LOCAL θ-CORRESPONDENCE FOR REAL UNITARY DUAL PAIRS

KAZUKO KONNO AND TAKUYA KONNO

ABSTRACT. This note summarizes the results of our paper [KK]. Motivated by applications to automorphic forms, we consider Weil representations of a unitary dual pair over $\mathbb{R}$ (not of its "determinant" covering) constructed by the doubling argument [HKS96]. We construct its Fock model and deduce the $\mathbf{K}$-type correspondence under this. As a consequence, we obtain the local θ-correspondence (variant of the Howe duality correspondence under this Weil representation) between limit of discrete series representations for unitary dual pairs of the same size. It is described in terms of the sign of the functional equation for certain automorphic L-factors. This can be viewed as an archimedean analogue of the ε-dichotomy property of the local θ-correspondence of unitary dual pairs over non-archimedean local fields [HKS96].

CONTENTS

1. HOWE DUALITY FOR UNITARY DUAL PAIRS

Let $(\mathbb{W}, \langle\langle \cdot, \cdot \rangle\rangle)$ be a $2N$-dimensional symplectic space over $\mathbb{R}$. We write $Sp(\mathbb{W})$ for its symplectic group. For each non-trivial character $\psi : \mathbb{R} \to \mathbb{C}^\times$, the *metaplectic group* of $(\mathbb{W}, \psi(\langle\langle \cdot, \cdot \rangle\rangle))$ is an extension:

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp_\psi(\mathbb{W}) \xrightarrow{p_{\mathbb{W}}} Sp(\mathbb{W}) \longrightarrow 1. \tag{1.1}$$

Its definition is given as follows. The *Heisenberg group* of $(\mathbb{W}, \psi(\langle\langle \cdot, \cdot \rangle\rangle))$ is given by $\mathcal{H}_\psi(\mathbb{W}) = \mathbb{W} \times \mathbb{C}^1$ with the multiplication law

$$(w; z)(w'; z') := (w + w'; zz'\psi\Big(\frac{\langle\langle w, w' \rangle\rangle}{2}\Big)), \quad w, w' \in \mathbb{W},\ z, z' \in \mathbb{C}^1.$$

The Stone-von-Neumann theorem asserts that there exists a unique isomorphism class $\rho_\psi^{\mathbb{W}} = \rho_\psi$ of irreducible unitary representations of $\mathcal{H}_\psi(\mathbb{W})$ on which the center $\mathbb{C}^1$ acts by multiplication. $Mp(\mathbb{W})$ is the unique extension (1.1) such that ρ_ψ extends to a unitary representation (also

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denoted by ρ_ψ) of the semidirect product $\mathcal{J}_\psi(\mathbb{W}) := \mathbb{W} \rtimes Mp_\psi(\mathbb{W})$ (the metaplectic *Jacobi group*). Here, the $Mp_\psi(\mathbb{W})$-action on $\mathcal{H}(\mathbb{W})$ is the composite of the $Sp(\mathbb{W})$-action on $\mathbb{W}$ with $p_{\mathbb{W}}$. Note that $\mathcal{J}_\psi(\mathbb{W})$ fits into the extension

$$1 \longrightarrow \mathcal{H}_\psi(\mathbb{W}) \longrightarrow \mathcal{J}_\psi(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1.$$

The restriction $\omega_{\mathbb{W},\psi} = \omega_{\mathbb{W}}$ of ρ_ψ to $Mp_\psi(\mathbb{W})$ is called the *Weil representation*.

For a direct sum decomposition $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$ of symplectic spaces, we write $i_{\mathbb{W}_1,\mathbb{W}_2} : Sp(\mathbb{W}_1) \times Sp(\mathbb{W}_2) \to Sp(\mathbb{W})$ for the associated (diagonal) embedding. This lifts to a homomorphism $\tilde{i}_{\mathbb{W}_1,\mathbb{W}_2} : Mp_\psi(\mathbb{W}_1) \times Mp_\psi(\mathbb{W}_2) \to Mp_\psi(\mathbb{W})$:

$$\begin{array}{ccc} Mp_\psi(\mathbb{W}_1) \times Mp_\psi(\mathbb{W}_2) & \xrightarrow{\tilde{i}_{\mathbb{W}_1,\mathbb{W}_2}} & Mp_\psi(\mathbb{W}) \\ {\scriptstyle p_{\mathbb{W}_1} \times p_{\mathbb{W}_2}}\big\downarrow & & \big\downarrow{\scriptstyle p_{\mathbb{W}}} \\ Sp(\mathbb{W}_1) \times Sp(\mathbb{W}_2) & \xrightarrow{i_{\mathbb{W}_1,\mathbb{W}_2}} & Sp(\mathbb{W}) \end{array}$$

and we have

$$\omega_{\mathbb{W}} \circ \tilde{i}_{\mathbb{W}_1,\mathbb{W}_2} \simeq \omega_{\mathbb{W}_1} \otimes \omega_{\mathbb{W}_2}. \tag{1.2}$$

If we write $-\mathbb{W}$ for the symplectic space $(\mathbb{W}, -\langle\langle \cdot, \cdot \rangle\rangle)$, then $\omega_{-\mathbb{W},\psi} = \omega_{\mathbb{W},\bar{\psi}}$ is isomorphic to the contragredient representation $\omega_{\mathbb{W},\psi}^\vee$ of $\omega_{\mathbb{W},\psi}$.

Let $(V, (\cdot, \cdot))$ and $(W, \langle \cdot, \cdot \rangle)$ be hermitian and skew-hermitian spaces over $\mathbb{C}$. Fix a square root i of -1. We can choose a basis $\boldsymbol{v}$, $\boldsymbol{w}$ of V, W, with respect to which we have

$$(v, v') = v^* I_{p,q} v', \quad \langle w, w' \rangle = iw I_{p',q'} w'^*.$$

Here, writing $\mathbf{1}_m$ for the $m \times m$ identity matrix,

$$I_{p,q} := \begin{pmatrix} \mathbf{1}_p & \\ & -\mathbf{1}_q \end{pmatrix}.$$

By abuse of terminology, we call (p', q') the *signature* of the skew-hermitian space $(W, \langle \cdot, \cdot \rangle)$. We write $n := p + q$, $n' := p' + q'$. The $\mathbb{R}$-vector space $\mathbb{W} := V \otimes_{\mathbb{C}} W$ with

$$\langle\langle v \otimes w, v' \otimes w' \rangle\rangle := \Re\Big((v, v')\overline{\langle w, w' \rangle}\Big)$$

is a symplectic space of dimension $2(N := nn')$ over $\mathbb{R}$. Here, $\Re z$ denotes the real part of $z \in \mathbb{C}$. We write $\boldsymbol{G}_V$, $\boldsymbol{G}_W$ for the unitary groups of $(V, (\cdot, \cdot))$, $(W, \langle \cdot, \cdot \rangle)$, respectively. We have a homomorphism

$$\iota_{V,W} = \iota_W \times \iota_V : \boldsymbol{G}_V \times \boldsymbol{G}_W \ni (g, g') \longmapsto g \otimes g' \in Sp(\mathbb{W}),$$

so that $\boldsymbol{G}_V$, $\boldsymbol{G}_W$ form a reductive dual pair in $Sp(\mathbb{W})$.

If we take $\psi(x) = e^{iax}$ with $a > 0$, the inverse images of $\iota_W(\boldsymbol{G}_V)$, $\iota_V(\boldsymbol{G}_W)$ under $p_{\mathbb{W}} : Mp_\psi(\mathbb{W}) \twoheadrightarrow Sp(\mathbb{W})$ are isomorphic to

$$\widetilde{\boldsymbol{G}}_V = \{(g, z) \in \boldsymbol{G}_V \times \mathbb{C}^1 \,|\, z^2 = \det g^{q'-p'}\},$$
$$\widetilde{\boldsymbol{G}}_W = \{(g, z) \in \boldsymbol{G}_W \times \mathbb{C}^1 \,|\, z^2 = \det g^{p-q}\}.$$

We write $\mathscr{R}(\widetilde{\boldsymbol{G}}_V, \omega_{\mathbb{W}})$ for the set of isomorphism classes of irreducible Harish-Chandra modules of $\widetilde{\boldsymbol{G}}_V$ which appear as quotients of $\omega_{\mathbb{W}}|_{\widetilde{\boldsymbol{G}}_V}$. The *Howe duality correspondence* asserts that the relation $\mathrm{Hom}_{\widetilde{\boldsymbol{G}}_V\times\widetilde{\boldsymbol{G}}_W}(\omega_{\mathbb{W}},\pi_V\otimes\pi_W)\neq 0$ determines an well-defined bijection [How89]:

$$\mathscr{R}(\widetilde{\boldsymbol{G}}_V,\omega_{\mathbb{W}})\ni \begin{array}{ccc}\pi_V & \longmapsto & \theta(\pi_V,W)\\ \theta(\pi_W,V) & \longleftarrow & \pi_W\end{array}\in\mathscr{R}(\widetilde{\boldsymbol{G}}_W,\omega_{\mathbb{W}}).$$

Explicit description of this correspondence in the cases $n=n'$, $n'+1$ are obtained in [Pau98], [Pau00], respectively.

