Degenerate Whittaker models for p-adic groups after C. Mœglin and J-L. Waldspurger *

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Abstract

In this note we present a detailed account for a work of Mœglin-Waldspurger on the dimension of the space of degenerate Whittaker vectors [MW]. We hope to extend this to the twisted case. This is in order to prove that certain twisted endoscopy for $GL(n)$ implies the (local) generic packet conjecture for many reductive groups.

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1 Degenerate Whittaker models

1.1 Some estimations

Let $F \supset \mathcal{O} \supset \mathfrak{p}_{\mathcal{F}} = (\varpi)$ be the usual notation for a non-archimedean local field of characteristic zero. Let p be the residual characteristic of F, and write $[F : \mathbb{Q}_p] = ef$ where e is the order of ramification and f is the modular degree. For $x \in \mathbb{Q}_p$, we have

$$\mathrm{ord}_F(x) = -\log_{p^f} |x|_p^{ef} = e \cdot \mathrm{ord}_{\mathbb{Q}_p}(x).$$

On the other hand, if $a \in \mathbb{N}$ then

$$\begin{aligned}\mathrm{ord}_{\mathbb{Q}_p}(a!) &= \sum_{k=1}^{[\log_p a]} k\Big(\Big[\frac{a}{p^k}\Big] - \Big[\frac{a}{p^{k+1}}\Big]\Big) = \sum_{k=1}^{[\log_p a]} \Big[\frac{a}{p^k}\Big] \\ &\leq \sum_{k=1}^{\infty} \frac{a}{p^k} = \frac{a}{p-1}.\end{aligned}$$

These yield

$$\mathrm{ord}_F(a!) \leq \frac{ae}{p-1}. \tag{1.1.1}$$

Let us consider a connected reductive F-group G. Let $\mathfrak{g}$ be its Lie algebra. We fix a non-degenerate symmetric bilinear form $B(\, , \,)$ on $\mathfrak{g}(F)$. We need an F-group embedding $\iota : G \hookrightarrow GL(n)$. This gives a Lie F-algebra embedding $\iota : \mathfrak{g}(F) \hookrightarrow \mathfrak{gl}(n, F) = \mathbb{M}_n(F)$. Let Λ^ι be the $\mathcal{O}$-lattice $\iota^{-1}(\mathbb{M}_n(\mathcal{O}))$ of $\mathfrak{g}(F)$. Clearly we have $[\Lambda^\iota, \Lambda^\iota] \subset \Lambda^\iota$. Choose $A' \in \mathbb{N}$ such that $B(X, Y) \in \mathcal{O}$, for any X, $Y \in \varpi^{A'}\Lambda^\iota$, and set $\Lambda := \varpi^{A'}\Lambda^\iota$, $\widetilde{\Lambda} := \varpi^{A'}\mathbb{M}_n(\mathcal{O})$. Λ is an $\mathcal{O}$-lattice in $\mathfrak{g}(F)$ satisfying

$$\text{(i) } B(\Lambda, \Lambda) \subset \mathcal{O}, \quad \text{(ii) } [*, *] \subset *, \quad \text{(iii) } \iota(*)\iota(*) \subset \widetilde{*}.$$

We shall use the followings.

Lemma . *(a) There exists $A \in \mathbb{N}$ such that $\exp|_{\varpi^A\Lambda}$ is injective.*
(b) For $c \in \mathbb{N}$ there exists $A_1 := \sup(A, \frac{3e}{p-1} + c + 2)$ such that

$$\forall X \in \varpi^n\Lambda, \ \forall Y \in \varpi^m\Lambda, \ \text{with } n, \ m \geq A_1 \tag{1.1.2}$$

$$\log(\exp X \exp Y) - (X + Y + \frac{1}{2}[X, Y]) \in \varpi^{n+m+c}\Lambda,$$

$$\forall X \in \varpi^n\Lambda \text{ with } n \geq A_1, \ \forall Y \in \varpi^m\Lambda \tag{1.1.3}$$

$$\mathrm{Ad}(\exp X)Y - (Y + \mathrm{ad}(X)Y) \in \varpi^{2n+m-\left[\frac{2e}{p-1}\right]}\Lambda.$$

The term $-\left[\frac{2e}{p-1}\right]$ is unnecessary if the residual characteristic of F is not 2.

Remark . *In [MW, 1.1.3], it was stated that*

$$\mathrm{Ad}(\exp X)Y - (Y + \mathrm{ad}(X)Y) \in \varpi^{n+m+c}\Lambda, \quad \forall X \in \varpi^n\Lambda, \forall Y \in \varpi^m\Lambda,$$

instead of (1.1.3). *But this is insufficient.*

Proof. (a) is well-known (cf. [H]).

To prove (1.1.2) we identify $X \in \mathfrak{g}(F)$ with $\iota(X) \in \mathbb{M}_n(F)$. By symmetricity, we may assume that $n \geq m \geq A_1$. We fix $k \in \mathbb{N}$ such that

$$(k+1)\big(m - \frac{e}{p-1}\big) \geq n + m + c + 1.$$

It follows from (1.1.1) that

$$\frac{X^a}{a!} \in \varpi^{na - \left[\frac{ae}{p-1}\right]}\widetilde{\Lambda} \subseteq \varpi^{\left[(n - \frac{e}{p-1})a\right]}\widetilde{\Lambda}.$$

In particular we have

$$\exp X \exp Y \in \left(1 + X + \frac{X^2}{2} + \varpi^{\left[3(n-\frac{e}{p-1})\right]}\widetilde{\Lambda}\right)\left(\sum_{j=0}^{k} \frac{Y^j}{j!} + \varpi^{\left[(k+1)(m-\frac{e}{p-1})\right]}\widetilde{\Lambda}\right).$$

$2n - m \geq n \geq A_1$ gives

$$\left[3(n - \frac{e}{p-1})\right] > 3\left(n - \frac{e}{p-1}\right) - 1 > n + m + c,$$

and our choice of k implies

$$\left[(k+1)(m - \frac{e}{p-1})\right] > (k+1)(m - \frac{e}{p-1}) - 1 \geq n + m + c.$$

Thus we obtain

$$\begin{aligned}
\exp X \exp Y &\in \left(1 + X + \frac{X^2}{2} + \varpi^{n+m+c}\widetilde{\Lambda}\right)\left(1 + Y + \frac{Y^2}{2} + \cdots + \frac{Y^k}{k!} + \varpi^{n+m+c}\widetilde{\Lambda}\right) \\
&= 1 + X + Y + \frac{X^2}{2} + XY + \frac{Y^2}{2} + \frac{X^2Y}{2} + \frac{XY^2}{2} + \frac{Y^3}{3!} + \dots \\
&\qquad + \frac{X^2Y^{k-2}}{2(k-2)!} + \frac{XY^{k-1}}{(k-1)!} + \frac{Y^k}{k!} + \varpi^{n+m+c}\widetilde{\Lambda} \\
&= 1 + X + Y + \frac{X^2}{2} + XY + \frac{Y^2}{2} + \sum_{j=3}^{k} \frac{Y^j}{j!} + \varpi^{n+m+c}\widetilde{\Lambda}.
\end{aligned} \tag{*}$$

Note that if $j \geq 1$, $\ell \geq 2$, the choice of A_1 gives

$$\begin{aligned}
\left[j\left(n - \frac{e}{p-1}\right)\right] + \left[\ell\left(m - \frac{e}{p-1}\right)\right] &> j\left(n - \frac{e}{p-1}\right) + \ell\left(m - \frac{e}{p-1}\right) - 2 \\
\geq n - \frac{e}{p-1} + 2\left(m - \frac{e}{p-1}\right) - 2 &= (n+m+c) + m - \left(\frac{3e}{p-1} + c + 2\right) \\
&\geq n + m + c.
\end{aligned}$$

On the other hand, similar estimation gives

$$\begin{aligned}
&\exp(X+Y+\frac{1}{2}[X,Y]) \\
&\quad \in 1+X+Y+\frac{[X,Y]}{2}+\sum_{j=2}^{k}\frac{1}{j}\left(X+Y+\frac{[X,Y]}{2}\right)^{j}+\varpi^{n+m+c}\widetilde{\Lambda} \\
&\quad = 1+X+Y+\frac{XY-YX}{2}+\frac{X^2+XY+YX+Y^2}{2}+\sum_{j=3}^{k}\frac{Y^j}{j!}+\varpi^{n+m+c}\widetilde{\Lambda}.
\end{aligned} \tag{$**$}$$

$(*)$ and $(**)$ imply

$$\exp X \exp Y \in \exp(X+Y+\frac{[X,Y]}{2}+\varpi^{n+m+c}\widetilde{\Lambda}),$$

and the assertion follows.

For (1.1.3), we still identify X with $\iota(X)$. Note that $n \geq A_1$ implies

$$\left[3(n-\frac{e}{p-1})\right] > 3n-\frac{3e}{p-1}-1 \geq 2n+c+1.$$

Thus we have

$$\begin{aligned}
\mathrm{Ad}(\exp X)Y &\in \left(Y+XY+\frac{X^2Y}{2}+\varpi^{2n+m+c+1}\widetilde{\Lambda}\right)\left(1-X+\frac{X^2}{2}+\varpi^{2n+c+1}\widetilde{\Lambda}\right) \\
&= Y+XY-YX+\frac{X^2Y}{2}-XYX+\frac{YX^2}{2}+\varpi^{2n+m+c+1}\widetilde{\Lambda} \\
&= Y+[X,Y]+\frac{X}{2}[X,Y]-[X,Y]\frac{X}{2}+\varpi^{2n+m+c+1}\widetilde{\Lambda}.
\end{aligned}$$

Noting $X/2 \in \varpi^{n-[\frac{2e}{p-1}]}\widetilde{\Lambda}$, $[X,Y] \in \varpi^{n+m}\widetilde{\Lambda}$, we get (1.1.3). □

1.2 The lattice Λ'

Let $N \in \Lambda$ be a nilpotent element. We write $\mathfrak{g}^N$ and G^N for the centralizer of N in $\mathfrak{g}$ and G, respectively.

Lemma . *$X \in \mathfrak{g}^N(F)$ if and only if $B(N,[X,Y]) = 0$ for any $Y \in \mathfrak{g}(F)$.*

Proof. We may assume that $\mathfrak{g} = \mathfrak{gl}(n)$ and $B(X,Y) = \mathrm{tr}(XY)$. Then the statement becomes that $XN = NX$ if and only if $\mathrm{tr}(NXY) = \mathrm{tr}(XNY)$ $(= \mathrm{tr}(NYX))$ for any $Y \in \mathbb{M}_n(F)$. This is clear since the latter condition reads

$$B(NX,Y) = B(XN,Y), \quad \forall Y \in \mathfrak{gl}(n,F)$$

and B is non-degenerate. □

Let ϕ be arbitrary element in $X_*(G)_F$ such that

$$\mathrm{Ad}(\phi(t))N = t^{-2}N, \quad \forall t \in \mathbb{G}_m.$$

We put $\mathfrak{g}_i := \{X \in \mathfrak{g} \,|\, \mathrm{Ad}(\phi(t))X = t^i X, \forall t \in \mathbb{G}_m\}$ so that

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i.$$

Since $\mathfrak{g}^N$ is $\mathrm{Ad}(\phi(\mathbb{G}_m))$-stable, we may choose $\mathfrak{m} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}^N \oplus \mathfrak{m}, \tag{1.2.1}$$

$$\mathfrak{m} \text{ is } \mathrm{Ad}(\phi)\text{-stable}. \tag{1.2.2}$$

We put $\mathfrak{m}_i = \mathfrak{m} \cap \mathfrak{g}_i$: $\mathfrak{m} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{m}_i$.

