CAP automorphic representations of $U_{E/F}(4)$ I. Local A-packets

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Abstract

This is the first part of our two papers devoted to an explicit description of the CAP automorphic representations of quasisplit unitary groups in four variables over number fields. We describe them along the lines of Arthur's conjecture [Art89]. In this first part, we calculate candidates of the local A-packets consisting of the local components of CAP automorphic representations, by means of the local θ-correspondence from unitary groups in two variables. For this, we need a version of the ε-dichotomy property of the local θ-correspondence for unitary groups in two variables, involving the Langlands-Shahidi local factors. We also verify that Hiraga's conjecture [Hir] on a relation between ZASS duality and A-packets is compatible with our candidates.

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1 Introduction

This is the first one of two articles devoted to an explicit description of the CAP automorphic representations of the quasisplit unitary group in four variables $U_{E/F}(4)$ associated to a quadratic extension E/F of number fields. The term "CAP" is a short hand for the phrase "Cuspidal but Associated to Parabolic subgroups". This is the name given by Piatetski-Shapiro [PS83] to those cuspidal automorphic representations which apparently contradict the generalized Ramanujan conjecture. More precisely, let G be a connected reductive group defined over a number field F and G^* be its quasisplit inner form. We write $\mathbb{A} = \mathbb{A}_F$ for the adele ring of F. An irreducible cuspidal representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A})$ is a *CAP form* if there exists a residual discrete automorphic representation $\pi^* = \bigotimes_v \pi_v^*$ of $G^*(\mathbb{A})$ such that, at all but finite number of v, π_v and π_v^* share the same absolute values of Hecke eigenvalues.

It is a consequence of results of Jacquet-Shalika [JS81b], [JS81a] and Moeglin-Waldspurger [MW89] that there are no CAP forms on the general linear groups. For a central division algebra D of dimension n^2 over $F^\times$, however, the trivial representation of $D^\times(\mathbb{A})$ is a CAP form, since it shares the same local component with the residual representation $\mathbf{1}_{GL(n,\mathbb{A})}$ at any place v where D is unramified. Also it is known that even the quasisplit rank one unitary group $U_{E/F}(3)$ of three variables has non-trivial CAP forms [GR90], [GR91]. But the first and most well-known examples of CAP forms are Howe-Piatetski-Shapiro's analogue of the θ_{10} representation [Sou88] and the Saito-Kurokawa representations [PS83] of $Sp(2)$ (for their local structures see also [Wal91]).

In any case, local components of CAP forms at almost all places are non-trivial Langlands quotients by definition, and hence non-tempered. In order to describe such forms, J. Arthur proposed a series of conjectures [Art89]. The conjectural description is through the so-called *A-parameters*, homomorphisms ψ from the direct product of the hypothetical Langlands group $\mathcal{L}_F$ of F with $SL(2,\mathbb{C})$ to the L-group ${}^LG$ of G:

$$\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \longrightarrow {}^LG,$$

considered modulo $\widehat{G}$-conjugation. We write $\Psi(G)$ for the set of $\widehat{G}$-conjugacy classes of A-parameters for G. By restriction, we obtain the local component

$$\psi_v : \mathcal{L}_{F_v} \times SL(2,\mathbb{C}) \to {}^LG_v$$

of ψ at each place v of F. Here $\mathcal{L}_{F_v}$ is the local Langlands group defined in [Kot84, §12], and ${}^LG_v$ is the L-group of the scalar extension $G_v = G \otimes_F F_v$. The local conjecture, among other things, associates to ψ_v a finite set $\Pi_{\psi_v}(G_v)$ of isomorphism classes of irreducible unitarizable representations of $G(F_v)$, called an *A-packet*. At all but finite number of v, $\Pi_{\psi_v}(G_v)$ is expected to contain a unique unramified element π_v^1. Using such elements, we can form the global A-packet associated to ψ:

$$\Pi_\psi(G) := \left\{ \bigotimes_v \pi_v \;\middle|\; \begin{array}{ll} \text{(i)} & \pi_v \in \Pi_{\psi_v}(G_v), \text{ at any } v; \\ \text{(ii)} & \pi_v = \pi_v^1, \text{ at almost all } v \end{array} \right\}.$$

Arthur's global conjecture predicts the multiplicity of each element of $\Pi_\psi(G)$ in the discrete spectrum of the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}))$.

Let us call an A-parameter ψ is of *CAP type* if

(i) ψ is *elliptic*. This is the condition for $\Pi_\psi(G^*)$ to contain a discrete automorphic representation.

(ii) $\psi|_{SL(2,\mathbb{C})}$ is non-trivial.

According to the conjecture, the CAP automorphic representations of $G(\mathbb{A})$ is contained in the global A-packets associated to an A-parameter of CAP type. In this paper, we construct candidates for the local A-packets associated to the local components of A-parameters of CAP type for the quasisplit unitary group $U_{E/F}(4)$ in four variables. Although the description tells us nothing about the character relations conjectured in [Art89], it is explicit and fairly complete. In the subsequent one [Kon], we shall develop a theory of L-functions of $U_{E/F}(4) \times GL(2)_E$, and obtain a complete description of CAP automorphic representations of $U_{E/F}(4)$ in terms of the local A-packets constructed in this article. Since the main result of the present article is a case-by-case description of the A-packets (see § 6), we present here an outline of our construction.

Parameter considerations In § 2, we classify the A-parameters of CAP type for $G = G_4 := U_{E/F}(4)$ together with their S-groups

$$\mathcal{S}_\psi(G) := \pi_0(\mathrm{Cent}(\psi, \widehat{G})/Z(\widehat{G})),$$

which plays a central role in the conjectural multiplicity formula. Assuming the existence of the Langlands group, we first classify the elliptic A-parameters for general unitary group G_n in n-variables (Prop. 2.2). In practice, we need to replace the role of the Langlands group by the classification of automorphic representations of general linear groups. Thanks to Rogawski's result on the base change for $G_2 = U_{E/F}(2)$, this replacement is available for the A-parameters of CAP type for G (see Cor. 2.3). Then we have a similar description for their local components in § 3.1.