2. Doubling construction

The Weil representation is the base of θ-correspondence in the theory of automorphic forms. In order to formulate the (local) θ-correspondence of the unitary dual pair $(\boldsymbol{G}_V,\boldsymbol{G}_W)$, we need a Weil representation not of $\widetilde{\boldsymbol{G}}_V\times\widetilde{\boldsymbol{G}}_W$ but of $\boldsymbol{G}_V\times\boldsymbol{G}_W$. This is achieved by M. Harris's doubling argument. Let us briefly review the construction from [HKS96].

Writing $(V^-,(\cdot,\cdot)^-):=(V,-(\cdot,\cdot))$, we introduce a hyperbolic hermitian space $(V^{\mathbb{H}},(\cdot,\cdot)^{\mathbb{H}}):=(V,(\cdot,\cdot))\oplus(V^-,(\cdot,\cdot)^-)$. ${}_{\triangle}V:=\{(v,v)\in V^{\mathbb{H}}\,|\,v\in V\}$, ${}_{\triangledown}V:=\{(v,-v)\in V^{\mathbb{H}}\,|\,v\in V\}$ are maximal isotropic subspaces of $V^{\mathbb{H}}$ dual to each other. We adopt similar notation for $(W,\langle\cdot,\cdot\rangle)$. These doublings yield the same doubled symplectic space $\mathbb{W}^{\mathbb{H}}:=V^{\mathbb{H}}\otimes_{\mathbb{C}}W=V\otimes_{\mathbb{C}}W^{\mathbb{H}}$ and the same polarization

$$\mathbb{W}^{\mathbb{H}}={}_{\triangledown}\mathbb{W}\oplus{}_{\triangle}\mathbb{W},\quad {}_{\diamondsuit}\mathbb{W}={}_{\diamondsuit}V\otimes_{\mathbb{C}}W=V\otimes_{\mathbb{C}}{}_{\diamondsuit}W,\ (\diamondsuit=\nabla,\Delta).$$

Both $\iota_V^{\mathbb{H}}:\boldsymbol{G}_{V^{\mathbb{H}}}\times\boldsymbol{G}_W\to Sp(\mathbb{W}^{\mathbb{H}})$ and $\iota_W^{\mathbb{H}}:\boldsymbol{G}_V\times\boldsymbol{G}_{W^{\mathbb{H}}}\to Sp(\mathbb{W}^{\mathbb{H}})$ define reductive dual pairs. Once such a polarization is fixed, we have an explicit description $Mp_\psi(\mathbb{W}^{\mathbb{H}})=Sp(\mathbb{W}^{\mathbb{H}})\times\mathbb{C}^1$ where the multiplication law is given by [RR93, Th.4.1]:

$$(g_1,z_1)(g_2,z_2)=(g_1g_2,c_{{}_{\triangle}\mathbb{W}}(g_1,g_2)),\quad g_i\in Sp(\mathbb{W}^{\mathbb{H}}),\ z_i\in\mathbb{C}^1.$$

The 2-cocycle $c_{{}_{\triangle}\mathbb{W}}(g_1,g_2)$ is the Weil constant $\gamma_\psi(L(g_1,g_2))$ of the Leray invariant of $L(g_1,g_2)=L({}_{\triangle}\mathbb{W},{}_{\triangle}\mathbb{W}.g_2^{-1},{}_{\triangle}\mathbb{W}.g_1)$.

First consider $\boldsymbol{G}_{W^{\mathbb{H}}}$. We write $\underline{\boldsymbol{w}}^-$ for $\underline{\boldsymbol{w}}$ viewed as a basis of W^-. We choose a Witt basis for the decomposition $W={}_{\triangledown}W\oplus{}_{\triangle}W$ to be

$$\underline{\mathbf{w}}':=\frac{\underline{\boldsymbol{w}}-\underline{\boldsymbol{w}}^-}{\sqrt{2}},\quad \underline{\mathbf{w}}\ =iI_{p',q'}\frac{\underline{\boldsymbol{w}}+\underline{\boldsymbol{w}}^-}{\sqrt{2}}.$$

Using this, the Siegel parabolic subgroup $P_{{}_{\triangle}W}:=\mathrm{Stab}({}_{\triangle}W,\boldsymbol{G}_{W^{\mathbb{H}}})=M_{{}_{\triangle}W}U_{{}_{\triangle}W}$ is given by

$$M_{{}_{\triangle}W}=\left\{m_{{}_{\triangle}W}(a):=\begin{pmatrix}a & \mathbf{0}_{n'}\\ \mathbf{0}_{n'} & a^{*,-1}\end{pmatrix}\,\middle|\,a\in GL(n',\mathbb{C})\right\},$$

$$U_{{}_{\triangle}W}=\left\{u_{{}_{\triangle}W}(b):=\begin{pmatrix}\mathbf{1}_{n'} & b\\ \mathbf{0}_{n'} & \mathbf{1}_{n'}\end{pmatrix}\,\middle|\,b=b^*\in\mathbb{M}_{n'}(\mathbb{C})\right\}.$$

We have the Bruhat decomposition $\boldsymbol{G}_{W^{\mathbb{H}}}=\coprod_{r=1}^{n'}P_{{}_{\triangle}W}\cdot w_r\cdot P_{{}_{\triangle}W}$ with

$$w_r:=\left(\begin{array}{cc|cc}\mathbf{0}_r & & -\mathbf{1}_r & \\ & \mathbf{1}_{n'-r} & & \\ \hline \mathbf{1}_r & & \mathbf{0}_r & \\ & & & \mathbf{1}_{n'-r}\end{array}\right).$$

For

$$g=\begin{pmatrix}a_1 & b_1\\ & a_1^{*,-1}\end{pmatrix}w_r\begin{pmatrix}a_2 & b_2\\ & a_2^{*,-1}\end{pmatrix}\in\boldsymbol{G}_{W^{\mathbb{H}}},$$

we set $d(g) := \det(a_1 a_2) \in \mathbb{C}^\times/\mathbb{R}_+^\times$, $r(g) := r = n' - \dim_{\mathbb{C}} {}_\Delta W.g \cap {}_\Delta W$. For any character ξ' of $\mathbb{C}^\times$ satisfying $\xi'|_{\mathbb{R}^\times} = \mathrm{sgn}^n$,

$$\beta^{\mathbb{H}}_{V,\xi'}(g) := \left(\gamma_\psi(1)^{2n}(-1)^q\right)^{-r(g)}\xi'(d(g)), \quad g \in \boldsymbol{G}_{W^{\mathbb{H}}},$$

splits the 2-cocycle $c_V(g_1, g_2) := c_{\Delta\mathbb{W}}(\iota^{\mathbb{H}}_V(g_1), \iota^{\mathbb{H}}_V(g_2))$, $(g_1, g_2 \in \boldsymbol{G}_{W^{\mathbb{H}}})$ [Kud94, Th.3.1]. That is,

$$\tilde{\iota}^{\mathbb{H}}_{V,\xi'} : \boldsymbol{G}_{W^{\mathbb{H}}} \ni g \longmapsto (g, \beta^{\mathbb{H}}_{V,\xi'}(g)) \in Mp_\psi(\mathbb{W}^{\mathbb{H}})$$

is an analytic homomorphism.

As for $\boldsymbol{G}_{V^{\mathbb{H}}}$, we choose a Witt basis of $V^{\mathbb{H}} = {}_\nabla V \oplus {}_\Delta V$ to be

$$\underline{\mathbf{v}}' := \frac{\boldsymbol{v} - \boldsymbol{v}^-}{\sqrt{2}}, \quad \underline{\mathbf{v}} := -i\frac{\boldsymbol{v} + \boldsymbol{v}^-}{\sqrt{2}} I_{p,q}.$$

We adopt similar definitions as in the W-side with respect to the Siegel parabolic subgroup $P_{\Delta V} := \mathrm{Stab}({}_\Delta V, \boldsymbol{G}_{V^{\mathbb{H}}}) = M_{\Delta V} U_{\Delta V}$. Note that this is realized as

$$M_{\Delta V} = \left\{ m_{\Delta V}(a) := \begin{pmatrix} a & \mathbf{0}_n \\ \mathbf{0}_n & a^{*,-1} \end{pmatrix} \,\middle|\, a \in GL(n, \mathbb{C}) \right\},$$
$$U_{\Delta V} = \left\{ u_{\Delta V}(b) := \begin{pmatrix} \mathbf{1}_n & \mathbf{0}_n \\ b & \mathbf{1}_n \end{pmatrix} \,\middle|\, b = b^* \in \mathbb{M}_n(\mathbb{C}) \right\}.$$

with respect to $\underline{\mathbf{v}}' \cup \underline{\mathbf{v}}$. Taking a character ξ of $\mathbb{C}^\times$ satisfying $\xi|_{\mathbb{R}^\times} = \mathrm{sgn}^{n'}$,

$$\beta^{\mathbb{H}}_{W,\xi}(g) := \left(\gamma_\psi(1)^{2n'}(-1)^{p'}\right)^{-r(g)}\xi(d(g)), \quad g \in \boldsymbol{G}_{V^{\mathbb{H}}}$$

splits $c_W(g_1, g_2) := c_{\Delta\mathbb{W}}(\iota^{\mathbb{H}}_W(g_1), \iota^{\mathbb{H}}_W(g_2))$, $(g_1, g_2 \in \boldsymbol{G}_{V^{\mathbb{H}}})$. Hence an analytic homomorphism

$$\tilde{\iota}^{\mathbb{H}}_{W,\xi} : \boldsymbol{G}_{V^{\mathbb{H}}} \ni g \longmapsto (g, \beta^{\mathbb{H}}_{W,\xi}(g)) \in Mp_\psi(\mathbb{W}^{\mathbb{H}})$$

is obtained.

