives the category equivalence

$$(\dagger) \qquad Alg(N_2(k_v)) \xrightarrow{\sim} \Big(\ell\text{-sheaves}/\widehat{N_2(k_v)}\Big).$$

If $\mathcal{E}$ is an ℓ-sheaf on $\widehat{N_2(k_v)}$ corresponding to an object E of $Alg(N_2(k_v))$ by the equivalence $(\dagger)$, then $E_{N_2(k_v),\psi} \simeq \mathcal{E}_\psi$ as $\mathcal{S}(\widehat{N_2(k_v)})$-modules for any $\psi \in \widehat{N_2(k_v)}$ (cf. [BZ] 5.9, 5.10).

Fix $\alpha \in k_v^\times - \mathrm{N}_{k'_w/k_v}(k'^\times_w)$ and set $\psi^+ := \psi_v$, $\psi^- := \psi_v^\alpha$. Then we have six functors

$$\begin{aligned}
&\Phi_\pm^- : Alg(B_2(k_v)) \ni E \longrightarrow E_{N_2(k_v),\psi^\pm} \in Alg(Z_2(k_v))\\
&\Phi_\pm^+ : Alg(Z_2(k_v)) \ni \rho \longrightarrow ind_{Z_2(k_v)N_2(k_v)}^{B_2(k_v)}(\rho \otimes \psi^\pm) \in Alg(B_2(k_v))\\
&\Psi^- : Alg(B_2(k_v)) \ni E \longrightarrow E_{N_2(k_v)} \in Alg(T_2(k_v))\\
&\Psi^+ : Alg(T_2(k_v)) \ni \mu \longrightarrow \mu \otimes \mathbf{1}_{N_2(k_v)} \in Alg(B_2(k_v)).
\end{aligned}$$

The main properties of these functors are as follows.

Lemma B.12. *(a) All functors are exact.*
(b) Ψ^- is left adjoint to Ψ^+, $\Phi_\pm^+$ is left adjoint to $\Phi_\pm^-$.
(c) $\Phi_\pm^- \circ \Psi^+ = 0$ and $\Psi^- \circ \Phi_\pm^+ = 0$.
(d) The adjonction map $i_\pm : \mathrm{id} \to \Phi_\pm^-\Phi_\pm^+$, $j : \Psi^-\Psi^+ \to \mathrm{id}$ are isomorphisms.
(e) We have an exact sequence

$$0 \longrightarrow \Phi_+^+\Phi_+^- \oplus \Phi_-^+\Phi_-^- \longrightarrow \mathrm{id} \longrightarrow \Psi^+\Psi^- \longrightarrow 0.$$

Proof. (a) is clear. (b) to (e) will be proved along the following considerations.

(1) For $(\pi, E) \in Alg(B_2(k_v))$, we can take an ℓ-sheaf $(\widehat{N_2(k_v)}, \mathcal{E})$ such that $E \simeq \mathcal{E}_c$ as $\mathcal{S}(\widehat{N_2(k_v)})$-modules. Using this isomorphism we transport the $B_2(k_v)$-module structure to $\mathcal{E}_c$. Then

$$\pi(n)\varphi(\theta) = \theta(n)\varphi(\theta), \quad (\theta \in \widehat{N_2(k_v)},\ n \in N_2(k_v),\ \varphi \in \mathcal{E}_c).$$

(2) Once we have realized (π, E) on $\mathcal{E}_c$, we analyze the $B_2(k_v)$-orbits in $\widehat{N_2(k_v)}$. We have three orbits;

$$Z := \{\mathbf{1}_{N_2(k_v)}\}, \quad Y^\pm := (\psi^\pm)^{B_2(k_v)}.$$

Let $\pi^\pm := \pi|_{\mathcal{E}_c(Y^\pm)}$. Then since the representation of $Z_2(k_v)N_2(k_v)$, the stabilizer of $\psi^\pm$ in $B_2(k_v)$, on the stalk $\mathcal{E}(Y^\pm)_{\psi^\pm} = \mathcal{E}_{\psi^\pm}$ is equivalent to $\Phi_\pm^-(\pi) \otimes \psi^\pm$, we have from [BZ] 2.23 (b) that

$$E(N_2(k_v), \mathbf{1}) = \pi^+ \oplus \pi^-, \quad \Phi_\pm^+\Phi_\pm^-(\pi) = \pi^\pm.$$

Thus from $(\tau \to \Phi_\pm^-(\pi)) \in \mathrm{Hom}_{Z_2(k_v)}(\tau, \Phi_\pm^-(\pi))$ we can construct an element of $\mathrm{Hom}_{B_2(k_v)}(\Phi_\pm^+(\tau), \pi)$ by the composition

$$\Phi_\pm^+(\tau) \longrightarrow [\Phi_\pm^+\Phi_\pm^-(\pi) = \pi^\pm] \hookrightarrow \pi.$$

(3) Next we construct the inverse $\mathrm{Hom}_{B_2(k_v)}(\Phi^+_\pm(\tau),\pi) \to \mathrm{Hom}_{Z_2(k_v)}(\tau,\Phi^-_\pm(\pi))$. By [BZ] 2.23 (a), we have an ℓ-sheaf $\mathcal{F}'_\pm$ over $Y^\pm$ such that $\Phi^+_\pm(\tau) \simeq \mathcal{F}'_{\pm,c}$ as $\mathcal{S}(Y^\pm)$-modules. We extend this to an ℓ-sheaf $\mathcal{F}_\pm$ on $\widehat{N_2(k_v)}$ by $\mathcal{F}_\pm|_{Z\cup Y^\mp} := \{0\}$. Then $\mathcal{F}_\pm$ is the ℓ-sheaf associated to $\Phi^+_\pm(\tau)$ by the construction of (1). Hence

$$\Phi^-_\pm\Phi^+_\pm(\tau) = (\Phi^+_\pm(\tau))_{N_2(k_v),\psi^\pm} \xrightarrow{\sim} \mathcal{F}_{\pm,\psi^\pm} = \tau.$$

Also we have $\Phi^-_\pm \circ \Phi^+_\mp = 0$. This gives the natural isomorphism $i_\pm : \mathrm{id} \xrightarrow{\sim} \Phi^-_\pm\Phi^+_\pm$, and for $\Phi^+_\pm(\tau) \to \pi$ we have an element

$$\tau \xrightarrow{\sim} \Phi^-_\pm\Phi^+_\pm(\tau) \to \Phi^-_\pm(\pi)$$

of $\mathrm{Hom}_{Z_2(k_v)}(\tau,\Phi^-_\pm(\pi))$.

