Unipotent automorphic representations of $U_{E/F}(4)$ *

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1 Motivation

The motivation of our study is to understand certain cohomology groups of Shimura varieties. Although the ultimate goal is to study the ℓ-adic cohomology, we start with the analytic one.

L^2-cohomology of Shimura varieties Write $\mathbb{S} := \mathrm{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. For the moment, we write $\mathbb{A} = \mathbb{R} \oplus \mathbb{A}_f$ for the ring of adeles for $\mathbb{Q}$. Recall that a Shimura variety is defined from a datum (G, X, K) consisting of

(1) G is a connected reductive group over $\mathbb{Q}$.

(2) X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the conditions in [Del79, 2.1.1].

(3) $K \subset G(\mathbb{A}_f)$ is an open compact subgroup, which we assume to be neat.

It is an arithmetic variety S_K defined over certain number field E such that

$$S_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \simeq G(\mathbb{Q})\backslash G(\mathbb{A})/L_h(\mathbb{R})K.$$

Here L_h is the centralizer of $h \in X$ in G. For each irreducible algebraic representation $\rho : G \to GL(V_\rho)$ of G, one can construct a local system

$$\mathcal{V}_\rho := V_\rho \underset{G(\mathbb{Q})}{\times} (X \times G(\mathbb{A}_f)/K).$$

We are interested in the L^2-cohomology groups

$$\mathrm{H}^i_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\rho)$$

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with coefficients in $\mathcal{V}_\rho$. (Notice that this is isomorphic to, via the Zucker conjecture proved by Saper-Stern and Looijenga, the intersection cohomology group of the Satake-Baily-Borel compactification of $S_K(\mathbb{C})$ with respect to the middle perversity.)

Matsushima-Murakami isomorphism The above cohomology groups are closely related to the L^2-automorphic forms on $G(\mathbb{A})$. Consider the right regular representation R of $G(\mathbb{A})$ on the Hilbert space

$$L^2(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A})) := \left\{ \begin{array}{c|ll} \phi : G(\mathbb{A}) \to \mathbb{C} & \text{(i)} & \phi(\gamma a g) = \phi(g),\ \gamma \in G(\mathbb{Q}),\ a \in \mathfrak{A}_G \\ \text{measurable} & \text{(ii)} & \int_{G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A})} |\phi(g)|^2\, dg < \infty \end{array} \right\},$$

$$[R(g)\phi](x) := \phi(xg), \quad g \in G(\mathbb{A}),\ \phi \in L^2(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})).$$

Here $\mathfrak{A}_G$ is the maximal $\mathbb{R}$-vector subgroup in $Z_G(\mathbb{R})$. By the general theory in spectral analysis, this has a direct sum decomposition

$$L^2(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A})) = L^2_{\mathrm{disc}}(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A})) \oplus L^2_{\mathrm{cont}}(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A})),$$

where $L^2_{\mathrm{disc}}(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A}))$ is a direct sum of the irreducible subrepresentations of R and $L^2_{\mathrm{cont}}(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A}))$ is its orthogonal complement.

An irreducible admissible representation of $G(\mathbb{A})$ is a restricted tensor product $\pi = \pi_\infty \otimes \bigotimes'_p \pi_p$ of irreducible admissible representations π_∞ of $G(\mathbb{R})$ and π_p of $G(\mathbb{Q}_p)$. Writing $m(\pi)$ for the multiplicity of such π in the representation $L^2_{\mathrm{disc}}(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A}))$, we have the Matsushima-Murakami formula [BW00, VII] (extended to the present case in [BC83]):

$$\mathrm{H}^i_{(2)}(S_K(\mathbb{C}), \mathcal{V}_\rho) \simeq \bigoplus_\pi \left(\mathrm{H}^i(\mathfrak{g}(\mathbb{C}), L_h(\mathbb{R}); \pi_\infty \otimes \rho) \otimes \pi_f^K \right)^{\oplus m(\pi)}.$$

Here π_f^K is the K-invariant part of the finite component $\pi_f := \bigotimes'_p \pi_p$. The relative Lie algebra cohomology $\mathrm{H}^i(\mathfrak{g}(\mathbb{C}), L_h(\mathbb{R}); \pi_\infty \otimes \rho)$ were completely described by Vogan-Zuckerman [VZ84]. Thus the remaining problem is to determine the multiplicity $m(\pi)$.