At this point, we need to assume some local assertions of Arthur's conjecture, which we review in § 3.2. In particular, for each local component ψ of a CAP type A-parameter, we have the associated non-tempered Langlands parameter ϕ_ψ. It is imposed that the L-packet $\Pi_{\phi_\psi}(G)$ corresponding to ϕ_ψ should be contained in $\Pi_\psi(G)$. In fact, we know from [Kon98], [Kon01] that $\Pi_{\phi_\psi}(G)$ consist exactly of the local components of residual discrete representations. We have the S-group $\mathcal{S}_\psi(G)$ as in the global case. Let us postulate the following slightly stronger assertion than Arthur's conjecture, which might not be true in general.

Assumption 1.1. *There exists a bijection* $\Pi_\psi(G) \ni \pi \longmapsto (\bar{s} \mapsto \langle \bar{s}, \pi\rangle_\psi) \in \Pi(\mathcal{S}_\psi(G))$. *Here* $\Pi(\mathcal{S}_\psi(G))$ *is the set of isomorphism classes of irreducible representations of* $\mathcal{S}_\psi(G)$.

There are six types of elliptic A-parameters appearing as local components of the A-parameters of CAP type. According to the assumption, for three types of the six, $\Pi_\psi(G)$ coincides with $\Pi_{\phi_\psi}(G)$ so that no construction beyond [Kon98] is necessary. But for the rest three types, $\Pi_{\phi_\psi}(G)$ contains only the half of the members of $\Pi_\psi(G)$.

Adams' conjecture We use the local θ-correspondence to construct the set $\Pi_\psi(G) \setminus \Pi_{\phi_\psi}(G)$ of missing members. Consider an m-dimensional hermitian space $(V, (\ ,\ ))$ and an n-dimensional skew-hermitian space $(W, \langle\ ,\ \rangle)$ over E. The unitary groups $G(V)$ and $G(W)$ for V and W form a type I dual reductive pair. For a pair $\underline{\xi} = (\xi, \xi')$ of characters of $E^\times$ satisfying $\xi|_{F^\times} = \omega_{E/F}^n$, $\xi'|_{F^\times} = \omega_{E/F}^m$, we have the Weil representation $\omega_{V,W,\underline{\xi}} = \omega_{W,\underline{\xi}}\omega_{V,\xi'}$ of $G(V) \times G(W)$ § 5.1 (cf. [HKS96]).

We write $\mathscr{R}(G(V), \omega_{W,\xi})$ for the set of isomorphism classes of irreducible admissible representations of $G(V)$ which appear as quotients of $\omega_{W,\xi}$. For $\pi_V \in \mathscr{R}(G(V), \omega_{W,\xi})$, the maximal π_V-isotypic quotient of $\omega_{V,W,\underline{\xi}}$ is of the form $\pi_V \otimes \Theta_{\underline{\xi}}(\pi_V, W)$ for some smooth representation $\Theta_{\underline{\xi}}(\pi_V, W)$ of $G(W)$. Similarly we have $\mathscr{R}(G(W), \omega_{V,\xi'})$ and $\Theta_{\underline{\xi}}(\pi_W, V)$ for each $\pi_W \in \mathscr{R}(G(W), \omega_{V,\xi'})$. The local Howe duality conjecture, which was proved by R. Howe himself if F is archimedean [How89] and by Waldspurger if F is a non-archimedean local field of odd residual characteristic [Wal90], asserts the following:

(i) $\Theta_{\underline{\xi}}(\pi_V, W)$ (resp. $\Theta_{\underline{\xi}}(\pi_W, V)$) is an admissible representation of finite length of $G(W)$ (resp. $G(V)$), so that it admits an irreducible quotient.

(ii) Moreover its irreducible quotient $\theta_{\underline{\xi}}(\pi_V, W)$ (resp. $\theta_{\underline{\xi}}(\pi_W, V)$) is unique.

(iii) $\pi_V \mapsto \theta_{\underline{\xi}}(\pi_V, W)$, $\pi_W \mapsto \theta_{\underline{\xi}}(\pi_W, V)$ are bijections between $\mathscr{R}(G(V), \omega_{W,\xi})$ and $\mathscr{R}(G(W), \omega_{V,\xi'})$ converse to each other.

A link between local θ-correspondence and A-packets is given by the following conjectures of J. Adams [Ada89], which will be taken as a leading principle in our construction. Suppose $n \geq m$. Then we have an L-embedding $i_{V,W,\underline{\xi}} : {}^L G(V) \to {}^L G(W)$ given by

$$i_{V,W,\underline{\xi}}(g \rtimes w) = \begin{cases} \xi'\xi^{-1}(w) \begin{pmatrix} g & \\ & \mathbf{1}_{n-m} \end{pmatrix} \times w & \text{if } w \in W_E \\ \begin{pmatrix} & g \\ J_{n-m}^{n-m-1} & \end{pmatrix} \rtimes w_\sigma & \text{if } w = w_\sigma, \end{cases}$$

where w_σ is a fixed element in $W_F \setminus W_E$ and $J_n = \mathrm{diag}(1, -1, \dots, (-1)^{n-1})$. Let $\mathrm{T} : SL(2, \mathbb{C}) \to \mathrm{Cent}(i_{V,W,\underline{\xi}}, \widehat{G}(W))$ be the homomorphism which corresponds to a regular unipotent element in $\mathrm{Cent}(i_{V,W,\underline{\xi}}, \widehat{G}(W)) \simeq GL(n-m, \mathbb{C})$ (the *tail representation*). Using this, we define the θ-lifting of A-parameters by

$$\theta_{V,W,\underline{\xi}} : \Psi(G(V)) \ni \psi \longmapsto (i_{V,W,\underline{\xi}} \circ \psi^\vee) \cdot T \in \Psi(G(W)).$$

Conjecture 1.2 ([Ada89] Conj.A). *The local θ-correspondence should be subordinated to the map of A-packets $\Psi_\psi(G(V)) \mapsto \Pi_{\theta_{V,W,\underline{\xi}}(\psi)}(G(W))$.*

Here we have said subordinated because $\Pi_\psi(G(V)) \cap \mathscr{R}(G(V), \omega_{W,\xi})$ is often strictly smaller than $\Pi_\psi(G(V))$. When these two are assured to coincide, we can expect more:

Conjecture 1.3 ([Ada89] Conj.B). *For V, W in the stable range, that is, the Witt index of W is larger than m, we have*

$$\Pi_{\theta_{V,W,\underline{\xi}}(\psi)}(G(W)) = \bigcup_{V;\, \dim_E V = m} \theta_{\underline{\xi}}(\Pi_\psi(G(V)), W).$$

ε-dichotomy Consider the special case $m = 2$ and $W = V \oplus -V$. It turns out that the three types of A-parameters in question are exactly those of the form $\psi = \theta_{V,W,\underline{\xi}}(\psi'_1)$ for some elliptic A-parameters ψ'_1 for G_2. Thus we should be able to construct the desired candidates by calculating $\theta(\tau_V, W)$, $\tau_V \in \Pi_{\psi'_1}(G(V))$ for various V. But for our global applications, we also need a parametrization of the members of $\Pi_\psi(G)$ well-suited to the conjectural multiplicity formula. For this, we need one more ingredient. For the purpose of this introduction, it is best to restrict ourselves to the non-archimedean case. For the corresponding result in the case $F = \mathbb{R}$, see § 5.3.

