CAP FORMS ON $U_{E/F}(4)$

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1. What are CAP forms ?

The term CAP in the title is a short hand for the phrase "Cuspidal automorphic representations which Associated to Parabolic subgroups". More precisely, let G be a connected reductive group defined over a number field F, and G^* be its quasisplit inner form. An irreducible automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$ is a *CAP form* if

(i) π is cuspidal;

(ii) There exists a residual discrete automorphic representation $\pi^* = \bigotimes_v \pi_v^*$ of $G^*(\mathbb{A})$ such that $\pi_v \simeq \pi_v^*$ at all but finite number of v.

Assuming the suitable generalization of Ramanujan's conjecture, the CAP forms should be exactly the non-tempered cusp forms. Thus such forms should be beautifully described in terms of Arthur's conjecture.

We write ${}^LG = \widehat{G} \rtimes_{\rho_G} W_F$ for the L-group of G where W_F is the Weil group of $\bar{F}/F$, $\bar{F}$ being an algebraic closure of F. For the purpose of stating the conjecture, we introduce the hypothetical Langlands group $\mathcal{L}_F$ of F. An A-parameter for G is a continuous homomorphism $\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \to {}^LG$ such that

(1) The following diagram commutes:

$$\begin{tikzcd}
\mathcal{L}_F \times SL(2,\mathbb{C}) \arrow[r, "\psi"] \arrow[d, "\mathrm{proj}"'] & {}^LG \arrow[d, "\mathrm{proj}"] \\
W_F \arrow[r, equal] & W_F
\end{tikzcd}$$

(2) $\psi|_{\mathcal{L}_F}$ is semisimple and $\psi(\mathcal{L}_F)$ is bounded;

(3) $\psi|_{SL(2,\mathbb{C})}$ is analytic.

We should also have some relevance condition (see Ex. 1.2 below). The set $\Psi(G)$ of $\widehat{G}$-conjugacy classes of A-parameters should parameterize the irreducible representations of $G(\mathbb{A})$ which appear in the L^2-automorphic spectrum of G.

The A-parameters ψ associated to CAP forms $\pi = \bigoplus_v \pi_v$ can be characterized as follows. The cuspidality of π implies, in particular, it appears in the discrete spectrum, so that ψ should be *elliptic*. That is, $\mathrm{Im}\psi$ is not contained in the L-group of any proper F-parabolic subgroup of G. We write $\Psi_0(G)$ for the set of elliptic elements in $\Psi(G)$. On the other hand the condition "associated to parabolics" is equivalent to the non-triviality of $\psi|_{SL(2,\mathbb{C})}$. $\Psi_{\mathrm{CAP}}(G)$ denotes the subset of $\psi \in \Psi(G)$ satisfying these two conditions. Let us look at some examples.

Date: 21 august, 2001.

2000 *Mathematics Subject Classification.* Primary 11F70; Secondary 11F85, 22E50.

Talk at the conference "Representations of p-adic Groups and Automorphic Forms", Takayama. The author is partially supported by the Grants-in-Aid for Scientific Research No. 12740018, the Ministry of Education, Science, Sports and Culture, Japan.

Example 1.1. *Consider $G = GL(n)$. ${}^LG$ is the direct product of $\widehat{G} = GL(n,\mathbb{C})$ and W_F, and an A-parameter is an n-dimensional representation of $\mathcal{L}_F \times SL(2,\mathbb{C})$. $\psi \in \Psi_0(G)$ is equivalent to its irreducibility. Thus each $\psi \in \Psi(G)$ is of the form*

$$\psi = \varphi \otimes \rho_d,$$

where φ is an m-dimensional irreducible representation of $\mathcal{L}_F$, ρ_d is the d-dimensional irreducible representation of $SL(2,\mathbb{C})$ and $n = dm$. φ should corresponds to an irreducible cuspidal representation π of $GL(m,\mathbb{A})$. Then it is a result of Moeglin-Waldspurger that the only automorphic representation attached to this is the global Langlands quotient

$$\Pi_\psi = \bigotimes_v J_{P_{m^d}}^{GL(n)}(\pi_v|\det|_v^{(d-1)/2} \otimes \pi_v|\det|_v^{(d-3)/2} \otimes \cdots \otimes \pi_v|\det|_v^{(1-d)/2}),$$

which appears in the residual spectrum. Here P_{m^d} denotes the standard parabolic subgroup of $GL(n)$ with a Levi component isomorphic to $GL(m)^d$. Thus there are no CAP forms on $GL(n)$.

