A note on Langlands' classification and irreducibility of induced representations of p-adic groups

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Abstract

In this note, we present a proof of the Langlands classification of the irreducible admissible representations of reductive p-adic groups. Then we deduce certain irreducibility result for parabolically induced modules from discrete series representations.

Contents

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1 Introduction

In this note, we shall prove two fundamental results in the representation theory of p-adic groups.

The first is the Langlands classification of irreducible admissible representations of connected reductive p-adic groups (Th. 3.5). This famous theorem had originally been proved by Langlands for real Lie groups [9], then its p-adic group analogue was treated independently in [5] and [11]. But the latter contains no proof (The argument suggested in [5, XI.2] does not work.). Silberger's article is well-written but the key lemma [11, Lem.5.3] is not true. Since the theorem plays a fundamental role in the harmonic analysis on p-adic reductive groups, I think it is of some value to writing out a complete proof, although it seems to be well-known to experts. The proof given in this note follows Langlands' original argument in the real case [9], while we rely in an essential way on the infinitesimal characters for p-adic groups introduced by Bernstein [4].

In [12, IV.1], Waldspurger proved that the standard intertwining operators are rational functions on the variety of representations. We combine this with the Langlands classification, and show that the parabolically induced modules from discrete series representations are irreducible on a Zariski open subset of the variety of such representations (Cor. 4.3). Again this is well-known to experts. For example, Waldspurger himself mentioned it in his definition of Harish-Chandra's j and μ-functions in [12, p. 48, IV.3]. Also this was used by Bernstein and Deligne in their analysis of components (under infinitesimal characters) of Hecke algebras [4, Prop. 3.14].

The contents of each section are as follows. In § 2, we collect elementary facts and results on the structure of connected reductive p-adic groups and their representations. Here we emphasize features caused by the discreteness of the valuation on the base field (see e.g. § 2.3). Also in § 2.2, the geometry of the restricted roots, which are essential in the proof of the Langlands classification, is reviewed from [9, § 4]. In § 3, we prove the Langlands classification. After reviewing Langlands' lemma on the growth behavior of the matrix coefficients of standard modules and two geometric lemmas, the proof is given in § 3.3. In § 4 we prove the irreducibility result. First in § 4.1, we recollect the proof of Waldspurger's irreducibility theorem (Th. 4.2) to emphasize the role played by the Langlands classification. Then the irreducibility on Zariski open subsets is proved in § 4.2.

Throughout the notes, we use only basic results in the harmonic analysis on p-adic groups, which were proved in [2], [3], [8] and §§ I.1-IV.2 of [12].

2 Preliminary

Let F be a non-archimedean local field of any characteristic. We write $\mathcal{O}$, $\mathfrak{p}_F$ and $|\ |_F$ for the maximal compact subring of F, its unique maximal ideal and the module of F, respectively. We write q for the cardinality of the residue field of $\mathcal{O}$.

2.1 Structure of $G(F)$

Let G be a connected reductive F-group. We fix a maximal F-split torus A_0 so that its centralizer M_0 is a minimal Levi subgroup of G. Write $\mathcal{L}$, $\mathcal{F}$ for the set of F-Levi and F-parabolic subgroups of G, respectively, containing M_0. Each $P \in \mathcal{F}$ has a unique Levi component M in $\mathcal{L}$, while the set $\mathcal{P}(M)$ of $P \in \mathcal{F}$ having M as a Levi component is finite. For $P \in \mathcal{P}(M)$, we write $\bar{P}$ for the element of $\mathcal{P}(M)$ which is opposite to P with respect to M.

Take $M \in \mathcal{L}$. As usual, we have the real vector spaces $\mathfrak{a}_M = \mathrm{Hom}(X^*(M)_F, \mathbb{R})$, $\mathfrak{a}_M^* = X^*(M)_F \otimes \mathbb{R}$ dual to each other, and the Harish-Chandra map $H_M : M(F) \to \mathfrak{a}_M$ given by

$$\exp\langle \chi, H_M(m)\rangle = |\chi(m)|_F, \quad \forall \chi \in X^*(M)_F.$$

Here, $X^*(M)_F$ is the group of F-rational characters of M. We write $M(F)^1$ for the kernel of H_M.

We write A_G for the maximal F-split torus in the center Z_G of G. The canonical isomorphism $X^*(G)_F \otimes \mathbb{Q} \xrightarrow{\sim} X^*(A_G) \otimes \mathbb{Q}$ (induced by restriction) combined with the commutative diagram

$$\begin{array}{ccc} X^*(M)_F & \longrightarrow & X^*(A_M) \\ \uparrow & & \downarrow \\ X^*(G)_F & \longrightarrow & X^*(A_G) \end{array}$$

shows that $\mathfrak{a}_G$ and $\mathfrak{a}_G^*$ are canonical direct summands of $\mathfrak{a}_M$ and $\mathfrak{a}_G$, respectively. If we write $\mathfrak{a}_M^G$ for the annihilator of $X^*(G)_F$ in $\mathfrak{a}_M$ and $\mathfrak{a}_M^{G,*} := X^*(A_M/A_G) \otimes \mathbb{R}$, then we have the direct sum decompositions

$$\mathfrak{a}_M = \mathfrak{a}_M^G \oplus \mathfrak{a}_G, \quad \mathfrak{a}_M^* = \mathfrak{a}_M^{G,*} \oplus \mathfrak{a}_G^*$$

dual to each other. We denote the $\mathfrak{a}_M^G$ and $\mathfrak{a}_G$-components of $H \in \mathfrak{a}_M$ by H^G and H_G, respectively. Similarly, $\lambda \in \mathfrak{a}_M^*$ admits a decomposition $\lambda = \lambda^G \oplus \lambda_G$, ($\lambda^G \in \mathfrak{a}_M^{G,*}$, $\lambda_G \in \mathfrak{a}_G^*$).

We fix a maximal compact subgroup $\mathbf{K}$ of $G(F)$ which is in good position relative to A_0. Then we have the Iwasawa decomposition $G(F) = U(F)M(F)\mathbf{K}$ for any $P = MU \in \mathcal{F}$. We write the corresponding decomposition of $g \in G(F)$ as $g = u_P(g)m_P(g)k_P(g)$, where $u_P(g) \in U(F)$, $m_P(g) \in M(F)$ and $k_P(g) \in \mathbf{K}$ are of course not unique. We fix various measures as in [12]. In particular, on any subgroup $H(F) \subset G(F)$, we fix an invariant measure which assigns 1 to the subgroup $H(F) \cap \mathbf{K}$. These satisfy the integration formulae

$$\begin{aligned}\int_{G(F)} f(g)\,dg &= \int_{\mathbf{K}} \int_{M(F)} \int_{U(F)} f(umk)\delta_P(m)^{-1}\,du\,dm\,dk \\ &= \gamma(G/M)^{-1} \int_{U(F)} \int_{M(F)} \int_{\bar{U}(F)} f(um\bar{u})\delta_P(m)^{-1}\,d\bar{u}\,dm\,du\end{aligned}$$

for any continuous compactly supported function f on $G(F)$ and $P = MU \in \mathcal{F}$. Here δ_P is the modular character of $P(F)$ and

$$\gamma(G/M) := \int_{\bar{U}(F)} \delta_P(m_P(\bar{u}))\,d\bar{u}.$$

This constant is independent of $P \in \mathcal{P}(M)$ as the notation suggests [12, I.1 (3)].

2.2 Restricted roots

We continue to take $M \in \mathcal{L}$. We review some elementary results on restricted roots from [1, § 1], [9, § 4]. We write $K^M := K \cap M$ for any subgroup $K \subset G$ and $M \in \mathcal{L}$.

The set of roots of A_M in G is denoted by Σ_M. $P \in \mathcal{P}(M)$ determines the subset Σ_P of P-positive elements in Σ_M. The set of reduced roots in Σ_P, that is, $\alpha \in \Sigma_P$ such that $\alpha/n \notin \Sigma_P$ for any $n \geq 2$, is denoted by $\Sigma_P^{\rm red}$. Notice that Σ_M spans $\mathfrak{a}_M^{G,*}$.

Suppose $M = M_0$. We know from [6, Cor. 5.8] that $(\Sigma_0 = \Sigma_{M_0}, \mathfrak{a}_0^{G,*})$ is a root system. In particular, the set of coroots $\Sigma_0^\vee \subset X_*(A_0)$ is defined. For $P_0 \in \mathcal{P}(M_0)$, we have the set of simple roots Δ_{P_0} in the positive system Σ_{P_0}, and write $\Delta_{P_0}^\vee$ for the set of corresponding simple coroots. Of course, Δ_{P_0} and $\Delta_{P_0}^\vee$ are basis of $\mathfrak{a}_0^{G,*}$ and $\mathfrak{a}_0^G$, respectively. We write $\widehat{\Delta}_{P_0} = \{\varpi_\alpha \,|\, \alpha \in \Delta_{P_0}\}$ and $\widehat{\Delta}_{P_0}^\vee = \{\varpi_\alpha^\vee \,|\, \alpha \in \Delta_{P_0}\}$ for the basis of $\mathfrak{a}_0^{G,*}$ and $\mathfrak{a}_0^G$ dual to $\Delta_{P_0}^\vee$ and Δ_{P_0}, respectively. We write $W = W^G$ for the Weyl group of A_0 in G, which is the Weyl group of the root system $(\mathfrak{a}_0^{G,*}, \Sigma_0)$. This acts on $\mathfrak{a}_0$ and hence on $\mathcal{F}$, $\mathcal{L}$. We identify W with its fixed system of representatives in $\mathrm{Norm}(A_0, G(F))$.

For general $P = MU \in \mathcal{F}$, we choose $P_0 \in \mathcal{P}(M_0)$ contained in P and define $\Delta_P := \{(\alpha_0|_{\mathfrak{a}_M}) \,|\, \alpha_0 \in \Delta_{P_0} \setminus \Delta_{P_0^M}\}$. This is independent of the choice of $P_0 \subset P$. The coroot attached to $\alpha = (\alpha_0|_{\mathfrak{a}_M}) \in \Delta_P$ is defined to be $(\alpha_0^\vee)_M \in \mathfrak{a}_M^G$. We write $\Delta_P^\vee := \{\alpha^\vee \,|\, \alpha \in \Delta_P\}$. It follows from the M_0-case that Δ_P and $\Delta_P^\vee$ are basis of $\mathfrak{a}_M^{G,*}$ and $\mathfrak{a}_M^G$, respectively. Notice that their dual basis are given by

$$\widehat{\Delta}_P^\vee := \{\varpi_\alpha^\vee = \varpi_{\alpha_0}^\vee \,|\, (\alpha = \alpha_0|_{\mathfrak{a}_M}) \in \Delta_P\},$$
$$\widehat{\Delta}_P := \{\varpi_\alpha = \varpi_{\alpha_0} \,|\, (\alpha = \alpha_0|_{\mathfrak{a}_M}) \in \Delta_P\}$$

respectively. In general $(\Sigma_M, \mathfrak{a}_M^{G,*})$ is not necessarily a root system. We write $W(M) = W^G(M) := \mathrm{Stab}(M, W)/W$.

We list some basic properties of restricted roots. First $(\Sigma_0, \mathfrak{a}_0^{G,*})$, a root system, satisfies the following properties.

$$\langle \alpha, \alpha^\vee \rangle = 2, \quad \langle \alpha, \beta^\vee \rangle \leq 0, \quad \forall \alpha \neq \beta \in \Delta_{P_0}. \tag{2.1}$$

If we set $\mathfrak{a}_{P_0}^{*,+} := \{\lambda \in \mathfrak{a}_0^* \,|\, \alpha^\vee(\lambda) > 0, \, \alpha \in \Delta_{P_0}\}$, ${}^+\mathfrak{a}_{P_0}^* := \{\lambda \in \mathfrak{a}_0^* \,|\, \varpi_\alpha(\lambda) > 0, \, \alpha \in \Delta_{P_0}\}$ then we have

$$\mathfrak{a}_{P_0}^{*,+} \subset {}^+\mathfrak{a}_{P_0}^*. \tag{2.2}$$

This is an easy consequence of (2.1).