(4) The left adjointness of Ψ^- to Ψ^+ is obvious. Since $\mathcal{F}_\pm|_Z = \{0\}$, we have $\Psi^-\Phi^+_\pm = 0$. $\Psi^+(\mu)|_{N_2(k_v)} = \mathbf{1}_{N_2(k_v)}$ gives $\Phi^-_\pm\Psi^+ = 0$. Finally the exact sequence in (e) is given by

$$0 \longrightarrow E(N_2(k_v),\mathbf{1}) \longrightarrow E \longrightarrow E_{N_2(k_v),\mathbf{1}} \longrightarrow 0.$$

□

Corollary B.13 (Kirillov models for supercuspidal τ_v). *Assume that τ_v is supercuspidal. Then for $W_v \in \mathcal{W}(\tau_v,\psi_v)$, $W_v(d(a))$ is in $\mathcal{S}(k'^\times_w)$ as a function of $a \in k'^\times_w$.*

Proof. Define $\pi_\pm := Ind^{B_2(k_v)}_{Z_2(k_v)N_2(k_v)}(\omega_{\tau_v}\otimes\psi^\pm)$ and $\pi^0_\pm := ind^{B_2(k_v)}_{Z_2(k_v)N_2(k_v)}(\omega_{\tau_v}\otimes\psi^\pm)$. By definition we have the natural embedding $\pi^0_\pm \hookrightarrow \pi_\pm$. On the other hand

$$\begin{aligned}\mathrm{Hom}_{B_2(k_v)}(\pi^0_\pm,\pi_\pm) &= \mathrm{Hom}_{Z_2(k_v)N_2(k_v)}(\pi^0_\pm|_{Z_2(k_v)N_2(k_v)},\omega_{\tau_v}\otimes\psi^\pm)\\ &= (\pi^0_\pm)^*_{N_2(k_v),\psi^\pm}\end{aligned}$$

is 1-dimensional.

Since τ_v is supercuspidal, it follows from Lemma B.12 that $\tau_v|_{B_2(k_v)} = \Phi^+_+\Phi^-_+(\tau_v) \oplus \Phi^+_-\Phi^-_-(\tau_v)$. But $\mathcal{W}(\tau_v,\psi_v) \ni W_v \to W_v|_{B_2(k_v)} \in \pi_+$ is non-zero by our assumption (recall $\psi_v = \psi^+$), and hence $\Phi^+_+\Phi^-_+(\tau_v) = \pi^0_+$.

We still have to show $\Phi^+_-\Phi^-_-(\tau_v) = \{0\}$. If this is not $\{0\}$, then it is a direct sum of copies of π^0_-. But since $\mathrm{Hom}_{B_2(k_v)}(\pi^0_-,\pi_+) = 0$, $\Phi^+_-\Phi^-_-(\tau_v)$ go to $\{0\}$ under the "restriction to $B_2(k_v)$"-functor. □

(5) Proof of Proposition B.10 in the supercuspidal case. From Corollary B.13, the G.C.D. of $H(1,s)$ is 1. On the other hand, from [Sh] Proposition 7.2 (b) we know that $L(s,\tau_v\times\chi_v)$ is also holomorphic on the region $\mathrm{Re}(s) > 0$. (q.e.d)

B.3.4. *K-types of the oscillator representation.* We now proceed to the archimedean inert case, $k'_w/k_v = \mathbb{C}/\mathbb{R}$. To calculate $Z_v(W_v,\Phi_v,\chi_v;s)$ explicitly, we need to know the K-types of the oscillator representation ω_{ψ_v,ξ_v}.

We begin with the explicit formulae for the differential $d\omega_{\psi_v,\xi_v}$. Noting $\eta_{\mathbb{C}/\mathbb{R}} = \mathrm{sgn}$, ξ_v can be written as

$$\xi_v(z) = (z/\overline{z})^{k/2}, \qquad (k \in 2\mathbb{Z}+1).$$

Write $\mathfrak{g}_{2,\mathbb{R}}$ for the $\mathbb{R}$-Lie algebra of $G_2(\mathbb{R})$. It is generated by the elements;

$$X_+ := \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad H := \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \quad U := \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad Z := \begin{pmatrix}\sqrt{-1}&0\\0&\sqrt{-1}\end{pmatrix}.$$

Lemma B.14. *The explicit formulae for the differential of* $(\omega_{\psi_v,\xi_v}, \mathcal{S}(\mathbb{C}))$ *are given as follows.*

$$d\omega_{\psi_v,\xi_v}(X_+)\Phi(z=x+\sqrt{-1}y) = \pm 2\pi\sqrt{-1}(x^2+y^2)\Phi(z), \quad (as\ \psi_v = \psi_{\mathbb{R}}^{\pm 1})$$

$$d\omega_{\psi_v,\xi_v}(H)\Phi(z) = \Phi(z) + x\frac{\partial}{\partial x}\Phi(z) + y\frac{\partial}{\partial y}\Phi(z)$$

$$d\omega_{\psi_v,\xi_v}(Z)\Phi(z) = \sqrt{-1}k\Phi(z) - y\frac{\partial}{\partial x}\Phi(z) + x\frac{\partial}{\partial y}\Phi(z)$$

$$d\omega_{\psi_v,\xi_v}(U)\Phi(z) = \pm\left[\frac{1}{8\pi\sqrt{-1}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(z) + 2\pi\sqrt{-1}(x^2+y^2)\Phi(z)\right],\ (as\ \psi_v = \psi_{\mathbb{R}}^{\pm 1}).$$

Proof. These can be easily deduced from (B.1), (B.2) and (B.3). □

Next we change the polarization.