Noting $\boldsymbol{G}_{V^-} = \boldsymbol{G}_V$, we have the (diagonal) embedding $i_V : \boldsymbol{G}_V \times \boldsymbol{G}_V \hookrightarrow \boldsymbol{G}_{V^{\mathbb{H}}}$ and the following commutative diagram:

$$\begin{array}{ccccc} \boldsymbol{G}_{V^{\mathbb{H}}} & \xrightarrow{\iota^{\mathbb{H}}_W} & Sp(\mathbb{W}^{\mathbb{H}}) & \xleftarrow{\iota^{\mathbb{H}}_V} & \boldsymbol{G}_{W^{\mathbb{H}}} \\ {\scriptstyle i_V}\uparrow & & \uparrow{\scriptstyle i_{\mathbb{W}}} & & \uparrow{\scriptstyle i_W} \\ \boldsymbol{G}_V \times \boldsymbol{G}_V & \xrightarrow{\iota_W \times \iota_W} & Sp(\mathbb{W}) \times Sp(\mathbb{W}) & \xleftarrow{\iota_V \times \iota_V} & \boldsymbol{G}_W \times \boldsymbol{G}_W \end{array}$$

Fix a pair $\underline{\xi} = (\xi, \xi')$ of characters of $\mathbb{C}^\times$ such that $\xi|_{\mathbb{R}^\times} = \mathrm{sgn}^{n'}$, $\xi'|_{\mathbb{R}^\times} = \mathrm{sgn}^n$. We define

$$\tilde{\iota}_{W,\xi} : \boldsymbol{G}_V \xrightarrow{\text{1st.}} \boldsymbol{G}_V \times \boldsymbol{G}_V \xrightarrow{i_V} \boldsymbol{G}_{V^{\mathbb{H}}} \xrightarrow{\tilde{\iota}^{\mathbb{H}}_{W,\xi}} Mp_\psi(\mathbb{W}^{\mathbb{H}}),$$
$$\tilde{\iota}_{V,\xi'} : \boldsymbol{G}_W \xrightarrow{\text{1st.}} \boldsymbol{G}_W \times \boldsymbol{G}_W \xrightarrow{i_W} \boldsymbol{G}_{W^{\mathbb{H}}} \xrightarrow{\tilde{\iota}^{\mathbb{H}}_{V,\xi'}} Mp_\psi(\mathbb{W}^{\mathbb{H}}),$$

where the left arrows are the embeddings to the first components. These yield homomorphisms

$$\tilde{\iota}_{W,\xi} : \boldsymbol{G}_V \longrightarrow Mp_\psi(\mathbb{W}), \quad \tilde{\iota}_{V,\xi'} : \boldsymbol{G}_W \longrightarrow Mp_\psi(\mathbb{W}).$$

These definition show that the following diagram commutes:

$$\begin{array}{ccccc}
\boldsymbol{G}_{V^{\mathbb{H}}} & \xrightarrow{\tilde{\iota}^{\mathbb{H}}_{W,\xi}} & Mp_{\psi}(\mathbb{W}^{\mathbb{H}}) & \xleftarrow{\tilde{\iota}^{\mathbb{H}}_{V,\xi'}} & \boldsymbol{G}_{W^{\mathbb{H}}} \\
{\scriptstyle i_V}\uparrow & & \uparrow{\scriptstyle \tilde{i}_{\mathbb{W}}} & & \uparrow{\scriptstyle i_W} \\
\boldsymbol{G}_V \times \boldsymbol{G}_V & \xrightarrow{\tilde{\iota}_{W,\xi} \times \xi(\det)^{-1}\tilde{\iota}_{W,\xi}} & Mp_{\psi}(\mathbb{W}) \times Mp_{\psi}(\mathbb{W}) & \xleftarrow{\tilde{\iota}_{V,\xi'} \times \xi'(\det)^{-1}\tilde{\iota}_{V,\xi'}} & \boldsymbol{G}_W \times \boldsymbol{G}_W.
\end{array}$$

Here, $\tilde{i}_{\mathbb{W}} : Mp_{\psi}(\mathbb{W}) \times Mp_{\psi}(\mathbb{W}) \ni ((g, z), (g', z')) \mapsto (i_{\mathbb{W}}(g, g'), z\bar{z}') \in Mp_{\psi}(\mathbb{W}^{\mathbb{H}})$.

Now we can define the Weil representation $\omega_{V,W,\underline{\xi}} = \omega_{W,\xi} \times \omega_{V,\xi'}$ of $\boldsymbol{G}_V \times \boldsymbol{G}_W$:

$$\omega_{W,\xi} := \omega_{\mathbb{W}} \circ \tilde{\iota}_{W,\xi}, \quad \omega_{V,\xi'} := \omega_{\mathbb{W}} \circ \tilde{\iota}_{V,\xi'}.$$

Also we have the Weil representations $\omega^{\mathbb{H}}_{W,\xi} := \omega_{\mathbb{W}^{\mathbb{H}}} \circ \tilde{\iota}^{\mathbb{H}}_{W,\xi}$, $\omega^{\mathbb{H}}_{V,\xi'} := \omega_{\mathbb{W}^{\mathbb{H}}} \circ \tilde{\iota}^{\mathbb{H}}_{V,\xi'}$ of $\boldsymbol{G}_{V^{\mathbb{H}}}$, $\boldsymbol{G}_{W^{\mathbb{H}}}$, respectively. The above diagram combined with (1.2) show

$$\omega^{\mathbb{H}}_{W,\xi} \circ i_V \simeq \omega_{W,\xi} \otimes \xi(\det)\omega^{\vee}_{W,\xi}, \quad \omega^{\mathbb{H}}_{V,\xi'} \circ i_W \simeq \omega_{V,\xi'} \otimes \xi'(\det)\omega^{\vee}_{V,\xi'}. \tag{2.1}$$

The explicit formula for the Schrödinger model $\mathcal{S}(_{\nabla}\mathbb{W})$ for $\omega^{\mathbb{H}}_{W,\xi}$, $\omega^{\mathbb{H}}_{V,\xi'}$ is given in [Kud94, § 5].

For completeness, we adopt the following convention. If $V = 0$ (resp. $W = 0$), we set $\boldsymbol{G}_V := \{1\}$ (resp. $\boldsymbol{G}_W = \{1\}$) and $\omega_{V,\xi'} = \xi'_u(\det)$ (resp. $\omega_{W,\xi} = \xi_u(\det)$). Here, for a character ξ of $\mathbb{C}^{\times}/\mathbb{R}^{\times}$, we write $\xi_u : U(1,\mathbb{R}) \ni z/\bar{z} \mapsto \xi(z) \in \mathbb{C}^{\times}$.

3. Doubling construction of the Fock model

We now study the Harish-Chandra module of $\omega_{V,W,\underline{\xi}}$. Notice that $(\underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{w}} \cup \underline{\boldsymbol{v}}^- \otimes \underline{\boldsymbol{w}}) \cup (i\underline{\boldsymbol{v}} I_{p,q} \otimes I_{p',q'}\underline{\boldsymbol{w}} \cup -i\underline{\boldsymbol{v}}^- I_{p,q} \otimes I_{p',q'}\underline{\boldsymbol{w}})$ is a Witt basis of $\mathbb{W}^{\mathbb{H}}$. We choose another Witt basis