Let us establish analogous properties for general Σ_M. (2.2) implies $\langle \varpi_\alpha, \varpi_\beta^\vee \rangle \geq 0$ for any α, $\beta \in \Delta_{P_0}$. This simply restricts to

$$\langle \varpi_\alpha, \varpi_\beta^\vee \rangle \geq 0, \quad \forall \alpha, \, \beta \in \Delta_P.$$

Setting $\mathfrak{a}_P^{*,+} := \{\lambda \in \mathfrak{a}_M^* \,|\, \alpha^\vee(\lambda) > 0, \, \alpha \in \Delta_P\}$, ${}^+\mathfrak{a}_P^* := \{\lambda \in \mathfrak{a}_M^* \,|\, \varpi_\alpha^\vee(\lambda) > 0, \, \alpha \in \Delta_P\}$, this amounts to the assertion

$$\bar{\mathfrak{a}}_P^{*,+} \subset {}^+\bar{\mathfrak{a}}_P^*. \tag{2.3}$$

As opposed to (2.2), the inclusion is between the closures. If $\alpha = \alpha_0|_{\mathfrak{a}_M}$, $(\alpha_0 \in \Delta_{P_0} \setminus \Delta_{P_0^M})$, $\alpha = (\alpha_0)_M$ and $\alpha_0^M \in \mathfrak{a}_0^{M,*}$ can be written as

$$\alpha_0^M = \sum_{\beta \in \Delta_{P_0^M}} x_\beta \beta, \quad x_\beta \in \mathbb{R}.$$

Here the coefficient x_β is given by $\langle \alpha_0^M, \varpi_\beta^{\vee,M}\rangle = \langle \alpha_0, \varpi_\beta^{\vee,M}\rangle$. Since $\varpi_\beta^{\vee,M} \in \bar{\mathfrak{a}}_{P_0^M}^{M,+} \subset {}^+\bar{\mathfrak{a}}_{P_0^M}^M$, $\varpi_\beta^{\vee,M} = \sum_{\gamma\in\Delta_{P_0}^M} y_\gamma \gamma^\vee$ with $y_\gamma \ge 0$. Thus (2.1) implies

$$x_\beta = \sum_{\gamma\in\Delta_{P_0^M}} y_\gamma \langle \alpha_0, \gamma^\vee\rangle \le 0.$$

In particular, we have for $(\beta = \beta_0|_{\mathfrak{a}_M}) \ne \alpha \in \Delta_P$

$$\langle \alpha, \beta^\vee\rangle = \langle \alpha_0, \beta_0^\vee\rangle - \sum_{\gamma\in\Delta_{P_0^M}} x_\gamma \langle \gamma, \beta_0^\vee\rangle \le 0. \tag{2.4}$$

Finally we have

$$\langle \alpha, \alpha^\vee\rangle > 0, \quad \alpha \in \Delta_P. \tag{2.5}$$

Otherwise we have $\langle \alpha, \beta^\vee\rangle \le 0$ for any $\beta \in \Delta_P$ so that $\alpha \in -\bar{\mathfrak{a}}_P^{G,*,+} \cap {}^+\bar{\mathfrak{a}}_P^{G,*} = \{0\}$. This contradicts $\alpha \ne 0$.

2.3 Unramified quasi-characters

H_G allows us to associate to each $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$ a quasi-character

$$e^\lambda : G(F) \ni g \longmapsto \exp\langle \lambda, H_G(g)\rangle \in \mathbb{C}^\times. \tag{2.6}$$

We write $X(G(F)) := \{e^\lambda \,|\, \lambda \in \mathfrak{a}_{G,\mathbb{C}}^*\} = \mathrm{Hom}_{\mathrm{cont}}(G(F)/G(F)^1, \mathbb{C}^\times)$ and $X_u(G(F))$ for its subgroup of unitary elements. If we write $\mathfrak{a}_{G(F)}$ for the lattice $H_G(G(F)) \subset \mathfrak{a}_G$ and $\mathfrak{a}_{G(F)}^* \subset \mathfrak{a}_G^*$ for its dual lattice, then we have the isomorphism

$$\mathfrak{a}_{G,\mathbb{C}}^*/2\pi i \mathfrak{a}_{G(F)}^* \ni \lambda \overset{\sim}{\longmapsto} e^\lambda \in X(G(F)).$$

This defines a $\mathbb{C}$-torus structure on $X(G(F))$. For $\chi \in X(G(F))$, we write $\Re\chi := |\chi|$ and $\Im\chi := (\Re\chi)^{-1}\chi$. $\Re\chi$ is identified with an element of $\mathfrak{a}_G^*$ by (2.6).

Now we take $M \in \mathcal{L}$ and consider the restriction homomorphism $X(G(F)) \to X(M(F))$.

Lemma 2.1. *For any $\chi \in X^*(G)_F$, $\chi(M(F)) = \chi(G(F))$.*

Proof. It suffices to check this in the case $M = M_0$. We write G_{der} for the derived group of G and $G_{\mathrm{ab}} := G/G_{\mathrm{der}}$ for its abelianization. If we write $M_0^{\mathrm{der}} := M_0 \cap G_{\mathrm{der}}$, we have the embedding of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{\mathrm{der}} & \longrightarrow & G & \longrightarrow & G_{\mathrm{ab}} & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & M_0^{\mathrm{der}} & \longrightarrow & M_0 & \longrightarrow & G_{\mathrm{ab}} & \longrightarrow & 1 \end{array}$$

Since $X^*(G)_F = X^*(G_{\mathrm{ab}})_F$, we have only to check that the images of $G(F)$ and $M_0(F)$ in $G_{\mathrm{ab}}(F)$ coincide. For this, we take Galois cohomology to have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{\mathrm{der}}(F) & \longrightarrow & G(F) & \longrightarrow & G_{\mathrm{ab}}(F) & \longrightarrow & \mathrm{H}^1(F, G_{\mathrm{der}}) \\ & & \uparrow & & \uparrow & & \| & & \uparrow \\ 1 & \longrightarrow & M_0^{\mathrm{der}}(F) & \longrightarrow & M_0(F) & \longrightarrow & G_{\mathrm{ab}}(F) & \longrightarrow & \mathrm{H}^1(F, M_0^{\mathrm{der}}) \end{array}$$

Then, what we have to show is

$$\ker(G_{\mathrm{ab}}(F) \to \mathrm{H}^1(F, G_{\mathrm{der}})) = \ker(G_{\mathrm{ab}}(F) \to \mathrm{H}^1(F, M_0^{\mathrm{der}})).$$

But since the left hand side equals the kernel of $G_{\mathrm{ab}}(F) \to \mathrm{H}^1(F, M_0^{\mathrm{der}}) \to \mathrm{H}^1(F, G_{\mathrm{der}})$, this follows from the injectivity of $\mathrm{H}^1(F, M_0^{\mathrm{der}}) \to \mathrm{H}^1(F, G_{\mathrm{der}})$. (Notice that this last statement is equivalent to [6, Th. 4.13, Prop. 4.7] which asserts that the minimal parabolic subgroups are all $G(F)$-conjugate to each other.) □

Since $X^*(G)_F$ injects into $X^*(M)_F$ by restriction, we can take a basis $\{\chi_i\}_{1\le i\le n}$ of $X^*(M)_F$ so that $\{d_i\chi_i\}_{1\le i\le r}$ for some $d_i \in \mathbb{N}$ and $0 \le r < n$ is a basis of $X^*(G)_F$. If $|\chi_i(M(F))|_F = q^{m_i\mathbb{Z}}$, $(1 \le i \le n,\ m_i \in \mathbb{N})$, then

$$\mathfrak{a}^*_{M(F)} = \sum_{i=1}^{n} \mathbb{Z}(m_i \log q)^{-1}\chi_i.$$

Now Lem. 2.1 asserts that $|d_i\chi_i(G(F))|_F = |d_i\chi_i(M(F))|_F = q^{d_i m_i\mathbb{Z}}$, $(1 \le i \le r)$, so that

$$\mathfrak{a}^*_{G(F)} = \sum_{i=1}^{r} \mathbb{Z}\frac{d_i}{d_i m_i \log q}\chi_i$$

is a direct summand of $\mathfrak{a}^*_{M(F)}$. Hence $X(G(F)) \to X(M(F))$ is injective. If we write $X^G(M(F))$ for the group of quasi-characters of $M(F) \cap G(F)^1$ trivial on $M(F)^1$, then we summarize the argument as the exact sequence of $\mathbb{C}$-tori

$$1 \longrightarrow X(G(F)) \longrightarrow X(M(F)) \longrightarrow X^G(M(F)) \longrightarrow 1. \tag{2.7}$$

The maps are restrictions.

2.4 Representations

We freely use the results of [2], [3] and [8] on algebraic (or smooth) representations of reductive p-adic groups, which are summarized in [12, I]. Let us recall some of them. We write $\mathrm{Alg}(G(F))$ for the category of algebraic representations of $G(F)$. We adopt the convention that the isomorphism class of (π, V) is denoted by π. If $\chi \in X(G(F))$, then we write (π_χ, V_χ) for the representation $\pi \otimes \chi$ on the space V. We write

$$\mathrm{Stab}(\pi, X(G(F))) = \{\chi \in X(G(F)) \,|\, \pi_\chi \simeq \pi\}.$$

By abuse of notation, we write (π_λ, V_λ) for $e^\lambda \otimes \pi$ on the space V for $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$. For $P = MU \in \mathcal{F}$, we have the parabolic induction functor $\mathrm{Alg}(M(F)) \ni (\pi, V) \mapsto (I_P^G(\pi), I_P^G(V)) \in \mathrm{Alg}(G(F))$ and the Jacquet functor $\mathrm{Alg}(G(F)) \ni (\pi, V) \mapsto (\pi_P, V_P) \in \mathrm{Alg}(M(F))$. They are related by the *Frobenius reciprocity*

$$\mathrm{Hom}_{G(F)}(\pi, I_P^G(\tau)) \simeq \mathrm{Hom}_{M(F)}(\pi_P, \tau).$$

As for the composition of these functors, we know the following result [3, 2.12]. For $P = MU$, $P' = M'U' \in \mathcal{F}$, we take a system of representatives ${}_{P'}W_P$ for $W^{M'}\backslash W/W^M$, so that we have the Bruhat decomposition $G = \coprod_{w\in {}_{P'}W_P} Pw^{-1}P'$. We fix a total order $w \le w'$ on ${}_{P'}W_P$ such that

- $G(F)_{\geq w} := \bigcup_{v \in {}_{P'}W_P,\, v \geq w} P(F)w^{-1}P'(F)$ is open in $G(F)$;
- $P(F)w^{-1}P'(F)$ is closed in $G(F)_{\geq w}$

with respect to the p-adic topology on $G(F)$. Then for $(\pi, V) \in \mathrm{Alg}(M(F))$, there is a $G(F)$-invariant decreasing filtration $\{\mathcal{F}_w\}_{w \in {}_{P'}W_P}$ of $I_P^G(V)_{P'}$, which we call the *Bruhat filtration*, such that

$$\mathcal{F}_w/\mathcal{F}_{>w} \simeq I_{w(P)^{M'}}^{M'}(w(\pi_{w^{-1}(P')^M}))$$

as an algebraic representation of $M'(F)$. Notice that the isomorphism class $w(\pi_{w^{-1}(P')^M})$ is independent of the choice of the representative for w. We write $(\pi^\vee, V^\vee)$ for the contragredient of $(\pi, V) \in \mathrm{Alg}(G(F))$. For $(\pi, V) \in \mathrm{Alg}(M(F))$, we have $I_P^G(\pi^\vee) \simeq I_P^G(\pi)^\vee$. If $(\pi, V) \in \mathrm{Alg}(G(F))$ is admissible, $(\pi_P)^\vee$ is isomorphic to $(\pi^\vee)_{\bar{P}}$. In fact, this is valid for B-admissible representations in the sense of [4].

If (π, V) is an admissible representation of finite length of $G(F)$, the set $\mathrm{JH}(\pi)$ of the elements of $\Pi(G(F))$ which appears as irreducible constituents of (π, V) is uniquely determined by π. Thus we may consider the Grothendieck group of the category of admissible representations of finite length of $G(F)$. For such representation (π, V), we write $[\pi]$ for its class in the Grothendieck group.

We write $\Pi(G(F))$ for the set of isomorphism classes of irreducible admissible representations of $G(F)$. We have the subsets $\Pi_{\mathrm{temp}}(G(F)) \supset \Pi_2(G(F))$ of tempered and square integrable elements of $\Pi(G(F))$, respectively. We write ω_π for the central character of $\pi \in \Pi(G(F))$. For any admissible representation (π, V) of $G(F)$, $\mathcal{E}xp(\pi) \subset \Pi(A_G(F))$ denotes the set of its central exponents [12, I.3]. Recall the *Langlands-Casselman criterion*:

(1) An admissible representation (π, V) of $G(F)$, having a unitary central character, is square integrable if and only if $\Re\mathcal{E}xp(\pi_P) \subset {}^+\mathfrak{a}_P^{G,*}$ for any $P \in \mathcal{F}$.