Lemma B.15 (Change of the polarization). *Define the partial Fourier transform* $\mathcal{F}_2$ *by*

$$\mathcal{F}_2\Phi(x+\sqrt{-1}y) = \int_{\mathbb{R}} \Phi(x+\sqrt{-1}v)\psi_v(yv)\,dv.$$

Then we have

(B.15)
$$\mathcal{F}_2\Big(d\omega_{\psi_v,\xi_v}(U)\Phi\Big)(z) = \pm\left[\frac{1}{8\pi\sqrt{-1}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\mathcal{F}_2\Phi(z) + 2\pi\sqrt{-1}(x^2+y^2)\mathcal{F}_2\Phi(z)\right]$$

(B.16)
$$\mathcal{F}_2\Big(d\omega_{\psi_v,\xi_v}(Z)\Phi\Big)(z) = \left[\sqrt{-1}k \pm \left(\frac{1}{4\pi\sqrt{-1}}\frac{\partial^2}{\partial x\partial y} + 4\pi\sqrt{-1}xy\right)\right]\mathcal{F}_2\Phi(z).$$

Here the signs $\pm$ *are according to* $\psi_v = \psi_{\mathbb{R}}^{\pm 1}$.

Proof. The proof is again a simple computation. □

The final preliminary is the change of variables $u := x+y$ and $v = x-y$. Then we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial u} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$
$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right), \quad \frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}.$$

Hence if we write $\phi(u,v) := \mathcal{F}_2\Phi(z) \in \mathcal{S}(\mathbb{R}^{\oplus 2})$ and set

$$D_t := \frac{1}{4\pi\sqrt{-1}}\frac{\partial^2}{\partial t^2} + \pi\sqrt{-1}t^2, \quad (t = u \text{ or } v),$$

then (B.15) and (B.16) become

(B.17)
$$d\omega_{\psi_v,\xi_v}(U)\phi(u,v) = \pm(D_u + D_v)\phi(u,v)$$

(B.18)
$$d\omega_{\psi_v,\xi_v}(Z)\phi(u,v) = \Big(\sqrt{-1}k \pm (D_u - D_v)\Big)\phi(u,v).$$

We now describe the $K_{2,v}$-types in ω_{ψ_v,ξ_v}. First we solve the differential equation

$$D_t\phi(t) = \sqrt{-1}\lambda\phi(t), \quad (\lambda \in \mathbb{R},\ \phi \in \mathcal{S}(\mathbb{R})).$$

Lemma B.16 (cf. [W] **Chapt. 5 §6).** *$D_t\phi(t) = \sqrt{-1}\lambda\phi(t)$ has a non-trivial solution in $\mathcal{S}(\mathbb{R})$ (or in $L^2(\mathbb{R})$) if and only if $\lambda \in \mathbb{Z}_{\geq 0} + 1/2$. If this is the case, the solution space is spanned by*

$$\phi_\lambda(t) = e^{-\pi t^2} H_{\lambda - 1/2}(\sqrt{2\pi}\cdot t).$$

Here H_n ($n \in Z_{\geq 0}$) denotes the classical Hermite function;

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}.$$

As usual we identify each irreducible representation of $K_{2,v}$ with a pair of integers;

$$(m,n) : K_{2,v} \ni e^{\sqrt{-1}\varphi} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longrightarrow e^{\sqrt{-1}m\theta} e^{\sqrt{-1}n\varphi} \in \mathbb{C}^1.$$

Proposition B.17 ($K_{2,v}$-types of ω_{ψ_v,ξ_v}). *Using the notations defined above, the $K_{2,v}$-types of ω_{ψ_v,ξ_v} are*

$$\begin{cases} \{(m,n) \in \mathbb{Z}\oplus\mathbb{Z} \,|\, m \equiv n \pmod 2,\ k-m < n < k+m\} & \text{if } \psi_v = \psi_{\mathbb{R}}, \\ \{(m,n) \in \mathbb{Z}\oplus\mathbb{Z} \,|\, m \equiv n \pmod 2,\ k+m < n < k-m\} & \text{if } \psi_v = \psi_{\mathbb{R}}^{-1}. \end{cases}$$

B.3.5. *$Z_v(W_v, \Phi_v, \chi_v; s)$ in the principal case.* We treat τ_v in the principal and complementary series at the same time. We may assume that

$$\tau_v = \mathrm{Ind}_{B_2(\mathbb{R})}^{G_2(\mathbb{R})}[\nu_v \otimes \mathbf{1}_{N_2(\mathbb{R})}], \quad \chi_v\xi_v(z) = |z|_{\mathbb{C}}^{\sigma_\mu}(z/\overline{z})^{l_\mu/2}, \quad \nu_v(z) = |z|_{\mathbb{C}}^{\sigma_\nu}(z/\overline{z})^{l_\nu/2},$$

where

$$\mathrm{Re}(\sigma_\mu) = 0, \quad 0 \leq \mathrm{Re}(\sigma_\nu) \leq 1, \quad l_\mu,\, l_\nu \in \mathbb{Z}.$$

As before we write $\mathcal{S}^0(\mathbb{C})$ for the space of standard functions with respect to ψ_v;

$$\mathcal{S}^0(\mathbb{C}) := \{e^{-2\pi(x^2+y^2)}P(x,y) \,|\, P \text{ is a polynomial function.}\}.$$

We may assume $\Phi_v \in \mathcal{S}^0(\mathbb{C})$.

Proposition B.18. *Under these assumptions, $Z_v(W_v, \Phi_v, \chi_v; s)$ is a linear combination of functions of the form*

$$\begin{aligned} (\text{B.19}) \quad & \Gamma_{\mathbb{R}}(s+\sigma_\mu+\sigma_\nu+j_1+\frac{n}{2}+1)\Gamma_{\mathbb{R}}(s+\sigma_\mu-\sigma_\nu+j_2+\frac{n}{2}+1) \\ & \times \Gamma_{\mathbb{R}}(s+\sigma_\mu+\sigma_\nu+j_1+\frac{n}{2})\Gamma_{\mathbb{R}}(s+\sigma_\mu-\sigma_\nu+j_2+\frac{n}{2})\Big/\Gamma_{\mathbb{R}}(2s+2\sigma_\mu+1) \end{aligned}$$

with $j_1,\, j_2,\, n \in \mathbb{Z}_{\geq 0}$. Moreover the term with $j_1 = j_2 = n = 0$ appears if and only if $k = l_\mu \pm l_\nu$ and l_μ is odd.

Assuming the proposition for a moment, we shall prove the following.

Corollary B.19. *The G.C.D. of $Z_v(W_v, \Phi_v, \chi_v; s)$ has the same set of poles in the region $\mathrm{Re}(s) > 0$ as*

$$(\text{B.20}) \qquad \frac{L_{\mathbb{C}}(s, \chi_v\nu_v)L_{\mathbb{C}}(s, \chi_v\nu_v^{-1})}{L_{\mathbb{R}}(2s, (\chi_v\xi_v)|_{\mathbb{R}^\times})}$$

has.