Now it follows from the above lemma that

$$B_N : \mathfrak{g} \times \mathfrak{g} \ni (X_1, X_2) \longmapsto B(N, [X_1, X_2]) \in \mathbb{G}_a$$

is an alternating form and $B_N|_{\mathfrak{m}}$ is non-degenerate. Since the definition of ϕ implies

$$\begin{aligned} B_N(\mathrm{Ad}(\phi(s))X_1, \mathrm{Ad}(\phi(s))X_2) &= B(N, \mathrm{Ad}(\phi(s))[X_1, X_2]) \\ &= B(\mathrm{Ad}(\phi(s^{-1}))N, [X_1, X_2]) = B(s^2N, [X_1, X_2]) = s^2 B_N(X_1, X_2), \end{aligned}$$

the duality B_N is between $\mathfrak{m}_i$ and $\mathfrak{m}_{2-i}$. Thus we can choose

- Basis $\{X_1, \dots, X_k\}$ of $\oplus_{i=0}^{-\infty} \mathfrak{m}_i$ and $\{X_1^*, \dots, X_k^*\}$ of $\oplus_{i=2}^{\infty} \mathfrak{m}_i$ dual to each other under B_N, and
- A Witt basis $\{Z_1, \dots, Z_\ell; Z_1^*, \dots, Z_\ell^*\}$ of $\mathfrak{m}_1$ for B_N,

so that

$$B_N(X_i, X_i^*) = 1, \quad 1 \le i \le k, \qquad B_N(Z_i, Z_i^*) = 1, \quad 1 \le i \le \ell.$$

We put

$$\mathfrak{m}^{\Lambda'} = \sum_{i=1}^{k} (\mathcal{O}\mathcal{X}_\rangle + \mathcal{O}\mathcal{X}_\rangle^*) + \sum_{\rangle=\infty}^{\ell} (\mathcal{O}\mathcal{Z}_\rangle + \mathcal{O}\mathcal{Z}_\rangle^*)$$

and let

$$\Lambda' := \mathfrak{m}^{\Lambda'} + \sum_{i \in \mathbb{Z}} \Lambda \cap \mathfrak{g}_i^N(F), \quad \mathfrak{g}_i^N := \mathfrak{g}_i \cap \mathfrak{g}^N.$$

This is a lattice in $\mathfrak{g}(F)$, and we can take $d \in \mathbb{N}$ such that

$$\varpi^d \Lambda \subset \Lambda' \subset \varpi^{-d} \Lambda.$$

1.3 The subgroup $\mathbf{K}_n$

Lemma . *Fix $C \geq d-1$. For $D \geq \sup(A_1 + d, C + 3d)$, we have the followings.*

(1) $\exp|_{\varpi^D\Lambda'} : \varpi^D\Lambda' \to \exp(\varpi^D\Lambda')$ *is a bijection.* $\mathbf{K}_n := \exp(\varpi^n\Lambda')$ *is a subgroup of* $G(F)$ *for* $n \geq D$.

(2) For n, $m \geq D$, $X \in \varpi^n\Lambda'$ and $Y \in \varpi^m\Lambda'$, we have

$$\log(\exp X \exp Y) - (X + Y + \frac{1}{2}[X,Y]) \in \varpi^{n+m+C}\Lambda.$$

(3) For $n \geq D$, $m \in \mathbb{Z}$, $X \in \varpi^n\Lambda'$ and $Y \in \varpi^m\Lambda'$, we have

$$\mathrm{Ad}(\exp X)Y - (Y + \mathrm{ad}(X)Y) \in \varpi^{2n+m-3d-\left[\frac{2e}{p-1}\right]}\Lambda.$$

The term $-\left[\frac{2e}{p-1}\right]$ is not necessary if the residual characteristic of F is odd.

Proof. Putting $c = 2d + C$, we deduce (2), (3) and the first assertion of (1) from the lemma in 1.1. We shall check that $\exp(\varpi^n\Lambda')$ is a group. For X, $Y \in \varpi^n\Lambda'$ with $n \geq D$, we need to show that $\log(\exp X \exp Y) \in \varpi^n\Lambda'$. We have from (2) that

$$\log(\exp X \exp Y) = X + Y + \frac{1}{2}[X,Y] + Z, \quad \exists Z \in \varpi^{2n+C-d}\Lambda'.$$

Then the assertion follows from

$$\frac{1}{2}[X,Y] \in \varpi^{\left[2\left(n-d-\frac{e}{p-1}\right)\right]}\Lambda \subset \varpi^{2\left(n-d-\frac{e}{p-1}\right)-d-1}\Lambda',$$

and

$$\begin{aligned} 2\left(n-d-\frac{e}{p-1}\right)-d-1 &= n+n-3d-\frac{2e}{p-1}-1 \\ &\geq n+\frac{e}{p-1}+C+1-d > n. \end{aligned}$$

□

1.4 The unipotent subgroup attached to (N, ϕ)

We now put

$$\mathfrak{u} := \sum_{i>0} \mathfrak{g}_i, \quad ,\overline{\mathfrak{u}} := \sum_{i<0} \mathfrak{g}_i, \quad \overline{\mathfrak{p}} := \sum_{i\leq 0} \mathfrak{g}_i,$$

and set $U := \exp(\mathfrak{u})$ and $\overline{P} := \mathrm{Stab}(\overline{\mathfrak{u}}, G)$. $\overline{P}$ is a parabolic subgroup of G such that $\mathrm{Lie}\overline{P} = \overline{\mathfrak{p}}$. For $n \geq D$, we set $\mathbf{K}'_n := \mathrm{Ad}(\phi(\varpi^{-n}))\mathbf{K}_n$.

Lemma .

$$\mathbf{K}_n = (\mathbf{K}_n \cap \overline{P}(F))(\mathbf{K}_n \cap U(F)) = (\mathbf{K}_n \cap U(F))(\mathbf{K}_n \cap \overline{P}(F)).$$

Proof. Lemma.1.3 implies

$$\mathbf{K}_n \cap U(F) = \exp(\varpi^n \Lambda' \cap \mathfrak{u}(F)), \quad \mathbf{K}_n \cap \overline{P}(F) = \exp(\varpi^n \Lambda' \cap \overline{\mathfrak{p}}(F)).$$

Also the definition of Λ' gives $\Lambda' = \Lambda' \cap \overline{\mathfrak{p}}(F) \oplus \Lambda' \cap \mathfrak{u}(F)$.

Now let $g = \exp X \in \mathbf{K}_n$ $(X \in \varpi^n \Lambda')$. We can choose $Y_0 \in \varpi^n \Lambda' \cap \overline{\mathfrak{p}}(F)$ and $Z_0 \in \varpi^n \Lambda' \cap \mathfrak{u}(F)$ such that $X = Y_0 + Z_0$. If we set $y_0 := \exp Y_0$ and $z_0 := \exp Z_0$, then Lemma.1.3 (2) asserts that $g_1 := y_0^{-1} g z_0^{-1} \in \mathbf{K}_{n+1}$. Thus we can take $X_1 \in \varpi^{n+1}\Lambda'$ such that $g_1 = \exp X_1$. Take $Y_1 \in \varpi^{n+1}\Lambda' \cap \overline{\mathfrak{p}}(F)$ and $Z_1 \in \varpi^{n+1}\Lambda' \cap \mathfrak{u}(F)$ satisfying $X_1 = Y_1 + Z_1$. Again Lemma.1.3 (2) gives

$$g_1 := y_1^{-1} g_1 z_1^{-1} \in \mathbf{K}_{n+2}, \quad y_1 := \exp(Y_1),\ z_1 := \exp(Z_1).$$

By repeating this process, we can take $\{y_m \in \mathbf{K}_{n+m} \cap \overline{P}(F)\}_{m\in\mathbb{N}}$ and $\{z_m \in \mathbf{K}_{n+m} \cap U(F)\}_{m\in\mathbb{N}}$ such that

$$g_{n+m} := (y_0 \cdots y_m)^{-1} g (z_m \cdots z_0)^{-1} \in \mathbf{K}_{n+m}, \quad \forall m \in \mathbb{N}.$$

We may choose subsequences of $\{(y_0 \cdots y_m)\}_{m\in\mathbb{N}}$ and $\{(z_m \cdots z_0)\}_{m\in\mathbb{N}}$ which are convergent. We write the limit of these sequences as y and z, respectively. Then

$$y^{-1} g z^{-1} \in \bigcap_{m\geq 0} \mathbf{K}_{n+m} = \{1\},$$

and hence $g = yz$ with $y \in \mathbf{K}_n \cap \overline{P}(F)$, $z \in \mathbf{K}_n \cap U(F)$. □

1.5 Remark

(1) Put

$$\begin{aligned} \overline{\mathbf{Q}}_n &:= \mathrm{Ad}(\phi(\varpi^{-n}))(\mathbf{K}_n \cap \overline{P}(F)), \quad n \geq D \\ \mathbf{V}_n &:= \mathrm{Ad}(\phi(\varpi^{-n}))\exp(\varpi^n \Lambda' \cap U(F)), \quad n \geq 0. \end{aligned}$$

Then we have

$$\mathbf{K}'_n = \overline{\mathbf{Q}}_n \mathbf{V}_n. \tag{1.5.1}$$

Consider the two groups on the right hand side.

It follows from the definition that

$$\begin{aligned} \varpi^n \Lambda' \cap \overline{\mathfrak{p}}(F) &= \varpi^n \mathfrak{m}^{\Lambda'} \cap \overline{\mathfrak{p}}(F) + \sum_{i\leq 0} \varpi^n \Lambda \cap \mathfrak{g}_i^N(F) \\ &= \varpi^n \Big(\sum_{j=1}^{k} \mathcal{O}\mathcal{X}_{|} + \sum_{\rangle \leq \prime} \ast \cap \mathfrak{g}_{\rangle}^{\mathcal{N}}(\mathcal{F}) \Big). \end{aligned}$$

Thus $\{\overline{\mathbf{Q}}_n\}_{n\geq D}$ is a system of fundamental neighborhoods of 1 in $\overline{P}(F)$.

Next comes $\mathbf{V}_n$. $\exp X$ belongs to $\mathbf{V}_n$ if and only if X lies in

$$\begin{aligned}&\mathrm{Ad}(\phi(\varpi^{-n}))(\varpi^n\Lambda'\cap\mathfrak{u}(F))\\ &=\mathrm{Ad}(\phi(\varpi^{-n}))\varpi^n\Big(\sum_{j=1}^{k}\mathcal{OX}_|^* + \sum_{|=\infty}^{\ell}(\mathcal{OZ}_| + \mathcal{OZ}_|^*) + \sum_{\rangle\geq\infty} * \cap \mathfrak{g}_\rangle^{\mathcal{N}}(\mathcal{F})\Big)\end{aligned}$$

which clearly contains

$$\begin{aligned}&\sum_{j=1}^{k}\varpi^{-n}\mathcal{OX}_|^* + \sum_{|=\infty}^{\ell}(\mathcal{OZ}_| + \mathcal{OZ}_|^*) + * \cap \mathfrak{g}_\infty^{\mathcal{N}}(\mathcal{F}) + \sum_{\rangle\geq\in}\varpi^{-\backslash} * \cap \mathfrak{g}_\rangle^{\mathcal{N}}(\mathcal{F})\\ &=\varpi^{-n}\left(\sum_{j=1}^{k}\mathcal{OX}_|^* + \sum_{\rangle\geq\in} * \cap \mathfrak{g}_\rangle^{\mathcal{N}}(\mathcal{F})\right) + \Lambda\cap\mathfrak{g}_1^N(F) + \sum_{j=1}^{\ell}(\mathcal{OZ}_| + \mathcal{OZ}_|^*).\end{aligned}$$

Thus $\mathbf{V}_n$ becomes bigger and bigger as n increases. Since

$$\sum_{j=1}^{\ell}(\mathcal{OZ}_| + \mathcal{OZ}_|^*) + * \cap \mathfrak{g}_\infty^{\mathcal{N}}(\mathcal{F}) = *' \cap \mathfrak{g}_\infty(\mathcal{F}),$$

we have

$$\bigcup_{n\geq 0}\mathbf{V}_n = \exp\big(\Lambda'\cap\mathfrak{g}_1(F)\big)\cdot U^2(F). \tag{1.5.2}$$

with $U^2(F) := \exp\left(\sum_{i\geq 2}\mathfrak{g}_i(F)\right)$.

(2) It follows form Lemma.1.3 (3) that $\mathbf{K}_m \lhd \mathbf{K}_n$ if $m\geq n\geq D$.

1.6 The character χ_n

We fix a non-trivial character ψ of F of order zero.

Lemma . *Suppose that $n\geq D$ and the residual characteristic of F is not 2.*

(1) $\chi_n : \mathbf{K}_n \ni \exp X \mapsto \psi(B(\varpi^{-2n}N, X)) \in \mathbb{C}^1$ is a character.

(2) Let $n\geq m\geq D$. Then $\mathrm{Stab}(\chi_n, \mathbf{K}_m)$ is contained in $\mathbf{K}_n\exp(\varpi^m\Lambda\cap\mathfrak{g}^N(F))$.