Theorem 1.4 (Th. 5.4). *Suppose* $\dim_E V = 2$ *and write* W_1 *for the hyperbolic skew-hermitian space* $(E^2, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$. *Take an L-packet* Π *of* $G_2(F) = G(W)$ *and* $\tau \in \Pi$ *[Rog90, Ch.11].*
(i) $\tau \in \mathscr{R}(G(W), \omega_{V,\xi'})$ *if and only if*

$$\varepsilon(1/2, \Pi \times \xi\xi'^{-1}, \psi_F)\omega_\Pi(-1)\lambda(E/F, \psi_F)^{-2} = \omega_{E/F}(-\det V).$$

Here the ε*-factor on the right hand side is the standard* ε*-factor for* G_2 *twisted by* $\xi\xi'^{-1}$. ω_Π *is the central character of the elements of* Π, *and* $\lambda(E/F, \psi_F)$ *is the Langlands* λ*-factor [Lan70].*
(ii) If this is the case, $\theta_{\underline{\xi}}(\tau, V) = (\xi^{-1}\xi')_{G(V)}\tau_V^\vee$. *Here* $(\xi^{-1}\xi')_{G(V)}$ *denotes the character* $G(V) \overset{\det}{\to} U_{E/F}(1, F) \ni z/\sigma(z) \mapsto \xi^{-1}\xi'(z) \in \mathbb{C}^\times$. *Recall the Jacquet-Langlands correspondence* $\Pi \mapsto \Pi_V$ *between the (discrete if* V *is anisotropic) L-packets of* $G_2(F)$ *and those of* $G(V)$ *[LL79].* $\Pi \ni \tau \mapsto \tau_V \in \Pi_V$ *is a certain bijection, which we cannot specify explicitly.*

This is a special case of the ε-*dichotomy* property of the local θ-correspondence for unitary groups over p-adic fields, which was proved for general unitary groups (at least for supercuspidal representations) in [HKS96]. But since we need to combine this with our description of the residual spectrum [Kon98], we have to use the Langlands-Shahidi ε-factors [Sha90] instead of Piatetski-Shapiro-Rallis's doubling ε-factors used in that paper. By this reason, we deduce the theorem from the analogous result for unitary similitude groups [Har93] in § 5.2. The key is the following description of the base change lifting of representations of $G_2(F)$ to $GL(2, E)$, which is proved at length in § 4.

Proposition 1.5 (Cor. 4.14). *Let* $\widetilde{\pi} = \omega \otimes \pi'$ *be an irreducible admissible representation of the unitary similitude group* $GU_{E/F}(2) \simeq E^\times \times GL(2, F)/\Delta F^\times$, *and write* $\Pi(\widetilde{\pi})$ *for the associated L-packet of* $G_2(F)$ *consisting of the irreducible components of* $\widetilde{\pi}|_{G_2(F)}$. *Then the standard base change of* $\Pi(\widetilde{\pi})$ *to* $GL(2, E)$ *[Rog90, 11.4] is given by* $\omega(\det)\pi'_E$, *where* π'_E *is the base change lift of* π' *to* $GL(2, E)$ *[Lan80].*

Candidates for the A-packets The candidates of the A-packets are constructed in § 6. Here we explain the construction in the non-archimedean case (§ 6.1), which is summarized in Fig. 1.

Each A-parameter of our concern (or its restriction to $\mathcal{L}_E \times SL(2, \mathbb{C})$) is of the form

$$\psi = \psi_1 \oplus (\xi'\xi^{-1} \otimes \rho_2),$$

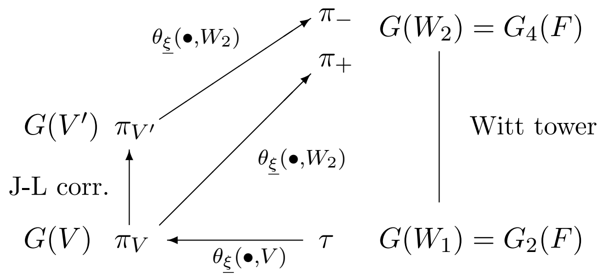

Figure 1: Construction of A-packets

where ψ_1 is an elliptic A-parameter for G_2. Take $\tau \in \Pi_{\psi_1}(G_2)$ and let $(V, (\ ,\ ))$ be the 2-dimensional hermitian space such that the condition of Th. 1.4 (i) holds. If we write $\pi_V := \theta_{\underline{\xi}}(\tau, V) \simeq (\xi\xi'^{-1})_{G(V)}\tau_V^\vee$, then the induction principle in local θ-correspondence [Kud86] (§ 6.1.1) implies that $\pi_+ := \theta_{\underline{\xi}}(\pi_V, W_2)$, $(\tau \in \Pi_{\psi_1}(G_2))$ form the local residual L-packet $\Pi_{\phi_\psi}(G_4)$. Next we take π'_V in the Jacquet-Langlands correspondent of the A-packet $\theta_{\underline{\xi}}(\Pi_{\psi_1}(G_2), V)$, an irreducible representation of the unitary group $G(V')$ of the other (isometry class of) 2-dimensional hermitian space V'. Then $\pi_- := \theta_{\underline{\xi}}(\pi_{V'}, W_2)$ is the so-called *early lift* or *first occurrence*. In particular, if $\pi_{V'}$ is supercuspidal then so isπ_-. Following Conj. 1.3, we define

$$\Pi_\psi(G) := \{\pi_\pm \,|\, \tau \in \Pi_\psi(G_2)\}.$$

This gives the sufficiently many members of the packet as predicted by Assumption 1.1. Analogous construction in the archimedean case is given in § 6.3.