2 Analytic contribution of boundary

In order to study $m(\pi)$, we start with the most accessible part. The discrete spectrum further decomposes as follows [Lan76], [MW94]:

$$L^2_{\mathrm{disc}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) = L^2_{\mathrm{cusp}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})) \oplus L^2_{\mathrm{res}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A})).$$

Here, $L^2_{\mathrm{cusp}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is just the completion of the space of cusp forms on $G(\mathbb{A})$ with respect to the L^2- (*i.e.* Petersson) norm. $L^2_{\mathrm{res}}(G(F)\mathfrak{A}_G\backslash G(\mathbb{A}))$ is spanned by the residues of certain Eisenstein series at their poles.

At this point, we restrict ourselves to the specific example. Take a quadratic imaginary field E and write σ for the generator of $\mathrm{Gal}(E/\mathbb{Q})$. Let G be the quasisplit unitary group

$$U_{E/F}(4) := \{ g \in \mathrm{R}_{E/F} GL(4) \,|\, \theta(g) = \sigma(g) \}$$

in 4 variables, where

$$\theta(g) := \mathrm{Ad}(I_4)\,{}^t g^{-1}, \quad I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

This has the following 4 conjugacy classes of parabolic subgroups.

- $\mathbf{B} = \mathbf{TU}$, Borel subgroup, $\mathbf{T} \simeq \mathrm{R}_{E/F}\mathbb{G}_m^2$;
- $P_1 = M_1U_1$, $M_1 \simeq \mathrm{R}_{E/F}\mathbb{G}_m \times U_{E/F}(2)$;
- $P_2 = M_2U_2$, $M_2 \simeq \mathrm{R}_{E/F}GL(2)$;
- G.

Theorem 2.1 ([Kon98]). *$L^2_{\mathrm{res}}(G(\mathbb{Q})\mathfrak{A}_G\backslash G(\mathbb{A}))$ for this G is the direct sum of the following 6 types of representations.*

(1) Stable case.

(a) $\eta_G : G(\mathbb{A}) \overset{\det}{\to} U_{E/F}(1,\mathbb{A}) \ni x\sigma(x)^{-1} \mapsto \eta(x) \in \mathbb{C}^1$, $\eta : \mathbb{A}_E^\times/E^\times\mathbb{A}^\times \to \mathbb{C}^1$.

(b) $J^G_{P_2}(\Pi_E|\det|^{1/2}_{\mathbb{A}_E}) \leftarrow I^G_{P_2}(\mu\Pi_E|\det|^{1/2}_{\mathbb{A}_E})$. Here $\mu : \mathbb{A}_E^\times/E^\times \to \mathbb{C}^\times$ satisfies $\mu|_{\mathbb{A}^\times} = \omega_{E/F}$, the sign character attached to $E/\mathbb{Q}$ by the classfield theory. Also $\Pi_E \subset L^2_{\mathrm{cusp}}(GL_2(E)\mathbb{A}^\times\backslash GL_2(\mathbb{A}_E))$ and $\sigma(\Pi_E) \simeq \Pi_E^\vee$ (contragredient representation).

(2) Endoscopic case.

(a) θ-lifts of the trivial representation of unitary groups in two variables.

(b) $J^G_{P_1}(\eta|\,|^{1/2}_{\mathbb{A}_E} \otimes \tau)$, τ has the base change lift Π_E as in (1.b). η is as in (1.a).

(c) $J^G_{P_1}(\eta|\,|^{1/2}_{\mathbb{A}_E} \otimes \eta'_{U(2)})$, $\eta \neq \eta'$ are as in (1.a).

(d) $J^G_{P_1}(\eta|\,|^{1/2}_{\mathbb{A}_E} \otimes \tau)$, τ has the base change lift in $L^2_{\mathrm{cont}}(GL_2(E)\mathbb{A}^\times\backslash GL_2(\mathbb{A}_E))$.

These discrete automorphic forms are associated to parabolic subgroups, so that they are expected to make the contribution from the boundary to the L^2-cohomology (Eisenstein cohomology classes in the sense of Harder).

3 L and A-indistinguishability

Unfortunately, the above result alone does not make any sense in arithmetic applications. This is because we need to describe the contribution in terms of the Hecke algebra action on the corresponding automorphic representation.

For irreducible representation $\pi = \bigotimes'_v \pi_v$ and $\pi' = \bigotimes'_v \pi'_v$ of $G(\mathbb{A})$, we have the following three equivalence relations.

Isomorphy $\pi \simeq \pi'$ iff $\pi_v \simeq \pi'_v$ at all v.