(2) An admissible representation (π, V) of $G(F)$ is tempered if and only if $\Re\mathcal{E}xp(\pi_P) \subset {}^+\bar{\mathfrak{a}}_P^{G,*}$ for any $P \in \mathcal{F}$.

In particular, parabolic induction preserves temperedness, while Jacquet functor does not. We also need the following weak classification of irreducible tempered representation.

Proposition 2.2 ([12] Prop. III.4.1). *(i) For any $\pi \in \Pi_{\mathrm{temp}}(G(F))$, there exist $P = MU \in \mathcal{F}$ and $\sigma \in \Pi_2(M(F))$ such that π is a direct summand of $I_P^G(\sigma)$.*
(ii) If both (P, σ) and (P', σ') satisfy (i), then there is $w \in W$ such that $w(M) = M'$ and $w(\sigma) \simeq \sigma'$.

Let (π, V) be an admissible representation of finite length of $M(F)$, $M \in \mathcal{L}$. For P, $P' \in \mathcal{P}(M)$, we have the intertwining integral

$$J_{P'|P}(\pi)\phi(g) := \int_{(U \cap U')(F) \backslash U'(F)} \phi(u'g)\, du', \quad \phi \in I_P^G(V).$$

For $\chi \in X(M(F))$ with $\alpha^\vee(\Re\chi) >> 0$, $\forall \alpha \in \Sigma_P \setminus \Sigma_{P'}$, the defining integral of $J_{P'|P}(\pi_\chi)$ converges absolutely. Moreover $J_{P'|P}$ defined in this way on some open subset of $\mathfrak{P} =$

$\{\pi_\chi \,|\, \chi \in X(M(F))\}$ becomes a rational function on $\mathfrak{P}$ [12, Th. IV.1.1]. Outside its poles, this defines an element of $\mathrm{Hom}_{G(F)}(I_P^G(V_\chi), I_{P'}^G(V_\chi))$. Moreover for any $\chi \in X(M(F))$, there exists $\phi \in I_P^G(V_\chi)$ such that $J_{P'|P}(\rho_\chi)\phi$ converges and is not zero [12, IV.1 (10)]. If further π is tempered, then $J_{P'|P}(\pi_\chi)\phi$ converges at $\chi \in X(M(F))$ satisfying $\alpha^\vee(\Re\chi) > 0$, $\forall \alpha \in \Sigma_P \setminus \Sigma_{P'}$ [12, Prop. IV.2.1].

2.5 Infinitesimal characters

Recall that an admissible representation (π, V) of $G(F)$ is cuspidal if its restriction to $G(F)^1$ is finite [2]. A theorem of Harish-Chandra asserts that this is equivalent to $\pi_P = \{0\}$ for $P \neq G, \in \mathcal{F}$. In particular, if (ρ, E) is a cuspidal representation of $M(F)$, then for any P, $P' \in \mathcal{P}(M)$, the Bruhat filtration simplifies to

$$[I_P^G(\rho)_{P'}] = \sum_{w \in W(M)} w(\rho).$$

We write $\Pi_0(G(F))$ for the subset of unitarizable cuspidal elements in $\Pi(G(F))$. This is contained in $\Pi_2(G(F))$. For each $\pi \in \Pi(G(F))$, there there exist $P_c = M_c U_c \in \mathcal{F}$ and an irreducible cuspidal representation (ρ, V_ρ) of $M_c(F)$ such that π is isomorphic to a subrepresentation of $I_{P_c}^G(\rho)$ [3, Th. 2.5]. Then π appears as a subquotient of $I_{P_c'}^G(\rho)$ for any $P_c' \in \mathcal{P}(M_c)$. Moreover the pair (M_c, ρ) is determined uniquely modulo W-conjugacy by π [3, Th. 2.9]. We call the W-conjugacy class the *infinitesimal character* of π and denote it by $\mathcal{X}_\pi$. Also we write $\mathcal{P}_c(\pi) := \bigcup_{(M_c,\rho)\in\mathcal{X}_\pi} \mathcal{P}(M_c)$.

If an admissible representation (π, V) of finite length of $G(F)$ admits an irreducible cuspidal subquotient ρ, then ρ appears both as a submodule and a quotient of π [2, 3.30]. From this property we deduce the following.

Lemma 2.3. *Let $P = MU \in \mathcal{F}$ and (ρ, V) be an irreducible cuspidal representation of $M(F)$. Then $\pi \in \Pi(G(F))$ is a submodule of $I_P^G(\rho)$ if and only if it is a quotient of $I_{\bar{P}}^G(\rho)$.*

Proof. π is a submodule of $I_P^G(\rho)$ if and only if

$$\{0\} \neq \mathrm{Hom}_{G(F)}(\pi, I_P^G(\rho)) \simeq \mathrm{Hom}_{M(F)}(\pi_P, \rho)$$

by Frobenius reciprocity. The above remark asserts that this is equivalent to

$$\begin{aligned}\{0\} \neq &\mathrm{Hom}_{M(F)}(\rho, \pi_P) \simeq \mathrm{Hom}_{M(F)}((\pi_P)^\vee, \rho^\vee)\\ \simeq &\mathrm{Hom}_{M(F)}((\pi^\vee)_{\bar{P}}, \rho^\vee) \simeq \mathrm{Hom}_{G(F)}(\pi^\vee, I_{\bar{P}}^G(\rho^\vee))\\ \simeq &\mathrm{Hom}_{G(F)}(I_{\bar{P}}^G(\rho), \pi)\end{aligned}$$

as desired. □

Using these, we can strengthen the Langlands-Casselman criterion as follows.

Lemma 2.4. *(i) $\pi \in \Pi(G(F))$ is square integrable if its central character is unitary and $\Re\mathcal{E}xp(\pi_{P_c}) \subset {}^+\mathfrak{a}_{P_c}^*$ for any $P_c \in \mathcal{P}_c(\pi)$.*
(ii) $\pi \in \Pi(G(F))$ is tempered if its central character is unitary and $\Re\mathcal{E}xp(\pi_{P_c}) \subset {}^+\bar{\mathfrak{a}}_{P_c}^*$ for any $P_c \in \mathcal{P}_c(\pi)$.

Proof. (i) The condition is obviously necessary. To see the sufficiency, we take $P = MU \in \mathcal{F}$ and an irreducible subquotient τ of π_P, and show $\Re\mathcal{E}xp(\tau) = \Re(\omega_\tau|_{A_M(F)}) \in {}^+\mathfrak{a}_P^*$. We take $(M_c, \rho) \in \mathcal{X}_\pi$ and $P_c \in \mathcal{P}(M_c)$ such that π is a submodule of $I_{P_c}^G(\rho)$. Since ρ is cuspidal, the Bruhat filtration simplifies

$$[I_{P_c}^G(\rho)_P] = \sum_{w \in {}_P W_{P_c}} [I_{w(P_c)^M}^M(w(\rho_{w^{-1}(P_c)^M}))] = \sum_{\substack{w \in W^M \backslash W \\ w(M_c) \subset M}} [I_{w(P_c)^M}^M(w(\rho))].$$

Thus τ is a subquotient of some $I_{w(P_c)^M}^M(w(\rho))$. In particular,

For any irreducible subquotient τ of π_P, $\mathcal{X}_\tau$ is a subset of $\mathcal{X}_\pi$.

Now take $P_c^M \in \mathcal{P}_c(\tau)$ such that $\tau_{P_c^M} \neq \{0\}$. If we write $P_c := P_c^M U$, then $\pi_{P_c} \neq \{0\}$ and

$$\begin{aligned}\mathcal{E}xp(\tau) =& \{(\chi|_{A_M(F)}) \,|\, \chi \in \mathcal{E}xp(\tau_{P_c^M})\} = \{(\omega_\sigma|_{A_M(F)}) \,|\, \sigma \in \mathrm{JH}(\tau_{P_c^M})\} \\ \subset& \{(\omega_\sigma|_{A_M(F)}) \,|\, \sigma \in \mathrm{JH}(\pi_{P_c})\} = \{(\chi|_{A_M(F)}) \,|\, \chi \in \mathcal{E}xp(\pi_{P_c})\}.\end{aligned}$$

But by the condition, $\Re\chi = \sum_{\beta_c \in \Delta_{P_c}} x_{\beta_c}\beta_c$, $(\exists x_{\beta_c} > 0)$ for $\chi \in \mathcal{E}xp(\pi_{P_c})$, so that

$$\Re(\chi|_{A_M(F)}) = (\Re\chi)_M = \sum_{\beta_c \in \Delta_{P_c} \backslash \Delta_{P_c^M}} x_\beta(\beta_c|_{\mathfrak{a}_M})$$

belongs to ${}^+\mathfrak{a}_P^{G,*}$. The sufficiency is proved. (ii) can be proved in the same way. $\square$

3 Langlands classification

In this section we prove the Langlands quotient theorem for p-adic reductive groups.

3.1 Standard modules and its matrix coefficients

Recall that a standard module of $G(F)$ is a representation of the form $(I_P^G(\pi_\lambda), I_P^G(V_\lambda))$, where $P = MU \in \mathcal{F}$, $\pi \in \Pi_{\mathrm{temp}}(M(F))$ and $\lambda \in \mathfrak{a}_P^{*,+}$. We write

$$\lim_{a \underset{P}{\to} \infty} f(a) = 0$$

if for arbitrary small ϵ, $\delta > 0$, there exists $R > 0$ such that $|f(a)| < \epsilon$ for any $a \in A_M(F) \cap G(F)^1$ satisfying

- $\alpha(H_M(a)) < -R$, $\forall \alpha \in \Sigma_P$;
- $\alpha(H_M(a))/\beta(H_M(a)) > \eta$, $\forall \alpha, \beta \in \Sigma_P$.

The following proposition is the analogue for p-adic groups of [9, Lem. 2.12]. The proof is completely the same and is omitted.

Proposition 3.1. *Let $(I_P^G(\pi_\lambda), I_P^G(V_\lambda))$ be a standard module. Then for $\phi \in I_P^G(V_\lambda)$ and $\phi^\vee \in I_P^G(V_{-\lambda}^\vee)$ we have*

$$\lim_{a \underset{\bar{P}}{\to} \infty} \delta_P(a)^{1/2}\omega_{\pi_\lambda}(a)^{-1}\langle I_P^G(\pi_\lambda, ma)\phi, \phi^\vee\rangle$$
$$=\gamma(G/M)^{-1}\langle (J_{\bar{P}|P}(\pi_\lambda)\phi)(m), \phi^\vee(1)\rangle, \quad m \in M(F).$$

This allows us to define the so called *Langlands quotient* of a standard module.

Corollary 3.2. *Suppose $(I_P^G(\pi_\lambda), I_P^G(V_\lambda))$ is a standard module.*
(i) Any $\phi \in I_P^G(V_\lambda)$ with $J_{\bar{P}|P}(\pi_\lambda)\phi \neq 0$ generates $I_P^G(V_\lambda)$ as a $G(F)$-module.
(ii) In particular, the representation $J_P^G(\pi_\lambda)$ on $J_P^G(V_\lambda) := \mathrm{im} J_{\bar{P}|P}(\pi_\lambda)$ is irreducible. It is the unique irreducible quotient of $I_P^G(\pi_\lambda)$.