Proof. (1) If we write X, $X'\in\varpi^n\Lambda'$ in the form

$$X = \sum_{j=1}^{k}\varpi^n(x_j^*X_j^* + x_jX_j) + \sum_{j=1}^{\ell}\varpi^n(z_jZ_j + z_j^*Z_j^*) + \varpi^n Z,$$

$$X' = \sum_{j=1}^{k}\varpi^n({x'}_j^*X_j^* + x_j'X_j) + \sum_{j=1}^{\ell}\varpi^n(z_j'Z_j + {z'}_j^*Z_j^*) + \varpi^n Z',$$

with x_j, x_j^*, z_j, z_j^*, x_j', ${x'}_j^*$, z_j', ${z'}_j^*\in\mathcal{O}$ and Z, $Z'\in\Lambda\cap\mathfrak{g}^N(F)$, then we have

$$B_N(X,X') = \varpi^{2n}\left(\sum_{j=1}^{k}(x_j{x'}_j^* - x_j^*x_j') + \sum_{j=1}^{\ell}(z_j{z'}_j^* - z_j^*z_j')\right)\in\mathfrak{p}_F^{2n}.$$

That is, $\psi(B(\varpi^{-2n}N, [X, X'])) = 1$ for $X,\ X' \in \varpi^n\Lambda'$. Now for $X,\ Y \in \varpi^n\Lambda'$, we have

$$\begin{aligned}\chi_n(\exp X \exp Y) &\overset{1.3\,(2)}{=} \psi \circ B(\varpi^{-2n}N, X + Y + \frac{1}{2}[X, Y] + \varpi^{2n+C}\Lambda) \\ &= \psi \circ B(\varpi^{-2n}N, X + Y)\psi \circ B(N, \varpi^C\Lambda).\end{aligned}$$

The assertion follows from $B(N, \Lambda) \in \mathcal{O}$.

(2) Suppose that $\exp X_0 \in \mathrm{Stab}(\chi_n, \mathbf{K}_m)$, $(X_0 \in \varpi^m\Lambda')$. We can write $X_0 = Y_0 + Z_0$, $Y_0 \in \varpi^m\mathfrak{m}^{\Lambda'}$, $Z_0 \in \varpi^m\Lambda \cap \mathfrak{g}^N(F)$. We have

$$\begin{aligned}\chi_n(\mathrm{Ad}(\exp(-Z_0)) \exp X) &= \psi \circ B(\varpi^{-2n}N, \mathrm{Ad}(\exp(-Z_0))X) \\ &= \psi \circ B(\varpi^{-2n}\mathrm{Ad}(\exp Z_0)N, X) = \psi \circ B(\varpi^{-2n}N, X) \\ &= \chi_n(\exp X)\end{aligned}$$

Thus we can replace $\exp X_0$ by $\exp X_0 \exp(-Z_0)$. It follows from 1.3 (2) that

$$\begin{aligned}\exp X_0 \exp(-Z_0) &= \exp\bigl(Y_0 + \frac{[Y_0, -Z_0]}{2} + Z), \quad \exists Z \in \varpi^{2m+C}\Lambda \\ &\in \exp(Y_0 + \varpi^{2m-3d}\Lambda').\end{aligned}$$

$m \geq C + 3d$ assures that $2m - 3d > m + C$. Thus if we set $X_1 := \log(\exp X_0 \exp(-Z_0))$, it can be written as $X_1 = Y_1 + Z_1$ with $Y_1 \in \varpi^m\mathfrak{m}^{\Lambda'}$ and $Z_1 \in \varpi^{m+C}\Lambda \cap \mathfrak{g}^N(F)$. Again we can replace $\exp X_1$ by

$$\begin{aligned}\exp X_1 \exp(-Z_1) &= \exp(Y_1 + \frac{[Y_1, -Z_1]}{2} + Z'), \quad \exists Z' \in \varpi^{2m+2C}\Lambda \\ &\in \exp(Y_1 + \varpi^{2m+C-3d}\Lambda').\end{aligned}$$

We have $X_2 := \log(\exp X_1 \exp(-Z_1)) = Y_2 + Z_2$ with $Y_2 \in \varpi^m\mathfrak{m}^{\Lambda'}$, $Z_2 \in \varpi^{m+2C}\Lambda \cap \mathfrak{g}^N(F)$. By repeating this process we obtain, for any $r \in \mathbb{N}$,

$$\begin{gathered}\exp X_0 = \exp X_r \exp Z_{r-1} \cdots \exp Z_0, \\ X_r = Y_r + Z_r, \quad Y_r \in \varpi^m\mathfrak{m}^{\Lambda'},\ Z_r \in \varpi^{m+rC}\Lambda \cap \mathfrak{g}^N(F).\end{gathered}$$

Thus, if we take $Z \in \varpi^m\Lambda \cap \mathfrak{g}^N(F)$ such that $\exp Z$ is the limit of a convergent subsequence of $\{\exp Z_r \cdots \exp Z_0\}_{r\in\mathbb{N}}$, then $Y := \log(\exp X_0 \exp(-Z))$ is in $\varpi^m\mathfrak{m}^{\Lambda'}$.

Now let $Y \in \varpi^m\mathfrak{m}^{\Lambda'}$ be such that $\exp Y \in \mathrm{Stab}(\chi_n, \mathbf{K}_m)$. Since 1.3 (3) gives

$$\chi_n(\mathrm{Ad}(\exp Y) \exp X) = \psi\bigl(B(\varpi^{-2n}N, X + [Y, X] + Z_{X,Y})\bigr), \quad \exists Z_{X,Y} \in \varpi^{2m+n-3d}\Lambda,$$

this is equivalent to

$$B_N(Y, X) + B(N, Z_{X,Y}) \in \mathfrak{p}_F^{2n}, \quad \forall X \in \varpi^n\Lambda'. \tag{$*$}$$

If $2m - 3d < n$, this forces us to have $B_N(Y, X) \in -B(N, Z_{X,Y}) + \varpi^{2n}\mathcal{O} \subset \mathfrak{p}_{\mathcal{F}}^{\in\Updownarrow+\backslash-\ni\lceil}$. $\mathfrak{m}^{\Lambda'}$ is self-dual with respect to B_N and this implies that $Y \in \varpi^{2m-3d}\mathfrak{m}^{\Lambda'}$. By repeating this argument, one may assume that $Y \in \varpi^k\mathfrak{m}^{\Lambda'}$ with $2k - 3d \geq n$. Then $B(N, Z_{X,Y}) \in B(N, \varpi^{2n}\Lambda) \subset \varpi^{2n}\mathcal{O}$. Thus $(*)$ becomes

$$B_N(Y, X) \in \mathfrak{p}_F^{2n}, \quad \forall X \in \varpi^n\Lambda'.$$

Now the self-duality of $\mathfrak{m}^{\Lambda'}$ gives $Y \in \varpi^n\Lambda'$. □

1.7 Degenerate Whittaker models

Let $N \in \mathfrak{g}(F)$ be a nilpotent element and take $\phi \in X_*(G)_F$ as above.

Put $V = \exp(\mathfrak{g}_1 \cap \mathfrak{g}^N)U^2$. We observe that

$$U(F) \rhd V(F) \rhd U^2(F). \tag{1.7.1}$$

(Clear because $U(F)/U^2(F)$ is abelian.) We define

$$\chi : V(F) \ni \exp X \longmapsto \psi(B(N, X)) \in \mathbb{C}^1.$$

Then

$$\chi \text{ is a character stable under } U(F). \tag{1.7.2}$$

Proof. Since B is $\mathrm{Ad}(\phi)$-invariant, its restriction to $\mathfrak{g}_i(F)\times\mathfrak{g}_j(F)$ is identically zero unless $i+j=0$. Writing $X = X_1 + X_{\geq 2}$, $X' = X'_1 + X'_{\geq 2} \in \mathrm{Lie}V(F)$, ($X_1$, $X'_1 \in \mathfrak{g}_1^N(F)$, $X_{\geq 2}$, $X'_{\geq 2} \in \sum_{i\geq 2} \mathfrak{g}_i(F)$), we have

$$\begin{aligned}\chi(\exp X \exp X') &= \psi\big(B(N, X + X' + \frac{[X_1, X'_1]}{2} + \sum_{j\geq 3} Y_j)\big), \exists Y_j \in \mathfrak{g}_j(F)\\ &\overset{N\in\mathfrak{g}_2(F)}{=} \chi(\exp X)\chi(\exp X')\psi\big(\frac{B_N(X_1, X'_1)}{2}\big)\\ &= \chi(\exp X)\chi(\exp X').\end{aligned}$$

Thus χ is a character.

Next take $X = X_1 + X_{\geq 2} \in \mathrm{Lie}V(F)$ as above and $Y = Y_1 + Y_{\geq 2} \in U(F)$, ($Y_1 \in \mathfrak{g}_1(F)$, $Y_{\geq 2} \in \mathrm{Lie}U^2(F)$). Then

$$\begin{aligned}\chi(\mathrm{Ad}(\exp Y)\exp X) &= \psi \circ B(N, \mathrm{Ad}(\exp Y)X) = \psi \circ B(N, \mathrm{Ad}\big(1 + Y + \frac{Y^2}{2} + \dots\big)X)\\ &= \psi \circ B(N, X + [Y, X] + Z), \quad \exists Z \in \sum_{i\geq 3} \mathfrak{g}_i(F)\\ &= \chi(\exp X)\psi(B_N(Y_1, X_1)) = \chi(\exp X),\end{aligned}$$

since $X_1 \in \mathfrak{g}^N(F)$. □

$N \in \mathfrak{g}_{-2}(F)$ gives

$$\chi|\exp(\mathfrak{g}_1^N(F)) = \mathbf{1}. \tag{1.7.3}$$

Now let (π, E) be an admissible representation of $G(F)$ [BZ]. We introduce the spaces [BZ]

$$\begin{aligned}E(V, \chi) &:= \mathrm{Span}\{\pi(v)\xi - \chi(v)\xi \,|\, v \in V(F),\, \xi \in E\}, \quad E_{V,\chi} := E/E(V, \chi),\\ E(U^2, \chi) &:= \mathrm{Span}\{\pi(u)\xi - \chi(u)\xi \,|\, u \in U^2(F),\, \xi \in E\}, \quad E_{U^2,\chi} := E/E(U^2, \chi).\end{aligned}$$

We set

$$\mathcal{H}_{\mathcal{N}} = \mathcal{H}_{\mathcal{N},\phi} := \begin{cases}\mathcal{H}(\mathfrak{m}_\infty(\mathcal{F}), \psi \circ \mathcal{B}_{\mathcal{N}}) & \text{if } \mathfrak{m}_1 \neq \{0\},\\ \mathbb{C}^1 & \text{otherwise.}\end{cases}$$

Here $\mathcal{H}(\mathfrak{m}_\infty(\mathcal{F}), \psi \circ \mathcal{B}_\mathcal{N})$ is the Heisenberg group $\mathfrak{m}_1(F) \times \mathbb{C}^1$ with the multiplication law:

$$(X, z)(Y, w) = (X + Y, zw\psi(B_N(X, Y))), \quad X,\, Y \in \mathfrak{m}_1(F),\ z,\ w \in \mathbb{C}^1.$$

Then the map

$$p_N : U(F) \ni \exp(X_1 + X_{\geq 2}) \longmapsto (X_1; \psi(B(N, X_{\geq 2}))) \in \mathcal{H}_\mathcal{N}$$

is a surjective homomorphism with the kernel $\ker \chi$. Write ρ_N for the Schrödinger representation of $\mathcal{H}(\mathfrak{m}_\infty(\mathcal{F}), \psi \circ \mathcal{B}_\mathcal{N})$ if $\mathfrak{m}_1 \neq 0$ and the identity representation of $\mathbb{C}^1$ otherwise. (1.7.2) assures that $U(F)$ acts on $E_{V,\chi}$ by $\rho_N \circ p_N$. We define the space of *degenerate Whittaker vectors with respect to* (N, ϕ) by

$$\mathcal{W}_{\mathcal{N},\phi}(\pi) := \mathrm{Hom}_{\mathcal{U}(\mathcal{F})}(\rho_\mathcal{N} \circ \surd^{\mathcal{N}}, \mathcal{E}_{\mathcal{V},\chi}).$$

We write $\mathcal{N}(\mathfrak{g}(\mathcal{F}))$ for the set of nilpotent $\mathrm{Ad}(G(F))$-orbits in $\mathfrak{g}(F)$. Define $\mathcal{N}_{\mathrm{Wh}}(\pi)$ for the subset of $\mathcal{N}(\mathfrak{g}(\mathcal{F}))$ consisting of those $\mathfrak{O}$ with $\mathcal{W}_{\mathcal{N},\phi}(\pi) \neq \{/\}$ for some $N \in \mathfrak{O}$ and ϕ.

From now on, we assume that N is in the lattice Λ. Set

$$\mathfrak{m}_1^{\Lambda'} := \mathfrak{m}_1(F) \cap \Lambda', \quad \mathbf{L} = \exp \mathfrak{m}_1^{\Lambda'}.$$

Since $\mathfrak{m}_1^{\Lambda'}$ is a self-dual lattice in $\mathcal{H}_\mathcal{N}/\mathbb{C}^\infty$, $\mathfrak{m}_1^{\Lambda'} \times \mathbb{C}^1$ is a maximal abelian subgroup in $\mathcal{H}_\mathcal{N}$. Hence it follows from the definition of ρ_N that

$$\mathcal{W}_{\mathcal{N},\phi}(\pi) \simeq \mathcal{E}_{\mathcal{V},\chi}^{\mathbf{L}}. \tag{1.7.4}$$

1.8 Character expansion

Now suppose that π is irreducible. The following theorem is due to Howe in the case of $GL(n)$ and Harish-Chandra in general.