Proof. Recall from [Sh2] §3 that

$$L_{\mathbb{C}}(s,\chi_v\nu_v) = L_{\mathbb{C}}(s,|z|_{\mathbb{C}}^{\sigma_\mu+\sigma_\nu}(z/\overline{z})^{(l_\mu+l_\nu-k)/2}) = \Gamma_{\mathbb{C}}\Big(s+\sigma_\mu+\sigma_\nu+\frac{|l_\mu+l_\nu-k|}{2}\Big)$$
$$L_{\mathbb{C}}(s,\chi_v\nu_v^{-1}) = \Gamma_{\mathbb{C}}\Big(s+\sigma_\mu-\sigma_\nu+\frac{|l_\mu-l_\nu-k|}{2}\Big)$$
$$L_{\mathbb{R}}(2s,\chi_v\xi_v|_{\mathbb{R}^\times}) = \Gamma_{\mathbb{R}}\left(s+\sigma_\mu+\frac{\epsilon}{2}\right),$$

where $\epsilon = 0$ or 1 is such that $\epsilon \equiv l_\mu \pmod 2$. Thus (B.20) has its only pole in the region $\mathrm{Re}(s) > 0$ at $s = \sigma_\mu - \sigma_\nu$ with

$$k = l_\mu - l_\nu, \quad 0 < \sigma_\nu < 1/2, \quad l_\nu \text{ is even.}$$

On the other hand (B.19) in Proposition B.18 has its only pole in the same region at $s = \sigma_\mu - \sigma_\nu$ with

$$j_1 = j_2 = n = 0.$$

We apply the last statement of Proposition B.18 to verify that these two conditions are equivalent. □

Proof of Proposition B.18. We may assume that $F_{s-1/2,v}(\chi_v\xi_v)$ and W_v are of the form

$$F_{s-1/2,v}^{(m_1)}\Big(\chi_v\xi_v; e^{\sqrt{-1}\varphi}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\Big) = e^{\sqrt{-1}l_\mu\varphi}e^{\sqrt{-1}m_1\theta}, \quad m_1 \in l_\mu + 2\mathbb{Z}$$
$$W_v^{(m_2)}\Big(e^{\sqrt{-1}\varphi}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\Big) = e^{\sqrt{-1}l_\nu\varphi}e^{\sqrt{-1}m_2\theta}, \quad m_2 \in l_\nu + 2\mathbb{Z}.$$

Let (n_1,n_2) be a $K_{2,v}$-type in ω_{ψ_v,ξ_v} (cf. Proposition B.17). Using the notations of Lemma B.16 we set

$$\phi_{n_1,n_2}(x+\sqrt{-1}y) := \phi_{\pm\frac{n_1-n_2+k}{2}}(x+y)\phi_{\pm\frac{n_1+n_2-k}{2}}(x-y) \in \mathcal{S}^0(\mathbb{C})$$
$$\Phi_{n_1,n_2}(x+\sqrt{-1}y) := \mathcal{F}_2^{-1}\phi_{n_1,n_2}(x+\sqrt{-1}y),$$

where the inverse Fourier transform $\mathcal{F}_2^{-1}$ is defined by

$$\mathcal{F}_2^{-1}\phi(x+\sqrt{-1}y) := \int_{\mathbb{R}} \phi(x+\sqrt{-1}v)\overline{\psi_v(vy)}\,dv.$$

From Lemma B.16, it is clear that Φ_{n_1,n_2} is the (n_1,n_2)-weight vector in ω_{ψ_v,ξ_v}, and it is enough to calculate

$$Z_v(W_v^{(m_2)},\Phi_{n_1,n_2},\chi_v;s)$$
$$= \int_{K_{2,v}}\int_{\mathbb{C}^\times} F_{s-1/2,v}^{(m_1)}(\chi_v\xi_v;k)\overline{\omega_{\psi_v,\xi_v}(k)\Phi_{n_1,n_2}(a)}W_v^{(m_2)}(d(a)k)\chi_v(a)|a|_{\mathbb{C}}^{s-1/2}\,da^\times\,dk$$

writing $k(\theta) := \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$,

$$= \int_{\mathbb{R}_+^\times}\int_0^{2\pi}\int_0^{\pi} F_{s-1/2,v}^{(m_1)}(\chi_v\xi_v; d(e^{\sqrt{-1}\varphi})k(\theta))\overline{\omega_{\psi_v,\xi_v}(d(e^{\sqrt{-1}\varphi})k(\theta))\Phi_{n_1,n_2}(a)}$$
$$W_v^{(m_2)}(d(ae^{\sqrt{-1}\varphi})k(\theta))\chi_v(a)|a|_{\mathbb{R}}^{2s-1}\,d\varphi\,d\theta\,da^\times$$

$$= \int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(a)} W_v^{(m_2)}(d(a))\chi_v(a)|a|_{\mathbb{R}}^{2s-1}\, da^\times$$
$$\int_0^{2\pi}\int_0^{\pi} e^{\sqrt{-1}(l_\mu+l_\nu-n_2)\varphi} e^{\sqrt{-1}(m_1+m_2-n_1)\theta}\, d\varphi\, d\theta.$$

This equals

$$\int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(a)} W_v^{(m_2)}(d(a))\chi_v(a)|a|_{\mathbb{R}}^{2s-1}\, da^\times$$

if $n_2 = l_\mu + l_\nu$, $n_1 = m_1 + m_2$ and zero otherwise.

Now we set $n_2 = l_\mu + l_\nu$, and take n_1, m_1 and m_2 so that $n_1 = m_1 + m_2$ and that the conditions of Proposition B.17 are satisfied. Then

$$Z_v(W_v^{(m_2)}, \Phi_{n_1,n_2}, \chi_v; s) = \int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(a)} W_v^{(m_2)}(d(a))|a|_{\mathbb{R}}^{2(s+\sigma_\mu)-1}\, da^\times.$$

Next we extend the restriction $\tau_v|_{H(\mathbb{R})}$ to a representation $\widehat{\tau}_v := \mathrm{Ind}_{B(\mathbb{R})}^{GL(2,\mathbb{R})}[(\nu_v|_{\mathbb{R}^\times} \otimes \mathbf{1}) \otimes \mathbf{1}_{N(\mathbb{R})}]$ of $GL(2,\mathbb{R})$ just as in the proof of Lemma B.6. Accordingly $W_v^{(m_2)}$ restricted to $H(\mathbb{R})$ is extended to a Whittaker function of $\widehat{\tau}_v$ and we have

$$W_v^{(m_2)}(d(a)) = \nu_v(a)^{-1} W_v^{(m_2)}\Big(\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\Big).$$

Thus

$$Z_v(W_v^{(m_2)},\Phi_{n_1,n_2},\chi_v;s) = \int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(a)} W_v^{(m_2)}\Big(\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{2(s+\sigma_\mu-\sigma_\nu)-1}\, da^\times$$
$$= 2\int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(\sqrt{a})} W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s+\sigma_\mu-\sigma_\nu-1/2}\, da^\times.$$

To compute this we need following lemmas.