Proof. (i) Take ϕ as in the statement. It suffices to show that if $\phi^\vee \in I_P^G(V_{-\lambda}^\vee)$ satisfies $\langle I_P^G(\pi_\lambda, g)\phi, \phi^\vee\rangle = 0$, $\forall g \in G(F)$, then $\phi^\vee = 0$. Applying the proposition to

$$0 = \langle I_P^G(\pi_\lambda, g^{-1}ah)\phi, \phi^\vee\rangle = \langle I_P^G(\pi_\lambda, ah)\phi, I_P^G(\pi_{-\lambda}^\vee, g)\phi^\vee\rangle,$$

we have

$$0 =\gamma(G/M)\lim_{a \underset{\bar{P}}{\to} \infty} \delta_P(a)^{1/2}\omega_{\pi_\lambda}(a)^{-1}\langle I_P^G(\pi_\lambda, ah)\phi, I_P^G(\pi_{-\lambda}^\vee, g)\phi^\vee\rangle$$
$$=\langle J_{\bar{P}|P}(\pi_\lambda)\phi(h), \phi^\vee(g)\rangle, \quad \forall h,\, g \in G(F).$$

By assumption, we can take $h \in G(F)$ such that $J_{\bar{P}|P}(\pi_\lambda)\phi(h) \neq 0$. Then we must have

$$0 = \langle J_{\bar{P}|P}(\pi_\lambda)\phi(mh), \phi^\vee(g)\rangle = \delta_P(m)^{-1/2}\langle \pi_\lambda(m)(J_{\bar{P}|P}(\pi_\lambda)\phi(h)), \phi^\vee(g)\rangle$$

for any $m \in M(F)$, $g \in G(F)$. Since π_λ is irreducible, this shows $\phi^\vee = 0$.

(ii) Take any proper maximal subrepresentation V' of $I_P^G(V_\lambda)$. If $V' \not\subset \ker J_{\bar{P}|P}(\pi_\lambda)$, (i) implies $V' = I_P^G(V_\lambda)$ which contradicts our choice of V'. Hence $V' \subset \ker J_{\bar{P}|P}(\pi_\lambda)$ and any irreducible quotient $I_P^G(V_\lambda)/V'$ has $J_P^G(V_\lambda)$ as a quotient. □

3.2 The order $\leq_P$ and a partition of $\mathfrak{a}_M^*$

We prepare some geometric properties of restricted roots which play an important role in the proof of Langlands classification [9, § 4].

Take $P = MU \in \mathcal{F}$. Define an order $\lambda \leq_P \mu$ on $\mathfrak{a}_M^*$ by $\mu \in \lambda + {}^+\bar{\mathfrak{a}}_P^{G,*}$. For $P_1 = M_1U_1 \supset P$, Δ_{P_1} is obtained by restricting $\Delta_P \setminus \Delta_{P^{M_1}}$. Thus for $\lambda,\, \mu \in \mathfrak{a}_{M_1}^*$, $\lambda \leq_P \mu$ is equivalent to $\lambda \leq_{P_1} \mu$. We also set

$$\mathfrak{a}_P^*(P_1) := \left\{ \lambda \in \mathfrak{a}_M^* \;\middle|\; \begin{array}{ll} \text{(i)} & \alpha^\vee(\lambda) > 0,\ \forall \alpha \in \Delta_{P_1} \\ \text{(ii)} & \varpi_\alpha^{\vee, M_1}(\lambda) \leq 0,\ \forall \alpha \in \Delta_{P^{M_1}} \end{array} \right\} = -{}^+\bar{\mathfrak{a}}_{P^{M_1}}^{M_1,*} \oplus \mathfrak{a}_{P_1}^{*,+}.$$

The simplest case is $\mathfrak{a}_P^*(P) = \mathfrak{a}_P^{*,+}$, and we have the disjoint decomposition

$$\mathfrak{a}_M^* = \coprod_{P_1 \in \mathcal{F}(M)} \mathfrak{a}_{P_1}^{*,+}. \tag{3.1}$$

Here $\mathcal{F}(M)$ is the set of $P_1 \in \mathcal{F}$ containing M.

Lemma 3.3. $\mathfrak{a}_M^* = \coprod_{P_1; G\supset P_1 \supset P} \mathfrak{a}_P^*(P_1)$.

Proof. First let us show that $\mathfrak{a}_P^*(P_1)$, $(P \subset P_1 \subset G)$ cover $\mathfrak{a}_M^*$ by an induction on $|\Delta_P|$. If P is maximal, $\mathfrak{a}_M^{G,*}$ is 1-dimensional and $\mathfrak{a}_P^*(P) = \mathfrak{a}_P^{*,+}$, $\mathfrak{a}_P^*(G) = -\bar{\mathfrak{a}}_P^{*,+}$, so that the assertion is clear. For general $P \in \mathcal{F}$, we take $\lambda \in \mathfrak{a}_M^*$. Suppose $\lambda \notin \mathfrak{a}_P^*(P)$ and take $\alpha \in \Delta_P$ such that $\alpha^\vee(\lambda) \le 0$. Let $P_\alpha = M_\alpha U_\alpha \supset P, \in \mathcal{F}$ be such that $\Delta_{P^{M_\alpha}} = \{\alpha\}$. Applying the induction hypothesis to λ_{M_α}, we find $P_1 = M_1 U_1 \supset P_\alpha$ such that

$$\lambda_{M_\alpha} = -\sum_{\beta \in \Delta_{P^{M_1}}, \neq \alpha} x_\beta (\beta|_{\mathfrak{a}_{M_\alpha}}) + \sum_{\gamma \in \Delta_{P_1}} y_\gamma \varpi_\gamma, \quad \exists x_\beta \ge 0,\, y_\gamma > 0.$$

On the other hand, since $\mathfrak{a}_M^{M_\alpha,*} = \mathbb{R}\alpha$,

$$\lambda^{M_\alpha} = \frac{\langle \lambda, \alpha^\vee\rangle}{\langle \alpha, \alpha^\vee\rangle}\alpha, \quad \beta|_{\mathfrak{a}_{M_\alpha}} = \beta - \frac{\langle \beta, \alpha^\vee\rangle}{\langle \alpha, \alpha^\vee\rangle}\alpha.$$

Thus we have

$$\begin{aligned}\lambda &= -\sum_{\beta\neq\alpha, \in \Delta_{P^{M_1}}} x_\beta \left(\beta - \frac{\langle \beta, \alpha^\vee\rangle}{\langle \alpha, \alpha^\vee\rangle}\alpha\right) + \sum_{\gamma \in \Delta_{P_1}} y_\gamma \varpi_\gamma + \frac{\langle \lambda, \alpha^\vee\rangle}{\langle \alpha, \alpha^\vee\rangle}\alpha \\ &= -\sum_{\beta\neq\alpha, \in \Delta_{P^{M_1}}} x_\beta \beta + \left(\frac{\langle \lambda, \alpha^\vee\rangle}{\langle \alpha, \alpha^\vee\rangle} + \sum_{\beta\neq\alpha, \in \Delta_{P^{M_1}}} x_\beta \frac{\langle \beta, \alpha^\vee\rangle}{\langle \alpha, \alpha^\vee\rangle}\right)\alpha + \sum_{\gamma \in \Delta_{P_1}} y_\gamma \varpi_\gamma.\end{aligned}$$

The coefficient of α is not positive by (2.4) and our assumption, hence $\lambda \in \mathfrak{a}_P^*(P_1)$. Next show that $\mathfrak{a}_P^*(P_1)$, $(P \subset P_1 \subset G)$ are disjoint. Suppose $P_1 \neq P_2$ contain P. We may assume that $\Delta_{P^{M_2}} \setminus \Delta_{P^{M_1}}$ is not empty. If $\alpha \in \Delta_{P^{M_2}} \setminus \Delta_{P^{M_1}}$, then

$$\begin{aligned}\langle \varpi_\alpha^{\vee, M_2} \mathfrak{a}_P^*(P_1)\rangle &= \langle \varpi_\alpha^{\vee, M_2}, \mathfrak{a}_{P_1^{M_2}}^{M_2,*,+}\rangle = \mathbb{R}_{>0}, \\ \langle \varpi_\alpha^{\vee, M_2}, \mathfrak{a}_P^*(P_2)\rangle &= \langle \varpi_\alpha^{\vee, M_2}, -{}^+\bar{\mathfrak{a}}_{P^{M_2}}^{M_2,*}\rangle = \mathbb{R}_{\le 0},\end{aligned}$$

hence $\mathfrak{a}_P^*(P_1)$ and $\mathfrak{a}_P^*(P_2)$ are disjoint. $\square$

Lemma 3.4. *Suppose P, $P' \in \mathcal{F}$ contain $P_c \in \mathcal{F}$. If $\lambda \in \mathfrak{a}_{P_c}^*(P)$, $\lambda' \in \mathfrak{a}_{P_c}^*(P')$ satisfy $\lambda \ge_{P_c} \lambda'$, then $\lambda_M \ge_{P_c} \lambda'_{M'}$.*

Proof. The hypothesis amounts to $\varpi_\alpha^\vee(\lambda) \ge \varpi_\alpha^\vee(\lambda')$, $\forall \alpha \in \Delta_{P_c}$.
(i) If $\alpha \in \Delta_{P_c} \setminus \Delta_{P_c^{M'}}$, $\lambda \in \mathfrak{a}_{P_c}^*(P)$ implies $\lambda_M \ge_{P_c} \lambda$ so that

$$\varpi_\alpha^\vee(\lambda_M) \ge \varpi_\alpha^\vee(\lambda) \ge \varpi_\alpha^\vee(\lambda') = \varpi_\alpha^\vee(\lambda'_{M'}).$$

(ii) Suppose $\alpha \in \Delta_{P_c^{M'}}$. Since $\lambda_M \in \mathfrak{a}_P^{*,+}$, $(\lambda_M)^{M'} \in \bar{\mathfrak{a}}_{P_c^{M'}}^{M',*,+} \subset {}^+\bar{\mathfrak{a}}_{P_c^{M'}}^{M',*}$ and

$$\langle \varpi_\alpha^{\vee, M'}, \lambda_M\rangle \ge 0. \tag{3.2}$$

If we expand $\varpi_{\alpha,M'}^\vee = \sum_{\beta \in \Delta_{P'}} x_\beta \varpi_\beta^\vee$, the coefficient of $\beta = \beta_c|_{\mathfrak{a}_{M'}}$, $(\beta_c \in \Delta_{P_c} \setminus \Delta_{P_c^{M'}})$ satisfies

$$\begin{aligned}x_\beta =& \langle \beta, \varpi_\alpha^\vee\rangle = \langle \beta_c - \beta_c^{M'}, \varpi_\alpha^\vee\rangle = -\langle \beta_c^{M'}, \varpi_\alpha^{\vee, M'}\rangle = -\langle \beta_c, \varpi_\alpha^{\vee, M'}\rangle \\ &\in -\langle \beta_c, \sum_{\gamma \in \Delta_{P_c^{M'}}} \mathbb{R}_{\ge 0}\gamma^\vee\rangle = \mathbb{R}_{\ge 0}\end{aligned}$$

by (2.4). Combining this with (3.2), we obtain

$$\begin{aligned}\langle \varpi_\alpha^\vee, \lambda_M - \lambda'_{M'} \rangle =& \langle \varpi_\alpha^{\vee, M'}, \lambda_M \rangle + \sum_{\beta \in \Delta_{P'}} x_\beta \langle \varpi_\beta^\vee, \lambda_M - \lambda'_{M'} \rangle \\ \geq & \sum_{\beta_c \in \Delta_{P_c} \setminus \Delta_{P_c^{M'}}} x_\beta \langle \varpi_{\beta_c}^\vee, \lambda_M - \lambda'_{M'} \rangle.\end{aligned}$$

As was seen in (i), the right hand side is non-negative. □

3.3 Langlands classification

Now we prove the result of this section.