(3.1)
$$\begin{pmatrix} {}^{\Re}\underline{e}' \\ {}^{\Im}\underline{e}' \\ {}^{\Re}\underline{e} \\ {}^{\Im}\underline{e} \end{pmatrix} := \begin{pmatrix} \underline{\mathbf{v}}' \otimes \underline{\boldsymbol{w}} \\ i\underline{\mathbf{v}}' \otimes \underline{\boldsymbol{w}} \\ -\underline{\mathbf{v}} \otimes I_{p',q'}\underline{\boldsymbol{w}} \\ -i\underline{\mathbf{v}} \otimes I_{p',q'}\underline{\boldsymbol{w}} \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} \mathbf{1}_N & -\mathbf{1}_N & \mathbf{0}_N & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{0}_N & \boldsymbol{I} & \boldsymbol{I} \\ \hline \mathbf{0}_N & \mathbf{0}_N & \mathbf{1}_N & -\mathbf{1}_N \\ -\boldsymbol{I} & -\boldsymbol{I} & \mathbf{0}_N & \mathbf{0}_N \end{array}\right) \begin{pmatrix} \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{w}} \\ \underline{\boldsymbol{v}}^- \otimes \underline{\boldsymbol{w}} \\ i\underline{\boldsymbol{v}} I_{p,q} \otimes I_{p',q'}\underline{\boldsymbol{w}} \\ -i\underline{\boldsymbol{v}}^- I_{p,q} \otimes I_{p',q'}\underline{\boldsymbol{w}} \end{pmatrix}$$
$$= \frac{1}{\sqrt{2}} \left(\begin{array}{cc|cc} \mathbf{1}_N & -\mathbf{1}_N & \mathbf{0}_N & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{0}_N & \boldsymbol{I} & \boldsymbol{I} \\ \hline \mathbf{0}_N & \mathbf{0}_N & \mathbf{1}_N & -\mathbf{1}_N \\ -\boldsymbol{I} & -\boldsymbol{I} & \mathbf{0}_N & \mathbf{0}_N \end{array}\right) \begin{pmatrix} \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{w}} \\ \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{w}}^- \\ i\underline{\boldsymbol{v}} I_{p,q} \otimes I_{p',q'}\underline{\boldsymbol{w}} \\ -i\underline{\boldsymbol{v}} I_{p,q} \otimes I_{p',q'}\underline{\boldsymbol{w}}^- \end{pmatrix} = \begin{pmatrix} \underline{\boldsymbol{v}} \otimes \underline{\mathbf{w}}' \\ i\underline{\boldsymbol{v}} \otimes \underline{\mathbf{w}}' \\ \underline{\boldsymbol{v}} I_{p,q} \otimes \underline{\mathbf{w}} \\ i\underline{\boldsymbol{v}} I_{p,q} \otimes \underline{\mathbf{w}} \end{pmatrix}$$

for $\mathbb{W}^{\mathbb{H}} = {}_{\nabla}\mathbb{W} \oplus {}_{\Delta}\mathbb{W}$, where $\boldsymbol{I} := I_{p,q} \otimes I_{p',q'}$. Notice that $\underline{\boldsymbol{v}}^- \otimes \underline{\boldsymbol{w}}$ and $\underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{w}}^-$ are identical in $\mathbb{W}^{\mathbb{H}}$. Let $\mathfrak{sp}(\mathbb{W}^{\mathbb{H}}) = \mathfrak{k}_{\mathbb{W}^{\mathbb{H}}} \oplus \mathfrak{p}_{\mathbb{W}^{\mathbb{H}}}$ be the Cartan decomposition of the Lie algebra of $Sp(\mathbb{W}^{\mathbb{H}})$ given by

$$\mathfrak{k}_{\mathbb{W}^{\mathbb{H}}} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle|\ \begin{array}{c} A = -{}^tA, \\ B = {}^tB \end{array} \in \mathbb{M}_{2N}(\mathbb{R}) \right\},$$
$$\mathfrak{p}_{\mathbb{W}} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \middle|\ \begin{array}{c} A = {}^tA \\ B = {}^tB \end{array} \in \mathbb{M}_{2N}(\mathbb{R}) \right\}$$

in the realization with respect to the basis (3.1). We choose the Cartan decompositions $\mathfrak{g}_V = \mathfrak{k}_V \oplus \mathfrak{p}_V$, $\mathfrak{g}_{V^{\mathbb{H}}} = \mathfrak{k}_{V^{\mathbb{H}}} \oplus \mathfrak{p}_{V^{\mathbb{H}}}$ for the Lie algebras of $\boldsymbol{G}_V$, $\boldsymbol{G}_{V^{\mathbb{H}}}$ to be the inverse image of the above decomposition under ι_W, $\iota_{W^{\mathbb{H}}}$, respectively. We also use the similar Cartan decompositions for the W-side. Since the basis transformation matrix in (3.1) is unitary, this coincides with the usual decomposition

$$\mathfrak{k}_V := \left\{ \begin{pmatrix} A & \\ & D \end{pmatrix} \in \mathfrak{g}_V \right\}, \quad \mathfrak{p}_V := \left\{ \begin{pmatrix} \mathbf{0}_p & B \\ B^* & \mathbf{0}_q \end{pmatrix} \in \mathfrak{g}_V \right\}$$

in the realization with respect to $\underline{\boldsymbol{v}}$. We write $\mathbf{K}_V \subset \boldsymbol{G}_V$, $\mathbf{K}_{\mathbb{W}} \subset Sp(\mathbb{W})$ for the maximal compact subgroups having the Lie algebras $\mathfrak{k}_V$, $\mathfrak{k}_{\mathbb{W}}$, respectively.

We write $\psi(x) = e^{d\psi x}$ with $d\psi \in i\mathbb{R}^\times$ and ε_ψ for the sign of $d\psi/i$. To obtain the Harish-Chandra modules of $\omega^{\mathbb{H}}_{W,\xi}$, $\omega^{\mathbb{H}}_{V,\xi'}$, we take the following totally complex polarization $\mathbb{W}^{\mathbb{H}}_{\mathbb{C}} = \mathbb{L}' \oplus \mathbb{L}$ of the complexification of $\mathbb{W}^{\mathbb{H}}$:

$$\begin{aligned} &\mathbb{L}' := \mathrm{span}_{\mathbb{C}}\,({}^{\Re}\underline{\mathbf{e}}' \cup {}^{\Im}\underline{\mathbf{e}}'), \quad \mathbb{L} := \mathrm{span}_{\mathbb{C}}\,{}^{\Re}\underline{\mathbf{e}} \cup {}^{\Im}\underline{\mathbf{e}}, \\ &{}^{\diamondsuit}\underline{\mathbf{e}}' := \frac{{}^{\diamondsuit}\underline{e}' - \varepsilon_\psi i {}^{\diamondsuit}\underline{e}}{\sqrt{2}}, \quad {}^{\diamondsuit}\underline{\mathbf{e}} := \frac{{}^{\diamondsuit}\underline{e} - \varepsilon_\psi i {}^{\diamondsuit}\underline{e}'}{\sqrt{2}}, \quad (\diamondsuit = \Re \text{ or } \Im). \end{aligned} \tag{3.2}$$

The universal enveloping algebra of the complexified Lie algebra $\mathfrak{h}_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ of $\mathcal{H}_\psi(\mathbb{W}^{\mathbb{H}})$ is the quantum algebra

$$\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = T(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})/(w \otimes w' - w' \otimes w - d\psi\langle\langle w, w'\rangle\rangle^{\mathbb{H}} \,|\, w,\, w' \in \mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$$

of $(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}, d\psi\langle\langle \cdot, \cdot\rangle\rangle^{\mathbb{H}})$. Here $T(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ is the tensor algebra of $\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}$. Thanks to the Poincaré-Birkoff-Witt theorem, we have the decomposition $\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = S(\mathbb{L}) \oplus \Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})\mathbb{L}'$. $S(\mathbb{L})$ stands for the symmetric algebra of $\mathbb{L}$.

$\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ carries a filtration $\Omega^0_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = \mathbb{C} \subset \Omega^1_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) \subset \cdots \subset \Omega^n_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) \subset \dots$ induced from the grading $T(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}) = \bigoplus_{n\in\mathbb{N}} T^n(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$. One can easily check that $\Omega^2_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ is isomorphic to the complexified Lie algebra $\mathfrak{j}_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ of $\mathcal{J}_\psi(\mathbb{W}^{\mathbb{H}})$. We define the representation $(r_\psi = r_\psi^{\mathbb{W}^{\mathbb{H}}}, S(\mathbb{L}))$ of $j_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ by

$$r_\psi(X)P = X.P, \quad X \in j(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}),\ P \in S(\mathbb{L}) = \Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})/\Omega_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})\mathbb{L}'.$$

It is known that this yields the $(\mathfrak{sp}(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}}), \mathbf{K}_{\mathbb{W}^{\mathbb{H}}})$-module of the Weil representation $\omega_{\mathbb{W}}$.

We identify ${}^{\Re}\underline{\mathbf{e}} = \{{}^{\Re}\mathbf{e}_{j,k}\}$, ${}^{\Im}\underline{\mathbf{e}} = \{{}^{\Im}\mathbf{e}_{j,k}\}$ with variables $\{u_{j,k}\}$, $\{\epsilon_j\epsilon'_k v_{j,k}\}$, respectively. Then $S(\mathbb{L}^{\mathbb{H}}) = \mathbb{C}[(u_{j,k}), (v_{j,k})]$ (polynomial ring over $\mathbb{M}_{n,n'}(\mathbb{C})^2$) on which $\mathfrak{h}_\psi(\mathbb{W}^{\mathbb{H}}_{\mathbb{C}})$ acts by

$$\begin{aligned} r_\psi({}^{\Re}\mathbf{e}_{j,k}) &= u_{j,k}, \quad r_\psi({}^{\Im}\mathbf{e}_{j,k}) = \epsilon_j\epsilon'_k v_{j,k}, \\ r_\psi({}^{\Re}\mathbf{e}'_{j,k}) &= d\psi\frac{\partial}{\partial u_{j,k}}, \quad r_\psi({}^{\Im}\mathbf{e}'_{j,k}) = \epsilon_j\epsilon'_k d\psi\frac{\partial}{\partial v_{j,k}}. \end{aligned}$$

Here ϵ_j (resp. ϵ'_j) denotes the (j,j)-entry of $I_{p,q}$ (resp. $I_{p',q'}$). Set $w_{j,k} := \sqrt{2}^{-1}(u_{j,k} + iv_{j,k})$, $\bar{w}_{j,k} := \sqrt{2}^{-1}(u_{j,k} - iv_{j,k})$, $(1 \le j \le n,\, 1 \le k \le n')$. Now we can state our first result.