L-indistinguishability $\pi \sim_L \pi'$ iff π_v and π'_v share the same L-factors at all v.

A-indistinguishability $\pi \sim_A \pi'$ iff π_v and π'_v share the same L-factors at all but finite number of v.

For GL_n, the strong multiplicity one theorem [JS81, Th. 4.4] assures that these three notions are actually equivalent for automorphic representations. In general, the Hecke algebra action on a automorphic form depends only on its A-indisinguishable class. Thus we need to describe:

- the A-indistinguishable classes of each representations in the Th. 2.1 (local question);
- the necessary and sufficient condition for each member of the A-indistinguishable class to be automorphic (global question).

4 Local A-packets

We start with the local question. To each of the residual representations in Th. 2.1, we associate a 4-dimensional representation of the group $\mathcal{A}_E := \mathcal{L}_E \times SL(2,\mathbb{C})$ as follows. Here $\mathcal{L}_E$ is the hypothetical Langlands group (a variant of the conjectural automorphic Galois group) of E. We write ρ_n is the n-dimensional irreducible representation of $SL(2,\mathbb{C})$ (n-th symmetric power of the standard representation). We also identify an idele class character χ of E with the character $\mathcal{L}_E \to W_E \xrightarrow{\chi} \mathbb{C}^\times$ by the classfield theory.

(1) Stable cases.

  (a) $\psi_\eta := \eta \otimes \rho_4$.

  (b) $\psi_{\Pi_E,\mu} := [\mu\varphi_{\Pi_E} \otimes \rho_2]$. Here $\varphi_{\Pi_E} : \mathcal{L}_E \to GL(2,\mathbb{C})$ is the irreducible representation associated to the cuspidal representation Π_E of $GL(2,\mathbb{A}_E)$.

(2) Endoscopic cases.

  (a) $\psi_{\underline{\mu}} = (\mu \otimes \rho_3) \oplus \mu'$. $\underline{\mu} = (\mu,\mu')$ and μ may be μ'.

  (b) $\psi_{\Pi_E,\eta} = (\eta \otimes \rho_2) \oplus \varphi_{\Pi_E}$. φ_{Π_E} is irreducible.

  (c) $\psi_{\underline{\eta}} = (\eta \otimes \rho_2) \oplus (\eta' \otimes \rho_2)$, where $\underline{\eta} = (\eta,\eta')$ and $\eta \neq \eta'$.

  (d) $\psi_{\eta,\underline{\mu}} = (\eta \otimes \rho_2) \oplus \mu \oplus \mu'$, where $\underline{\mu} = (\mu,\mu')$ are as in (2.a) but $\mu \neq \mu'$.

These are essentially the Arthur parameters [Art89] of the representations in Th. 2.1.

Let p be a prime which is inert in E, that is, $E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a field. Write $\mathcal{A}_{E_p} = \mathcal{L}_{E_p} \times SL(2,\mathbb{C})$ with $\mathcal{L}_{E_p} := W_{E_p} \times SU(2)$, a variant of the Weil-Deligne group. By "restriction", we have the local parameters $\psi_p : \mathcal{A}_{E_p} \to GL(4,\mathbb{C})$ associated to ψ in the above list.

Proposition 4.1. *For each ψ_p, we can construct the following local A-indistinguishable class $\Pi_{\psi_p}(G)$ associated to it.*

(1) Stable cases.

(a) $\Pi_{\psi_{\eta,p}}(G) = \{\pi := \eta_{p,G}\}$.

(b) $\Pi_{\psi_{\Pi_E,\mu,p}}(G) = \{\pi := J^G_{P_2}(\mu_p \Pi_{E,p} |\det|^{1/2}_{E_p})\}$.

(2) Endoscopic cases.

(a) $\Pi_{\psi_{\underline{\mu},p}}(G) = \{\pi^{\pm} := J^G_{P_1}(\mu_p |\ |_{E_p} \otimes \tau_{\pm})\}$. *$\tau_{\pm}$ are the irreducible representations having the base change lift* $I_B^{GL(2,E_p)}(\mu_p \otimes \mu'_p)$.