Theorem ([HC]). *There exists a neighborhood U_π of 1 in $G(F)$ and a finite set of complex numbers $\{c_\mathfrak{O}(\pi)\}_{\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(\mathcal{F}))}$ such that*

$$\mathrm{tr}\pi(f) = \sum_{\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(\mathcal{F}))} c_\mathfrak{O}(\pi) \int_\mathfrak{O} f\widehat{(\exp X)}\, d_{\mu_\mathfrak{O}}(X), \quad \forall f \in C_c^\infty(U_\pi).$$

Here, $f\widehat{(\exp X)}$ is the Fourier transform

$$\mathcal{F}_{\psi\circ\mathcal{B}}(\{ \circ \exp)(\mathcal{X}) := \int_{\mathfrak{g}(\mathcal{F})} \{(\exp \mathcal{Y})\psi(\mathcal{B}(\mathcal{X}, \mathcal{Y}))\, \lceil \mathcal{Y}.$$

The constants $c_\mathfrak{O}(\pi)$ depend on the choice of various measures. We fix the relevant measures as follows. We fix the measure on $\mathfrak{g}(F)$ self-dual with respect to $\psi \circ B$. The measure on $G(F)$ is chosen so that the Jacobian of exp has absolute value one near 0. For each $\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(\mathcal{F}))$ and $N \in \mathfrak{O}$, the tangent space of $\mathfrak{O}$ at N is

$$T_N\mathfrak{O} \simeq \mathrm{Span}\{\mathrm{ad}(X)N \mid X \in \mathfrak{g}(F)\} \simeq \mathfrak{g}(F)/\mathfrak{g}^N(F) \simeq \mathfrak{m}(F).$$

On this we fix a measure self-dual with respect to $\psi \circ B_N$. This determines the invariant measure $\mu_\mathfrak{O}$ on $\mathfrak{O}$.

We write $\mathcal{N}_\mathcal{B}(\pi)$ for the set of nilpotent orbits $\mathfrak{O} \in \mathcal{N}(\mathfrak{g}(\mathcal{F}))$ such that $c_\mathfrak{O}(\pi) \neq 0$.

1.9 System of isotypic spaces

From now on, we suppose that the residual characteristic of F is not 2. For $n \geq D$, we introduce

$$\chi'_n : \mathbf{K}'_n \xrightarrow{\mathrm{Ad}(\phi(\varpi^n))} \mathbf{K}_n \xrightarrow{\chi_n} \mathbb{C}^1$$

and

$$\begin{aligned} E[\chi_n] &:= \{\xi \in E \,|\, \pi(k)\xi = \chi_n(k)\xi \,\forall k \in \mathbf{K}_n\} \\ E[\chi'_n] &:= \{\xi \in E \,|\, \pi(k)\xi = \chi'_n(k)\xi \,\forall k \in \mathbf{K}'_n\}. \end{aligned}$$

Of course we have $E[\chi'_n] = \pi(\phi(\varpi^{-n}))E[\chi_n]$. We also define for m, $n \geq D$

$$\begin{aligned} I_{n,m} : E[\chi_n] \ni \xi &\longmapsto \frac{1}{\mathrm{meas}\mathbf{K}_m} \int_{\mathbf{K}_m} \overline{\chi_m(k)}\pi(k\phi(\varpi^{m-n}))\xi \, dk \in E[\chi_m], \\ I'_{n,m} : E[\chi'_n] \ni \xi &\longmapsto \frac{1}{\mathrm{meas}\mathbf{K}'_m} \int_{\mathbf{K}'_m} \overline{\chi'_m(k)}\pi(k)\xi \, dk \in E[\chi'_m], \\ I_n : E \ni \xi &\longmapsto \frac{1}{\mathrm{meas}\mathbf{K}_n} \int_{\mathbf{K}_n} \overline{\chi_n(k)}\pi(k)\xi \, dk \in E[\chi_n], \\ I'_n : E \ni \xi &\longmapsto \frac{1}{\mathrm{meas}\mathbf{K}'_n} \int_{\mathbf{K}'_n} \overline{\chi'_n(k)}\pi(k)\xi \, dk \in E[\chi'_n]. \end{aligned}$$

The following diagram obviously commutes:

$$\begin{array}{ccc} E[\chi_n] & \xrightarrow{I_{n,m}} & E[\chi_m] \\ {\scriptstyle \pi(\phi(\varpi^{-n}))}\downarrow & & \downarrow{\scriptstyle \pi(\phi(\varpi^{-m}))} \\ E[\chi'_n] & \xrightarrow{I'_{n,m}} & E[\chi'_m] \end{array} \tag{1.9.1}$$

Also we remark that

$$\begin{aligned} &\chi'_n|_{\overline{\mathbf{Q}}_n} = \mathbf{1}, \quad \chi_n|_{\overline{P}(F)\cap\mathbf{K}_n} = \mathbf{1}, \\ &\chi'_n|_{\mathbf{L}} = \mathbf{1} \quad (\text{note that } \mathbf{L} \subset \mathbf{V}_n, \, \forall n) \\ &\chi'_n|_{V(F)\cap\mathbf{K}'_n} = \chi|_{V(F)\cap\mathbf{K}'_n}. \end{aligned} \tag{1.9.2}$$

These are clear from above.

Now we have for $m > n$, $\xi \in E[\chi'_n]$

$$\begin{aligned} I'_{n,m}(\xi) &= \frac{1}{\mathrm{meas}\mathbf{K}'_m} \int_{\mathbf{K}'_m} \overline{\chi'_m(k)}\pi(k)\xi \, dk = \frac{1}{\mathrm{meas}\mathbf{K}'_m} \int_{\overline{\mathbf{Q}}_m\mathbf{V}_m} \overline{\chi'_m(k)}\pi(k)\xi \, dk \\ &= \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{\mathbf{V}_m} \overline{\chi'_m(k)} \left(\frac{1}{\mathrm{meas}\overline{\mathbf{Q}}_m} \int_{\overline{\mathbf{Q}}_m} \overline{\chi'_m(k')}\pi(kk')\xi \, dk' \right) dk \end{aligned}$$

since $\overline{\chi'_m(k')} = 1$ by (1.9.2) and $\xi \in E[\chi'_n]$ is invariant under $\overline{\mathbf{Q}}_m \subset \overline{\mathbf{Q}}_n$,

$$= \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{\mathbf{V}_m} \overline{\chi'_m(k)}\pi(k)\xi \, dk.$$

Noting

$$\mathbf{V}_m = \mathbf{K}'_m \cap U(F), \quad V = \exp(\mathfrak{g}_1^N) \cdot U^2, \quad \mathbf{V}_m = (V(F) \cap \mathbf{K}'_m) \rtimes \mathbf{L}, \tag{1.9.3}$$

we see that the above becomes

$$\begin{aligned} I'_{n,m}(\xi) &= \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{V(F)\cap\mathbf{K}'_m} \int_{\mathbf{L}} \overline{\chi'_m(k\ell)}\pi(k\ell)\xi \, dk \, d\ell \\ &\overset{(1.9.2)}{=} \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{V(F)\cap\mathbf{K}'_m} \int_{\mathbf{L}} \overline{\chi'_m(k)}\pi(k)\xi \, dk \, d\ell \quad (\xi \in E[\chi'_n]) \\ &= \frac{1}{\mathrm{meas}(V(F)\cap\mathbf{K}'_m)} \int_{V(F)\cap\mathbf{K}'_m} \overline{\chi'_m(k)}\pi(k)\xi \, dk \\ &\overset{(1.9.2)}{=} \frac{1}{\mathrm{meas}(V(F)\cap\mathbf{K}'_m)} \int_{V(F)\cap\mathbf{K}'_m} \overline{\chi(k)}\pi(k)\xi \, dk, \quad \forall \xi \in E[\chi'_n]. \end{aligned} \tag{1.9.4}$$

Since $V(F)\cap\mathbf{K}'_m$ grows larger as m increases, we see that $\ker I'_{n,m} \subset \ker I'_{n,\ell}$ if $m \le \ell$. We put

$$E'_{n,\chi} := \bigcup_{m>n} \ker I'_{n,m}.$$

Then it follows from (1.9.4) and [BZ, 2.33] that $E'_{n,\chi} \subset E(V,\chi)$ and the map

$$j : E/E'_{n,\chi} \longrightarrow E_{V,\chi} \tag{1.9.5}$$

is well-defined.

1.10

Lemma . *If $\mathcal{W}_{\mathcal{N},\phi}(\pi) \neq 0$, then $E[\chi_n]$ and $E[\chi'_n]$ are non-zero for sufficiently large n.*

Proof. Since $(\mathcal{W}_{\mathcal{N},\phi}(\pi) \simeq \mathcal{E}^{\mathbf{L}}_{\mathcal{V},\chi}) \neq 0$, we can choose $\xi \in E$ whose image in $E_{V,\chi}$ is non-zero and belongs to $E^{\mathbf{L}}_{V,\chi}$. Take $M \in \mathbb{N}$ such that

- $M \ge D$, and
- ξ is $\overline{\mathbf{Q}}_M$-invariant.

Then we have for $m \ge M$,

$$\begin{aligned} I'_m(\xi) &= \frac{1}{\mathrm{meas}\mathbf{K}'_m} \int_{\mathbf{K}'_m} \overline{\chi'_m(k)}\pi(k)\xi \, dk \overset{(1.9.2)}{=} \frac{1}{\mathrm{meas}\mathbf{K}'_m} \int_{\mathbf{V}_m} \int_{\overline{\mathbf{Q}}_m} \overline{\chi'_m(v)}\pi(vq) \, dq \, dv \\ &= \frac{1}{\mathrm{meas}\mathbf{K}'_m} \int_{\mathbf{V}_m} \overline{\chi'_m(v)}\pi(v) \left(\int_{\overline{\mathbf{Q}}_m} \pi(q)\xi \, dq \right) dv \\ &\overset{m\ge M}{=} \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{\mathbf{V}_m} \overline{\chi'_m(v)}\pi(v)\xi \, dv \\ &\overset{(1.9.3)}{=} \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{\mathbf{L}} \int_{V(F)\cap\mathbf{K}'_m} \overline{\chi'_m(k)}\pi(\ell k)\xi \, dk \, d\ell \\ &\overset{(1.9.2)}{=} \frac{1}{\mathrm{meas}\mathbf{V}_m} \int_{\mathbf{L}} \pi(\ell) \left(\int_{V(F)\cap\mathbf{K}'_m} \overline{\chi(k)}\pi(k)\xi \, dk \right) d\ell. \end{aligned} \tag{\dagger}$$

Our choice of ξ assures that $\pi(\ell)\xi - \xi$ belongs to $E(V,\chi) \subset E(U^2,\chi)$ for any $\ell \in \mathbf{L}$. By [BZ, 2.33], we can take M sufficiently large so that

$$\int_{U^2(F)\cap \mathbf{K}'_m} \overline{\chi(k)}\pi(k)(\pi(\ell)\xi - \xi)\, dk = 0, \quad \forall m \geq M,\, \forall \ell \in \mathbf{L},$$

since $U^2(F)\cap \mathbf{K}'_m$ exhausts $U^2(F)$ as m increases and $\mathrm{Span}\{\pi(\ell)\xi \,|\, \ell \in \mathbf{L}\}$ is finite dimensional. This gives

$$\begin{aligned}
&\int_{V(F)\cap \mathbf{K}'_m} \overline{\chi(k)}\pi(k)(\pi(\ell)\xi - \xi)\, dk \\
&\qquad = \int_{\Lambda\cap \mathfrak{g}_1^N(F)} \int_{U^2(F)\cap \mathbf{K}'_m} \overline{\chi(\exp X\cdot k)}\pi(\exp X\cdot k)(\pi(\ell)\xi - \xi)\, dk\, dX \\
&\qquad = \int_{\Lambda\cap \mathfrak{g}_1^N(F)} \pi(\exp X) \int_{U^2(F)\cap \mathbf{K}'_m} \overline{\chi(k)}\pi(k)(\pi(\ell)\xi - \xi)\, dk\, dX = 0
\end{aligned}$$

for any $m \geq M$ and $\ell \in \mathbf{L}$. Since $\mathbf{L}$ normalizes $V(F)\cap \mathbf{K}'_m$, this is equivalent to saying that the inner integral on the right hand side of (†) is $\pi(\mathbf{L})$-invariant. We conclude

$$I'_m(\xi) = \frac{1}{\mathrm{meas}(V(F)\cap \mathbf{K}'_m)} \int_{V(F)\cap \mathbf{K}'_m} \overline{\chi(k)}\pi(k)\xi\, dk.$$