Theorem 3.5. *(i) For any irreducible admissible representation* (π, V) *of* $G(F)$*, there exist* $P = MU \in \mathcal{F}$*,* $\tau \in \Pi_{\mathrm{temp}}(M(F))$ *and* $\lambda \in \mathfrak{a}_P^{*,+}$ *such that* $\pi \simeq J_P^G(\tau_\lambda)$*.*
(ii) The triple (P, τ, λ) *is uniquely determined by* π *up to* W*-conjugacy.*

Proof. (i) We first choose P and λ. We fix $P_0 \in \mathcal{P}(M_0)$ and write $\mathcal{F}(P_0)$ for the set of P_0-standard parabolic subgroups of G. Also set $\mathcal{P}_c(\pi, P_0) := \mathcal{P}_c(\pi) \cap \mathcal{F}(P_0)$. For each $\mu \in \bigcup_{P_c \in \mathcal{P}_c(\pi, P_0)} \Re\mathcal{E}xp(\pi_{\bar{P}_c})$ there exists a unique $P_\mu \in \mathcal{F}(P_0)$ such that $\mu \in \mathfrak{a}_{P_0}^*(P_\mu)$ (Lem. 3.3). Take $\Lambda \in \bigcup_{P_c \in \mathcal{P}_c(\pi, P_0)} \Re\mathcal{E}xp(\pi_{\bar{P}_c})$ such that $\Lambda_{M_\Lambda} \in \mathfrak{a}_{P_\Lambda}^{*,+}$ is maximal with respect to the order $\geq_{P_0}$, and set $P := P_\Lambda$, $\lambda := \Lambda_M \in \mathfrak{a}_P^{*,+}$. Next choose τ. Take $P_c \in \mathcal{P}_c(\pi, P_0)$ such that $\Re\mathcal{E}xp(\pi_{\bar{P}_c})$ contains Λ. Since $P \supset P_c$, we have $\mathcal{E}xp(\pi_{\bar{P}}) \supset \{(\chi|_{A_M(F)}) \,|\, \chi \in \mathcal{E}xp(\pi_{\bar{P}_c})\}$. Notice that these two sets might not coincide because Jacquet modules along $\bar{P}_c^M$ of some irreducible constituents of $\pi_{\bar{P}}$ can be zero. Anyway we find $\chi \in \mathcal{E}xp(\pi_{\bar{P}})$ such that $\Re\chi = \Lambda|_{\mathfrak{a}_M} = \lambda$. The weak χ-isotypic subspace $V_{\bar{P},\chi}$ of $V_{\bar{P}}$ is an $M(F)$-submodule. Let (τ, V_τ) be such that $(\tau_\lambda, V_{\tau,\lambda})$ is an irreducible subrepresentation of $V_{\bar{P},\chi}$.

Let us prove that (P, τ, λ) satisfies the condition of (i). Combining Frobenius reciprocity and duality for Jacquet modules, we have

$$\begin{aligned}\{0\} \neq & \mathrm{Hom}_{M(F)}(\tau_\lambda, \pi_{\bar{P}}) \simeq \mathrm{Hom}_{M(F)}((\pi^\vee)_P, \tau_{-\lambda}^\vee) \simeq \mathrm{Hom}_{G(F)}(\pi^\vee, I_P^G(\tau_{-\lambda}^\vee)) \\ \simeq & \mathrm{Hom}_{G(F)}(I_P^G(\tau_\lambda), \pi).\end{aligned}$$

Thus π is an irreducible quotient of $I_P^G(\tau_\lambda)$. We still have to prove that (τ, V_τ) is tempered. By construction its central character ω_τ is unitary. Thanks to Lem. 2.4, it suffices to verify $\Re\mathcal{E}xp(\tau_{\bar{P}_c'^M}) \subset -^+\bar{\mathfrak{a}}_{P_c'^M}^{M,*}$ for any $P_c'^M \in \mathcal{P}_c(\tau, P_0^M)$. For this, we write $P_c' := P_c'^M U \in \mathcal{P}_c(\pi, P_0)$ and take $\chi \in \mathcal{E}xp(\tau_{\bar{P}_c'^M})$. We need to check $\Re\chi = \sum_{\beta \in \Delta_{P_c'^M}} x_\beta \beta$ for some $x_\beta \leq 0$. Take the parabolic subgroup $P \supset (Q = LN) \supset P_c'$ such that $\Delta_{P_c'^L} = \{\beta \in \Delta_{P_c'^M} \,|\, x_\beta > 0\}$. Also we find $P \supset P_1 \supset P_c'$ for which $\Re\chi + \lambda \in \mathfrak{a}_{P_c'}^*(P_1)$. From definition, we have

$$\Re\chi + \lambda \geq_{P_c'} \sum_{\beta \in \Delta_{P_c'^M} \setminus \Delta_{P_c'^L}} x_\beta \beta + \lambda.$$

Thus Lem. 3.4 gives

$$(\Re\chi + \lambda)_{M_1} \geq_{P_c'} \Big(\sum_{\beta \in \Delta_{P_c'^M} \setminus \Delta_{P_c'^L}} x_\beta \beta + \lambda \Big)_M = \lambda = \Lambda_M.$$

But since $e^\lambda\chi \in \mathcal{E}xp(\tau_{\lambda,\bar{P}'^M_c}) \subset \mathcal{E}xp(\pi_{\bar{P}'_c})$, our choice of Λ implies

$$(\Re\chi)_{M_1} + \lambda = (\Re\chi + \lambda)_{M_1} \leq_{P'_c} \Lambda_M = \lambda.$$

Hence $(\Re\chi)_{M_1} = 0$ so that $x_\beta \leq 0$ for any $\beta \in \Delta_{P'^M_c}$.

(ii) Suppose two triples (P, τ, λ) and (P', τ', λ') as in the theorem satisfy $J^G_P(\tau_\lambda) \simeq \pi \simeq J^G_{P'}(\tau'_{\lambda'})$. We may assume both P and P' contain P_0. By Prop. 2.2, we have $P_d = M_dU_d \subset P$ and $\sigma \in \Pi_2(M_d(F))$ such that τ is a direct summand of $I^M_{P^M_d}(\sigma)$. Moreover [3, Th. 2.5] assures that there exists $P_c = M_cU_c \subset P_d$, $\rho \in \Pi_0(M_c(F))$ and $\mu \in \mathfrak{a}^{M_d,*}_{M_c}$ such that σ is a submodule of $I^{M_d}_{\bar{P}^{M_d}_c}(\rho_\mu)$, equivalently (Lem. 2.3), a quotient of $I^{M_d}_{P^{M_d}_c}(\rho_\mu)$.

$$\{0\} \neq \mathrm{Hom}_{M_d(F)}(\sigma, I^{M_d}_{\bar{P}^{M_d}_c}(\rho_\mu)) \simeq \mathrm{Hom}_{M_c(F)}(\sigma_{\bar{P}^{M_d}_c}, \rho_\mu)$$

combined with the Langlands-Casselman criterion gives $\mu \in -{}^+\mathfrak{a}^{M_d,*}_{P^{M_d}_c}$. Also writing $\Lambda := \lambda + \mu$, π is a quotient of $I^G_{P_c}(\rho_\Lambda)$. Similarly for (P', τ', λ') we take $P'_c \subset P'_d \subset P'$, $\sigma' \in \Pi_2(M'_d(F))$, $\rho' \in \Pi_0(M'_c(F))$ and $\mu' \in -{}^+\mathfrak{a}^{M'_d,*}_{P'^{M'_d}_c}$. Since $I^G_{P_c}(\rho_\mu)$ and $I^G_{P'_c}(\rho'_{\mu'})$ share the irreducible constituent π, there is $w_1 \in W$ such that

$$w_1(M_c) = M'_c, \quad w_1(\rho) \simeq \rho', \quad w_1(\Lambda) = \Lambda'. \tag{3.3}$$

Next the Bruhat filtration gives

$$\begin{aligned}[I^G_{P_d}(\sigma_\lambda)_{\bar{P}'_c}] &= \sum_{w \in {}_{\bar{P}'_c}W_{P_d}} [I^{M'_c}_{w(P_d)^{M'_c}}(w(\sigma_{\lambda, w^{-1}(\bar{P}'_c)^{M_d}}))] \\ &= \sum_{\substack{w \in W/W^{M_d} \\ w(M_d) \supset M'_c}} [w(\sigma_{w^{-1}(\bar{P}'_c)^{M_d}})_{w(\lambda)}].\end{aligned}$$

Notice that, thanks to (3.3) and the vanishing of the Jacquet modules of cuspidal representations, the terms of w with $w(P_d)^{M'_c} \neq M'_c$ vanish. On the other hand,

$$\begin{aligned}\{0\} \neq \mathrm{Hom}_{G(F)}(I^G_{P'_c}(\rho'_{\Lambda'}), \pi) &\simeq \mathrm{Hom}_{G(F)}(\pi^\vee, I^G_{P'_c}(\rho'^\vee_{-\Lambda'})) \simeq \mathrm{Hom}_{M'_c(F)}(\pi^\vee_{P'_c}, \rho'^\vee_{-\Lambda'}) \\ &\simeq \mathrm{Hom}_{M_{c'}(F)}(\rho'_{\Lambda'}, \pi_{\bar{P}'_c}),\end{aligned}$$

so that $\rho'_{\Lambda'} \in \mathrm{JH}(\pi_{\bar{P}'_c}) \subset \mathrm{JH}(I^G_{P_d}(\sigma_\lambda)_{\bar{P}'_c})$. Thus $\rho'_{\Lambda'} \in v(\mathrm{JH}(\sigma_{v^{-1}(\bar{P}'_c)^{M_d}}))_{v(\lambda)}$ for some $v \in W/W^{M_d}$ with $v(M_d) \supset M'_c$. But since $I^G_{P_d}(\sigma)$ is tempered,

$$\Re\mathcal{E}xp(I^G_{P_d}(\sigma)) = \bigcup_{\substack{w \in W/W^{M_d} \\ w(M_d) \supset M'_c}} \Re\mathcal{E}xp(w(\sigma_{w^{-1}(\bar{P}'_c)^{M_d}}))$$

is contained in $-{}^+\bar{\mathfrak{a}}^{G,*}_{P'_c}$. Thus $\Lambda' \leq_{P'_c} v(\lambda)$. Moreover, taking $P_1 \supset P'_c$ such that $v(\lambda) \in \mathfrak{a}^*_{P'_c}(P_1)$, we obtain

$$\lambda' \leq_{P'_c} v(\lambda)_{M_1}, \quad \lambda' \leq_{P_0} v(\lambda)_{M_1} \tag{3.4}$$

from Lem. 3.4. Now we claim the following. We fix a W-invariant positive definite symmetric bilinear form $(\,|\,)$ and write $\|\ \|$ for the associated norm.

Claim 3.5.1. *If λ, $\lambda' \in \bar{\mathfrak{a}}_{P_0}^{G,*,+}$ satisfy $\lambda \leq_{P_0} \lambda'$, then $\|\lambda\| \leq \|\lambda'\|$.*

Proof. Recall that the coroot $\alpha^\vee$ of $\alpha \in \Delta_{P_0}$ is identified with $2\alpha/\|\alpha\|^2$ by $(\,|\,)$ [7, VI.1.1 Lem. 2], so that

$$(\alpha|\varpi_\beta) = \frac{\|\alpha\|^2}{2}\delta_{\alpha,\beta}, \quad \alpha,\, \beta \in \Delta_{P_0}. \tag{3.5}$$

λ, λ' are written as

$$\lambda = \sum_{\beta\in\Delta_{P_0}} y_\beta \varpi_\beta, \quad \lambda' = \sum_{\beta\in\Delta_{P_0}} y'_\beta \varpi_\beta, \quad y_\beta,\, y'_\beta \geq 0,$$

and the assumption on them is

$$\lambda' = \lambda + \sum_{\alpha\in\Delta_{P_0}} x_\alpha \alpha, \quad x_\alpha \geq 0.$$

Hence the claim follows from

$$\begin{aligned}(\lambda|\lambda') &= \|\lambda\|^2 + \sum_{\alpha,\,\beta\in\Delta_{P_0}} x_\alpha y_\beta (\alpha|\varpi_\beta) = \|\lambda\|^2 + \sum_{\alpha\in\Delta_{P_0}} x_\alpha y_\alpha \frac{\|\alpha\|^2}{2} \geq \|\lambda\|^2,\\ (\lambda|\lambda') &= \|\lambda'\|^2 + \sum_{\alpha,\,\beta\in\Delta_{P_0}} x_\alpha y'_\beta (\alpha|\varpi_\beta) = \|\lambda\|^2 - \sum_{\alpha\in\Delta_{P_0}} x_\alpha y'_\alpha \frac{\|\alpha\|^2}{2} \leq \|\lambda'\|^2.\end{aligned}$$

□

Since $\mathfrak{a}_M^{G,*}$ and $\mathfrak{a}_0^{M,*}$ are spanned by $\widehat{\Delta}_{P_0} \setminus \widehat{\Delta}_{P_0^M}$ and $\Delta_{P_0^M}$, respectively, (3.5) in the above proof assures that $\mathfrak{a}_M^{G,*}$ and $\mathfrak{a}_0^{G,*}$ are orthogonal to each other under $(\,|\,)$. Applying this and the claim to (3.4), we obtain

$$\|\lambda'\| \leq \|v(\lambda)_{M_1}\| \leq \|v(\lambda)\| = \|\lambda\|. \tag{3.6}$$

Replacing the role of (P,τ,λ) and (P',τ',λ'), we obtain the reverse inequality and hence $\|\lambda\| = \|\lambda'\|$. This together with (3.6) implies $v(\lambda) = v(\lambda)_{M_1} \in \mathfrak{a}_{P_1}^{G,*,+}$. Thanks to (3.1), this and $\lambda \in \mathfrak{a}_P^*$ force $P_1 = v(P)$. Since P_1 and P are both P_0-standard, we conclude $P = P_1$ and $v \in W^M$. In particular (3.4) reads $\lambda' \leq_{P_0} \lambda$. Again replacing (P,τ,λ) and (P',τ',λ'), we also have $\lambda \leq_{P_0} \lambda'$, hence $\lambda = \lambda'$, $P = P'$.