In § 6.2, we show that our candidates of the A-packets are compatible with Hiraga's conjecture [Hir] on a relation between ZASS (Zelevinsky-Aubert-Schneider-Stühler) duality and the A-packets. Although the global A-parameters of CAP type are elliptic, some of their local components are not. For later use, we also describe these non-elliptic A-packets in terms of local θ-correspondence in § 6.4. Also the split case $E \simeq F \oplus F$, $G \simeq GL(4)_F$ is treated in § 6.5.

In the final section § 7, we give an example of the half of the multiplicity formula in order to motivate the reader for the subsequent article. The result (Th. 7.1) is not satisfactory but it clarifies the role of global root numbers in the multiplicity formula, which is one of the main features in Arthur's conjecture.

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Notation For a subset S of a group G, we write $\mathrm{Cent}(S, G)$ and $\mathrm{Norm}(S, G)$ for the centralizer and normalizer of S in G, respectively. For a topological group or an algebraic group G, write G^0 for its connected component of the identity. $\mathrm{diag}(x_1, \ldots, x_r)$ or

$\operatorname{diag}(\{x_i\}_{i=1}^r)$ stands for the matrix

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_r \end{pmatrix},$$

where x_i are square matrices. We write $\mathbb{M}_{n,m}(R)$ and $\mathbb{M}_n(R)$ for the R-modules of $n \times m$ and $n \times n$ matrices with entries in a ring R, respectively.

2 A-parameters of CAP type for unitary groups

We start with a classification of the elliptic A-parameters for unitary groups associated to a quadratic extension of number fields. The parameters which we shall treat in this paper are listed in Cor. 2.3

2.1 Settings

Throughout this section let E be a quadratic extension of a number field F. We write σ for the generator of the Galois group $\Gamma_{E/F}$ of this extension. We fix an algebraic closure $\bar{F}$ of F containing E. Γ and W_F denote the Galois and Weil group of $\bar{F}$ over F, respectively. Also we need the hypothetical Langlands group $\mathcal{L}_F$ of F. We use this only to describe A-parameters, and in practical considerations we use the classification of automorphic representations of $GL(n)$ instead. As for the theory of endoscopy and the notation related to it, we follow the book [KS99].

Writing

$$I_n := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & ⋰ & & \\ (-1)^{n-1} & & & \end{pmatrix},$$

we introduce the outer automorphism $\theta_n(g) := \operatorname{Ad}(I_n)^t g^{-1}$ of $GL(n)$. Let us realize the quasisplit unitary group G_n associated to E/F in such a way that

$$G_n(R) = \{g \in GL(n, R \otimes_F E) \,|\, \theta_n(\sigma(g)) = g\}$$

for any commutative F-algebra R. By the choice of θ_n, the usual splitting of $GL(n)_E \simeq G_n \otimes_F E$ gives a splitting $\mathbf{spl}_{G_n} = (\mathbf{B}_n, \mathbf{T}_n, \{X\})$ stable under $\Gamma_{E/F}$ (an *F-splitting*). We fix the following set of L-group data $(\widehat{G}_n, \rho_{G_n}, \eta_{G_n})$ for G_n: $\widehat{G}_n = GL(n, \mathbb{C})$,

$$\rho_{G_n}(w) = \begin{cases} \mathrm{id} & \text{if } w \in W_E, \\ \theta_n & \text{otherwise,} \end{cases}$$

and η_{G_n} is the obvious identification between the dual of the based root datum for G_n and the based root datum of $\widehat{G}_n$. The L-group of G_n is the semidirect product ${}^L G_n = \widehat{G}_n \rtimes_{\rho_{G_n}} W_F$.

Also we let $H_n := \mathrm{R}_{E/F}GL(n)$, where $\mathrm{R}_{E/F}$ stands for Weil's restriction of scalars. The double copy of the standard splitting of $GL(n)$ defines a splitting $\mathbf{spl}_{H_n} = (\mathbf{B}_n^H, \mathbf{T}_n^H, \{X^{H_n}\})$ of $H_n \otimes_F E \simeq GL(n)_E^2$, which is actually an F-splitting of H_n. The L-group ${}^L H_n = \widehat{H}_n \rtimes_{\rho_{H_n}} W_F$ is given by $\widehat{H}_n = GL(n,\mathbb{C})^2$ and

$$\rho_{H_n}(w)(h_1,h_2) = \begin{cases} (h_1,h_2) & \text{if } w \in W_E, \\ (h_2,h_1) & \text{otherwise,} \end{cases}$$

for $(h_1,h_2) \in GL(n,\mathbb{C})^2$.

For the moment, consider a general connected reductive group G defined over F with the L-group ${}^L G$. We define A-parameters for G as in the introduction [Art89]. In practice, we need to drop the boundedness of $\psi(\mathcal{L}_F)$ because of the lack of the generalized Ramanujan conjecture. We shall give an alternative condition which is accessible but more technical in § 3.2. Note that the classification results given below is not affected by this change. Two A-parameters are *equivalent* if they are $\widehat{G}$-conjugate. The set of equivalence classes of A-parameters for G is denoted by $\Psi(G)$. For $\psi \in \Psi(G)$, we write $S_\psi(G)$ for the centralizer of the image of ψ in $\widehat{G}$, and set $\mathcal{S}_\psi(G) := S_\psi(G)/S_\psi(G)^0 Z(\widehat{G})^\Gamma$. An A-parameter ψ is *elliptic* if $S_\psi(G)^0$ is contained in $Z(\widehat{G})^\Gamma$, or equivalently, $S_\psi(G)^0 \subset \widehat{A}_G$ with $\widehat{A}_G := (Z(\widehat{G})^\Gamma)^0$. For $G = G_n$, $\widehat{A}_G$ is trivial so that this condition is equivalent to the discreteness of $S_\psi(G)$. We write $\Psi_0(G)$ for the subset elliptic elements in $\Psi(G)$. $\Psi_{\mathrm{CAP}}(G)$ denotes the subset of elements of *CAP type*, defined in the introduction, of $\Psi_0(G)$.

In what follows, we write $\mathcal{A}_F := \mathcal{L}_F \times SL(2,\mathbb{C})$ for brevity. We fix $w_\sigma \in W_F \setminus W_E$ so that $\mathcal{A}_F = \mathcal{A}_E \cup w_\sigma \mathcal{A}_E$.