Example 1.2. *Next look at the multiplicative group $G = D^\times$ of a central division algebra D of dimension n^2 over F. $\psi \in \Psi_{\mathrm{CAP}}(G)$ is of the same form as in the previous example. But there should be a relevance condition on ψ so that only the parameters of the form*

$$\psi = \chi \otimes \rho_n, \quad \dim\chi = 1$$

is relevant. $\chi : \mathcal{L}_F \to W_F \to \mathbb{C}^\times$ is identified with an idele class character of F. Certainly the corresponding representation $\Pi_\psi = \chi \circ \nu_{D/F}$ is a CAP form, where $\nu_{D/F}$ is the reduced norm of D.

Example 1.3. *Finally we describe $\Psi_{\mathrm{CAP}}(G)$ for $G = G_n := U(n)_{E/F}$, the quasisplit unitary group in n-variables attached to a quadratic extension E of F. We write σ for the generator of $\mathrm{Gal}(E/F)$, and define θ_n by*

$$\theta_n(g) := \mathrm{Ad}\left(\begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}\right){}^tg^{-1}.$$

We realize G_n so that

$$G_n(R) := \{g \in GL(n, R\otimes_F E) \,|\, \theta_n({}^\sigma g) = g\}$$

for any commutative F-algebra R. ${}^LG_n = GL(n,\mathbb{C}) \rtimes_{\rho_{G_n}} W_F$, where

$$\rho_{G_n}(w) = \begin{cases} \mathrm{id} & \text{if } w \in W_E; \\ \theta_n & \text{otherwise.} \end{cases}$$

ψ is known to be determined by its restriction to $\mathcal{L}_E \times SL(2,\mathbb{C})$. Now each $\psi \in \Psi_{\mathrm{CAP}}(G)$ is of the form

$$\psi|_{\mathcal{L}_E\times SL(2,\mathbb{C})} = \bigoplus_{i=1}^r (\omega_i\varphi_i) \otimes \rho_{d_i}.$$

Here,

- *φ_i is an irreducible m_i-dimensional representation of $\mathcal{L}_E$, which corresponds to an irreducible cuspidal representation π_i of $GL(m_i, \mathbb{A}_E)$ such that ${}^\sigma\pi \simeq \pi^\vee$ and $\omega_\pi|_{\mathbb{A}^\times}$, the central character of π restricted to $\mathbb{A}^\times$, is trivial.*
- *ω_i is an idele class character of E whose restriction to $\mathbb{A}^\times$ equals $\omega_{E/F}^{n-d_i-m_i+1}$. $\omega_{E/F}$ is the quadratic character of $\mathbb{A}^\times/F^\times$ associated to E/F by the classfield theory.*
- *$n = \sum_{i=1}^r d_i m_i$, and the $\omega_i\varphi_i \otimes \rho_{d_i}$ are not equivalent to each other.*

What are the CAP forms associated to the A-parameters in Ex. 1.3 ? The cases $n \le 2$ is trivial. When $n = 3$, it was proven by Gelbart-Rogawski that such representations are the theta liftings from $G_1 = U(1)_{E/F}$. Today we shall report the result of the joint work with Kazuko Konno in the case $n = 4$. There are 6 types of parameters for G_4 according to the following list of $\{(d_i, m_i)\}_i$:

(1) Stable parameters: (1.a) $\{(4,1)\}$, (1.b) $\{(2,2)\}$.
(2) Endoscopic parameters: (2.a) $\{(3,1),(1,1)\}$, (2.b) $\{(2,1),(1,2)\}$, (2.c) $\{(2,1),(2,1)\}$, (2.d) $\{(2,1),(1,1),(1,1)\}$.