Theorem 3.1. *The decomposition* (2.1) *of the* $(\mathfrak{g}_{V,\mathbb{C}} \oplus \mathfrak{g}_{W,\mathbb{C}}, \mathbf{K}_V \times \mathbf{K}_W)$*-module* $S(\mathbb{L}^{\mathbb{H}})$ *is given by*

$$\begin{aligned} &\big((\omega^{\mathbb{H}}_{W,\xi} \circ i_V)\times(\omega^{\mathbb{H}}_{V,\xi'} \circ i_W), \mathbb{C}[(w_{j,k}), (\bar{w}_{j,k})]\big) \\ &\qquad = \begin{cases} (\omega_{V,W,\underline{\xi}}, \mathbb{C}[(w_{j,k})]) \otimes (\xi(\det)\xi'(\det)\omega^\vee_{V,W,\underline{\xi}}, \mathbb{C}[(\bar{w}_{j,k})]) & \text{if } \varepsilon_\psi > 0, \\ (\omega_{V,W,\underline{\xi}}, \mathbb{C}[(\bar{w}_{j,k})]) \otimes (\xi(\det)\xi'(\det)\omega^\vee_{V,W,\underline{\xi}}, \mathbb{C}[(w_{j,k})]) & \text{if } \varepsilon_\psi < 0. \end{cases} \end{aligned}$$

Thus the Fock model $\mathscr{P}_{V,W,\underline{\xi}}$ *for* $\omega_{V,W,\underline{\xi}}$ *is* $\mathbb{C}[(w_{j,k})]$ *if* $\varepsilon_\psi > 0$ *and* $\mathbb{C}[(\bar{w}_{j,k})]$ *otherwise.*

We also have the following explicit formulae for $(\omega_{V,W,\underline{\xi}}, \mathscr{P}_{V,W,\underline{\xi}})$. We use the basis

$$\begin{aligned} U_{j,k} &:= E_{j,k}, \quad (1 \le j,k \le p), \quad X_{j,k} := E_{p+j,k}, \quad (1 \le j \le q, 1 \le k \le p),\\ V_{j,k} &:= E_{p+j,p+k}, \quad (1 \le j,k \le q), \quad Y_{j,k} := E_{j,p+k}, \quad (1 \le j \le p, 1 \le k \le q),\\ U'_{j,k} &:= E_{j,k}, \quad (1 \le j,k \le p'), \quad X'_{j,k} := E_{j,p'+k}, \quad (1 \le j \le p', 1 \le k \le q'),\\ V'_{j,k} &:= E_{p'+j,p'+k}, \quad (1 \le j,k \le q'), \quad Y'_{j,k} := E_{p'+j,k}, \quad (1 \le j \le q', 1 \le k \le p') \end{aligned} \tag{3.3}$$

of $\mathfrak{g}_{V,\mathbb{C}}$, $\mathfrak{g}_{W,\mathbb{C}}$ realized with respect to $\underline{\boldsymbol{v}}$, $\underline{\boldsymbol{w}}$, respectively. Here $E_{j,k}$ denotes the (j,k)-elementary matrix. We write

$$\xi(z) = \left(\frac{z}{\bar{z}}\right)^{m/2}, \quad \xi'(z) = \left(\frac{z}{\bar{z}}\right)^{m'/2}, \quad m \equiv n',\ m' \equiv n \pmod 2, \in \mathbb{Z}.$$

Proposition 3.2. *(a) When $\varepsilon_\psi > 0$, we have the following explicit formulae for $(\omega_{V,W,\underline{\xi}}, \mathscr{P}_{V,W,\underline{\xi}})$.*

$$\begin{aligned}
\omega_{W,\xi}(U_{j,k}) =& \frac{m+q'-p'}{2}\delta_{j,k} - \sum_{\ell=1}^{p'} w_{k,\ell}\frac{\partial}{\partial w_{j,\ell}} + \sum_{\ell=p'+1}^{n'} w_{j,\ell}\frac{\partial}{\partial w_{k,\ell}},\\
\omega_{W,\xi}(V_{j,k}) =& \frac{m+p'-q'}{2}\delta_{j,k} + \sum_{\ell=1}^{p'} w_{p+j,\ell}\frac{\partial}{\partial w_{p+k,\ell}} - \sum_{\ell=p'+1}^{n'} w_{p+k,\ell}\frac{\partial}{\partial w_{p+j,\ell}},\\
\omega_{W,\xi}(X_{j,k}) =& -\frac{1}{|d\psi|}\sum_{\ell=1}^{p'} w_{p+j,\ell}w_{k,\ell} + |d\psi|\sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial w_{p+j,\ell}\partial w_{k,\ell}},\\
\omega_{W,\xi}(Y_{j,k}) =& |d\psi|\sum_{\ell=1}^{p'} \frac{\partial^2}{\partial w_{j,\ell}\partial w_{p+k,\ell}} - \frac{1}{|d\psi|}\sum_{\ell=p'+1}^{n'} w_{j,\ell}w_{p+k,\ell},\\
\omega_{V,\xi'}(U'_{j,k}) =& \frac{m'+p-q}{2}\delta_{j,k} + \sum_{\ell=1}^{p} w_{\ell,j}\frac{\partial}{\partial w_{\ell,k}} - \sum_{\ell=p+1}^{n} w_{\ell,k}\frac{\partial}{\partial w_{\ell,j}},\\
\omega_{V,\xi'}(V'_{j,k}) =& \frac{m'+q-p}{2}\delta_{j,k} - \sum_{\ell=1}^{p} w_{\ell,p'+k}\frac{\partial}{\partial w_{\ell,p'+j}} + \sum_{\ell=p+1}^{n} w_{\ell,p'+j}\frac{\partial}{\partial w_{\ell,p'+k}},\\
\omega_{V,\xi'}(X'_{j,k}) =& \frac{1}{|d\psi|}\sum_{\ell=1}^{p} w_{\ell,j}w_{\ell,p'+k} - |d\psi|\sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial w_{\ell,j}\partial w_{\ell,p'+k}},\\
\omega_{V,\xi'}(Y'_{j,k}) =& -|d\psi|\sum_{\ell=1}^{p} \frac{\partial^2}{\partial w_{\ell,p'+j}\partial w_{\ell,k}} + \frac{1}{|d\psi|}\sum_{\ell=p+1}^{n} w_{\ell,p'+j}w_{\ell,k}.
\end{aligned}$$

(iii) Similarly if $\varepsilon_\psi < 0$, we have

$$\begin{aligned}
\omega_{W,\xi}(U_{j,k}) =& \frac{m+p'-q}{2}\delta_{j,k} + \sum_{\ell=1}^{p'} \bar{w}_{j,\ell}\frac{\partial}{\partial \bar{w}_{k,\ell}} - \sum_{\ell=p'+1}^{n'} \bar{w}_{k,\ell}\frac{\partial}{\partial \bar{w}_{j,\ell}},\\
\omega_{W,\xi}(V_{j,k}) =& \frac{m+q'-p'}{2}\delta_{j,k} - \sum_{\ell=1}^{p'} \bar{w}_{p+k,\ell}\frac{\partial}{\partial \bar{w}_{p+j,\ell}} + \sum_{\ell=p'+1}^{n'} \bar{w}_{p+j,\ell}\frac{\partial}{\partial \bar{w}_{p+k,\ell}},
\end{aligned}$$

$$\omega_{W,\xi}(X_{j,k}) =|d\psi| \sum_{\ell=1}^{p'} \frac{\partial^2}{\partial \bar{w}_{p+j,\ell}\partial \bar{w}_{k,\ell}} - \frac{1}{|d\psi|}\sum_{\ell=p'+1}^{n'} \bar{w}_{p+j,\ell}\bar{w}_{k,\ell},$$

$$\omega_{W,\xi}(Y_{j,k}) =-\frac{1}{|d\psi|}\sum_{\ell=1}^{p'} \bar{w}_{j,\ell}\bar{w}_{p+k,\ell} + |d\psi|\sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial \bar{w}_{j,\ell}\partial \bar{w}_{p+k,\ell}},$$