Since $\xi \notin E(V,\chi)$, [BZ, 2.33] implies

$$0 \neq \int_{V(F)\cap \mathbf{K}'_m} \overline{\chi(k)}\pi(k)\xi\, dk = \mathrm{meas}(V(F)\cap \mathbf{K}'_m) I'_m(\xi).$$

Thus $I'_m(\xi) \in E[\chi'_n]$ cannot be zero. $\square$

1.11

Proposition . *If $\mathcal{W}_{\mathcal{N},\phi}(\pi) \neq$ ʼ, then there exists $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)$ (cf. 1.8) such that $N \in \overline{\mathfrak{O}}$ (the closure in the usual p-adic topology).*

Proof. By Lemma 1.10 we may suppose that $E[\chi_n] \neq 0$, for sufficiently large n. Put

$$\varphi_n(x) := \begin{cases} \chi_n(x)^{-1} & \text{if } x \in \mathbf{K}_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the theory of elementary idempotents implies

$$\dim E[\chi_n] = \frac{1}{\mathrm{meas}\mathbf{K}_n} \mathrm{tr}\pi(\varphi_n).$$

We shall apply 1.8 to this. We note

$$\begin{aligned}
\widehat{\varphi_n \circ \exp}(X) &= \int_{\mathfrak{g}(F)} \varphi_n(\exp Y)\psi(B(X,Y))\,dY \\
&= \int_{\varpi^n\Lambda'} \chi_n(\exp(-Y))\psi(B(X,Y))\,dY \\
&= \int_{\varpi^n\Lambda'} \psi(B(\varpi^{-2n}N,-Y)+B(X,Y))\,dY \\
&= \int_{\varpi^n\Lambda'} \psi(B(X-\varpi^{-2n}N,Y))\,dY \\
&= \begin{cases} \mathrm{meas}(\varpi^n\Lambda') & \text{if } X \in \varpi^{-2n}N + (\varpi^n\Lambda')^*, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Here $(\varpi^n\Lambda')^*$ denotes the dual lattice of $\varpi^n\Lambda'$. Then Theorem 1.8 gives

$$\sum_{\mathfrak{O}\in\mathcal{N}_{\mathcal{B}}(\pi)} c_{\mathfrak{O}}(\pi)\int_{\mathfrak{O}} \widehat{\varphi_n \circ \exp}(X)\,d\mu_{\mathfrak{O}}(X) = \mathrm{tr}\pi(\varphi_n) = \mathrm{meas}\mathbf{K}_n \dim E[\chi_n] \neq 0.$$

Thus there must be $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)$ such that $\mathfrak{O} \cap \varpi^{-2n}N + (\varpi^n\Lambda')^*$ is non-empty. Or equivalently,

$$\mathfrak{O} \cap N + \varpi^{2n}(\varpi^n\Lambda')^* \neq \emptyset, \quad \forall n >> 0$$

since $\mathfrak{O}$ is invariant under multiplication by $(F^\times)^2$. But because $\{\varpi^{2n}(\varpi^n\Lambda')^* \,|\, n \in \mathbb{N}\}$ form a fundamental system of neighborhoods of 0 in $\mathfrak{g}(F)$, we deduce $N \in \overline{\mathfrak{O}}$. □

1.12

We define a partial order $\mathfrak{O} \geq \mathfrak{O}'$ on $\mathcal{N}(\mathfrak{g}(\mathcal{F}))$ by $\overline{\mathfrak{O}} \supset \overline{\mathfrak{O}}'$. For each subset $S \subset \mathcal{N}(\mathfrak{g}(\mathcal{F}))$, we write $S^{\sup}$ for the set of maximal elements in S with respect to this partial order.

Lemma . *Suppose $N \in \mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\sup}$. Then $\dim E[\chi_n] = c_{\mathfrak{O}}(\pi)$ for sufficiently large n.*

Proof. As in the proof of Prop.1.11, we have

$$\dim E[\chi_n] = \frac{1}{\mathrm{meas}\mathbf{K}_n}\mathrm{tr}\pi(\varphi_n) = \sum_{\mathfrak{O}'\in\mathcal{N}_{\mathcal{B}}(\pi)} c_{\mathfrak{O}'}(\pi)\int_{\mathfrak{O}'\cap(\varpi^{-2n}N+(\varpi^n\Lambda')^*)} d\mu_{\mathfrak{O}'}(X). \qquad (\dagger)$$

The last integral does not vanish only if $N \in \overline{\mathfrak{O}}'$. If $N \in \overline{\mathfrak{O}}'$ then $\overline{\mathfrak{O}}' \cap \mathfrak{O} \neq \emptyset$ and the maximality of $\mathfrak{O}$ implies $\mathfrak{O}' = \mathfrak{O}$. Thus (†) becomes

$$\dim E[\chi_n] = c_{\mathfrak{O}}(\pi)\mu_{\mathfrak{O}}(\mathfrak{O} \cap (\varpi^{-2n}N + (\varpi^n\Lambda')^*)).$$

Let us calculate $\mu_{\mathfrak{O}}(\mathfrak{X}_n)$ for sufficiently large n, where $\mathfrak{X}_n = \mathfrak{O} \cap (\varpi^{-2n}N + (\varpi^n\Lambda')^*)$. The map $G(F)/G^N(F) \ni g \xrightarrow{\sim} \mathrm{Ad}(g)N \in \mathfrak{O}$ gives

$$G(F)\phi(\varpi^n)/G^N(F) \ni g\phi(\varpi^n) \xmapsto{\sim} \mathrm{Ad}(g)\varpi^{-2n}N \in \mathfrak{O}.$$

Since $\mathrm{Ad}(g)\varpi^{-2n}N \in \varpi^{-2n}N + (\varpi^n\Lambda')^*$ if and only if

$$\mathrm{Ad}(g)N - N \in \varpi^{2n}(\varpi^n\Lambda')^* = \varpi^n\Lambda'^*,$$

we have

$$\mathfrak{X}_n = \{\mathrm{Ad}(g\phi(\varpi^n))N \,|\, g \in G(F)/G^N(F),\ \mathrm{Ad}(g)N - N \in \varpi^n\Lambda'^*\}.$$

Next we note that $\{\varpi^n\Lambda'^*\}_{n\in\mathbb{N}}$ is a system of fundamental neighborhoods of 0 in $\mathfrak{g}(F)$. It follows that

$$\{g \in G(F)/G^N(F) \,|\, \mathrm{Ad}(g)N - N \in \varpi^n\Lambda'^*\}, \quad n >> 0$$

form a system of fundamental neighborhoods of 1 in $G(F)/G^N(F)$. Thus noting the decomposition $\mathfrak{g}(F) = \mathfrak{g}^N(F) \oplus \mathfrak{m}(F)$, for any $a \geq D$, we can take $M \in \mathbb{N}$ such that

$$\mathfrak{X}_n \subset \mathrm{Ad}\big(\exp(\varpi^a\mathfrak{m}^{\Lambda'})\phi(\varpi^n)\big)N, \quad \forall n \geq M.$$

We shall calculate $\mathfrak{X}_n$ with $n \geq M$. Suppose $X \in \varpi^a\mathfrak{m}^{\Lambda'}$ satisfies $\mathrm{Ad}(\exp(X)\phi(\varpi^n))N \in \mathfrak{X}_n$. Then $Z := \mathrm{Ad}(\exp X)N - N \in \varpi^n\Lambda'^*$ and

$$\begin{aligned}\psi \circ B(\varpi^{-2n}\mathrm{Ad}(\exp X)N, Y) &= \psi \circ B(\varpi^{-2n}(N+Z), Y)\\ &= \psi \circ B(\varpi^{-2n}N, Y)\psi \circ B(\varpi^{-2n}Z, Y)\\ &= \psi \circ B(\varpi^{-2n}N, Y), \quad \forall Y \in \varpi^n\Lambda'.\end{aligned}$$

In other words, $\exp X$ is in $\mathrm{Stab}(\chi_n, \mathbf{K}_a) \subset \mathbf{K}_n \exp(\varpi^a\Lambda \cap \mathfrak{g}^N(F))$ (Lem.1.6 (2)). Thus we may write $\exp X = \exp Y \exp Z$ with $Y \in \varpi^n\Lambda'$, $Z \in \varpi^a\Lambda \cap \mathfrak{g}^N(F)$. But then it follows from (1.1.2) with $c = 2d + C$ that

$$X \in Y + Z + \frac{1}{2}[Y, Z] + \varpi^{n+a+C+d}\Lambda \cap \varpi^a\mathfrak{m}^{\Lambda'} \subset \varpi^n\mathfrak{m}^{\Lambda'}.$$

That is, $\mathfrak{X}_n \subset \{\mathrm{Ad}(\exp X\phi(\varpi^n))N \,|\, X \in \varpi^n\mathfrak{m}^{\Lambda'}\}$. Conversely, if $X \in \varpi^n\mathfrak{m}^{\Lambda'}$, then (1.1.3) gives

$$\mathrm{Ad}(\exp X)N - N \in [X, N] + \varpi^{2n-2d-[\frac{2e}{p-1}]}\Lambda.$$

Since $(\Lambda')^*$ is a fixed lattice of $\mathfrak{g}(F)$, we have $\varpi^{n-2d-[\frac{2e}{p-1}]}\Lambda \subset (\Lambda')^*$ for sufficiently large n. Moreover in each simple component of $\mathfrak{g}(F)$, we have

$$\begin{aligned}B(\varpi^{-n}[X, N], Y) &= \varpi^{-n}\mathrm{tr}(XNY - NXY) = \varpi^{-n}\mathrm{tr}(NYX - NXY)\\ &= \varpi^{-n}\mathrm{tr}(N \cdot [Y, X]) = B(N, [Y, \varpi^{-n}X])\\ &= B_N(Y, \varpi^{-n}X) \in B_N(\mathfrak{m}^{\Lambda'}, \mathfrak{m}^{\Lambda'})\\ &\subset \mathcal{O}, \quad \forall \mathcal{Y} \in *'.\end{aligned}$$

That is, $\varpi^{-n}[X, N] \in (\Lambda')^*$ and we obtain $\mathrm{Ad}(\exp X)N - N \in \varpi^n(\Lambda')^*$. We conclude

$$\mathfrak{X}_n = \{\mathrm{Ad}(\exp X\phi(\varpi^n))N \,|\, X \in \varpi^n\mathfrak{m}^{\Lambda'}\}. \tag{1.12.1}$$

We now have

$$\dim E[\chi_n] = c_{\mathfrak{O}}(\pi)\mu_{\mathfrak{O}}(\mathfrak{X}_n) = c_{\mathfrak{O}}(\pi)\mathrm{meas}[\mathbf{K}_n\phi(\varpi^n)G^N(F)/G^N(F)]$$

$$
\begin{aligned}
&= c_{\mathfrak{O}}(\pi)\mathrm{meas}[(\mathbf{K}_n/\mathbf{K}_n \cap G^N(F))\phi(\varpi^n)] \\
&= c_{\mathfrak{O}}(\pi)\mathrm{meas}[\mathrm{Ad}(\phi(\varpi^{-n}))(\mathbf{K}_n/\mathbf{K}_n \cap G^N(F))] \\
&= c_{\mathfrak{O}}(\pi)|\det(\mathrm{Ad}(\phi(\varpi^{-n}))|\mathfrak{g}(F)/\mathfrak{g}^N(F))|\mathrm{meas}(\mathbf{K}_n/\mathbf{K}_n \cap G^N(F)).
\end{aligned}
$$

Noting that $\dim \mathfrak{m} = \dim \mathfrak{m}_1 + \sum_{i\geq 2} 2\dim \mathfrak{m}_i$, we have

$$
\begin{aligned}
\det(\mathrm{Ad}(\phi(\varpi^{-n}))|\mathfrak{g}(F)/\mathfrak{g}^N(F)) &= \det(\mathrm{Ad}(\phi(\varpi^{-n}))|\mathfrak{m}) \\
&= \left|\varpi^{-n\dim \mathfrak{m}_1} \cdot \prod_{i\geq 2} \varpi^{-ni\dim \mathfrak{m}_i} \cdot \varpi^{-n(2-i)\dim \mathfrak{m}_i}\right| \\
&= |\varpi^{-n\dim \mathfrak{m}}|.
\end{aligned}
$$