In this talk, we shall concentrate to the interesting cases (2.b) and (2.c).

2. Local theory

Let us write η and η' for characters of $\mathbb{A}_E^\times/E^\times$ whose restriction to $\mathbb{A}^\times$ is trivial. Then the parameters to be considered are given by

$$(2.b)\ \psi_{\eta,\pi}|_{\mathcal{L}_E \times SL(2,\mathbb{C})} = (\eta \otimes \rho_2) \oplus \varphi_\pi, \quad (2.c)\ \psi_{\underline{\eta}}|_{\mathcal{L}_E \times SL(2,\mathbb{C})} = (\eta \otimes \rho_2) \oplus (\eta' \otimes \rho_2),$$

where $\underline{\eta} = (\eta, \eta')$ and π is an irreducible cuspidal representation of $GL(2, \mathbb{A}_E)$ satisfying the condition of Ex. 1.3. At each place v of F, these give rise to the local parameters $\psi_v : \mathcal{L}_{F_v} \times SL(2,\mathbb{C}) \to {}^L G_v$. In this section, we shall construct the local A-packet $\Pi_{\psi_v}(G)$ attached to such ψ_v. Again for brevity, we consider only those v where $E_v := E \otimes_F F_v$ is a quadratic extension of F_v and π_v is (super) cuspidal in (2.b) and $\eta_v \ne \eta'_v$ in (2.c). Also to keep in accordance with the theme of this conference, we restrict ourselves to non-archimedean v. (The archimedean case is more explicit and needs some case-by-case treatment.)

2.1. **Parabolic induction.** Recall some assertion from Arthur's conjecture. For $\psi_v \in \Psi_0(G_v)$ we set $\mathcal{S}_{\psi_v}(G) := \mathrm{Cent}(\psi_v, \widehat{G})/Z(\widehat{G})^{\Gamma_v}$, where $\Gamma_v = \mathrm{Gal}(\bar{F}_v/F_v)$. In the case of $G = G_n$, we have $\mathcal{S}_{\psi_v}(G) \simeq (\mathbb{Z}/2\mathbb{Z})^{r-1}$ where r is as in Ex. 1.3. Moreover since F is non-archimedean, we may postulate that there exists a perfect duality

$$\langle\ ,\ \rangle : \mathcal{S}_{\psi_v}(G) \times \Pi_{\psi_v}(G) \longrightarrow \mathbb{C}^\times.$$

Next set

$$\mu_\psi : \mathbb{G}_m(\mathbb{C}) \ni t \longmapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in SL(2,\mathbb{C}) \xrightarrow{\psi} \widehat{G}.$$

There is an F-parabolic subgroup $P_\psi = M_\psi U_\psi$ such that $\widehat{M_\psi} = \mathrm{Cent}(\mu_\psi, \widehat{G})$ and μ_ψ is a $\widehat{P}_\psi$-dominant cocharacter of $\widehat{M}_\psi$. Associated to $\psi_v|_{\mathcal{L}_{F_v}} \in \Psi(M_\psi)$ is the tempered L-packet $\Pi_{\psi_v|\mathcal{L}_{F_v}}(M_\psi)$. Then Arthur imposed

$$\Pi'_{\psi_v}(G) := \{ J^G_{P_\psi}(\tau \otimes e^{\mu_\psi}) \,|\, \tau \in \Pi_{\psi_v|\mathcal{L}_{F_v}}(M_\psi) \} \subset \Pi_{\psi_v}(G).$$

Here, again $J^G_P(\tau \otimes e^\lambda)$ denotes the Langlands quotient of $I^G_P(\tau \otimes e^\lambda) = \mathrm{ind}^{G(F_v)}_{P(F_v)}[(\tau \otimes e^\lambda) \otimes \mathbf{1}_{U(F_v)}]$.