$$\omega_{V,\xi'}(U'_{j,k}) =\frac{m'+q-p}{2}\delta_{j,k} - \sum_{\ell=1}^{p} \bar{w}_{\ell,k}\frac{\partial}{\partial \bar{w}_{\ell,j}} + \sum_{\ell=p+1}^{n} \bar{w}_{\ell,j}\frac{\partial}{\partial \bar{w}_{\ell,k}},$$

$$\omega_{V,\xi'}(V'_{j,k}) =\frac{m'+p-q}{2}\delta_{j,k} + \sum_{\ell=1}^{p} \bar{w}_{\ell,p'+j}\frac{\partial}{\partial \bar{w}_{\ell,p'+k}} - \sum_{\ell=p+1}^{n} \bar{w}_{\ell,p'+k}\frac{\partial}{\partial \bar{w}_{\ell,p'+j}},$$

$$\omega_{V,\xi'}(X'_{j,k}) =-|d\psi|\sum_{\ell=1}^{p} \frac{\partial^2}{\partial \bar{w}_{\ell,j}\partial \bar{w}_{\ell,p'+k}} + \frac{1}{|d\psi|}\sum_{\ell=p+1}^{n} \bar{w}_{\ell,j}\bar{w}_{\ell,p'+k},$$

$$\omega_{V,\xi'}(Y'_{j,k}) =\frac{1}{|d\psi|}\sum_{\ell=1}^{p} \bar{w}_{\ell,p'+j}\bar{w}_{\ell,k} - |d\psi|\sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial \bar{w}_{\ell,p'+j}\partial \bar{w}_{\ell,k}}.$$

Following the argument of [How89], one can deduce the local θ-correspondence under $\omega_{V,W,\underline{\xi}}$ from this. Let $\mathscr{R}(\boldsymbol{G}_V, \omega_{W,\xi})$ be the set of isomorphism classes of irreducible $(\mathfrak{g}_{V,\mathbb{C}}, \mathbf{K}_V)$-modules π_V such that $\mathrm{Hom}_{(\mathfrak{g}_{V,\mathbb{C}},\mathbf{K}_V)}(\mathscr{P}_{V,W,\underline{\xi}}, \pi_V) \neq 0$. For $\pi_V \in \mathscr{R}(\boldsymbol{G}_V, \omega_{W,\xi})$, let $\mathscr{N}(\pi_V)$ be the intersection of $\ker\phi$, $\phi \in \mathrm{Hom}_{(\mathfrak{g}_{V,\mathbb{C}},\mathbf{K}_V)}(\mathscr{P}_{V,W,\underline{\xi}}, \pi_V)$.

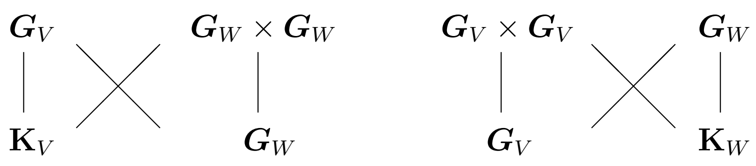
Theorem 3.3 (local θ-correspondence)**.** *Take* $\pi_V \in \mathscr{R}(\boldsymbol{G}_V, \omega_{W,\xi})$.
(i) The quotient $\mathscr{P}_{V,W,\underline{\xi}}/\mathscr{N}(\pi_V)$ *is of finite length, hence has an irreducible quotient.*
(ii) $\mathscr{P}_{V,W,\underline{\xi}}/\mathscr{N}(\pi_V)$ *admits a unique irreducible quotient* $\theta_{\underline{\xi}}(\pi_V, W)$.
(iii) This and the analogous construction in the W*-side give a bijection*

$$\mathscr{R}(\boldsymbol{G}_V, \omega_{W,\xi}) \ni \begin{array}{ccc} \pi_V & \longmapsto & \theta_{\underline{\xi}}(\pi_V, W) \\ \theta_{\underline{\xi}}(\pi_W, V) & \longleftarrow & \pi_W \end{array} \in \mathscr{R}(\boldsymbol{G}_W, \omega_{V,\xi'}).$$

4. $\mathbf{K}$-TYPE CORRESPONDENCE

The next problem is to compute the bijection in Th.3.3 explicitly. We first prepare some more notation.

We write $V_+ := \mathrm{span}_{\mathbb{C}}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_p\}$, $V_- := \mathrm{span}_{\mathbb{C}}\{\boldsymbol{v}_{p+1}, \dots, \boldsymbol{v}_n\}$ and $(\cdot,\cdot)_\pm$ for the restrictions of $(\cdot,\cdot)$ to $V_\pm$, respectively: $(V, (\cdot,\cdot)) = (V_+, (\cdot,\cdot)_+) \oplus (V_-, (\cdot,\cdot)_-)$. Then $\mathbf{K}_V = \boldsymbol{G}_{V_+} \times \boldsymbol{G}_{V_-}$. We adopt the similar notation for $\boldsymbol{G}_W$. Thus we have the seesaw dual pairs



in $Sp(\mathbb{W})$.

Take any decompositions $\xi = \xi_+ \cdot \xi_-$, $\xi' = \xi'_+ \cdot \xi'_-$ such that $\xi_\pm|_{\mathbb{R}^\times} = \mathrm{sgn}^{\dim W_\pm}$, $\xi'_\pm|_{\mathbb{R}^\times} = \mathrm{sgn}^{\dim V_\pm}$, respectively. The Weil representations $\omega_{V,W_+,(\xi_+,\xi')} \otimes \omega_{V,W_-,(\xi_-,\xi')}$ of $(\boldsymbol{G}_V \times \boldsymbol{G}_V) \times \mathbf{K}_W$ and $\omega_{V_+,W,(\xi,\xi'_+)} \otimes \omega_{V_-,W,(\xi,\xi'_-)}$ of $\mathbf{K}_V \times (\boldsymbol{G}_W \times \boldsymbol{G}_W)$ share the same Fock model $\mathscr{P}_{V,W,\underline{\xi}}$.

Using the basis (3.3), we take the Harish-Chandra decompositions $\mathfrak{g}_{V,\mathbb{C}} = \mathfrak{k}_{V,\mathbb{C}} \oplus \mathfrak{p}^+_{V,W_\pm} \oplus \mathfrak{p}^-_{V,W_\pm}$, $\mathfrak{g}_{W,\mathbb{C}} = \mathfrak{k}_{W,\mathbb{C}} \oplus \mathfrak{p}^+_{W,V_\pm} \oplus \mathfrak{p}^-_{W,V_\pm}$ as

$$\mathfrak{p}^{\varepsilon_\psi}_{V,W_+} = \mathfrak{p}^{-\varepsilon_\psi}_{V,W_-} = \mathrm{span}_\mathbb{C}\{X_{j,k}\}^{1\le k\le p}_{1\le j\le q}, \quad \mathfrak{p}^{-\varepsilon_\psi}_{V,W_+} = \mathfrak{p}^{\varepsilon_\psi}_{V,W_-} = \mathrm{span}_\mathbb{C}\{Y_{j,k}\}^{1\le k\le q}_{1\le j\le p},$$
$$\mathfrak{p}^{\varepsilon_\psi}_{W,V_+} = \mathfrak{p}^{-\varepsilon_\psi}_{W,V_-} = \mathrm{span}_\mathbb{C}\{X'_{j,k}\}^{1\le k\le q'}_{1\le j\le p'}, \quad \mathfrak{p}^{-\varepsilon_\psi}_{W,V_+} = \mathfrak{p}^{\varepsilon_\psi}_{W,V_-} = \mathrm{span}_\mathbb{C}\{Y'_{j,k}\}^{1\le k\le p'}_{1\le j\le q'}.$$

We define the spaces of $\mathbf{K}_W$ and $\mathbf{K}_V$-*harmonics* to be

$$\mathscr{H}_V(\mathbf{K}_W) := \{P \in \mathscr{P}_{V,W,\underline{\xi}} \mid \omega_{W_\pm,\xi_\pm}(\mathfrak{p}^-_{V,W_\pm})P = 0\},$$
$$\mathscr{H}_W(\mathbf{K}_V) := \{P \in \mathscr{P}_{V,W,\underline{\xi}} \mid \omega_{V_\pm,\xi'_\pm}(\mathfrak{p}^-_{W,V_\pm})P = 0\},$$

respectively. Their intersection $\mathscr{J}_{V,W,\underline{\xi}} := \mathscr{H}_V(\mathbf{K}_W) \cap \mathscr{H}_W(\mathbf{K}_V)$ is called the space of *joint harmonics*. Prop.3.2 applied to $(V, W_\pm)$, $(V_\pm, W)$ in place of (V, W) shows that $\mathscr{J}_{V,W,\underline{\xi}}$ consists of $P \in \mathscr{P}_{V,W,\underline{\xi}}$ killed by

$$\text{(4.1)} \quad \begin{aligned} &\sum_{\ell=1}^{p'} \frac{\partial^2}{\partial w_{j,\ell}\partial w_{p+k,\ell}}, \quad \sum_{\ell=p'+1}^{n'} \frac{\partial^2}{\partial w_{j,\ell}\partial w_{p+k,\ell}}, \quad (1 \le j \le p, 1 \le k \le q),\\ &\sum_{\ell=1}^{p} \frac{\partial^2}{\partial w_{\ell,j}\partial w_{\ell,p'+k}}, \quad \sum_{\ell=p+1}^{n} \frac{\partial^2}{\partial w_{\ell,j}\partial w_{\ell,p'+k}}, \quad (1 \le j \le p', 1 \le k \le q'),\end{aligned}$$

if $d\psi i < 0$. When $d\psi i > 0$, we have the same description with $w_{j,k}$ replaced by $\bar{w}_{j,k}$.