We still have to show $\tau \simeq \tau'$. Since $\pi \simeq J_P^G(\tau'_\lambda)$ is a submodule of $I_{\bar{P}}^G(\tau'_\lambda)$, we have

$$\mathrm{Hom}_{M(F)}(I_P^G(\tau_\lambda)_{\bar{P}}, \tau'_\lambda) \simeq \mathrm{Hom}_{G(F)}(I_P^G(\tau_\lambda), I_{\bar{P}}^G(\tau'_\lambda)) \neq \{0\}.$$

Comparing this with the Bruhat filtration formula

$$[I_P^G(\tau_\lambda)_{\bar{P}}] = \sum_{w\in{}_{\bar{P}}W_P} [I_{w(P)^M}^M(w(\tau_{w^{-1}(\bar{P})^M})_{w(\lambda)})],$$

we see

$$\tau'_\lambda \in \mathrm{JH}\big(I_{w(P)^M}^M(w(\tau_{w^{-1}(\bar{P})^M})_{w(\lambda)})\big), \quad \exists w \in {}_{\bar{P}}W_P. \tag{3.7}$$

But the temperedness of $I_P^G(\tau)$ implies $\Re\mathcal{E}xp(I_{w(P)^M}^M(w(\tau_{w^{-1}(\bar{P})^M})))$ is contained in $-^+\overline{\mathfrak{a}}_P^{G,*} \subset -^+\overline{\mathfrak{a}}_{P_0}^{G,*}$, that is, $\lambda \le_{P_0} w(\lambda)$. Again applying the above claim and the argument just after it, we have

$$\|\lambda\| \le \|w(\lambda)_{M_w}\| \le \|w(\lambda)\| = \|\lambda\|,$$

where $P_w = M_w U_w \in \mathcal{F}(P_0)$ is such that $w(\lambda) \in \mathfrak{a}_{P_0}^*(P_w)$. But this shows $w(\lambda) = w(\lambda)_{M_w} \in \mathfrak{a}_{P_w}^{G,*,+}$ and thus $w(P) = P_w$. Again noting that P and P_w are standard, this forces $P_w = P$, $w \in W^M$. Now (3.7) reads $\tau'_\lambda \in \mathrm{JH}(\tau_\lambda)$, hence $\tau' \simeq \tau$. (Q.E.D)

4 Some irreducibility results

In this section we give several applications of Th. 3.5 or rather Prop. 3.1. We first deduce a theorem of Waldspurger on irreducibility of parabolically induced modules.

4.1 A theorem of Waldspurger

We need the following lemma.

Lemma 4.1. *Suppose $P = MU \in \mathcal{F}$ is maximal and $\rho \in \Pi_0(M(F))$. Then there exists a Zariski open dense subset $\Xi_P(\rho) \subset X(M(F))$ such that $I_P^G(\rho_\chi)$ is irreducible for $\chi \in \Xi_P(\rho)$.*

Proof. It suffices to construct a proper Zariski closed subset $Z(\rho) \in X(M(F))$ which contains any $\chi \in X(M(F))$ such that $I_P^G(\rho_\chi)$ is reducible. Recall the exact sequence (2.7). We have $I_P^G(\rho_{\chi\omega}) \simeq I_P^G(\rho_\chi)_\omega$ for $\omega \in X(G(F))$, so that the reducibility of $I_P^G(\rho_\chi)$ depends only on the image χ^G of $\chi \in X(M(F))$ in $X^G(M(F))$. Let us construct a Zariski closed proper subset $Z^G(\rho) \subset X^G(M(F))$ which contains all χ^G such that $I_P^G(\rho_\chi)$ is reducible. We consider the cases $\chi^G \in X_{\mathrm{unit}}^G(M(F))$ and $\chi^G \notin X_{\mathrm{unit}}^G(M(F))$ separately.

First suppose $\chi^G \in X_{\mathrm{unit}}^G(M(F))$. By multiplying $\Re\chi^{-1} \in X(G(F))$ to χ if necessary, we may assume that $\chi \in X_{\mathrm{unit}}(M(F))$. Since $\rho_\chi \in \Pi_0(M(F)) \subset \Pi_2(M(F))$, $I_P^G(\rho_\chi)$ is reducible only if $w(\rho_\chi) \simeq \rho_\chi$ for some element $w \neq 1, \in W(M)$ [12, Prop. IV.2.2]. Such w is unique if it exists because P is maximal, and we must have $w(P) = \bar{P}$. If $I_P^G(\rho_{\chi\mu})$ is reducible at some other $\chi\mu \in X_{\mathrm{unit}}(M(F))$, then we must have $w(\rho_{\chi\mu}) \simeq (\rho_\chi)_{w(\mu)} \simeq \rho_{\chi\mu}$ so that

$$w(\mu)\mu^{-1} \in \mathrm{Stab}(\rho_\chi, X(M(F))). \tag{4.1}$$

We know from [4, 2.2] that $\mathrm{Stab}(\rho_\chi, X(M(F)))$ is a finite group, and hence $\chi\mu \in X(M(F))$ satisfying (4.1) form a proper Zariski closed subset $Z^0(\rho) \subset X(M(F))$.

Next assume $\chi^G \notin X_{\mathrm{unit}}^G(M(F))$. If we write α for the unique element of Δ_P, then we have $\alpha^\vee(\Re\chi) = \alpha^\vee(\Re\chi^G) \neq 0$. First consider the case $\alpha^\vee(\Re\chi) > 0$. Then $I_P^G(\rho_\chi)$ admits a unique irreducible quotient $J_P^G(\rho_\chi) = \mathrm{im}J_{\bar{P}|P}(\rho_\chi)$ (Cor. 3.2), so that $I_P^G(\rho_\chi)$ is reducible if and only if $\delta_P^G(\rho_\chi) := \ker J_{\bar{P}|P}(\rho_\chi)$ is non-trivial. At such $\chi \in X(G(F))$, the Jacquet modules of $J_P^G(\rho_\chi)$ and $\delta_P^G(\rho_\chi)$ are calculated as follows. Note that the infinitesimal characters of these representations consist of the W-conjugates of (M, ρ_χ), so that at least one of the Jacquet modules along P and $\bar{P}$ does not vanish.

(1) If P and $\bar{P}$ are conjugate, $w(P) = \bar{P}$ assures both $\delta_P^G(\rho_\chi)_P$ and $J_P^G(\rho_\chi)_P$ do not vanish. From the Bruhat filtration formula $[I_P^G(\rho_\chi)_P] = \rho_\chi + w(\rho)_{w(\chi)}$, either

$$(\delta_P^G(\rho_\chi)_P, J_P^G(\rho_\chi)_P) \simeq (\rho_\chi, w(\rho)_{w(\chi)})$$

or $(w(\rho)_{w(\chi)}, \rho_\chi)$ occurs. Since the latter combined with Lem. 2.4 forces $J_P^G(\rho_\chi) \in \Pi_2(G(F))$ and contradicts the uniqueness of the Langlands data (Th. 3.5 (ii)), we must have the former.

(2) Otherwise, $W(M)$ is trivial and the Bruhat filtration formula becomes $I_P^G(\rho_\chi)_P \simeq I_P^G(\rho_\chi)_{\bar{P}} \simeq \rho_\chi$. Thus we have either

$$\delta_P^G(\rho_\chi)_P \simeq J_P^G(\rho_\chi)_{\bar{P}} \simeq \rho_\chi$$

or $\delta_P^G(\rho_\chi)_{\bar{P}} \simeq J_P^G(\rho_\chi)_P \simeq \rho_\chi$. The latter possibility does not occur by the same reasoning as in (1), hence the former holds.

Now we recall some construction from the proof of [12, Th. IV.1.1]. If we write B_M for the polynomial ring of the $\mathbb{C}$-torus $X(M(F))$, each $m \in M(F)$ determines an element $b_m : X(M(F)) \ni \chi \mapsto \chi(m) \in \mathbb{C}^\times$ of $B_M^\times$. This gives the *universal character*:

$$\mu_{B_M} : M(F) \ni m \longmapsto b_m \in B_M^\times.$$

of $M(F)$. Using this, we define the *B_M-admissible representations* (in the sense of [4, 1.7])

$$(\rho_B := \rho \otimes \mu_{B_M}, V_B := V \otimes_{\mathbb{C}} B_M)$$

of $M(F)$, and $(I_P^G(\rho_B), I_P^G(V_B))$ of $G(F)$. Now the rationality of $J_{\bar{P}|P}(\rho_\chi)$ is equivalent to the existence of a homomorphism $J \in \mathrm{Hom}_{G(F),B_M}(I_P^G(V_B), I_{\bar{P}}^G(V_B))$ of B_M-admissible representations and $b \in B_M$ such that

$$b(\chi) J_{\bar{P}|P}(\rho_\chi)\phi_\chi = J(\phi)_\chi, \quad \forall \chi \in X(M(F)),\ \phi \in I_P^G(V_B).$$

Here ϕ_χ denotes the localization of ϕ at χ.

Applying the Jacquet functor to this J, we obtain $J_P \in \mathrm{Hom}_{M(F),B_M}(I_P^G(V_B)_P, I_{\bar{P}}^G(V_B)_P)$. The Bruhat filtration $\mathcal{F}_{\bullet,P}$ of $I_P^G(V_B)_P$ is defined. We take a non-zero element $v \in \mathcal{F}_{1,P} \simeq V_B$ and consider $J_P(v)$. We note

(3) $\chi \mapsto J_P(v)_\chi$ is a polynomial function on $X(M(F))$. Moreover, (1), (2) above show that $J_P(v)$ has a zero at χ with $\alpha^\vee(\Re\chi) > 0$ if $I_P^G(\rho_\chi)$ is reducible.

(4) $J_{\bar{P}|P}(\rho_\chi)$ is an isomorphism at $\chi \in X_{\mathrm{unit}}(M(F))$ where ρ_χ is G-regular. Thus $J_P(v) \in I_{\bar{P}}^G(V_B)_P$ is not zero.

(4) combined with the commutative diagram

$$\begin{array}{ccc} I_P^G(V_{\chi\omega}) & \xrightarrow{J_{\bar{P}|P}(\rho_{\chi\omega})} & I_{\bar{P}}^G(V_{\chi\omega}) \\ \downarrow & & \downarrow \\ I_P^G(V_\chi)_\omega & \xrightarrow{\omega\circ J_{\bar{P}|P}(\rho_\chi)\circ\omega^{-1}} & I_{\bar{P}}^G(V_\chi)_\omega \end{array}$$

shows that the set of $\chi^G \in X^G(M(F))$ such that χ is a zeros of $\chi \mapsto J_P(v)_\chi$ is a proper Zariski closed subset $Z^+(\rho)^G$ of $X^G(M(F))$. By (3), the set of $\chi^G \in X(M(F))$ such that $\alpha^\vee(\Re\chi) > 0$ and $I_P^G(\rho_\chi)$ is reducible is contained in this $Z^+(\rho)^G$.

Finally, we consider the case $\alpha^\vee(\Re\chi) < 0$. Since the reducibility of $I_P^G(\rho_\chi)$ is equivalent to that of the contragredient $I_P^G(\rho^\vee_{\chi^{-1}})$, the proper Zariski closed subset $Z^-(\rho)^G := (Z^+(\rho^\vee)^G)^{-1} \subset X^G(M(F))$ contains those χ with $\alpha^\vee(\Re\chi) < 0$ such that $I_P^G(\rho_\chi)$ is reducible.

We conclude that the set of $\chi^G \in X^G(M(F))$ where $I_P^G(\rho_\chi)$ is reducible, is contained in the Zariski closed subset

$$Z(\rho)^G := Z^0(\rho)^G \cup Z^+(\rho)^G \cup Z^-(\rho)^G.$$

□

Now we are able to prove the following irreducibility theorem of Waldspurger.