We now examine these constructions in our G_4-case. The standard parabolic subgroups of G_4 are G, $P_i = M_iU_i$ ($i = 1,\ 2$) and the upper triangular Borel subgroup $\mathbf{B} = \mathbf{TU}$, where

$$M_1 = \left\{ m_1(a,g) = \left(\begin{array}{c|c|c} a & & \\ \hline & g & \\ \hline & & \theta_1({}^\sigma a) \end{array}\right) \;\middle|\; \begin{array}{c} a \in \mathrm{Res}_{E/F}\mathbb{G}_m \\ g \in G_2 \end{array} \right\},$$

$$U_1 = \left\{ u_1(y,\beta) = \left(\begin{array}{c|cc|c} 1 & y'' & y' & \beta - \langle y, y\rangle/2 \\ \hline & 1 & & -{}^\sigma y' \\ & & 1 & {}^\sigma y'' \\ \hline & & & 1 \end{array}\right) \;\middle|\; \begin{array}{c} y = (y'', y') \in W_1 \\ \beta \in \mathbb{G}_a \end{array} \right\},$$

$$M_2 = \left\{ m_2(a) = \left(\begin{array}{c|c} a & \\ \hline & \theta_2({}^\sigma a) \end{array}\right) \;\middle|\; a \in H_2 \right\},$$

$$U_2 = \left\{ u_2(b) = \left(\begin{array}{c|c} \mathbf{1}_2 & b \\ \hline & \mathbf{1}_2 \end{array}\right) \;\middle|\; \begin{array}{c} b \in (\mathrm{Res}_{E/F}\mathbb{M}_2) \\ {}^t({}^\sigma b) = -b \end{array} \right\}.$$

Here $(W_1, \langle\ ,\ \rangle)$ is the hyperbolic skew-Hermitian space $(E^2, \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right))$ over E.

Proposition 2.1. *The group $\mathcal{S}_{\psi_v}(G)$ and $\Pi'_{\psi_v}(G)$ for parameters (2.b), (2.c) are given as follows.*

A-parameter	$\mathcal{S}_{\psi_v}(G)$	$\Pi'_{\psi}(G)$
(2.b) ψ_{π_v,η_v}	$\mathbb{Z}/2\mathbb{Z}$	$\{J^G_{P_1}(\eta_v\lvert\ \rvert^{1/2}_{E_v} \otimes \tau_v) \mid \tau_v \in T_v\}$
(2.c) $\psi_{\underline{\eta}_v}$	$\mathbb{Z}/2\mathbb{Z}$	$\{J^G_{P_2}(I(\eta_v \otimes \eta'_v)\lvert \det \rvert^{1/2}_{E_v})\}$

T_v in (2.b) is the L-packet of $G_2(F_v)$ whose standard base change to $GL(2, E_v)$ is π_v. $I(\eta_v \otimes \eta'_v)$ in (2.c) is the obvious notation for the principal series representation of $GL(2, E_v)$.

2.2. **Local theta correspondence.** We still need to construct the members of $\Pi_{\psi_v}(G) \setminus \Pi'_{\psi_v}(G)$. For this we use the theta correspondence. We write $G_2^+ := G_2$ and G_2^- for its anisotropic inner form. Fix a non-trivial additive character ψ_{F_v} of F_v. For a character η_v of $E_v^\times$ having the trivial restriction to $F_v^\times$, we can construct the Weil representation $(\omega_{\eta_v,\pm}, S)$ of the unitary dual pair $G_2^\pm(F_v) \times G_{2n}(F_v)$. Again for simplicity, we assume that the residual characteristic of F_v is odd. Then for an irreducible admissible representation π of $G_{2n}(F_v)$ (resp. τ of $G_2^\pm(F_v)$), we have its (possibly zero) local Howe correspondent $\theta^\pm_{2n,\eta_v}(\pi)$ (resp. $\theta^{2n}_{\eta_v}(\tau)$), an irreducible representation of $G_2^\pm(F_v)$ (resp. $G_{2n}(F_v)$). Then we can prove the following.