Lemma B.20. *(i) The Mellin transform of* $\overline{\Phi_{n_1,n_2}(\sqrt{a})}$*;*

$$\int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(\sqrt{a})}|a|_{\mathbb{R}}^{s-1/2}\, da^\times$$

is a linear combination of terms $\Gamma_{\mathbb{R}}(s+\frac{n+1}{2})\Gamma_{\mathbb{R}}(s+\frac{n-1}{2})$ $(n \in \mathbb{Z}_{\geq 0})$.
(ii) The term $\Gamma_{\mathbb{R}}(s+\frac{1}{2})\Gamma_{\mathbb{R}}(s-\frac{1}{2})$ *appears if and only if* $k = l_\mu + l_\nu$.

Proof. Recall that

$$\Phi_{n_1,n_2}(r) = \mathcal{F}_2^{-1}\phi_{n_1,n_2}(r) = \int_{\mathbb{R}} \phi_{n_1,n_2}(r+\sqrt{-1}y)\, dy$$
$$= \int_{\mathbb{R}} \phi_{\pm\frac{n_1-n_2+k}{2}}(r+y)\phi_{\pm\frac{n_1+n_2-k}{2}}(r-y)\, dy$$
$$= \int_{\mathbb{R}} e^{-2\pi(r^2+y^2)} H_{\pm\frac{n_1-n_2+k}{2}}(\sqrt{2\pi}(r+y)) H_{\pm\frac{n_1+n_2-k}{2}}(\sqrt{2\pi}(r-y))\, dy$$
$$= e^{-2\pi r^2}\int_{\mathbb{R}} e^{-x^2} H_{\pm\frac{n_1-n_2+k}{2}}(x+\sqrt{2\pi}r) H_{\pm\frac{n_1+n_2-k}{2}}(\sqrt{2\pi}r-x)\,\frac{dx}{\sqrt{2\pi}}$$
$$= (-1)^{\pm\frac{n_1+n_2-k}{2}} e^{-2\pi r^2}\int_{\mathbb{R}} e^{-x^2} H_{\pm\frac{n_1-n_2+k}{2}}(x+\sqrt{2\pi}r) H_{\pm\frac{n_1+n_2-k}{2}}(x-\sqrt{2\pi}r)\,\frac{dx}{\sqrt{2\pi}}.$$

Clearly this is a linear combination of terms of the form $e^{-2\pi r^2} r^n$ $(n \in \mathbb{Z}_{\geq 0})$. The orthogonality relation of Hermite polynomials asserts that the term $e^{-2\pi r^2}$ appears if and

only if $\pm\frac{n_1-n_2+k}{2} = \pm\frac{n_1+n_2-k}{2}$, or equivalently $k = n_2 = l_\mu + l_\nu$. Now our lemma follows from the well-known formula

$$\int_0^\infty e^{-2\pi a}|a|_{\mathbb{R}}^{s+n/2-1/2}\, da^\times = \pi^{-s-n/2}\Gamma\Big(\frac{s+(n+1)/2}{2}\Big)\Gamma\Big(\frac{s+(n-1)/2}{2}\Big).$$

□

Lemma B.21. *(i) The Mellin transform of $W_v^{(m_2)}$;*

$$\int_{\mathbb{R}^\times} W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s-1/2}\, da^\times$$

is a linear combination of terms $\Gamma_{\mathbb{R}}(s+j_1+2\sigma_\nu)\Gamma_{\mathbb{R}}(s+j_2)$ $(j_1,\, j_2 \in \mathbb{Z}_{\geq 0})$.
(ii) The term $\Gamma_{\mathbb{R}}(s+2\sigma_\nu)\Gamma_{\mathbb{R}}(s)$ appears if and only if m_2 is even.

Proof. We may assume $\mathrm{Re}(\sigma_\nu) \gg 0$ and $\mathrm{Re}(s) \gg 0$ throughout the proof. Let

$$\Phi^{(m_2)}(x,y) := e^{-\pi(x^2+y^2)}(x+\sqrt{-1}y)^{m_2} \in \mathcal{S}^0(\mathbb{R}^2)$$
$$f_{\nu_v}^{(m_2)}(g) := \nu_v(\det(g))|\det(g)|_{\mathbb{R}}^{1/2}\int_{\mathbb{R}^\times} \Phi^{(m_2)}([0,t]g)\nu_v(t)|t|_{\mathbb{R}}\, dt^\times.$$

This integral converges absolutely for $\mathrm{Re}(\sigma_\nu) > -m_2+1$, and $f_{\nu_v}^{(m_2)}$ is a m_2-weight vector in $\widehat{\tau}_v$. Thus from [JL] §5, we may take $W_v^{(m_2)}$ to be

$$W_v^{(m_2)}(g) = \int_{\mathbb{R}} f_{\nu_v}^{(m_2)}\Big(w^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\Big)\overline{\psi_v(x)}\, dx$$

from definition,

$$= \nu_v(\det(g))|\det(g)|_{\mathbb{R}}^{1/2}\int_{\mathbb{R}}\int_{\mathbb{R}^\times} \Phi^{(m_2)}([t,tx]g)\nu_v(t)|t|_{\mathbb{R}}\overline{\psi_v(x)}\, dt^\times\, dx$$

putting $x := tx$,

$$= \nu_v(\det(g))|\det(g)|_{\mathbb{R}}^{1/2}\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \Phi^{(m_2)}([t,x]g)\overline{\psi_v(t^{-1}x)}\, dx\, \nu_v(t)\, dt^\times$$

writing $R(g)$ for the right translation by g,

$$= \nu_v(\det(g))|\det(g)|_{\mathbb{R}}^{1/2}\int_{\mathbb{R}^\times} \mathcal{F}_2^{-1}[R(g)\Phi^{(m_2)}](t,t^{-1})\nu_v(t)\, dt^\times$$

using Proposition 1.6 (ii) in [JL],

$$= \nu_v(\det(g))|\det(g)|_{\mathbb{R}}^{1/2}\int_{\mathbb{R}^\times} [\omega_\psi(g)\mathcal{F}_2^{-1}\Phi^{(m_2)}](t,t^{-1})\nu_v(t)\, dt^\times.$$