2.2 Global A-parameters for unitary groups

Here we give a description of $\Psi_0(G_n)$. This begins with an irreducible decomposition.

Let $\psi \in \Psi_0(G_n)$. Thanks to the semisimplicity of Langlands parameter, the representation $\psi|_{\mathcal{A}_E} : \mathcal{A}_E \to GL(n,\mathbb{C})$ admits an irreducible decomposition. Moreover looking at the action of $\mathcal{A}_F/\mathcal{A}_E \simeq \Gamma_{E/F}$, this decomposition must be of the form

$$\psi|_{\mathcal{A}_E} \simeq \bigoplus_{i=1}^{r} \tau_i^{\oplus m_i} \oplus \bigoplus_{j=r+1}^{s} (\tau_j \oplus \sigma(\tau_j))^{\oplus m_j}, \tag{2.1}$$

where $\sigma(\tau_j) := \tau_j \circ \mathrm{Ad}(w_\sigma)$, and we have imposed

$$\tau_i \not\simeq \tau_j, \quad \text{if } i \neq j, \qquad \begin{cases} \tau_i \simeq \sigma(\tau_i) & \text{for } 1 \le i \le r, \\ \tau_j \not\simeq \sigma(\tau_j) & \text{for } r+1 \le j \le s. \end{cases}$$

Let V_i be the τ_i- (resp. $\tau_i \oplus \sigma(\tau_i)$-) isotypic subspace of $\psi|_{\mathcal{A}_E}$ for $1 \le i \le r$ (resp. $r+1 \le i \le s$). We may assume that $\prod_{i=1}^s GL(V_i)$ is the standard Levi subgroup $\prod_{i=1}^s GL(n_i)$ with $n_i := \dim V_i$. Write $\psi(w_\sigma) = \psi(w_\sigma)^0 \rtimes w_\sigma$ with $\psi(w_\sigma)^0 \in \widehat{G}_n$. For this to preserve each $GL(n_i)$-component, $\psi(w_\sigma)^0$ must be of the form

$$\psi(w_\sigma)^0 = \begin{pmatrix} & & & x_1 \\ & & x_2 & \\ & \iddots & & \\ x_s & & & \end{pmatrix}, \quad x_i \in GL(n_i,\mathbb{C}).$$

Thus writing $\hat{n}_i := \sum_{j=i+1}^{s} n_j$, $\check{n}_i := \sum_{j=1}^{i-1} n_j$, we have

$$\begin{aligned}
&\rho_{G_n}(w_\sigma)\psi(w_\sigma)^0 \\
&= \begin{pmatrix} & & & I_{n_s} \\ & & \iddots & \\ & (-1)^{\hat{n}_2} I_{n_2} & & \\ (-1)^{\hat{n}_1} I_{n_1} & & & \end{pmatrix} {}^t(\psi(w_\sigma)^0)^{-1} \begin{pmatrix} & & & (-1)^{\check{n}_s} I_{n_s}^{-1} \\ & & \iddots & \\ & (-1)^{\check{n}_2} I_{n_2}^{-1} & & \\ I_{n_1}^{-1} & & & \end{pmatrix} \\
&= \begin{pmatrix} & & & (-1)^{\check{n}_s}\theta_{n_s}(x_s) \\ & & \iddots & \\ & (-1)^{\check{n}_2+\hat{n}_2}\theta_{n_2}(x_2) & & \\ (-1)^{\hat{n}_1}\theta_{n_1}(x_1) & & & \end{pmatrix}
\end{aligned}$$

noting $\check{n}_i + \hat{n}_i = n - n_i$,

$$= \begin{pmatrix} & & & (-1)^{n-n_s}\theta_{n_s}(x_s) \\ & & \iddots & \\ & (-1)^{n-n_2}\theta_{n_2}(x_2) & & \\ (-1)^{n-n_1}\theta_{n_1}(x_1) & & & \end{pmatrix}$$

This gives

$$\begin{aligned}
&\mathrm{diag}(\{(-1)^{n-n_i} x_i \theta_{n_i}(x_i)\}_{i=1}^s) \times w_\sigma^2 = \psi(w_\sigma)^0 \rho_{G_n}(w_\sigma)(\psi(w_\sigma)^0) \times w_\sigma^2 = \psi(w_\sigma^2) \\
&= \psi(w_\sigma)^2 = \mathrm{diag}(\{\tau_i(w_\sigma^2)^{\oplus m_i}\}_{i=1}^r, \{(\tau_i \oplus \sigma(\tau_i))(w_\sigma^2)^{\oplus m_i}\}_{i=r+1}^s) \times w_\sigma^2.
\end{aligned} \tag{2.2}$$

We write $\mathbb{A}$ and $\mathbb{A}_E$ for the adele rings of F and E, respectively. $\omega_{E/F}$ denotes the quadratic idele class character of F associated to E/F by the classfield theory. For each $1 \le i \le s$, we shall fix a character ω'_i of $\mathbb{A}_E^\times / E^\times$ whose restriction to $\mathbb{A}^\times$ equals to $\omega_{E/F}^{n-n_i}$. Define $\psi_i : \mathcal{A}_F \to {}^L G_{n_i}$, $(1 \le i \le s)$ by

$$\psi_i|_{\mathcal{A}_E} = \begin{cases} {\omega'}_i^{-1} \tau_i^{\oplus m_i} \times p_{W_E} & \text{for } 1 \le i \le r, \\ {\omega'}_i^{-1} (\tau_i \oplus \sigma(\tau_i))^{\oplus m_i} \times p_{W_E} & \text{for } r+1 \le i \le s, \end{cases} \qquad \psi_i(w_\sigma) = x_i \rtimes w_\sigma,$$

where p_{W_E} denotes the conjectural homomorphism $\mathcal{L}_E \to W_E$. ω'_i are identified with characters of W_E by the classfield theory.

Lemma 2.1. *(i) ψ_i is an well-defined A-parameter for G_{n_i} for each $1 \le i \le s$.*
(ii) $\psi \in \Psi_0(G_n)$ if and only if $\psi_i \in \Psi_0(G_{n_i})$ for all $1 \le i \le s$.