Also it follows from the choice of our measures in 1.8 that

$$
\mathrm{meas}(\mathbf{K}_n/\mathbf{K}_n \cap G^N(F)) = \mathrm{meas}(\varpi^n \mathfrak{m}^{\Lambda'}) = |\varpi^{n\dim \mathfrak{m}}|.
$$

We obtain $\dim E[\chi_n] = c_{\mathfrak{O}}(\pi)$. □

1.13

Lemma . *Let $N \in \mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$ be as above. Take $b > D$ and $X \in \varpi^{[n/2]+b}\Lambda \cap \mathfrak{g}^N(F)$. Then*

(i) $\exp X \in \mathrm{Norm}(\mathbf{K}_n, G(F))$*,* $\exp X \in \mathrm{Stab}(\chi_n, G(F))$*.*

(ii) In particular, $\pi(\exp X)$ *preserves* $E[\chi_n]$*. Moreover* $\pi(\exp X)|E[\chi_n] = \mathrm{id}$ *for sufficiently large n.*

Proof. (i) If $Y \in \varpi^n\Lambda'$, we have

$$
\mathrm{Ad}(\exp X)\exp Y = \exp(Y + [X,Y] + Z),
$$

where $Z \in \varpi^{2[n/2]+2b-d+n}\Lambda \subset \varpi^n\Lambda'$. Also we have $[X,Y] \in [\varpi^{[n/2]+b}\Lambda, \varpi^{n-d}\Lambda] \subset \varpi^{n+[n/2]+b-d}\Lambda \subset \varpi^n\Lambda'$. Thus $\exp X \in \mathrm{Norm}(\mathbf{K}_n, G(F))$. The latter assertion follows from

$$
\begin{aligned}
\chi_n(\mathrm{Ad}(\exp X)\exp Y) &= \psi(B(\varpi^{-2n}N, Y + [X,Y] + Z)) \\
&= \psi(N(\varpi^{-2n}N, Y))\psi(\varpi^{-2n}B_N(X,Y))\psi(B(\varpi^{-2n}N, Z)) \\
&= \chi_n(\exp Y),
\end{aligned}
$$

for $Y \in \varpi^n\Lambda'$. Note that $B_N(X,Y) = 0$ because $X \in \mathfrak{g}^N(F)$. Also $B(\varpi^{-2n}N, Z) \in \varpi^{-2n+2[n/2]+2b-d+n}\mathcal{O} \subset \varpi^{\in[\backslash/\in]-\backslash+\in\mathcal{C}+\triangledown\lceil}\mathcal{O}$.

(ii) We first modify the assertion. Take $H \in \mathfrak{g}(F)$ such that $\mathrm{ad}(H)N = 2N$.

Claim . $B([H,N],X) = 0$.

Proof. We may decompose $\mathfrak{g}$ as $\mathfrak{g} = \bigoplus_{i=0}^{r} \mathfrak{g}_i$, where $\mathfrak{g}_0 = \mathfrak{z}$ is the center and each $\mathfrak{g}_i$ is a simple Lie algebra. We note that this decomposition is orthogonal with respect to $B(\ ,\ )$. In fact, we have

$$B([X_i, Y_i], X_j) = B(X_i, [Y_i, X_j]) = B(X_i, 0) = 0$$

for any $X_i,\ Y_i \in \mathfrak{g}_i(F)$, $X_j \in \mathfrak{g}_j(F)$, $1 \le i \le r$, $0 \le j \le r$, $i \ne j$. But since $\mathfrak{g}_i$ $(1 \le i \le r)$ are simple, it is spanned by the elements $[X_i, Y_i]$ $(X_i,\ Y_i \in \mathfrak{g}_i(F))$. Thus the decomposition is orthogonal.

We write $N = \sum_{i=1}^{r} N_i$, $X = \sum_{i=0}^{r} X_i$ and $H = \sum_{i=0}^{r} H_i$, where N_i, X_i and H_i belong to $\mathfrak{g}_i(F)$. Then

$$\begin{aligned} B([H,N],X) &= B(\sum_{i,j=0}^{r} [H_i, N_j], \sum_{i=0}^{r} X_i) = B(\sum_{i=1}^{r} [H_i, N_i], \sum_{i=0}^{r} X_i) \\ &= \sum_{i=1}^{r} B([H_i, N_i], X_i). \end{aligned}$$

Thus we are reduced to the case when $\mathfrak{g}$ is simple.

Now we fix an embedding of F-Lie algebras $\mathfrak{g}(F) \ni X \hookrightarrow \widetilde{X} \in \mathfrak{gl}(n, F)$ in such a way that

$$B(X, Y) = \mathrm{tr}\widetilde{X}\widetilde{Y}$$

holds. We conclude

$$\begin{aligned} B([H,N],X) &= \mathrm{tr}\widetilde{[H,N]}\widetilde{X} = \mathrm{tr}\widetilde{H}\widetilde{N}\widetilde{X} - \mathrm{tr}\widetilde{N}\widetilde{H}\widetilde{X} \\ &= \mathrm{tr}\widetilde{N}\widetilde{X}\widetilde{H} - \mathrm{tr}\widetilde{N}\widetilde{H}\widetilde{X} = B(N, [X, H]) = B_N(X, H) \\ &= 0 \end{aligned}$$

□

This claim implies $B(N, X) = \frac{1}{2} B([H, N], X) = 0$ and hence

$$\psi(B(\varpi^{-2n} N, X)) = 1. \tag{1.13.1}$$

Next we note that if n is sufficiently large, $\exp X$ belongs to a compact open subgroup $\mathbf{K}_m$ for some m. Considering $\pi|_{\mathbf{K}_m}$ as a unitary representation of $\mathbf{K}_m$, we see that $\pi(\exp X)|E[\chi_n]$ is a unitary operator. In particular, $\pi(\exp X)|E[\chi_n] = \mathrm{id}$ if and only if

$$\mathrm{tr}\pi(\exp X)|E[\chi_n] = \dim E[\chi_n]. \tag{†}$$

We recall the function φ_n from the proof of Prop. 1.11. Usual calculation shows that $\pi(\varphi_n)$ is the $\mathrm{meas}\mathbf{K}_n$-multiple of the $\mathbf{K}_n$-equivariant projector onto $E[\chi_n]$. Moreover we have ($\delta_{\exp X}$ is the Dirac distribution at $\exp X$)

$$\pi(\varphi_n * \delta_{\exp X})\xi = \mathrm{meas}\mathbf{K}_n \pi(\exp X)\xi, \quad \xi \in E[\chi_n]. \tag{1.13.2}$$

Using (1.13.2), the formula (†) to be shown becomes

$$\frac{1}{\mathrm{meas}\mathbf{K}_n} \mathrm{tr}(\pi(\varphi_n * \delta_{\exp X})|E[\chi_n]) = \dim E[\chi_n]. \tag{‡}$$

To prove this, we calculate $\mathrm{tr}\pi(\varphi_n * \delta_{\exp X})|E[\chi_n]$ by means of Th.1.8. Write R_c^∞ for the right regular representation of $G(F)$ on $C_c^\infty(G(F))$. We know that [BZ, 1.25]

$$\varphi_n * \delta_{\exp X} = R_c^\infty(\exp(-X))\varphi_n.$$

Calculate the Fourier transform:

$$((R_c^\infty(\exp(-X))\varphi_n) \circ \exp)^\wedge(Y) = \int_{\mathfrak{g}(F)} \varphi_n(\exp T \exp(-X))\psi(B(T,Y))\, dT.$$

The integrand is not zero only if $\exp T \in \mathbf{K}_n \exp X$. Lem.1.3 (2) gives

$$\log(\exp Y \exp X) = X + Y + \frac{1}{2}[Y,X] + Z, \quad Z \in \varpi^{[n/2]+b-d+n+C}\Lambda,$$

and hence

$$\varpi^n\Lambda' \ni S \longmapsto \exp(X+S) \in \mathbf{K}_n \exp X$$

is a bijection. (Note that $\exp|_{X+\varpi^n\Lambda'}$ is injective for sufficiently large n.) Thus we have

$$\begin{aligned} &((R_c^\infty(\exp(-X))\varphi_n) \circ \exp)^\wedge(Y) \\ &= \psi(B(X,Y)) \int_{\mathfrak{g}(F)} \varphi_n(\exp(X+S)\exp(-X))\psi(B(S,Y))\, dS. \end{aligned}$$

As in the proof of (1.1.2), we have

$$\exp(X+S) = 1 + X + S + \frac{X^2 + XS + SX + S^2}{2} + \frac{X^3}{6} + \varpi^{2n+1}\Lambda,$$
$$\exp(-X) = 1 - X + \frac{X^2}{2} - \frac{X^3}{6} + \varpi^{2n+1}\Lambda,$$

and hence

$$\begin{aligned} \exp(X+S)\exp(-X) &= 1 - X + X + S + \frac{X^2}{2} - X^2 - SX + \frac{X^2}{2} + \frac{XS+SX}{2} + \frac{S^2}{2} \\ &\quad - \frac{X^3}{6} + \frac{X^3}{2} + \frac{SX^2}{2} - \frac{X^3}{2} - \frac{XSX}{2} - \frac{SX^2}{2} - \frac{S^2X}{2} + \frac{X^3}{6} + \varpi^{2n+1}\Lambda \\ &= 1 + S + \frac{[X,S]}{2} + \frac{S^2}{2} + \varpi^{2n+1}\Lambda. \end{aligned}$$

Also we have

$$\exp\big(S + \frac{[X,S]}{2}\big) \in 1 + S + \frac{[X,S]}{2} + \frac{S^2}{2} + \varpi^{2n+1}\Lambda.$$

This enables us to take $T' \in \varpi^{2n+1}\Lambda$ such that

$$\exp(X+S)\exp(-X) = \exp(S + \frac{[X,S]}{2} + T').$$

In particular we have

$$\begin{aligned} \varphi_n(\exp(X+S)\exp(-X)) &= \psi(B(\varpi^{-2n}N, S + \frac{[X,S]}{2} + T'))^{-1} \\ &= \psi(B(\varpi^{-2n}N,S))^{-1}\psi(\varpi^{-2n}2^{-1}B_N(X,S))^{-1} = \varphi_n(\exp S), \end{aligned}$$

and

$$((R_c^\infty(\exp(-X))\varphi_n)\circ\exp)^\wedge(Y) = \psi(B(X,Y))\int_{\mathfrak{g}(F)}\varphi_n(\exp S)\psi(B(S,Y))\,dS$$
$$= \psi(B(X,Y))\widehat{\varphi_n\circ\exp}(Y).$$

Now Th.1.8 gives

$$\operatorname{tr}\pi(\varphi_n * \delta_{\exp X}) = \operatorname{tr}\pi(R_c^\infty(\exp(-X))\varphi_n)$$
$$= \sum_{\mathfrak{O}'\in\mathcal{N}(\mathfrak{g}(\mathcal{F}))} c_{\mathfrak{O}'}(\pi)\int_{\mathfrak{O}'}((R_c^\infty(\exp(-X))\varphi_n)\circ\exp)^\wedge(Y)\,d\mu_{\mathfrak{O}'}(Y)$$

since $N\in\mathfrak{O}\in\mathcal{N}_{\mathcal{B}}(\pi)^{\sup}$ (cf. proof of 1.12)

$$= c_{\mathfrak{O}}(\pi)\int_{\mathfrak{O}}\psi(B(X,Y))(\widehat{\varphi_n\circ\exp})(Y)\,d\mu_{\mathfrak{O}}(Y)$$

using the formula for $\widehat{\varphi_n\circ\exp}$ (see 1.11)

$$= c_{\mathfrak{O}}(\pi)\int_{\mathfrak{O}\cap\varpi^{-2n}N+(\varpi^n\Lambda')^*}\psi(B(X,Y))\mathrm{meas}\mathbf{K}_n\,d\mu_{\mathfrak{O}}(Y)$$
$$= \mathrm{meas}\mathbf{K}_n\cdot c_{\mathfrak{O}}(\pi)\int_{\mathfrak{X}_n}\psi(B(X,Y))\,d\mu_{\mathfrak{O}}(Y),$$

where $\mathfrak{X}_n := \mathfrak{O}\cap\varpi^{-2n}N+(\varpi^n\Lambda')^*$. Using the description (1.12.1) of $\mathfrak{X}_n$, we may write $Y\in\mathfrak{X}_n$ as $Y=\varpi^{-2n}\mathrm{Ad}(\exp Z)N$ with $Z\in\varpi^n\mathfrak{m}^{\Lambda'}$. Then

$$\psi(B(X,Y)) = \psi(B(X,\varpi^{-2n}\mathrm{Ad}(\exp Z)N))$$
$$\overset{(1.1.3)}{=}\psi(\varpi^{-2n}B(X,N+[Z,N]+Z')),\quad \exists Z'\in\varpi^{2n-2d-\left[\frac{2e}{p-1}\right]}\Lambda$$

noting $b\geq D\geq\frac{3e}{p-1}+3d+C+2$,

$$= \psi(\varpi^{-2n}B(X,N))\psi(\varpi^{-2n}B(X,[Z,N]))$$

noting $B(X,[Z,N])=B(N,[X,Z])$

$$= \psi(\varpi^{-2n}B(X,N))\psi(\varpi^{-2n}B_N(X,Z))$$
$$= \psi(\varpi^{-2n}B(X,N)).$$

Hence we conclude

$$\frac{1}{\mathrm{meas}\mathbf{K}_n}\operatorname{tr}(\pi(\varphi_n*\delta_{\exp X})|E[\chi_n])$$
$$= c_{\mathfrak{O}}(\pi)\int_{\mathfrak{X}_n}\psi(B(X,Y))\,d\mu_{\mathfrak{O}}(Y) = \psi(\varpi^{-2n}B(X,N))c_{\mathfrak{O}}(\pi)\mathrm{meas}\mathfrak{X}_n$$
$$\overset{(1.13.1)}{=}\dim E[\chi_n]$$

which is (‡). □

1.14 Key injectivity

Corollary . *For sufficiently large n, $j : E[\chi'_n]/E'_{n,\chi} \to E_{V,\chi}$ in 1.9 is injective and its image is $E^{\mathbf{L}}_{V,\chi}$.*

Proof. Suppose that n is sufficiently large so that Lemma 1.13 holds.