Here ω_{ψ_v} denotes the Weil representation of $SL(2,\mathbb{R})$ defined in [JL] Proposition 1.3 extended to $GL(2,\mathbb{R})$ by Proposition 1.6 *loc.cit.*

Using this we have

$$W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big) = \int_{\mathbb{R}^\times} \mathcal{F}_2^{-1}\Phi^{(m_2)}(at,t^{-1})\nu_v(at)|a|_{\mathbb{R}}^{1/2}\, dt^\times.$$

Hence the Mellin transform becomes

$$\int_0^\infty W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s-1/2}\, da^\times$$

$$= \frac{1}{2}\left(\int_{\mathbb{R}^\times} W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s-1/2}\,da^\times + \int_{\mathbb{R}^\times} W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s-1/2}\mathrm{sgn}(a)\,da^\times\right)$$

$$= \frac{1}{2}\Big(\int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} \mathcal{F}_2^{-1}\Phi^{(m_2)}(at,t^{-1})\nu_v(at)|a|_{\mathbb{R}}^s\,dt^\times\,da^\times$$
$$+ \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} \mathcal{F}_2^{-1}\Phi^{(m_2)}(at,t^{-1})\nu_v(at)|a|_{\mathbb{R}}^s\mathrm{sgn}(a)\,dt^\times\,da^\times\Big)$$

$$= \frac{1}{2}\Big(\int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} \mathcal{F}_2^{-1}\Phi^{(m_2)}(x,y)\nu_v(x)|xy|_{\mathbb{R}}^s\,dx^\times\,dy^\times$$
$$+ \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} \mathcal{F}_2^{-1}\Phi^{(m_2)}(x,y)\nu_v(x)|xy|_{\mathbb{R}}^s\mathrm{sgn}(xy)\,dx^\times\,dy^\times\Big)$$

On the other hand

$$\mathcal{F}_2^{-1}\Phi^{(m_2)}(x,y) = \int_{\mathbb{R}} e^{-\pi(x^2+u^2)}(x+\sqrt{-1}u)^{m_2}\psi_v(-uy)\,du$$
$$= \sum_{j=0}^{m_2}\sqrt{-1}^j x^{m_2-j}\binom{m_2}{j}e^{-\pi x^2}\int_{\mathbb{R}} e^{-\pi u^2}u^j\psi_v(-uy)\,du.$$

This is a linear combination of terms $e^{-\pi(x^2+y^2)}x^{j_1}y^{j_2}$ ($j_1,\ j_2 \in \mathbb{Z}_{\geq 0}$), and $e^{-\pi(x^2+y^2)}$ appears if and only if m_2 is even. Hence the Mellin transform is a linear combination of

$$\frac{1}{2}\Big(\int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} e^{-\pi(x^2+y^2)}x^{j_1}y^{j_2}|x|_{\mathbb{R}}^{2\sigma_\nu+s}\mathrm{sgn}(x)^{l_\nu}|y|_{\mathbb{R}}^s\,dx^\times\,dy^\times$$
$$+ \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} e^{-\pi(x^2+y^2)}x^{j_1}y^{j_2}|x|_{\mathbb{R}}^{2\sigma_\nu+s}\mathrm{sgn}(x)^{l_\nu+1}|y|_{\mathbb{R}}^s\mathrm{sgn}(y)\,dx^\times\,dy^\times\Big)$$
$$= \frac{1}{2}\Big(\int_{\mathbb{R}^\times} e^{-\pi x^2}|x|_{\mathbb{R}}^{2\sigma_\nu+j_1+s}\mathrm{sgn}(x)^{l_\nu+j_1}\,dx^\times\int_{\mathbb{R}^\times} e^{-\pi y^2}|y|_{\mathbb{R}}^{j_2+s}\mathrm{sgn}(y)^{j_2}\,dy^\times$$
$$+ \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|_{\mathbb{R}}^{2\sigma_\nu+j_1+s}\mathrm{sgn}(x)^{l_\nu+j_1+1}\,dx^\times\int_{\mathbb{R}^\times} e^{-\pi y^2}|y|_{\mathbb{R}}^{j_2+s}\mathrm{sgn}(y)^{j_2+1}\,dy^\times\Big)$$

This equals

$$2\int_0^\infty e^{-\pi x^2}x^{2\sigma_\nu+j_1+s}\,dx^\times\int_0^\infty e^{-\pi y^2}y^{j_2+s}\,dy^\times = \frac{1}{2}\Gamma_{\mathbb{R}}(s+j_1+2\sigma_\nu)\Gamma_{\mathbb{R}}(s+j_2)$$

if $j_1 + l_\nu \equiv j_2 \pmod 2$ and zero otherwise. The term with $j_1 = j_2 = 0$ appears if and only if l_ν (or equivalently m_2) is even. □

We now return to the proof of Proposition B.18. We apply the Barnes-Mellin lemma ([Ja] Lemma 17.3.2) to the result of Lemmas B.20 and B.21 to see that

$$\int_{\mathbb{R}_+^\times}\overline{\Phi_{n_1,n_2}(\sqrt{a})}W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s-1}\,da^\times$$

is a linear combination of terms

$$\frac{\Gamma_{\mathbb{R}}(s+j_1+2\sigma_\nu+\frac{n+1}{2})\Gamma_{\mathbb{R}}(s+j_2+\frac{n+1}{2})\Gamma_{\mathbb{R}}(s+j_1+2\sigma_\nu+\frac{n-1}{2})\Gamma_{\mathbb{R}}(s+j_2+\frac{n-1}{2})}{\Gamma_{\mathbb{R}}(2s+j_1+j_2+2\sigma_\nu+n)},$$

and the term corresponding to $j_1 = j_2 = n = 0$ appears if and only if $k = l_\mu + l_\nu$ and l_ν is even. Since k is odd, this is equivalent to $k = l_\mu + l_\nu$ and l_μ is odd. Replacing s with $s+\sigma_\mu-\sigma_\nu+1/2$ will yield the proposition. The statement for $k = l_\mu - l_\nu$ will be obtained by replacing ν_v with $(\nu_v\circ\sigma)^{-1}$. (q.e.d.)