Theorem 4.2 ([10] Th. 3.2). *For $P = MU \in \mathcal{F}$ and an irreducible admissible representation (π, V) of $M(F)$, there exists a neighborhood $\mathcal{U}$ of 0 in $\mathfrak{a}^*_{M,\mathbb{C}}$ such that the following holds. $I_P^G(\pi_\lambda)$ is irreducible for any $\lambda \in \mathcal{U}$ such that $\alpha^\vee(\Re\lambda) \neq 0$, $\forall\alpha \in \Sigma_M$.*

Proof. For $(M_c, \rho) \in \mathcal{X}_\pi$, there exists $P_c \in \mathcal{P}(M_c)$ contained in P such that π is isomorphic to a submodule of $I^M_{P_c^M}(\rho)$. Then $[I_P^G(\pi_\lambda)]$ is contained in $I^G_{P_c}(\rho_\lambda)$ so that $(M_c, \rho_\lambda) \in \mathcal{X}_\tau$ for any irreducible subquotient τ of $I_P^G(\pi_\lambda)$. In particular we can take $P_c' \in \mathcal{P}(M_c)$ such that

$$\mathrm{Hom}_{M_c(F)}(\tau_{P_c'}, \rho_\lambda) \simeq \mathrm{Hom}_{G(F)}(\tau, I^G_{P_c'}(\rho_\lambda)) \neq 0.$$

On the other hand Bruhat filtration asserts that

$$\begin{aligned} [I_P^G(\pi_\lambda)_{P_c'}] &= \sum_{w \in {}_{P_c'}W_P} [I^{M_c}_{w(P)^{M_c}}(w(\pi_{\lambda, w^{-1}(P_c')^M}))] \\ &= \sum_{\substack{w \in W/W^M \\ w(M) \supset M_c}} [w(\pi_{\lambda, w^{-1}(P_c')^M})]. \end{aligned} \tag{4.2}$$

Claim 4.2.1. *We can take a neighborhood $\mathcal{U}$ of 0 in $\mathfrak{a}^*_{M,\mathbb{C}}$ sufficiently small, so that the set of irreducible constituents of $w(\pi_{\lambda, w^{-1}(P_c')^M})$, ($w \in W/W^M$, $w(M) \supset M_c$) are disjoint to each other at any $\lambda \in \mathcal{U}$ satisfying $\alpha^\vee(\Re\lambda) \neq 0$, $\forall\alpha \in \Sigma_M$.*

Proof. If there are no such neighborhood, we can take a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in $\mathfrak{a}^*_{M,\mathbb{C}}$ which converges to 0 such that

- $\alpha^\vee(\Re\lambda_n) \neq 0$, $\forall\alpha \in \Sigma_M$;
- For any $n \in \mathbb{N}$, there exist $w_1 \neq w_2 \in W/W^M$ which satisfies the condition of (4.2) such that $w_1(\pi_{\lambda, w_1^{-1}(P_c')^M})$ and $w_2(\pi_{\lambda, w_2^{-1}(P_c')^M})$ share a same irreducible constituent.

By replacing $\{\lambda_n\}_{n\in\mathbb{N}}$ with its subsequence, we may assume that w_1, w_2 are independent of n. Since the Bruhat filtration gives

$$\begin{aligned}[w_i(I^M_{P^M_c}(\rho)_{w_i^{-1}(P'_c)^M})_{w_i(\lambda)}] &= \sum_{\substack{w\in W^M/W^{M_c}\\ w(M_c)=w_i^{-1}(M_c)}} [w_i(w(\rho))_{w_i(\lambda)}]\\ &= \sum_{w\in W^M(M_c)} [w(\rho)_{w_i(\lambda)}],\end{aligned}$$

for each $n\in\mathbb{N}$, we find v_1, $v_2\in W^M(M_c)$ such that

$$v_1(\rho)_{w_1(\lambda_n)}\simeq v_2(\rho)_{w_2(\lambda_n)}.$$

Again taking some subsequence, we can assume v_i are independent of n. The set of $w_1(\lambda_n)-w_2(\lambda_n)$ satisfying this equality is finite modulo $2\pi i\mathfrak{a}^*_{M_c(F)}$ by [4, 1.6], and hence discrete in $\mathfrak{a}^*_{M,\mathbb{C}}$. As $\{\lambda_n\}_{n\in\mathbb{N}}$ tends to 0, this implies $v_1(\rho)\simeq v_2(\rho)$ and in particular, for sufficiently large $n\in\mathbb{N}$, we have

$$w_1(\lambda_n)=w_2(\lambda_n).$$

Now since $\alpha^\vee(\Re\lambda_n)\neq 0$, $\forall\alpha\in\Sigma_M$, $\Re\lambda_n$ belongs to a uniquely determined chamber $\mathfrak{a}^{*,+}_{P'}$ in $\mathfrak{a}^*_M$. Then the above equality implies that $w_1^{-1}w_2$ preserves $\mathfrak{a}^{*,+}_{P'}$, or equivalently $w_1^{-1}w_2\in W^M$. This is a contradiction. □

Now let $\lambda\in\mathcal{U}$ be as in the claim. For $(I^G_P(\pi_\lambda), I^G_P(V_\lambda))$ to be irreducible, it suffices that its irreducible submodule (τ, V_τ) is isomorphic to $I^G_P(\pi_\lambda)$.

Claim 4.2.2. *$I^G_P(\pi_\lambda)\simeq\tau$ if and only if $[w(\pi_{w^{-1}(P'_c)^M})_{w(\lambda)}]\subset[\tau_{P'_c}]$ for any $P'_c\in\mathcal{P}(M_c)$ and $w\in W/W^M$ satisfying $w(M)\supset M_c$.*

Proof. Since we have chosen $\lambda\in\mathcal{U}$ as in the claim, the condition is equivalent to $[\tau_{P'_c}]\supset[I^G_P(\pi_\lambda)_{P'_c}]$, $\forall P'_c\in\mathcal{P}(M_c)$. This amounts to the vanishing of the Jacquet modules of $I^G_P(V_\lambda)/V_\tau$ along any $P'_c\in\mathcal{P}(M_c)$. Thanks to [4, 2.5], this implies $I^G_P(V_\lambda)/V_\tau=\{0\}$. □

Since the condition of this claim can be rewritten as $[w^{-1}(\tau_{P'_c})=\tau_{w^{-1}(P'_c)}]\supset[\pi_{\lambda,w^{-1}(P_{c'})^M}]$, replacing P'_c for $w^{-1}(P'_c)$, it suffices to show the following.

Claim 4.2.3. *If $\lambda\in\mathcal{U}$ satisfies $\alpha^\vee(\Re\lambda)\neq 0$, $\forall\alpha\in\Sigma_M$, then $[\tau_{P'_c}]\supset[\pi_{\lambda,P'^M_c}]$ for any $M'_c\in\mathcal{L}^M$ which is $G(F)$-conjugate to M_c and $P'_c\in\mathcal{P}(M'_c)$.*

Let us prove this by induction on $d(P'_c, P'^M_cU)$, where $d(P,P')$ denotes the number of walls between the chambers $\mathfrak{a}^{*,+}_P$ and $\mathfrak{a}^{*,+}_{P'}$. When $P'_c=P'^M_cU$,

$$\{0\}\neq\mathrm{Hom}_{G(F)}(\tau, I^G_P(\pi_\lambda))\simeq\mathrm{Hom}_{M(F)}(\tau_P,\pi_\lambda)$$

gives $\pi_\lambda\subset[\tau_P]$ and hence $[\pi_{\lambda,P'^M_c}]\subset[(\tau_P)_{P'^M_cU}]=[\tau_{P'_c}]$.

Next suppose $d(P'_c, P'^M_cU)=d>0$ and take $P''_c\in\mathcal{P}(M'_c)$ such that $d(P'_c,P''_c)=1$, $d(P''_c,P'^M_cU)=d-1$. Then $\Sigma_{P'^M_cU}\setminus\Sigma_{P'_c}=(\Sigma_{P'^M_cU}\setminus\Sigma_{P''_c})\sqcup(\Sigma_{P''_c}\setminus\Sigma_{P'_c})$ is disjoint to

$\Sigma_{P_c'^M}$ so that $\Sigma_{P_c'^M} = \Sigma_{P_c'^M} \cap \Sigma_{P_c''} = \Sigma_{P_c''^M}$. That is, $P_c'^M = P_c''^M$. This combined with the induction hypothesis gives

$$[\tau_{P_c''}] \supset [\pi_{\lambda, P_c''^M}] = [\pi_{\lambda, P_c'^M}].$$

What is left to show is that we can replace P_c'' with P_c' in the left hand side. Noting that the appearing representations are all cuspidal, we have only to show that if $(M_c', \rho') \in \mathcal{X}_{I_P^G(\pi)} = W.\mathcal{X}_\pi$, the multiplicities of ρ'_λ in $[\tau_{P_c''}]$ and $[\tau_{P_c'}]$ are equal.

Claim 4.2.4. *If $(M_c', \rho'_\lambda) \in \mathcal{X}_\tau$ with $\rho' \in \Pi_0(M_c'(F))$ (i.e. ρ' is unitarizable) and $\lambda \in \mathcal{U}$ satisfies the condition of the theorem, then the multiplicities of ρ'_λ in $[\tau_{P_c''}]$ and $[\tau_{P_c'}]$ are equal.*

Proof. We write $\alpha \in \Delta_{P_c'}$ for the unique element of $\Sigma^{\mathrm{red}}_{P_c'} \setminus \Sigma^{\mathrm{red}}_{P_c''}$. Let $M_1 \in \mathcal{L}(M_c')$ be such that $\Delta_{P_c'^{M_1}} = \{\alpha\}$ and set $P_1 := M_1 \cdot U_c'$. Then $P_1 \in \mathcal{F}(M_1)$ contains P_c'. Also noting $\Sigma^{\mathrm{red}}_{P_c''} = (\Sigma^{\mathrm{red}}_{P_c'} \setminus \{\alpha\}) \sqcup \{-\alpha\}$, we have

$$\{(\beta|_{\mathfrak{a}_{M_1}}) \neq 0 \,|\, \beta \in \Sigma^{\mathrm{red}}_{P_c''}\} = \{(\beta|_{\mathfrak{a}_{M_1}}) \,|\, \beta \neq -\alpha, \in \Sigma^{\mathrm{red}}_{P_c''}\} = \Sigma^{\mathrm{red}}_{P_1}$$

so that $P_c'' \subset P_1$. It is enough to prove that

> The multiplicities of ρ'_λ in $[\sigma_{P_c'^{M_1}}]$ and $[\sigma_{P_c''^{M_1}}]$ are equal for any irreducible constituent σ of τ_{P_1}.

Since $\mathcal{P}^{M_1}(M_c') = \{P_c'^{M_1}, P_c''^{M_1}\}$, either $\sigma_{P_c'^{M_1}}$ or $\sigma_{P_c''^{M_1}}$ does not vanish so that we may assume $\sigma_{P_c'^{M_1}} \neq 0$. Any irreducible quotient of $\sigma_{P_c'^{M_1}}$ is of the form $w(\rho'_\lambda)$, $(\exists w \in W(M_c'))$, and Frobenius reciprocity shows that σ is a submodule of $I^{M_1}_{P_c'^{M_1}}(w(\rho')_{w(\lambda)})$. Thanks to Lemma 4.1, $I^{M_1}_{P_c'^{M_1}}(w(\rho')_\chi)$ is irreducible at χ in some Zariski open dense subset of $X(M_c'(F))$, and hence the theorem is valid for $I^{M_1}_{P_c'^{M_1}}(w(\rho')_\mu)$, $(\mu \in \mathfrak{a}^*_{M_c',\mathbb{C}})$. Now since $P_c'^M = P_c''^M$ and $P_c'^{M_1} \neq P_c''^{M_1}$, $\alpha_M = \alpha|_{\mathfrak{a}_M} \neq 0$ belongs to Σ_M. In particular, for $\lambda \in \mathcal{U}$ as in the theorem, $w(\alpha)^\vee(w(\lambda)) = \alpha^\vee(\lambda) = \alpha^\vee_M(\lambda) \neq 0$. This means, by choosing $\mathcal{U}$ appropriately small, that if $\lambda \in \mathcal{U}$ satisfies the condition of the theorem then so is $w(\lambda)$ but $I_P^G(\pi_\lambda)$ replaced with $I^{M_1}_{P_{c'}{}^{M_1}}(w(\rho')_{w(\lambda)})$. Hence $I^{M_1}_{P_{c'}{}^{M_1}}(w(\rho')_{w(\lambda)})$ is irreducible and isomorphic to σ. Thus we conclude

$$[\sigma_{P_c'^{M_1}}] = [I^{M_1}_{P_c'^{M_1}}(w(\rho')_{w(\lambda)})_{P_c'^{M_1}}] = [I^{M_1}_{P_c'^{M_1}}(w(\rho')_{w(\lambda)})_{P_c''^{M_1}}] = [\sigma_{P_c''^{M_1}}].$$

□

This finishes the proof of the theorem.