Let $\mathfrak{j}_n := \varpi^{[n/2]+b}\Lambda \cap \mathfrak{g}_1^N(F)$, $\mathbf{J}_n := \exp \mathfrak{j}_n$. Then

$$\mathbf{J}_n\mathbf{K}_n \subset G(F) \text{ is a subgroup.} \tag{1.14.1}$$

In fact for $X_i \in \mathfrak{j}_n$ and $Y_i \in \varpi^n\Lambda'$ $(i = 1, 2)$ we have $Y \in \varpi^n\Lambda'$ such that

$$\begin{aligned} \exp X_1 \exp Y_1 \exp X_2 \exp Y_2 &\overset{1.13(\mathrm{i})}{=} \exp X_1 \exp Y \exp X_2 \\ &= \exp X_1 \exp X_2 \exp(\mathrm{Ad}(\exp(-X_2))Y). \end{aligned}$$

Since 1.13 (i) assures that $\exp(\mathrm{Ad}(\exp(-X_2))Y)$ is in $\mathbf{K}_n$, we have only to show that $\exp X_1 \exp X_2 \in \mathbf{J}_n\mathbf{K}_n$. For this we use (1.1.2) to have

$$\exp X_1 \exp X_2 = \exp(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + X'), \quad \exists X' \in \varpi^{2[n/2]+2b}\Lambda \cap \mathfrak{g}^N(F),$$

and

$$\begin{aligned} &\exp(-(X_1 + X_2)) \exp X_1 \exp X_2 = \exp(-(X_1 + X_2)) \exp(X_1 + X_2 + \frac{1}{2}[X_1, X_2] + X') \\ &\overset{(1.1.2)}{=} \exp\left(\frac{[X_1, X_2]}{2} + X' - \frac{1}{2}[X_1 + X_2, \frac{[X_1, X_2]}{2} + X'] + X''\right), \quad \exists X'' \in \varpi^{2[n/2]+2b}\Lambda \cap \mathfrak{g}^N(F). \end{aligned}$$

But since

$$\left.\begin{array}{l} \frac{[X_1,X_2]}{2} \in \varpi^{2[n/2]+2b}\Lambda \cap \mathfrak{g}^N(F) \subset \varpi^{n+b}\Lambda \cap \mathfrak{g}^N(F) \subset \varpi^{n+2d}\Lambda' \cap \mathfrak{g}^N(F), \\ X', X'' \in \varpi^{n+2d}\Lambda' \cap \mathfrak{g}^N(F), \\ \frac{1}{2}[X_1 + X_2, \frac{[X_1,X_2]}{2} + X'] \in \varpi^{3[n/2]+3b}\Lambda \cap \mathfrak{g}^N(F) \subset \varpi^{n+2d}\Lambda' \cap \mathfrak{g}^N(F) \end{array}\right\} \subset \varpi^n\Lambda',$$

we see that

$$\exp X_1 \exp X_2 \in \exp(X_1 + X_2) \exp(\varpi^n\Lambda' \cap \mathfrak{g}^N(F)) \subset \mathbf{J}_n\mathbf{K}_n. \tag{$\clubsuit$}$$

Now 1.13 (i) allows us to extend χ_n trivially on $\mathbf{J}_n$ to a character of $\mathbf{J}_n\mathbf{K}_n$. This is well-defined by ($\clubsuit$). Then 1.13 (ii) gives

$$\begin{aligned} E[\chi_n] &= \{\xi \in E \mid \pi(j)\pi(k)\xi = \chi_n(k)\xi, \forall jk \in \mathbf{J}_n\mathbf{K}_n\} \\ &= \{\xi \in E \mid \pi(x)\xi = \chi_n(x)\xi, \forall x \in \mathbf{J}_n\mathbf{K}_n\}. \end{aligned}$$

This amounts to

$$\begin{aligned} E[\chi'_n] &= \pi(\phi(\varpi^{-n}))E[\chi_n] \\ &= \{\xi \in E \mid \pi(\mathrm{Ad}(\phi(\varpi^{-n}))x)\xi = \chi_n(x)\xi, x \in \mathbf{J}_n\mathbf{K}_n\} \\ &= \{\xi \in E \mid \pi(x)\xi = \chi'_n(x)\xi, \forall x \in \mathrm{Ad}(\phi(\varpi^{-n}))(\mathbf{J}_n) \cdot \mathbf{K}'_n\}. \end{aligned}$$

Here $\chi'_n : \mathrm{Ad}(\phi(\varpi^{-n}))(\mathbf{J}_n)\mathbf{K}'_n \to \mathbb{C}^1$ is the trivial extension of χ'_n on $\mathrm{Ad}(\phi(\varpi^{-n}))(\mathbf{J}_n)$. Thus for $m \geq n$, we have

$$\begin{aligned} I'_{n,m}(\xi) &\overset{(1.9.4)}{=} \frac{1}{\mathrm{meas}(V(F)\cap \mathbf{K}'_m)} \int_{V(F)\cap\mathbf{K}'_m} \chi(k^{-1})\pi(k)\xi\, dk \\ &\overset{(1.7.4)}{=} \frac{1}{\mathrm{meas}(V(F)\cap \mathbf{K}'_m)} \cdot (\mathrm{const}) \int_{\mathrm{Ad}(\phi(\varpi^{-m}))(\mathbf{J}_m)\mathbf{K}'_m\cap V(F)} \chi(k^{-1})\pi(k)\xi\, dk. \end{aligned}$$

Suppose that $\xi \in E$ is contained in $\ker j \subset E(V,\chi)$. Since $\{\mathrm{Ad}(\phi(\varpi^{-m}))(\mathbf{J}_m)\mathbf{K}'_m \cap V(F)\}_m$ is an increasing series of compact open subgroup of $V(F)$ which exhausts $V(F)$, [BZ, 2.33] implies that there exists $m \in \mathbb{N}$ such that

$$\int_{\mathrm{Ad}(\phi(\varpi^{-m}))(\mathbf{J}_m)\mathbf{K}'_m\cap V(F)} \chi(k^{-1})\pi(k)\xi\, dk = 0.$$

That is, $\xi \in \mathrm{Ker} I'_{n,m} \subset E'_{n,\chi}$ and j is injective.

Next we calculate $j(E[\chi'_n])$. We have already seen that it is contained in $E^{\mathbf{L}}_{V,\chi}$. Take $\overline{\xi} \in E^{\mathbf{L}}_{V,\chi}$ and let $\xi \in j^{-1}(\overline{\xi})$. Then as in the proof of 1.10, we have $M(\xi) \in \mathbb{N}$ such that

$$I'_m(\xi) - \xi = \frac{1}{\mathrm{meas}(V(F)\cap \mathbf{K}'_m)} \int_{V(F)\cap\mathbf{K}'_m} (\overline{\chi(k)}\pi(k)\xi - \xi)\, dk, \quad \forall m \geq M(\xi).$$

Let $\mathbf{K}_\xi := \mathrm{Stab}(\xi, V(F)\cap\mathbf{K}'_m) \cap \mathrm{Ker}\chi$ and write

$$V(F)\cap\mathbf{K}'_m = \coprod_{i=1}^{r} k_i \mathbf{K}_\xi$$

for the coset decomposition. Then we have

$$\begin{aligned} I'_m(\xi) - \xi &= \frac{1}{r\cdot \mathrm{meas}\mathbf{K}_\xi} \sum_{i=1}^{r} \int_{\mathbf{K}_\xi} (\overline{\chi(k_i x)}\pi(k_i x)\xi - \xi)\, dx \\ &= \frac{1}{r\cdot \mathrm{meas}\mathbf{K}_\xi} \sum_{i=1}^{r} \int_{\mathbf{K}_\xi} (\overline{\chi(k_i)}\pi(k_i)\xi - \xi)\, dx \\ &= \frac{1}{r} \sum_{i=1}^{r} \overline{\chi(k_i)}(\pi(k_i)\xi - \chi(k_i)\xi), \quad \forall m \geq M(\xi). \end{aligned}$$

Since the last line is contained in $E(V,\chi)$, we see that $j(I'_m(\xi)) = \overline{\xi}$ for any $m \geq M(\xi)$.

Now let $\overline{\xi}_1, \ldots, \overline{\xi}_T$ be linearly independent elements in $E^{\mathbf{L}}_{V,\chi}$. Putting $M := \sup_{1\leq i\leq T} M(\xi_i)$, we set $\eta_i := I'_n(\xi_i) \in E[\chi'_n]/E'_{n,\chi}$ for a choice of inverse images ξ_i for $\overline{\xi}_i$. Then we have

(1) $\dim E^{\mathbf{L}}_{V,\chi} \leq \dim E[\chi'_n]/E'_{n,\chi} \leq \dim E[\chi_n] = c_{\mathfrak{O}}(\pi)$ is finite.

(2) (By taking a basis of $E^{\mathbf{L}}_{V,\chi}$ as $\{\xi_i\}$) for $n \geq M$, $E[\chi'_n]/E'_{n,\chi} \xrightarrow{j_n} E^{\mathbf{L}}_{V,\chi}$ is surjective.

□

1.15 Injectivity continued

Lemma . *$I_{n,n+1}$ is injective for sufficiently large n.*

Proof. For the brevity, write $d\mu_n$ for the measure on $\mathbf{K}_n$ such that $\mu_n(\mathbf{K}_n) = 1$. Then one has

$$
\begin{aligned}
& I_{n+1,n} \circ I_{n,n+1} \circ I_n(\xi) \\
&= \int_{\mathbf{K}_n} \int_{\mathbf{K}_{n+1}} \int_{\mathbf{K}_n} \overline{\chi}_n(\gamma)\overline{\chi}_{n+1}(h)\overline{\chi}_n(\gamma')\pi(\gamma\phi(\varpi)^{-1}h\phi(\varpi)\gamma')\xi \, d_{\mu_n}(\gamma') \, d_{\mu_{n+1}}(h) \, d_{\mu_n}(\gamma) \\
&= \int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \overline{\chi}_n(\gamma)\pi(\gamma) \int_{\mathbf{K}_{n+1}} \overline{\chi}_{n+1}(h)\overline{\chi}_n(\gamma')\pi(\phi(\varpi)^{-1}h\phi(\varpi)\gamma')\xi \, d_{\mu_{n+1}}(h) \, d\mu_n(\gamma') \, d\mu_n(\gamma).
\end{aligned}
$$

The inner integral reads

$$
\begin{aligned}
& \int_{\mathbf{K}_{n+1}} \overline{\chi}_{n+1}(h)\overline{\chi}_n(\gamma')\pi(\phi(\varpi)^{-1}h\phi(\varpi)\gamma')\xi \, d_{\mu_{n+1}}(h) \\
&= \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \int_{\mathbf{K}_{n+1}\cap\mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi}_{n+1}(\dot\gamma h)\overline{\chi}_n(\gamma')\pi(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma h)\gamma')\xi \, d_{\mu_{n+1}}(h)
\end{aligned}
$$

putting $x := \mathrm{Ad}(\phi(\varpi^{-1}))h$,

$$
\begin{aligned}
&= \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \int_{\mathrm{Ad}(\phi(\varpi^{-1}))(\mathbf{K}_{n+1})\cap\mathbf{K}_n} \overline{\chi}_{n+1}(\dot\gamma\mathrm{Ad}(\phi(\varpi))x)\overline{\chi}_n(\gamma') \\
&\qquad\qquad \pi(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma)x\gamma')\xi \, d\mu_{n+1}(\mathrm{Ad}(\phi(\varpi))(x)).
\end{aligned}
$$