Proof. (i) follows immediately from (2.2). As for (ii), we have only to note that Schur's lemma implies

$$S_\psi(G) = \mathrm{Cent}(\psi, \mathrm{Cent}(\psi(\mathcal{A}_E), \widehat{G}_n)) = \mathrm{Cent}(\psi, \prod_{i=1}^s GL(n_i, \mathbb{C})) = \prod_{i=1}^s S_{\psi_i}(G_{n_i}).$$

□

Thus we are reduced to study the ellipticity of each ψ_i. First consider the case $1 \leq i \leq r$. Writing V_{τ_i} for the space of τ_i, we have $V_i = V_{\tau_i} \otimes \mathbb{C}^{m_i}$ and

$$S_{\psi_i}(G_{n_i}) = \mathrm{Cent}(\psi_i(w_\sigma), \mathbf{1}_{V_{\tau_i}} \otimes GL(m_i, \mathbb{C})).$$

Then $\iota := \mathrm{Ad}(\psi_i(w_\sigma))$ acts on $GL(m_i, \mathbb{C})$ as an automorphism of order two, because $\psi_i(w_\sigma^2) = \tau_i(w_\sigma^2) \otimes \mathbf{1}_{m_i}$. For ψ_i to be in $\Psi_0(G_{n_i})$, it is necessary and sufficient that $GL(m_i, \mathbb{C})^\iota$ is finite. Since $\mathrm{Aut}(GL(m_i)) = \mathrm{Int}(GL(m_i)) \rtimes \langle \theta_{m_i} \rangle$, this forces $m_i = 1$. Thus $n_i = \dim \tau_i$ and $\psi_i(w_\sigma)$ acts as $\mathrm{Ad}(\psi_i(w_\sigma)^0) \circ \theta_{n_i}$ on τ_i. This also implies $\sigma(\tau_i)$ is isomorphic to $\tau_i^\vee$, the contragredient of τ_i.

Next consider $r + 1 \leq i \leq s$. Similar argument as above shows $m_i = 1$. We have an irreducible decomposition $V_i = V_{\tau_i} \oplus V_{\sigma(\tau_i)}$ as an $\mathcal{A}_E$-module and

$$S_{\psi_i}(G_{n_i}) = \mathrm{Cent}(\psi_i(w_\sigma), \mathbb{C}^\times \mathrm{id}_{V_{\tau_i}} \times \mathbb{C}^\times \mathrm{id}_{V_{\sigma(\tau_i)}}).$$

Since $\mathrm{Ad}(\psi_i(w_\sigma))\psi_i|_{\mathcal{A}_E} = \psi_i|_{\mathcal{A}_E} \circ \mathrm{Ad}(w_\sigma)$, $\psi_i(w_\sigma)$ must interchange V_{τ_i} and $V_{\sigma(\tau_i)}$. These show that $S_{\psi_i}(G_{n_i})$ equals the diagonal subgroup $\mathbb{C}^\times(\mathrm{id}_{V_{\tau_i}} \oplus \mathrm{Ad}(\psi_i(w_\sigma))\mathrm{id}_{V_{\tau_i}})$. Hence such ψ_i cannot be elliptic.

Define $\Phi_0^{\mathrm{st}}(G_n)$ to be the set of isomorphism classes of Langlands parameters $\varphi : \mathcal{L}_F \to {}^L G_n$ such that

- $\mathrm{Im}\varphi$ is bounded.
- $\varphi|_{\mathcal{A}_E}$ is irreducible.

For $m \in \mathbb{N}$, let ρ_m be the m-dimensional irreducible representation of $SL(2, \mathbb{C})$.

Proposition 2.2. $\Psi_0(G_n)$ *is in bijection with the set of finite families of quadruples* $(d_i, m_i, \omega_i, \varphi_i)_{i=1}^r$*:*

(i) d_i, m_i *are positive integers satisfying* $n = \sum_{i=1}^r d_i m_i$.

(ii) ω_i *is an idele class character of* E *such that* $\omega_i|_{\mathbb{A}^\times} = \omega_{E/F}^{n-d_i-m_i+1}$.

(iii) $\varphi_i \in \Phi_0^{\mathrm{st}}(G_{m_i})$.

such that $\omega_i\varphi_i \otimes \rho_{d_i} \not\simeq \omega_j\varphi_j \otimes \rho_{d_j}$ *for* $1 \leq i \neq j \leq r$. *The bijection is given by the relation*

$$\psi|_{\mathcal{A}_E} = \begin{pmatrix} \omega_1(\varphi_1|_{\mathcal{L}_E}) \otimes \rho_{d_1} & & \\ & \ddots & \\ & & \omega_r(\varphi_r|_{\mathcal{L}_E}) \otimes \rho_{d_r} \end{pmatrix},$$

$$\psi(w_\sigma) = \begin{pmatrix} & & \varphi_1(w_\sigma) J_{m_1}^{d_1-1} \otimes \mathbf{1}_{d_1} \\ & \iddots & \\ \varphi_r(w_\sigma) J_{m_r}^{d_r-1} \otimes \mathbf{1}_{d_r} & & \end{pmatrix} \rtimes w_\sigma.$$

Here $J_m = \mathrm{diag}(1, -1, \ldots, (-1)^{m-1})$.

Proof. We retain the notation of the above argument. We may write

$$\psi_i|_{\mathcal{A}_E} \simeq \varphi_i' \otimes \rho_{d_i}.$$

Here φ_i' is an irreducible representation of $\mathcal{L}_E$ with bounded image whose dimension we denote by m_i. We obviously have $n = \sum_{i=1}^r d_i m_i$.

Since $\sigma(\psi_i|_{\mathcal{A}_E}) \simeq (\psi_i|_{\mathcal{A}_E})^\vee$, we may choose an isomorphism $\varphi_i'(w_\sigma) : \theta_{m_i} \circ \varphi_i' \xrightarrow{\sim} \varphi_i' \circ \mathrm{Ad}(w_\sigma)$. This must satisfy

$$\mathrm{Ad}(\varphi_i'(w_\sigma)\theta_{m_i}(\varphi_i'(w_\sigma))) \circ \varphi_i' = \varphi_i' \circ \mathrm{Ad}(w_\sigma^2) = \mathrm{Ad}(\varphi_i'(w_\sigma^2)) \circ \varphi_i'.$$