Proposition 2.2 (local θ-correspondence for $U(2)$)**.** *Consider the case $n = 1$.*
(i) $\theta^\epsilon_{2,\eta_v}(\pi)$ does not vanish if and only if $\epsilon(1/2, \pi \times \eta_v, \psi_{F_v})\omega_\pi(-1)\lambda(E_v/F_v, \psi_{F_v})^2 = \epsilon$.
(ii) If this is the case, we have

$$\theta^\epsilon_{2,\eta_v}(\pi) = \begin{cases} \pi^\vee & \text{if } \epsilon = 1, \\ \mathrm{JL}(\pi^\vee) & \text{otherwise.} \end{cases}$$

Here $\pi^\vee$ is the contragredient of π and $\mathrm{JL}(\pi^\vee)$ is the "Jacquet-Langlands correspondent" of $\pi^\vee$.

Remark 2.3. *This is the ϵ-*dichotomy *property which is proved by Harris-Kudla-Sweet for general unitary dual pairs. But their result uses the ϵ-factor defined by the doubling* *method of Piatetskii-Shapiro-Rallis. The comparison conjecture between their ϵ-factor and that defined by the Langlands-Shahidi method is not yet established in the present case. This is the reason why we rely on the construction of M. Harris using the Shimizu-Jacquet-Langlands correspondence.*

In general, we rarely have a good understanding of the ramified local theory of theta correspondences. Except for few examples such as Waldspurger's study on the Shimura correspondence, all such examples put some strong restriction on the representations treated (e.g. trivial in the example of Siegel-Weil formula, and one-dimensional in the study of Shalika-Tanaka).

As was remarked above, we deduce the proposition from the similar result for $GU(2)$ obtained by M. Harris. His method deduces the ramified local theory from the Shimizu-Jacquet-Langlands correspondence combined with the result of Waldspurger-Tunnel-Saito-Prasad on the restriction of a representation of $GL(2, F_v)$ to elliptic Cartan subgroups. Our task is to deduce the unitary group case from the similitude group setting. This is achieved with the help of the following lemma. Let us identify the unitary similitude group $\widetilde{G}_2 = GU(2)$ with $(\mathrm{Res}_{E/F}\mathbb{G}_m \times GL(2))/\mathbb{G}_m$ by

$$(\mathrm{Res}_{E/F}\mathbb{G}_m \times GL(2))/\mathbb{G}_m \ni (z, g) \longmapsto z^{-1}g \in \widetilde{G}_2 \subset \mathrm{Res}_{E/F}GL(2).$$

Thus each irreducible representation $\widetilde{\pi}$ of $\widetilde{G}_2(F_v)$ is of the form $\omega\otimes\pi$, where ω is a character of $E_v^\times$ and π is an irreducible representation of $GL(2, F_v)$ such that $(\omega|_{F_v^\times})\omega_\pi = \mathbf{1}$.

Lemma 2.4. *We have $\widetilde{\pi}|_{G_2(F_v)} = \bigoplus_{\tau\in T} \tau$, where T is the unique L-packet of $G_2(F_v)$ whose standard base change lift is $\omega(\det)\pi_{E_v}$. π_{E_v} denotes the base change of π to $GL(2, E_v)$.*

We are now ready to complete the A-packets Π_{ψ_v}. Consider the following diagram of local theta correspondences.

$$\begin{array}{ccc} G_2^{-\epsilon}(F_v) & \xrightarrow{\theta^4_{\eta_v}} & G_4(F_v) \\ & \theta^4_{\eta_v} & \\ \mathrm{JL} \updownarrow & \nearrow & | \\ G_2^{\epsilon}(F_v) & \underset{\theta^{\epsilon}_{2,\eta_v}}{\longleftarrow} & G_2(F_v) \end{array}$$