Moreover if we write $x = \exp X$ $(X \in \varpi^n\Lambda')$, we have

$$
\begin{aligned}
\chi_{n+1}(\mathrm{Ad}(\phi(\varpi))(x)) &= \psi(B(\varpi^{-2n-2}N, \mathrm{Ad}(\phi(\varpi))X)) \\
&= \psi(B(\varpi^{-2n-2}\mathrm{Ad}(\phi(\varpi^{-1}))N, X)) = \psi(B(\varpi^{-2n}N, X)) \\
&= \chi_n(x).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
& I_{n+1,n} \circ I_{n,n+1} \circ I_n(\xi) \\
&= \int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \overline{\chi}_n(\gamma)\pi(\gamma) \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \int_{\mathrm{Ad}(\phi(\varpi^{-1}))(\mathbf{K}_{n+1})\cap\mathbf{K}_n} \overline{\chi}_{n+1}(\dot\gamma)\overline{\chi}_n(x\gamma') \\
&\qquad\qquad \pi(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma)x\gamma')\xi \, d\mu_{n+1}(\mathrm{Ad}(\phi(\varpi))x) \, d\mu_n(\gamma') \, d\mu_n(\gamma) \\
&= \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi}_{n+1}(\dot\gamma) \int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \varphi_n(\gamma)\pi(\gamma)\pi(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma) \\
&\qquad \int_{\mathrm{Ad}(\phi(\varpi^{-1}))(\mathbf{K}_{n+1})\cap\mathbf{K}_n} \varphi_n(x\gamma')\pi(x\gamma')\xi \, d\mu_{n+1}(\mathrm{Ad}(\phi(\varpi))x) \, d\mu_n(\gamma') \, d\mu_n(\gamma)
\end{aligned}
$$

putting γ' for $x\gamma'$

$$= \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi}_{n+1}(\dot\gamma) \int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \varphi_n(\gamma)\pi(\gamma)\pi(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)$$
$$\int_{\mathrm{Ad}(\phi(\varpi^{-1}))(\mathbf{K}_{n+1})\cap\mathbf{K}_n} \varphi_n(\gamma')\pi(\gamma')\xi \, d\mu_{n+1}(\mathrm{Ad}(\phi(\varpi))x) \, d\mu_n(\gamma') \, d\mu_n(\gamma)$$
$$= \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi}_{n+1}(\dot\gamma)\mu_{n+1}(\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n)$$
$$\int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \varphi_n(\gamma)\pi(\gamma)\pi(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)\varphi_n(\gamma')\pi(\gamma')\xi \, d\mu_n(\gamma') \, d\mu_n(\gamma)$$
$$= \mu_{n+1}(\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n) \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi_{n+1}(\dot\gamma)}$$
$$\times \int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \varphi_n(\gamma)\varphi_n(\gamma')\pi(\gamma \mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma\cdot\gamma')\xi \, d\mu_n(\gamma') \, d\mu_n(\gamma).$$

noting $\mathrm{Ad}(\phi(\varpi^{-1}))\mathbf{K}_{n+1} \subset \mathrm{Norm}(\mathbf{K}_n, G(F))$ for $n >> 0$ (cf. 1.5 (2)), we put $\gamma := \mathrm{Int}(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)^{-1}(\gamma)$

$$= \mu_{n+1}(\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n) \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi_{n+1}(\dot\gamma)}$$
$$\int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \varphi_n(\mathrm{Int}(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)(\gamma))\varphi_n(\gamma')\pi(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma)\cdot\gamma\gamma')\xi \, d\mu_n(\gamma') \, d\mu_n(\gamma)$$
$$= \mu_{n+1}(\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n) \sum_{\dot\gamma \in \mathbf{K}_{n+1}/\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n} \overline{\chi_{n+1}(\dot\gamma)}\pi(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma))$$
$$\int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \overline{\chi_n(\mathrm{Int}(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)(\gamma)\cdot\gamma')}\pi(\gamma\gamma')\xi \, d\mu_n(\gamma') \, d\mu_n(\gamma).$$

Putting $\gamma\gamma' = \gamma'$, this double integral becomes

$$\int_{\mathbf{K}_n} \int_{\mathbf{K}_n} \overline{\chi_n(\mathrm{Int}(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)\gamma\cdot\gamma^{-1}\gamma')}\pi(\gamma')\xi \, d\mu_n(\gamma') \, d\mu_n(\gamma)$$
$$= \int_{\mathbf{K}_n} \mathrm{Int}(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma^{-1}))\overline{\chi}_n \cdot \chi_n(\gamma) \, d\mu_n(\gamma) \cdot \int_{\mathbf{K}_n} \overline{\chi_n(\gamma')}\pi(\gamma')\xi \, d\mu_n(\gamma')$$
$$= \begin{cases} I_n(\xi) & \text{if } \mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma) \in \mathrm{Stab}(\chi_n, G(F)), \\ 0 & \text{otherwise.} \end{cases}$$

Thus we conclude

$$I_{n+1,n}\circ I_{n,n+1}\circ I_n(\xi)$$
$$= \mu_{n+1}(\mathbf{K}_{n+1}\cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n) \sum_{\substack{\dot\gamma \in \mathbf{K}_{n+1}/\mathrm{Ad}(\phi(\varpi))\mathbf{K}_n\cap\mathbf{K}_{n+1} \\ \dot\gamma\in \mathrm{Ad}(\phi(\varpi))\mathrm{Stab}(\chi_n, G(F))}} \overline{\chi_{n+1}(\dot\gamma)}\pi(\mathrm{Ad}(\phi(\varpi^{-1}))(\dot\gamma))I_n(\xi).$$

In the sum above we see

$$\begin{aligned}\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma &\in \mathrm{Ad}(\phi(\varpi^{-1}))(\mathbf{K}_{n+1}) \cap \mathrm{Stab}(\chi_n, G(F)) \\ &= \exp(\mathrm{Ad}(\phi(\varpi^{-1}))(\varpi^{n+1}\Lambda')) \cap \mathrm{Stab}(\chi_n, G(F)) \\ &\subset \exp(\varpi^{[n/2]+b}\Lambda') = \mathbf{K}_{[n/2]+b} \cap \mathrm{Stab}(\chi_n, G(F)) \\ &\overset{1.6(2)}{\subset} \exp(\varpi^{[n/2]+b}\Lambda \cap \mathfrak{g}^N(F))\mathbf{K}_n.\end{aligned}$$

We may suppose that $\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma = \exp X$ with $X \in \varpi^{[n/2]+b}\Lambda \cap \mathfrak{g}^N(F)$. We apply 1.13 (ii) to have $\pi(\mathrm{Ad}(\phi(\varpi^{-1}))\dot\gamma)I_n(\xi) = I_n(\xi)$. Moreover

$$\chi_{n+1}(\dot\gamma) = \psi(B(\varpi^{-2n-2}N, \mathrm{Ad}(\phi(\varpi))X)) = \psi(B(\varpi^{-2n}N, X)) \overset{(1.13.1)}{=} 1$$

because $X \in \mathfrak{g}^N(F)$. We obtain

$$\begin{aligned}I_{n+1} \circ I_{n,n+1} \circ I_n(\xi) &= \mu_{n+1}(\mathbf{K}_{n+1} \cap \mathrm{Ad}(\phi(\varpi))\mathbf{K}_n) \times \sum_{\substack{\dot\gamma \in \mathbf{K}_{n+1}/\mathrm{Ad}(\phi(\varpi))(\mathbf{K}_n)\cap\mathbf{K}_{n+1} \\ \dot\gamma \in \mathrm{Ad}(\phi(\varpi))\mathrm{Stab}(\chi_n, G(F))}} I_n(\xi) \\ &= \kappa I_n(\xi)\end{aligned}$$

for some non-zero constant κ. Thus $I_{n+1,n} \circ I_{n,n+1}$ is injective and so is $I_{n,n+1}$. □

1.16 The result

Theorem . *Let (π, E) be an irreducible admissible representation of $G(F)$. Then*

$$\mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}} = \mathcal{N}_{\mathrm{Wh}}(\pi)^{\mathrm{sup}}.$$

Proof. Take $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$, $N \in \mathfrak{O} \cap \Lambda$ and $\phi : \mathbb{G}_m \to G$ such that $\mathrm{Ad}(\phi(t))N = t^{-2}N$. Then 1.14 implies

$$\mathcal{W}_{\mathcal{N},\phi}(\pi) \overset{(1.7.4)}{\simeq} \mathcal{E}^{\mathbf{L}}_{\mathcal{V},\chi} \neq \prime$$

if and only if $E[\chi'_n]/E'_{n,\chi} \neq 0$ for any sufficiently large n. Since $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$, we have

$$0 \neq c_{\mathfrak{O}}(\pi) \overset{1.12}{=} \dim E[\chi_n] = \dim E[\chi'_n].$$

Now we take n sufficiently large so that 1.15 holds. Then noting (1.9.1),

$$I'_{n,m} = I'_{n,n+1} \circ \cdots \circ I'_{m-1,m}$$

is injective and hence $E'_{n,\chi} = \{0\}$. Thus $E[\chi'_n]/E'_{n,\chi} \neq 0$, and we have shown that $\mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}} \subset \mathcal{N}_{\mathrm{Wh}}(\pi)$.

Let $\mathfrak{O} \in \mathcal{N}_{\mathrm{Wh}}(\pi)$. Prop.1.11 assures that for $N \in \mathfrak{O}$, there exists $\mathfrak{O}' \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$ such that $N \in \overline{\mathfrak{O}'}$. This implies that $\overline{\mathfrak{O}} \subset \overline{\mathfrak{O}'}$. The previous paragraph asserts that $\mathfrak{O}' \in \mathcal{N}_{\mathrm{Wh}}(\pi)$. In particular, if $\mathfrak{O} \in \mathcal{N}_{\mathrm{Wh}}(\pi)^{\mathrm{sup}}$, the maximality implies $\mathfrak{O} = \mathfrak{O}'$. Hence $\mathcal{N}_{\mathrm{Wh}}(\pi)^{\mathrm{sup}} \subset \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$.

Conversely, take $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$. Then it is contained in $\mathcal{N}_{\mathrm{Wh}}(\pi)$, and we can take $\mathfrak{O}' \in \mathcal{N}_{\mathrm{Wh}}(\pi)^{\mathrm{sup}} \subset \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$ such that $\overline{\mathfrak{O}} \subset \overline{\mathfrak{O}'}$. Since $\mathfrak{O}$, $\mathfrak{O}' \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$, we have $\mathfrak{O} = \mathfrak{O}' \in \mathcal{N}_{\mathrm{Wh}}(\pi)^{\mathrm{sup}}$. □

1.17

Corollary . *Let* $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}}$. *Then we have* $c_{\mathfrak{O}}(\pi) = \dim \mathcal{W}_{\mathcal{N},\phi}(\pi)$ *for any choice of* $N \in \mathfrak{O}$ *and* ϕ.

Proof. Let $n \in \mathbb{N}$ be sufficiently large and $\mathfrak{O} \in \mathcal{N}_{\mathcal{B}}(\pi)^{\mathrm{sup}} = \mathcal{N}_{\mathrm{Wh}}(\pi)^{\mathrm{sup}}$. Then

$$\begin{aligned}\dim \mathcal{W}_{\mathcal{N},\phi}(\pi) &\overset{(1.7.4)}{=} \dim E^{\mathbf{L}}_{V,\chi} \overset{1.14}{=} \dim E[\chi'_n]/E'_{\chi,n} \\ &= \dim E[\chi'_n]/\bigcup_{m>n} \ker(I'_{n,m}) \overset{1.15}{=} \dim E[\chi_n] \\ &\overset{1.12}{=} c_{\mathfrak{O}}(\pi).\end{aligned}$$

□

References

[BZ] I.N. Bernstein and A.V. Zelevinskii, *Representations of the group $GL(n, F)$ where F is a non-archimedean local field*, Russian Math. Surveys 31:3 (1976), pp. 1–68.

[HC] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Collected Works, vol. 4.

[H] Howe, R. *Kirillov theory for compact p-adic groups*, Pacific J. Math. **73** (1977), no. 2, 365–381.

[MW] C. Mœglin and J.-L. Waldspurger, *Modèles de Whittaker dégénérés pour des groupes p-adiques*, Math. Zeit **196** (1987) pp. 427–452.