B.3.6. *$Z_v(W_v, \Phi_v, \chi_v; s)$ in the discrete case.* Here we take care of τ_v in the discrete series. We embed such τ_v into a generalized principal series:

$$\tau_v \hookrightarrow \mathrm{Ind}_{B_2(\mathbb{R})}^{G_2(\mathbb{R})}[\nu_v \otimes \mathbf{1}_{N_2(\mathbb{R})}].$$

Here $\nu_v(z) = |z|_{\mathbb{C}}^{k_\nu/2}(z/\overline{z})^{l_\nu/2}$ satisfies if $\epsilon_\nu = 0$ or 1 is such that $l_\nu \equiv \epsilon_\nu \pmod 2$, then $k_\nu - \epsilon_\nu \in 2\mathbb{N} - 1$.

Proposition B.22. *(i) Under these assumptions, the G.C.D. of $Z_v(W_v, \Phi_v, \chi_v; s)$ is given by*

$$\frac{\Gamma_{\mathbb{C}}(s + \sigma_\mu + k_\nu/2)\Gamma_{\mathbb{C}}(s + \sigma_\mu + k_\nu/2 + 1)}{\Gamma_{\mathbb{R}}(2s + 2\sigma_\mu + k_\nu + 2)}.$$

(ii) In particular $Z_v(W_v, \Phi_v, \chi_v; s)$ is holomorphic in the region $\mathrm{Re}(s) > 0$.

Proof. Let the notations be as in the proof of Proposition B.18. Then the local zeta integral $Z_v(W_v^{(m_2)}, \Phi_{n_1,n_2}, \chi_v; s)$ equals

$$\int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(a)} W_v^{(m_2)}(d(a))\chi_v(a)|a|_{\mathbb{R}}^{2s-1}\, da^\times$$

if $n_2 = l_\mu + l_\nu$, $n_1 = m_1 + m_2$ and zero otherwise. We concentrate on the former case. We again extend $W_v^{(m_2)}$ to a Whittaker function of $\widehat{\tau}_v = \mathrm{Ind}_{B(\mathbb{R})}^{GL(2,\mathbb{R})}[(\nu_v|_{\mathbb{R}^\times} \otimes \mathbf{1}) \otimes \mathbf{1}_{N(\mathbb{R})}]$ and have

$$Z_v(W_v^{(m_2)}, \Phi_{n_1,n_2}, \chi_v; s) = 2\int_{\mathbb{R}_+^\times} \overline{\Phi_{n_1,n_2}(\sqrt{a})} W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s+\sigma_\mu-(k_\nu+1)/2}\, da^\times.$$

The analogue of Lemma B.21 in this case is much simpler.

Lemma B.23. *The Mellin transform of $W_v^{(m_2)}$;*

$$\int_{\mathbb{R}_+^\times} W_v^{(m_2)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)|a|_{\mathbb{R}}^{s-1/2}\, da^\times$$

is a linear combination of terms $\Gamma_{\mathbb{R}}(s + k_\nu + j + 1)\Gamma_{\mathbb{R}}(s + k_\nu + j)$ $(j \in \mathbb{Z}_{\geq 0})$.

Proof. From the explicit calculation in [JL] pp. 187–189, we know that

$$W_v^{(2n+k_\nu+1)}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big) = \begin{cases} \begin{cases} P_n(a)a^{k_\nu+1/2}e^{-2\pi a} & \text{if } a > 0, \\ 0 & \text{if } a \leq 0, \end{cases} & \text{if } \psi_v = \psi_{\mathbb{R}}, \\ \begin{cases} 0 & \text{if } a \geq 0, \\ P_n(-a)(-a)^{k_\nu+1/2}e^{2\pi a} & \text{if } a < 0, \end{cases} & \text{if } \psi_v = \psi_{\mathbb{R}}^{-1}. \end{cases}$$

Here $P_n(a)$ is a certain polynomial. The formulae for $W_v^{(-2n-k_\nu-1)}$ are given by replacing $\psi_v = \psi_{\mathbb{R}}$ and $\psi_v = \psi_{\mathbb{R}}^{-1}$ in the above. Thus our Mellin transform is a linear combination of

$$\int_{\mathbb{R}^\times} e^{-2\pi a}a^{s+k_\nu+j}\, da^\times = \Gamma_{\mathbb{R}}(s + k_\nu + j + 1)\Gamma_{\mathbb{R}}(s + k_\nu + j), \quad (j \in \mathbb{Z}_{\geq 0}).$$

□

We again apply the Barnes-Mellin lemma to the result of Lemmas B.20 and B.23 to see that $Z_v(W_v^{(m_2)}, \Phi_{n_1,n_2}, \chi_v; s)$ is a linear combination of

$$\frac{\Gamma_{\mathbb{R}}(s+\sigma_\mu+\frac{k_\nu}{2}+j+\frac{n}{2}+2)\Gamma_{\mathbb{R}}(s+\sigma_\mu+\frac{k_\nu}{2}+j+\frac{n}{2}+1)^2\Gamma_{\mathbb{R}}(s+\sigma_\mu+\frac{k_\nu}{2}+j+\frac{n}{2})}{\Gamma_{\mathbb{R}}(2s+2\sigma_\mu+k_\nu+2j+n+2)}$$

$(j, n \in \mathbb{Z}_{\geq 0})$. Finally we note

$$\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(s) = \int_0^\infty e^{-2\pi x} x^s \, dx^\times = \frac{1}{2}\Gamma_{\mathbb{C}}(s)$$

to have the proposition. □

B.4. Poles of $L(s, \tau \times \chi)$. We finally come to the main result of this appendix.