4.2 Irreducibility of induced modules from discrete series

Finally we deduce the following irreducibility result for induced from discrete series representations.

Corollary 4.3. *For $P = MU \in \mathcal{F}$ and an irreducible square-integrable representation (σ, E) of $M(F)$, there exists a Zariski open dense subset $\Xi_P(\sigma) \subset X(M(F))$ such that $I_P^G(\sigma_\chi)$ is irreducible at any $\chi \in \Xi_P(\sigma)$.*

Remark 4.4. *The special case of cuspidal σ of the corollary was used in the proof of [4, 3.12], which describes the general structure of isotypic components of a Hecke algebras under infinitesimal characters. Also this result is used in the construction of j and hence μ-functions in [12].*

Proof. As in the proof of Lemma 4.1, it suffices to construct a proper Zariski closed subset $Z_P(\pi) \in X(M(F))$ which contains all the χ such that $I_P^G(\sigma_\chi)$ is reducible. Regarding the decomposition (3.1), we divide the argument according to $P_1 \in \mathcal{F}(M)$ for which $\Re\chi \in \mathfrak{a}_{P_1}^{*,+}$.

Writing $P_1 = M_1U_1$ for the Levi decomposition with $M_1 \in \mathcal{L}$, we take $P_1' \in \mathcal{P}(M_1)$ which contains P so that $I_P^G(\sigma_\chi) \simeq I_{P_1'}^G(I_{P^{M_1}}^{M_1}(\sigma_\chi))$. We first consider the reducibility of $I_{P^{M_1}}^{M_1}(\sigma_\chi)$. If we write $Z_w^{M_1}(\pi)$ for the set of $\chi \in X(M(F))$ such that $w(\sigma_\chi) \simeq \sigma_\chi$, then it follows from [12, Prop. IV.2.2] that $I_{P^{M_1}}^{M_1}(\sigma_\chi)$ is reducible only if χ belongs to

$$Z^{M_1}(\sigma) := \bigcup_{w \neq 1, \in W^{M_1}(M)} Z_w^{M_1}(\sigma).$$

As in the proof of Lemma 4.1, if $\chi \in Z_w^{M_1}(\sigma)$ then $\mu\chi \in Z_w^{M_1}(\sigma)$ if and only if $w(\mu)\mu^{-1} \in \mathrm{Stab}(\sigma, X(M(F)))$. Since $\mathrm{Stab}(\sigma, X(M(F)))$ is contained in the finite set $\mathrm{Stab}(\mathcal{X}_\sigma, X(M(F)))$, we find that $Z_w^{M_1}(\sigma)$ and $Z^{M_1}(\sigma)$ are proper Zariski closed subset of $X(M(F))$.

Next consider $\chi \notin Z^{M_1}(\sigma)$ (always with $\Re\chi \in \mathfrak{a}_{P_1}^{*,+}$). We can write it as $\mu\lambda$, ($\mu \in X_{\mathrm{unit}}(M(F))$, $\lambda \in X(M_1(F))$). Set $(\tau, V) := (I_{P^{M_1}}^{M_1}(\sigma_\mu), I_{P^{M_1}}^{M_1}(E_\mu))$.

(i) If $P_1' = P_1$ then $I_P^G(\pi_\chi) \simeq I_{P_1}^G(\tau_\lambda)$. Since $\tau \in \Pi_{\mathrm{temp}}(M_1(F))$ and $\Re\lambda \in \mathfrak{a}_{P_1}^{*,+}$, $\mathrm{im}J_{\bar{P}_1|P_1}(\tau_\lambda)$ is irreducible (Cor. 3.2). In particular, $I_{P_1}^G(\tau_\lambda)$ is reducible if and only if $\ker J_{\bar{P}_1|P_1}(\tau_\lambda)$ is non-trivial.

(ii) For general P_1', we have $d(P_1, P_1') + d(P_1', \bar{P}_1) = d(P_1, \bar{P}_1)$ so that the functional equation

$$J_{\bar{P}_1|P_1}(\tau_\lambda) = J_{\bar{P}_1|P_1'}(\tau_\lambda) \circ J_{P_1'|P_1}(\tau_\lambda)$$

holds [12, IV.1 (12)]. As was remarked in (i), $\mathrm{im}J_{\bar{P}_1|P_1}(\tau_\lambda)$ is irreducible. Thus $I_P^G(\sigma_\chi) \simeq I_{P_1'}^G(\tau_\lambda)$ is reducible only if either

(1) $\mathrm{im}J_{P_1'|P_1}(\tau_\lambda)$ is a proper submodule of $I_{P_1'}^G(V_\lambda)$, or

(2) $\ker J_{\bar{P}_1|P_1'}(\tau_\lambda) \neq \{0\}$.

Let us prove that (1) is equivalent to

(1)′ $\ker J_{P_1'|P_1}(\tau_\lambda) \neq \{0\}$.

In fact, for $P_c = M_cU_c \in \mathcal{P}_c(\tau_\lambda)$, Bruhat filtration gives

$$\begin{aligned}[I_{P_c}^G(I_{P_1'}^G(V_\lambda)_{P_c})] &= [I_{P_c}^G\Big(\sum_{\substack{w \in W/W^{M_1} \\ w(M_1) \supset M_c}} w(V_{\lambda, w^{-1}(P_c)^{M_1}})\Big)] \\ &= \sum_{\substack{w \in W/W^{M_1} \\ w(M_1) \supset M_c}} [I_{P_c}^G(w(V_{\lambda, w^{-1}(P_c)^{M_1}}))],\end{aligned}$$

and hence the functor $\varphi_{M_c}: \pi \mapsto \bigoplus_{P_c \in \mathcal{P}(M_c)} I_{P_c}^G(\sigma_{P_c})$ of [4, 2.5] applied to $I_{P_1'}^G(V_\lambda)$ yields

$$[\varphi_{M_c}(I_{P_1'}^G(V_\lambda))] = \sum_{P_c \in \mathcal{P}(M_c)} \sum_{\substack{w \in W/W^{M_1} \\ w(M_1) \supset M_c}} [I_{P_c}^G(w(V_{\lambda, w^{-1}(P_c)^{M_1}}))].$$

Notice that this is independent of $P_1' \in \mathcal{P}(M_1)$. Thanks to [4, 2.5], (1) amounts to the non-vanishing of $\varphi_{M_c}\big(I_{P_1'}^G(V_\lambda)/\mathrm{im}(J_{P_1'|P_1}(\tau_\lambda))\big)$. But since $[\varphi_{M_c}(I_{P_1'}^G(V_\lambda))] = [\varphi_{M_c}(I_{P_1}^G(V_\lambda))]$, this is equivalent to $\ker\varphi_{M_c}(J_{P_1'|P_1}(\tau_\lambda)) \neq \{0\}$, which again by [4, 2.5] amounts to $(1)'$.

In any case, we have seen that the reducibility of $I_P^G(\pi_\chi)$ implies $\ker J_{\bar{P}_1|P_1}(\tau_\lambda) \neq \{0\}$. If we write $P' := P^{M_1}U_1$, $P'' := P^{M_1}\bar{U}_1$, we have the commutative diagram

$$\begin{array}{ccc} I_{P_1}^G(V_\lambda) & \xrightarrow{J_{\bar{P}_1|P_1}(\tau_\lambda)} & I_{\bar{P}_1}^G(V_\lambda) \\ \downarrow & & \downarrow \\ I_P^G(E_\chi) & \xrightarrow{J_{P^{M_1}\bar{U}_1|P}(\sigma_\chi)} & I_{P^{M_1}\bar{U}_1}^G(E_\chi) \end{array}$$

from the definition of the intertwining operator. Then the non-vanishing of $\ker J_{\bar{P}_1|P_1}(\tau_\lambda)$ is equivalent to that of $\ker J_{P''|P'}(\sigma_\chi)$.

Let $(M_c, \rho) \in \mathcal{X}_\sigma$. As in the proof of Lem. 4.1, we take $J \in \mathrm{Hom}_{G(F), B_M}(I_{P'}^G(V_B), I_{P''}^G(V_B))$ and $b \in B_M$ such that

$$b(\chi)J_{P''|P'}(\sigma_\chi) = J(\phi)_\chi, \quad \forall \chi \in X(M(F)),\, \phi \in I_{P'}^G(V_B).$$

Consider the image

$$J_{P_c} \in \mathrm{Hom}_{M_c(F), B_M}(I_{P'}^G(V_B)_{P_c}, I_{P''}^G(V_B)_{P_c})$$

of J under the Jacquet functor along $P_c \in \mathcal{P}(M_c)$.

If $P_c' \in \mathcal{P}(M_c)$ is such that σ is a submodule of $I_{P_c'^M}^M(\rho)$, then each gradation of the Bruhat filtration $\{\mathcal{F}_{w, P_c}\}_{w \in_{P_c} W_{P'}}$ of $I_{P'}^G(V_B)_{P_c}$ is contained in the corresponding gradation of the Bruhat filtration $\{\widetilde{\mathcal{F}}_{v, P_c}\}_{v \in W(M_c)}$ of $I_P^G(I_{P_c'^M}^M(\rho)_\chi) \simeq I_{P_c'}^G(\rho_\chi)_{P_c}$:

$$[\mathrm{Gr}_w \mathcal{F}_{\bullet, P_c}] \subset \sum_{\substack{v \in W(M_c) \\ v \in W^{M_c} w W^M}} [\mathrm{Gr}_v \widetilde{\mathcal{F}}_{\bullet, P_c}], \quad \mathrm{Gr}_v \widetilde{\mathcal{F}}_{\bullet, P_c} \simeq v(\rho_\chi).$$

Put

$$\Omega_w := \{v \in W^{M_c} w W^M / W^{M_c} \,|\, \widetilde{\mathcal{F}}_{v, P_c} \cap \mathcal{F}_{w, P_c} \not\subset \widetilde{\mathcal{F}}_{>v, P_c}\}, \quad \Omega := \coprod_{w \in W^{M_c} \backslash W / W^M} \Omega_w.$$

For each $v \in \Omega$, we fix a representative $\xi_v \in \widetilde{\mathcal{F}}_{v, P_c} \setminus \widetilde{\mathcal{F}}_{>v, P_c}$ of $\mathrm{Gr}_v \widetilde{\mathcal{F}}_{\bullet, P_c}$, and consider $J_{P_c}(\xi_v) \in I_{P''}^G(V_B)_{P_c}$, $(P_c \in \mathcal{P}(M_c),\, v \in \Omega)$.

Suppose $\ker J_{P''|P'}(\sigma_\chi) \neq \{0\}$. Then at least one of its Jacquet modules $(\ker J_{P''|P'}(\sigma_\chi))_{P_c}$, $(\exists P_c \in \mathcal{P}(M_c))$ is not trivial [4, 2.5]. Since $\mathrm{Gr}_v \widetilde{\mathcal{F}}_{\bullet, P_c} \simeq v(\rho_\chi)$ is irreducible, this implies that $\xi_{v,\chi} \in \ker J_{P_c,\chi}$ for some $v \in \Omega$. In other words, the set of $\chi \in X(M(F))$ such that $\Re\chi \in \mathfrak{a}_{P_1}^{*,+}$ and at which $I_P^G(\sigma_\chi)$ is reducible is contained in

$$Z_P^+(\sigma) := \{\chi \in X(M(F)) \mid J_{P_c}(\xi_v)_\chi = 0,\, \exists P_c \in \mathcal{P}(M_c),\, v \in \Omega\}$$

Notice that this is independent of P_1. Furthermore, it follows from the rationality of $J_{P_c}(\xi_v)$ and Th. 4.2 that this is a proper Zariski closed subset of $X(M(F))$.

Finally we obtain a proper Zariski closed subset

$$Z_P(\sigma) := Z_P^+(\sigma) \cup \bigcup_{M_1 \in \mathcal{L}(M)} Z^{M_1}(\sigma)$$

which contains all χ at which $I_P^G(\sigma_\chi)$ is reducible. □

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