The following result is implicit in the proof of Th.3.3.

Proposition 4.1 (cf. [How89] §3)**.** *(1)* $\mathscr{J}_{V,W,\underline{\xi}}$ *is stable under* $\omega_{V,W,\underline{\xi}}(\mathbf{K}_V \times \mathbf{K}_W)$.
(2) We write $\mathscr{R}(\mathbf{K}_V, \mathscr{J}_{V,W,\underline{\xi}})$ *for the set of* $\mathbf{K}_V$*-types which appear as irreducible direct summands of* $\mathscr{J}_{V,W,\underline{\xi}}$*. Similarly we define* $\mathscr{R}(\mathbf{K}_W, \mathscr{J}_{V,W,\underline{\xi}})$ *in the* W*-side. Then* $\mathscr{J}_{V,W,\underline{\xi}}$ *is multiplicity free as a* $\mathbf{K}_V \times \mathbf{K}_W$*-module, so that it gives a bijection*

$$\mathscr{R}(\mathbf{K}_V, \mathscr{J}_{V,W,\underline{\xi}}) \ni \begin{matrix} \tau_V & \longmapsto & \theta_{\underline{\xi}}(\tau_V, \mathbf{K}_W) \\ \theta_{\underline{\xi}}(\tau_W, \mathbf{K}_V)) & \longleftarrow & \tau_W \end{matrix} \in \mathscr{R}(\mathbf{K}_W, \mathscr{J}_{V,W,\underline{\xi}}).$$

(3) For a $\mathbf{K}_V$*-type* τ_V*, we write* $\deg_{W,\xi}(\tau_V)$ *for the minimum degree of polynomials in the* τ_V*-isotypic subspace in* $\mathscr{P}_{V,W,\underline{\xi}}$*. (Set* $\deg_{W,\xi}(\tau_V) := \infty$ *if* τ_V *does not appear in* $\mathscr{P}_{V,W,\underline{\xi}}$*.)* $\mathbf{K}_V$*-type* τ_V *of* $\pi_V \in \mathscr{R}(\boldsymbol{G}_V, \omega_{W,\xi})$ *is of* minimal (W, ξ)-degree *if* $\deg_{W,\xi}(\tau_V)$ *is minimal among* $\det_{W,\xi}(\tau)$*, (*τ *runs over the set of* $\mathbf{K}_V$*-types in* π_V*). Similar definition applies to the* W*-side.*

(i) *Suppose* τ_V *is a* $\mathbf{K}_V$*-type in* $\pi_V \in \mathscr{R}(\boldsymbol{G}_V, \omega_{W,\xi})$ *of minimal* (W, ξ)*-degree. Then* $\tau_V \in \mathscr{R}(\mathbf{K}_V, \mathscr{J}_{V,W,\underline{\xi}})$*.*

(ii) *Furthermore,* $\theta_{\underline{\xi}}(\tau_V, \mathbf{K}_W)$ *is a* $\mathbf{K}_W$*-type of minimal* (V, ξ')*-degree in* $\theta_{\underline{\xi}}(\pi_V, W)$*.*

Similar assertion holds in the W*-side.*

Because of the assertion (3), the $\mathbf{K}$-type correspondence (2) plays an important role in the explicit description of the Howe correspondence [AB95], [Pau98], [Pau00]. Following the calculation of [KV78, III.6], we obtain the following.

We write $\mathfrak{b}_V = \mathfrak{t}_{V,\mathbb{C}} \oplus \mathfrak{n}_V$, $\bar{\mathfrak{b}}_V = \mathfrak{t}_{V,\mathbb{C}} \oplus \bar{\mathfrak{n}}_V$ for the upper and lower triangular Borel subalgebras of $\mathfrak{k}_{V,\mathbb{C}}$ in the realization with respect to $\underline{\boldsymbol{v}}$, respectively. Using similar notation for $\mathfrak{k}_{W,\mathbb{C}}$, we set

$$(\mathfrak{b}_{V,\psi}, \mathfrak{b}_{W,\psi}) := \begin{cases} (\bar{\mathfrak{b}}_V, \mathfrak{b}_W) & \text{if } \varepsilon_\psi = 1, \\ (\mathfrak{b}_V, \bar{\mathfrak{b}}_W) & \text{if } \varepsilon_\psi = -1. \end{cases}$$

We take a basis $\{e_1, \dots, e_p, \bar{e}_1, \dots, \bar{e}_q\}$ of $\mathfrak{t}^*_{V,\mathbb{C}}$ as

$$e_i(\mathrm{diag}(t_1, \dots, t_n)) := t_i, \quad \bar{e}_i(\mathrm{diag}(t_1, \dots, t_n)) := t_{p+i}$$

and identify $\mathfrak{t}^*_{V,\mathbb{C}}$ with $\mathbb{C}^n$ by this. We write $\{e'_1, \dots, e'_{p'}; \bar{e}'_1, \dots, \bar{e}'_{q'}\}$ for the analogous basis for $\mathfrak{t}^*_{W,\mathbb{C}}$, which gives an identification $\mathfrak{t}^*_{W,\mathbb{C}} \simeq \mathbb{C}^{n'}$.

Theorem 4.2 (K-type correspondence)**.** *(i) Write the $\mathfrak{b}_{V,\psi}$-highest weight of a $\mathbf{K}_V$-type τ_V as*

$$\begin{aligned}(4.2)\qquad &\Big(\frac{m}{2}, \dots, \frac{m}{2}; \frac{m}{2}, \dots, \frac{m}{2}\Big) + \varepsilon_\psi\Big(\frac{q'-p'}{2}, \dots, \frac{q'-p'}{2}; \frac{p'-q'}{2}, \dots, \frac{p'-q'}{2}\Big)\\ &+\varepsilon_\psi\big(-a_1, \dots, -a_r, 0, \dots, 0, b_s, \dots, b_1; -d_1, \dots, -d_u, 0, \dots, 0, c_t, \dots, c_1\big),\end{aligned}$$

for some $a_1 \geq \dots \geq a_r$, $b_1 \geq \dots \geq b_s$, $c_1 \geq \dots \geq c_t$, $d_1 \geq \dots \geq d_u \in \mathbb{Z}_{>0}$. Then τ_V belongs to $\mathscr{R}(\mathbf{K}_V, \mathscr{I}_{V,W,\underline{\xi}})$ if and only if $r + t \leq p'$, $s + u \leq q'$. In that case, the $\mathfrak{b}_{W,\psi}$-highest weight of $\theta_{\underline{\xi}}(\tau_V, \mathbf{K}_W)$ is given by

$$\begin{aligned}(4.3)\qquad &\Big(\frac{m'}{2}, \dots, \frac{m'}{2}; \frac{m'}{2}, \dots, \frac{m'}{2}\Big) + \varepsilon_\psi\Big(\frac{p-q}{2}, \dots, \frac{p-q}{2}; \frac{q-p}{2}, \dots, \frac{q-p}{2}\Big)\\ &+\varepsilon_\psi\big(a_1, \dots, a_r, 0, \dots, 0, -c_t, \dots, -c_1; d_1, \dots, d_u, 0, \dots, 0, -b_s, \dots, -b_1\big).\end{aligned}$$

(ii) Conversely, a $\mathbf{K}_W$-type τ_W with the $\mathfrak{b}_{W,\psi}$-highest weight (4.3) *belongs to $\mathscr{R}(\mathbf{K}_W, \mathscr{I}_{V,W,\underline{\xi}})$ if and only if $r + s \leq p$, $t + u \leq q$. In that case, $\theta_{\underline{\xi}}(\tau_W, \mathbf{K}_V)$ has the $\mathfrak{b}_{V,\psi}$-highest weight* (4.2).

The same argument as in the proof of [Pau98, Prop.1.4.10] shows:

Corollary 4.3. *Let $(V, (\cdot,\cdot))$ be an n-dimensional hermitian space and τ_V be a $\mathbf{K}_V$-type. Fix a pair $\underline{\xi} = (\xi, \xi')$ of characters of $\mathbb{C}^\times$ such that $\xi|_{\mathbb{R}^\times} = \xi'|_{\mathbb{R}^\times} = \mathrm{sgn}^n$. Then there exists a unique (up to isometry) n-dimensional skew-hermitian space $(W, \langle\cdot,\cdot\rangle)$ such that $\tau_V \in \mathscr{R}(\mathbf{K}_V, \mathscr{I}_{V,W,\underline{\xi}})$.*

5. Local θ-correspondence for limit of discrete series

As a consequence of our calculation, we compute the Howe correspondence between limit of discrete series representations of unitary dual pairs of the same size (cf. [Li90], [Pau98, 5.2]).