The left vertical arrow is the "Jacquet-Langlands correspondence", while the right vertical line indicates the Witt tower. For $\tau_v \in T_v$ in Prop. 2.1, write

$$\epsilon := \epsilon(1/2, \pi \times \eta_v, \psi_{F_v})\omega_\pi(-1)\lambda(E_v/F_v, \psi_{F_v})^2$$

so that $\theta^\epsilon_{2,\eta_v}(\tau_v) \neq 0$. The tower property of theta correspondence implies

$$\theta^4_{\eta_v} \circ \theta^\epsilon_{2,\eta_v}(\tau_v) \simeq \pi^+_{\psi,v} := J^G_{P_1}(\eta_v|\ |^{1/2}_{E_v} \otimes \tau_v).$$

On the other hand, we know from Prop. 2.2 that $\theta^2_{\eta_v} \circ \mathrm{JL} \circ \theta^\epsilon_{2,\eta_v}(\tau_v)$ is zero. This again combined with the tower property says that

$$\pi^-_{\psi,v} := \theta^4_{\eta_v} \circ \mathrm{JL} \circ \theta^\epsilon_{2,\eta_v}(\tau_v)$$

is a cuspidal representation of $G_4(F_v)$. We argue similarly in the case (2.c) and obtain the following.

Theorem 2.5. $\Pi_{\psi_v}(G) = \{\pi^\pm_{\psi,v}\}$, *where* $\pi^\pm_{\psi,v}$ *are defined above in the case (2.b) and*

$$\pi^+_{\psi,v} = J^G_{P_2}(I(\eta_v \otimes \eta'_v)|\det|^{1/2}_{E_v}), \quad \pi^-_{\psi,v} := \theta^4_{\eta_v}((\eta_v {\eta'_v}^{-1})_{G_2^-}),$$

in the case (2.c). Here $(\eta_v {\eta'_v}^{-1})_{G_2^-} : G_2^-(F_v) \xrightarrow{\det} G_1(F_v) \ni x\sigma(x)^{-1} \mapsto (\eta_v {\eta'_v}^{-1})(x) \in \mathbb{C}^\times$.

Remark 2.6. *(i) Similar results hold for other types of* π_v *in the case (2.b). In particular, when* π_v *is the special representation,* $\pi^-_{\psi,v}$ *coincides with that in the case (2.c).*
(ii) The archimedean case is treated in a similar way. In the case (2.b) with T_v *is endoscopic,* $|T_v| = 2$ *while its Jacquet-Langlands correspondent is a singleton. But since* $\epsilon(1/2, \tau_v \times \eta_v, \psi_\mathbb{R})\omega_{\tau_v}(-1)\lambda(\mathbb{C}/\mathbb{R}, \psi_\mathbb{R})^2 = 1$ *in this case, the theorem is still valid.*

3. Global theory

We now define the duality (multiplicity pairing) $\langle\ ,\ \rangle : \mathcal{S}_{\psi_v}(G) \times \Pi_{\psi_v}(G) \to \{\pm 1\}$ so that $\pi^-_{\psi,v}$ corresponds to the sign character of $\mathcal{S}_{\psi_v}(G)$. The following verifies the multiplicity formula for the global A-packet $\Pi_\psi(G) = \bigoplus_v \Pi_{\psi_v}(G)$ conjectured by Arthur.

Theorem 3.1. *If we write* $m(\pi)$ *for the multiplicity of an irreducible representation* $\pi = \bigoplus_v \pi_v \in \Pi_\psi(G)$ *of* $G_4(\mathbb{A})$ *in the* L^2*-automorphic spectrum, then*

$$m(\pi) = \frac{1}{|\mathcal{S}_\psi(G)|} \sum_{\bar{s} \in \mathcal{S}_\psi(G)} \epsilon_\psi(\bar{s}) \prod_v \langle \bar{s}, \pi_v \rangle.$$

Here

$$\epsilon_\psi(s) = \begin{cases} \mathrm{sgn}_{\mathcal{S}_\psi(G)} & \textit{in the case (2.b) with } \epsilon(1/2, \pi \times \eta) = -1, \\ \mathbf{1} & \textit{otherwise.} \end{cases}$$

We show the inequality $\geq$ by the global theta correspondence. Also in some cases, this gives the exact equality. But in other cases including (2.c), we need the analysis of the Fourier coefficients to have the converse inequality.

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