The irreducibility of φ_i' implies $\varphi_i'(w_\sigma)\theta_{m_i}(\varphi_i'(w_\sigma)) = z\varphi_i'(w_\sigma^2)$, for some $z \in \mathbb{C}^\times$. Moreover, applying $\mathrm{Ad}(\varphi_i'(w_\sigma)) \circ \theta_{m_i}$ to this, we see

$$\begin{aligned} z\varphi_i'(w_\sigma^2) =& \varphi_i'(w_\sigma)\theta_{m_i}(\varphi_i'(w_\sigma)) = \mathrm{Ad}(\varphi_i'(w_\sigma)) \circ \theta_{m_i}(\varphi_i'(w_\sigma)\theta_{m_i}(\varphi_i'(w_\sigma))) \\ =& z^{-1}\mathrm{Ad}(\varphi_i'(w_\sigma)) \circ \theta_{m_i}(\varphi_i'(w_\sigma^2)) = z^{-1}\varphi_i'(\mathrm{Ad}(w_\sigma)w_\sigma^2) \\ =& z^{-1}\varphi_i'(w_\sigma^2), \end{aligned}$$

so that $z = \pm 1$. Take an idele class character ω_i'' of E satisfying $\omega_i''|_{\mathbb{A}^\times} = \omega_{E/F}^{\epsilon_i}$, where $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ is such that $(-1)^{\epsilon_i} = z$. Now one can easily check that

$$\varphi_i(w) := \begin{cases} \omega_i''^{-1}(w)\varphi_i'(w) \times p_{W_E}(w) & \text{if } w \in \mathcal{L}_E, \\ \varphi_i'(w_\sigma) \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases}$$

defines an well-defined element $\varphi_i \in \Phi_0^{\mathrm{st}}(G_{m_i})$.

We still have to determine ϵ_i. For this, we note the tensor product decomposition $\theta_{n_i} = \mathrm{Ad}(J_{m_i}^{d_i-1})\theta_{m_i} \otimes \theta_{d_i}$. If we write $\psi_i(w_\sigma) = g_i(\varphi_i^0(w_\sigma)J_{m_i}^{d_i-1} \otimes \mathbf{1}_{d_i}) \rtimes w_\sigma$ for some $g_i \in GL(n_i, \mathbb{C})$, we have

$$\begin{aligned} (\omega_i''\varphi_i^0)(w_\sigma w w_\sigma^{-1}) \otimes \rho_{d_i}(g) = & \mathrm{Ad}(\psi_i(w_\sigma))\psi_i^0(w, g) \\ =& \mathrm{Ad}(g_i)(\mathrm{Ad}(\varphi_i(w_\sigma))(\omega_i''\varphi_i^0)(w) \otimes \rho_{d_i}(g)) \\ =& \mathrm{Ad}(g_i)((\omega_i''\varphi_i^0)(w_\sigma w w_\sigma^{-1}) \otimes \rho_{d_i}(g)), \end{aligned}$$

for any $w \in \mathcal{L}_E$, $g \in SL(2, \mathbb{C})$. Thus g_i must be a scalar matrix which, after a suitable conjugation in $\widehat{G}_{n_i}$, we may assume to be $\mathbf{1}_{n_i}$. We now use the equality between

$$\begin{aligned} \psi_i^0(w_\sigma)^2 =& (\varphi_i(w_\sigma)^0 J_{m_i}^{d_i-1} \otimes \mathbf{1}_{d_i})\theta_{n_i}(\varphi_i(w_\sigma)^0 J_{m_i}^{d_i-1} \otimes \mathbf{1}_{d_i}) \\ =& \varphi_i(w_\sigma)^0 J_{m_i}^{d_i-1} \cdot J_{m_i}^{d_i-1}\theta_{m_i}(\varphi_i(w_\sigma)^0)\theta_{m_i}(J_{m_i}^{d_i-1})J_{m_i}^{1-d_i} \otimes \mathbf{1}_{d_i} \\ =& (-1)^{(m_i-1)(d_i-1)}\varphi_i(w_\sigma^2)^0 \otimes \mathbf{1}_{d_i} \end{aligned}$$

and $\psi_i^0(w_\sigma^2) = \omega_i''(w_\sigma^2)\varphi_i^0(w_\sigma^2) \otimes \mathbf{1}_{d_i}$ to see that $\epsilon_i \equiv (d_i - 1)(m_i - 1) \mod 2$. We set $\omega_i := \omega_i'\omega_i''$. □

We specialize this to the case $n = 4$. $\Phi_0(H_n)$ denotes the set of equivalence (*i.e.* $\widehat{H}_n$-conjugacy) classes of elliptic Langlands parameters with bounded image. If we embed $\widehat{H}_n$ diagonally into $GL(2n, \mathbb{C})$, $\Phi_0(H_n)$ is exactly the set of induced representations $\mathrm{ind}_{\mathcal{L}_E}^{\mathcal{L}_F} \varphi_E$

where φ_E runs over the set of isomorphism classes of irreducible n-dimensional representations of $\mathcal{L}_E$ with bounded image. Then conjecturally, this is in bijection with the set of irreducible cuspidal representations of $H_n(\mathbb{A})$.

Recall the standard base change map:

$$\xi_{\mathbf{1}} : {}^L G_n \ni g \rtimes w \longmapsto (g, \theta_n(g)) \rtimes w \in {}^L H_n.$$

The map $\Phi_0^{\mathrm{st}}(G_n) \ni \varphi \mapsto \xi_{\mathbf{1}} \circ \varphi \in \Phi_0(H_n)$ is an well-defined injection. For $n \leq 3$, its image consists of $\varphi = \mathrm{ind}_{\mathcal{L}_E}^{\mathcal{L}_F} \varphi_E$ such that the corresponding irreducible cuspidal rerpesentation Π_E of $H_n(\mathbb{A})$ satisfies:

(i) $\sigma(\Pi_E) := \Pi_E \circ \sigma \simeq \Pi_E^\vee$. This forces that its central character ω_{Π_E} restricted to $\mathrm{N}_{E/F}(\mathbb{A}_E^\times)$ is trivial.

(ii) ω_{Π_E} is trivial on $\mathbb{A}^\times$.

(iii) The twisted tensor (generalized Asai-Oda) L-function $L_{\mathrm{Asai}}(s, \Pi_E)$ does not have a pole at $s = 1$ in the case $n = 2$.

In fact, the stable cuspidal L-packets of $G_n(\mathbb{A})$ lift exactly to these cuspidal representations by the base change lift corresponding to $\xi_{\mathbf{1}}$ [Rog90, 11.5, Th.13.3.3]. Thus, in practice, we can parametrize each element $\varphi_\Pi \in \Phi_0^{\mathrm{st}}(G_n)$, $(n \leq 3)$ by the irreducible cuspidal representation $\Pi_E = \xi_{\mathbf{1}}(\Pi)$ of $H_n(\mathbb{A})$ satisfying the above conditions.