(b) $\Pi_{\psi_{\Pi_E,\eta,p}}(G) = \{\pi_+ := J^G_{P_1}(\eta_p |\ |^{1/2}_{E_p} \otimes \tau), \pi_-\}$. *$\tau$ has the base change lift Π_{E_p} and π_- is supercuspidal.*

(c) $\Pi_{\psi_{\underline{\eta},p}}(G) = \{\pi^+ := J^G_{P_2}(I_B^{GL(2,E_p)}(\eta_p \otimes \eta'_p)|\det|^{1/2}_{E_p}), \pi^-\}$. *$\pi^-$ is supercuspidal.*

(d) $\Pi_{\psi_{\eta,\underline{\mu},p}}(G) = \{\pi^{\pm,+} := J^G_{P_1}(\eta_p |\ |^{1/2}_{E_p} \otimes \tau_{\pm}), \pi^{\pm,-}\}$. *$\tau_{\pm}$ are as in (2.a) and $\pi^{\pm,-}$ are both supercuspidal.*

Here we have discussed only the case where ψ_p is "elliptic". That is, its image is not contained in any proper parabolic subgroup of the L-group. But of course, we have a similar result for non-elliptic ψ_p. Also we calculated the corresponding results at the archimedean places.

5 Half of the multiplicity formula

Now we define the global A-packet (A-indistinguishable class) simply as the restricted tensor product

$$\Pi_\psi(G) := \bigotimes_v{}' \Pi_{\psi_v}(G) = \{\pi = \bigotimes_v{}' \pi_v \,|\, \pi_v = \pi_v^+ \text{ or } \pi_v^{+,+}, \forall' v\}.$$

Notice that at all but finite number of p, π_p^+ or $\pi_p^{+,+}$ is unramified so that the restricted tensor product makes sense. We define

$$\mathcal{S}_\psi(G) := [\mathrm{Aut}(\psi)/Z(GL(4,\mathbb{C}))]^\theta \simeq \begin{cases} \{1\} & \text{in (1.a,b)} \\ \mathbb{Z}/2\mathbb{Z} & \text{in (2.a,b,c)} \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{in (2.d).} \end{cases}$$

Also we define the *multiplicty pairing* $\langle\ ,\ \rangle : \Pi_{\psi_v}(G) \times \mathcal{S}_\psi(G) \to \mathbb{C}^\times$ so that π_v^+, $\pi_v^{+,+}$ correspond to the trivial character, π_v^-, $\pi_v^{+,-}$, $\pi_v^{-,+}$ and $\pi_v^{-,-}$ correspond to sgn, $\mathbf{1} \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \mathbf{1}$ and $\mathrm{sgn} \otimes \mathrm{sgn}$, respectively. Now we can state our main result.

Theorem 5.1. *For $\pi = \bigotimes_v{}' \pi_v \in \Pi_\psi(G)$, we have*

$$m(\pi) \geq \frac{1}{|\mathcal{S}_\psi(G)|} \sum_{s \in \mathcal{S}_\psi(G)} \epsilon_\psi(s) \prod_v \langle s, \pi_v \rangle.$$

Here

$$\epsilon_\psi = \begin{cases} \mathrm{sgn}_{\mathcal{S}_\psi(G)} & \text{in (2.b) with } \varepsilon(1/2, \Pi_E \times \eta^{-1}) = -1; \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Remark 5.2. *(1) The above inequality should be the equality, which is a special case of the multiplicity formula conjectured by Arthur [Art89, § 8]. This I hope to prove by considering the product L-function for $U_{E/F}(2) \times GL(2)_E$ in a near future.*
(2) This formula shows that sometimes A-indistinguishable automorphic representations are not L-indistinguishable. To compensate this problem, it is necessary to work with the theory of endoscopy as Arthur and Kottwitz suggested.
(3) Consider the example (2.b). In this case at $v = \infty$, and for certain ψ_∞, Π_{ψ_∞} consists of non-tempered cohomological representations $\pi_\infty^{+,\pm}$ and holomorphic and anti-holomorphic discrete series representations $\pi_\infty^{-,\pm}$. Thus

$$\mathrm{H}^i(\mathfrak{g}(\mathbb{C}), L_h; \pi_\infty \otimes \rho_\psi) \neq 0, \quad \text{if } \begin{cases} i = 3,\ 5 \text{ and } \pi_\infty \simeq \pi_\infty^{+,\pm}, \\ i = 4 \text{ and } \pi_\infty \simeq \pi_\infty^{-,\pm}. \end{cases}$$

Again to treat these we need the endoscopy. Notice that the weight of ρ_2 in the parameter is $1, -1$, which indicates the occurance of the cohomology at degree $5 = 4 + 1$, $3 = 4 - 1$.

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