Consider the group $\boldsymbol{G}_V = U(p, q)$ realized with respect to the basis $\underline{\boldsymbol{v}}$. $\boldsymbol{T}_V \subset \boldsymbol{G}_V$ denotes the diagonal fundamental Cartan subgroup with the Lie algebra $\mathfrak{t}_V$. The isomorphism classes of irreducible limit of discrete series representations of $\boldsymbol{G}_V$ are classified as follows [Vog84, §2]. Up to $\mathbf{K}_V$-conjugation, an *elliptic limit character* of $\boldsymbol{G}_V$ is a triple $\gamma = (\Psi, \lambda, \Lambda)$ consisting of

- (LC1) Ψ is a positive system in $R(\mathfrak{g}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$, the root system of $\mathfrak{t}_{V,\mathbb{C}}$ in $\mathfrak{g}_{V,\mathbb{C}}$.
- (LC2) $\lambda \in \sum_{j=1}^p (n + 1 + 2\mathbb{Z})e_j \oplus \sum_{j=1}^q (n + 1 + 2\mathbb{Z})\bar{e}_j \subset \mathfrak{t}^*_{V,\mathbb{C}}$, satisfying the following conditions.
  - (a) $\alpha^\vee(\lambda) \geq 0$ for any $\alpha \in \Psi$.
  - (b) If a simple root α of Ψ satisfies $\alpha^\vee(\lambda) = 0$, it must be non-compact: $\alpha \in R(\mathfrak{p}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$.
- (LC3) Λ is a character of $\boldsymbol{T}_V$ such that $d\Lambda = \lambda + \rho(\Psi_{\mathrm{ncpt}}) - \rho(\Psi_{\mathrm{cpt}})$. Here $\Psi_{\mathrm{ncpt}} := \Psi \cap R(\mathfrak{p}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$, $\Psi_{\mathrm{cpt}} := \Psi \cap R(\mathfrak{k}_{V,\mathbb{C}}, \mathfrak{t}_{V,\mathbb{C}})$, and $\rho(\Sigma)$ denotes the half of the sum of roots in Σ.

For such $\gamma = (\Psi, \lambda, \Lambda)$, we have an irreducible limit of discrete series representation $\pi_V(\lambda, \Psi)$ of $\boldsymbol{G}_V$. This is characterized by its unique minimal $\mathbf{K}_V$-type with the Ψ_{cpt}-highest weight Λ and the infinitesimal character λ (or its Weyl group orbit) [KV95, Ch.11]. Two such representations

$\pi_V(\lambda,\Psi)$ and $\pi_V(\lambda',\Psi')$ associated to $\gamma=(\Psi,\lambda,\Lambda)$ and $\gamma'=(\Psi',\lambda',\Lambda')$ are isomorphic if and only if γ and γ' are $\mathbf{K}_V$-conjugate.

Now we consider a unitary dual pair $(\boldsymbol{G}_V,\boldsymbol{G}_W)$ with $n=n'$. Fix a character pair $\underline{\xi}=(\xi,\xi')$ as before:

$$\xi(z)=\left(\frac{z}{\bar z}\right)^{m/2},\quad \xi'(z)=\left(\frac{z}{\bar z}\right)^{m'/2},\quad m\equiv m'\equiv n \pmod 2).$$

For each odd algebraic character $\mu^a(z):=(z/\bar z)^{a/2}$, $(a\in 2\mathbb{Z}+1)$ of $\mathbb{C}^\times$, we introduce the sign

$$\varepsilon_\psi(a)=\varepsilon_\psi(\mu^a):=\varepsilon(1/2,\mu^a,\psi_{\mathbb{C}})\mu^a(-i)=\varepsilon_\psi\cdot\operatorname{sgn}(a).$$

Here $\varepsilon(s,\mu^a,\psi_{\mathbb{C}})$ denotes the Artin ε-factor for μ^a [Tat79, (3.2)] and $\psi_{\mathbb{C}}:=\psi\circ\operatorname{Tr}_{\mathbb{C}/\mathbb{R}}$.

Let $\pi_V(\lambda,\Psi)$ be a limit of discrete series representation of $\boldsymbol{G}_V$. Taking a suitable $\mathbf{K}_V$-conjugation, we may assume

(i) λ is of the form ($[a]_k$ denotes the k-tuple $(a,\dots,a)$.)

$$\lambda=\Big(\overbrace{\frac{m}{2},\dots,\frac{m}{2}}^{p};\overbrace{\frac{m}{2},\dots,\frac{m}{2}}^{q}\Big) - \frac{1}{2}\Big(\underbrace{[a_1]_{k_1},\dots,[a_r]_{k_r},[b_s]_{\ell_s},\dots,[b_1]_{\ell_1}}_{p};\underbrace{[a_1]_{\bar k_1},\dots,[a_r]_{\bar k_r},[b_s]_{\bar\ell_s},\dots,[b_1]_{\bar\ell_1}}_{q}\Big), \tag{5.1}$$

where $a_i,b_j\in 2\mathbb{Z}+1$ satisfy

$$\varepsilon_\psi(a_i)>0,\quad \varepsilon_\psi(b_j)<0, \tag{5.2}$$

$$|a_1|>|a_2|>\dots>|a_r|,\quad |b_1|>|b_2|>\dots>|b_s|. \tag{5.3}$$

(ii) $\Psi_{\text{cpt}}=R(\mathfrak{b}_{V,\psi},\mathfrak{t}_{V,\mathbb{C}})$.

Setting $k:=\sum_{i=1}^r k_i$, $\ell:=\sum_{i=1}^s \ell_i$, $\bar k:=\sum_{i=1}^r \bar k_i$, $\bar\ell:=\sum_{i=1}^r \bar\ell_i$, let $(W,\langle\cdot,\cdot\rangle)$ be the skew-hermitian space of signature $(p':=k+\bar\ell, q':=\bar k+\ell)$. Set

$$\lambda':=\Big(\overbrace{\frac{m'}{2},\dots,\frac{m'}{2}}^{p'};\overbrace{\frac{m'}{2},\dots,\frac{m'}{2}}^{q'}\Big) + \frac{1}{2}\Big(\underbrace{[a_1]_{k_1},\dots,[a_r]_{k_r},[b_s]_{\bar\ell_s},\dots,[b_1]_{\bar\ell_1}}_{p'};\underbrace{[a_1]_{\bar k_1},\dots,[a_r]_{\bar k_r},[b_s]_{\ell_s},\dots,[b_1]_{\ell_1}}_{q'}\Big), \tag{5.4}$$

$$\Psi'_{\text{cpt}}:=R(\mathfrak{b}_{W,\psi},\mathfrak{t}_{W,\mathbb{C}}), \tag{5.5}$$

$$e'_i-\bar e'_j\in\Psi'_{\text{ncpt}} \iff \begin{cases}\text{(i)} & 1\le i\le k,\ 1\le j\le \bar k \text{ and } e_i-\bar e_j\notin\Psi, \text{ or}\\ \text{(ii)} & k<i\le p',\ \bar k<j\le q' \text{ and } \bar e_{i-k+\bar k}-e_{j-\bar k+k}\notin\Psi, \text{ or}\\ \text{(iii)} & (e'_i-\bar e'_j)^\vee(\lambda')>0.\end{cases} \tag{5.6}$$

Using the characterization by the minimal $\mathbf{K}$-type and the infinitesimal character of limit of discrete series representations, we deduce the following from Th.4.2.

Theorem 5.1. *(1) A limit of discrete series representation $\pi_V(\lambda,\Psi)$ of $\boldsymbol{G}_V$ belongs to $\mathscr{R}(\boldsymbol{G}_V,\omega_{W,\xi})$ for an n-dimensional skew-hermitian space W if and only if W has the signature (p',q').*
(2) In that case, $\theta_{\underline{\xi}}(\pi_V(\lambda,\Psi),W)\simeq\pi_W(\lambda',\Psi')$.

Remark 5.2. *The signature (p', q') is determined by (p, q) and the signatures $\varepsilon_\psi(\xi^{-1}\omega_i)$, where $\underline{\omega} = (\omega_1, \dots, \omega_n)$ is the character of $\boldsymbol{T}_V$ with the differential λ. This is the archimedean analogue of the ε-*dichotomy property *of the local θ-correspondence of non-archimedean unitary dual pairs* [HKS96, Th.6.1].

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DEPT OF EDU., FUKUOKA UNIVERSITY OF EDUCATION, 1-1 BUNKYOMACHI AKAMA, MUNAKATA-CITY, FUKUOKA, 811-4192, JAPAN

E-mail address: kazukokonno@vesta.ocn.ne.jp

URL: http://www15.ocn.ne.jp/~tkonno/kkonno/Kazuko.html

GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, HAKOZAKI, HIGASHI-KU, FUKUOKA 812-8581, JAPAN

E-mail address: takuya@math.kyushu-u.ac.jp

URL: http://knmac.math.kyushu-u.ac.jp/~tkonno/