Theorem B.24 (Poles of $L(s, \tau \times \chi)$). *$L(s, \tau \times \chi)$ has its only possible pole in $\mathrm{Re}(s) > 0$ at $s = 1$, and it is a pole if and only if*

$$\textit{(i)}\ \chi|_{\mathbb{A}^\times} = \eta_{k'/k}, \qquad \textit{(ii)}\ \tau = \Theta(\chi^{-1}, \psi)_{\omega_\tau}.$$

Proof. Lemma B.6, Lemma B.7, Corollary B.9, Proposition B.10, Corollary B.19 and Proposition B.22 combined with Proposition B.5 assert that we have only to determine the poles of

$$Z(\varphi, \theta_\Phi, \chi; s) := \int_{G_2(k)\backslash G_2(\mathbb{A})} \varphi(g)\overline{\theta_\Phi(g)} E(F(\chi\xi), s-1/2)(g)\, dg.$$

First note that φ is rapidly decreasing while θ_Φ and $E(F(\chi\xi), s-1/2)(g)$ are slowly increasing. Hence the integral converges absolutely (if $E(F(\chi\xi), s-1/2)(g)$ is defined), and its poles come from those of $E(F(\chi\xi), s-1/2)(g)$. But it is well-known that

(1) The only pole of $E(F(\chi\xi), s-1/2)(g)$ in $\mathrm{Re}(s) > 0$ occurs if and only if $\chi\xi|_{\mathbb{A}^\times} = \mathbf{1}$ and it locates at $s = 1$.
(2) The residue of $E(F(\chi\xi), s-1/2)(g)$ at the pole spans the 1-dimensional representation $(\chi\xi)^{G_2}$.

Thus $Z(\varphi, \theta_\Phi, \chi; s)$ has its only possible pole in $\mathrm{Re}(s) > 0$ at $s = 1$, and it is a pole if and only if $\chi|_{\mathbb{A}^\times} = \eta_{k'/k}$ and

$$\int_{G_2(k)\backslash G_2(\mathbb{A})} \varphi(g)\overline{\theta_\Phi(g)}(\chi\xi)^{G_2}(g)\, dg \neq 0. \tag{B.21}$$

But the left hand side reads

$$\begin{aligned}
&\int_{G_2(k)Z_2(\mathbb{A})\backslash G_2(\mathbb{A})} \int_{Z_2(k)\backslash Z_2(\mathbb{A})} \varphi(zg)\overline{\theta_\Phi(zg)}(\chi\xi)^{G_2}(zg)\, dz\, dg \\
&\quad = \int_{G_2(k)Z_2(\mathbb{A})\backslash G_2(\mathbb{A})} \varphi(g)(\chi\xi)^{G_2}(g) \overline{\int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\omega_\tau\chi\xi(z)}\theta_\Phi(zg)\, dz}\, dg \\
&\quad = \int_{G_2(k)Z_2(\mathbb{A})\backslash G_2(\mathbb{A})} (\chi\xi)^{G_2}(g)\varphi(g)\overline{\theta_{\Phi,\omega_\tau\chi\xi}(g)}\, dg.
\end{aligned}$$

Of course this means that (B.21) is equivalent to

$$\tau = (\chi\xi)^{{G_2}^{-1}} \otimes \Theta(\xi, \psi)_{\omega_\tau\chi\xi}, \tag{B.22}$$

since the multiplicity one theorem is true for G_2.

We now analyze the representation $(\chi\xi)^{{G_2}^{-1}} \otimes \Theta(\xi,\psi)_{\omega_\tau\chi\xi}$. By definition this space is spanned by functions of the form

$$\begin{aligned}(\chi\xi)^{{G_2}^{-1}}(g) &\int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\omega_\tau\chi(z)}\theta(\omega_{\psi,\xi}(g)\Phi(\bullet z))\,dz \\ &= \int_{Z_2(k)\backslash Z_2(\mathbb{A})} (\chi\xi)^{{G_2}^{-1}}(zg)\chi\xi(z)\overline{\omega_\tau\chi(z)}\theta(\xi(z)^{-1}\omega_{\psi,\xi}(zg)\Phi)\,dz \\ &= \int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\omega_\tau(z)}\theta((\chi\xi)^{{G_2}^{-1}}(zg)\omega_{\psi,\xi}(zg)\Phi)\,dz.\end{aligned}$$

On the other hand it follows from the explicit formulae (B.8), (B.9), (B.10) that

$$((\chi\xi)^{{G_2}^{-1}} \otimes \omega_{\psi,\xi}, \mathcal{S}(\mathbb{A}_{k'})) \simeq (\omega_{\psi,\chi^{-1}}, \mathcal{S}(\mathbb{A}_{k'})).$$

Hence $(\chi\xi)^{{G_2}^{-1}} \otimes \Theta(\xi,\psi)_{\omega_\tau\chi\xi}$ is spanned by

$$\begin{aligned}\int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\omega_\tau(z)}\theta(\omega_{\psi,\chi^{-1}}(zg)\Phi)\,dz &= \int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\omega_\tau(z)}\theta(\omega_{\psi,\chi^{-1}}(z)\omega_{\psi,\chi^{-1}}(g)\Phi)\,dz \\ &= \int_{Z_2(k)\backslash Z_2(\mathbb{A})} \overline{\omega_\tau}\chi^{-1}(z)\theta(\omega_{\psi,\chi^{-1}}(g)\Phi(\bullet z))\,dz.\end{aligned}$$

These span $\Theta(\chi^{-1},\psi)_{\omega_\tau}$. □

We now apply Shahidi's non-vanishing theorem to $L(s,\pi,r_{w(P_2)})$ to have the following corollary.

Corollary B.25 (Proposition 5.2). *The poles of $r(w(P_2),\pi)$ in the region $\langle \mathrm{Re}\pi, \beta_2^\vee\rangle \geq 0$ locate at*

$$\begin{aligned}&\mathfrak{S}(P_2,\eta_{k'/k}) := \{\pi=\chi\otimes\tau\in\mathfrak{P}\,;\,\chi|_{\mathbb{A}^\times} = |\ |_{\mathbb{A}}^2\eta_{k'/k},\ \tau=\Theta(\chi^{-1}|\ |_{\mathbb{A}_{k'}},\psi)_{\omega_\tau}\} \\ &\mathfrak{S}(P_2,\mathbf{1}) := \{\pi=\chi\otimes\tau\in\mathfrak{P}\,;\,\chi|_{\mathbb{A}^\times} = |\ |_{\mathbb{A}},\ L(0,\tau\times\chi)\neq 0\}.\end{aligned}$$

Here in the definition of $\mathfrak{S}(P_2,\eta_{k'/k})$, ψ has been chosen so that τ_v is generic with respect to it.

Note that $\mathfrak{S}(P_2,\mathbf{1})$ is the only pole of $L(s,\pi,r_{w(P_2),2}) = L_k(s,\chi|_{\mathbb{A}^\times})$.

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