We save η and μ to denote idele class characters of E whose restriction to $\mathbb{A}^\times$ equal $\mathbf{1}$ and $\omega_{E/F}$, respectively.

Corollary 2.3. *The set $\Psi_{\mathrm{CAP}}(G_4)$ consists of the following elements.*
(1) Stable parameters.

Name	$\psi\|_{\mathcal{A}_E}$	$\psi(w_\sigma)$
(1.a) ψ_η	$(\eta \otimes \rho_4) \times p_{W_E}$	$\mathbf{1}_4 \rtimes w_\sigma$
(1.b) $\psi_{\Pi,\mu}$	$[\mu(\varphi_\Pi^0\|_{\mathcal{L}_E}) \otimes \rho_2] \times p_{W_E}$	$[\varphi_\Pi^0(w_\sigma) J_2 \otimes \mathbf{1}_2] \rtimes w_\sigma$

Here Π runs over the set of stable L-packets of $G_2(\mathbb{A})$ containing a cuspidal representation.
(2) Endoscopic parameters.

Name	$\psi\|_{\mathcal{A}_E}$	$\psi(w_\sigma)$
(2.a) $\psi_{\underline{\mu}}$	$((\mu \otimes \rho_3) \oplus \mu') \times p_{W_E}$	$\begin{pmatrix} & \mathbf{1}_3 \\ 1 & \end{pmatrix} \rtimes w_\sigma$
(2.b) $\psi_{\Pi,\eta}$	$((\eta \otimes \rho_2) \oplus \varphi_\Pi^0\|_{\mathcal{L}_E}) \times p_{W_E}$	$\begin{pmatrix} & \mathbf{1}_2 \\ \varphi_\Pi^0(w_\sigma) & \end{pmatrix} \rtimes w_\sigma$
(2.c) $\psi_{\underline{\eta}}$	$((\eta \otimes \rho_2) \oplus (\eta' \otimes \rho_2)) \times p_{W_E}$	$\begin{pmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{pmatrix} \rtimes w_\sigma$
(2.d) $\psi_{\eta,\underline{\mu}}$	$((\eta \otimes \rho_2) \oplus \mu \oplus \mu') \times p_{W_E}$	$\begin{pmatrix} & & \mathbf{1}_2 \\ & 1 & \\ 1 & & \end{pmatrix} \rtimes w_\sigma$

Here, in (2.a) $\underline{\mu} = (\mu, \mu')$ and μ' can be μ, in (2.b) Π and φ_Π are the same as in (1.b), in (2.c) $\underline{\eta} = (\eta, \eta')$ with $\eta \neq \eta'$, and in (2.d) $\underline{\mu} = (\mu, \mu')$ with $\mu \neq \mu'$.

3 Local A-parameters and induced representations

From now on, we turn to the local situation and study the local A-packets associated to the parameters listed in Cor. 2.3. Thus let E be a quadratic extension of a local field F of characteristic zero. We write $|\,|_F$ for the module of F. We adopt analogous notation as in the global setting such as $\Gamma_{E/F} := \mathrm{Gal}(E/F) = \langle\sigma\rangle$, $\Gamma = \mathrm{Gal}(\bar{F}/F)$, W_F and $w_\sigma \in W_F \setminus W_E$. The Langlands group of F is given by ([Kot84, § 12])

$$\mathcal{L}_F := \begin{cases} W_F & \text{if } v \text{ is archimedean,} \\ W_F \times SU(2) & \text{if } v \text{ is non-archimedean.} \end{cases}$$

3.1 Local A-parameters for G_4

First we recall some general construction from [Art89]. Let G be a quasisplit connected reductive group over F and ${}^LG$ be its L-group. An *A-parameter* for G is a continuous homomorphism $\psi : \mathcal{L}_F \times SL(2,\mathbb{C}) \to {}^LG$ such that

(i) $\psi|_{W_F}$ is semisimple and has bounded image.

(ii) The composite $W_F \xrightarrow{\psi} {}^LG \xrightarrow{\mathrm{pr}_2} W_F$ is the identity.

(iii) ψ restricted to $SL(2,\mathbb{C})$ or $SU(2) \times SL(2,\mathbb{C})$ is analytic.

Again we remark that the boundedness condition in (i) might be too strong for our purpose, since the generalized Ramanujan conjecture is not yet known. An appropriate alternative condition will be given in § 3.2 (Rem. 3.5). The notion of equivalence and ellipticity for A-parameters are defined in the same manner as in the global case. Also the groups $S_\psi(G)$ and $\mathcal{S}_\psi(G)$ are defined. Although we consider only elliptic global parameters, the same is not always true for their local components. Thus we have to classify the whole $\Psi(G)$ in the local case. For this, we use the reduction argument of [Art89, § 7].

Take a maximal torus A^ψ in $S_\psi(G)$ and set $\widehat{M}^\psi := \mathrm{Cent}(A^\psi, \widehat{G})$. Recall that we fixed a splitting $\mathbf{spl}_{\widehat{G}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ of $\widehat{G}$ in the description of ${}^LG$. Taking a suitable $\widehat{G}$-conjugate of ψ, we may assume $A^\psi \subset \mathcal{T}$ and that $\widehat{M}^\psi$ is a standard Levi subgroup of $\widehat{G}$ with respect to $(\mathcal{B}, \mathcal{T})$. Thus we have a standard parabolic subgroup $\widehat{P}^\psi = \widehat{M}^\psi \widehat{U}^\psi$ of $\widehat{G}$. One can check that this is stable under the Γ-action ρ_G. In fact, since $\psi(W_F)$ commutes with A^ψ, $\mathrm{Ad}(\psi(W_F))$ preserves the weight decomposition under $\mathrm{Ad}(A^\psi)$ so that

$$\mathrm{Ad}(\psi(w))\widehat{P}^\psi = \widehat{P}^\psi, \quad \forall w \in W_F.$$

Writing $\psi(w) = \psi^0(w) \rtimes w$ with $\psi^0(w) \in \widehat{G}$, this becomes

$$\rho_G(w)(\widehat{P}^\psi) = \mathrm{Ad}(\psi^0(w)^{-1})\widehat{P}^\psi, \quad \forall w \in W_F.$$

Since $\rho_G(w)(\widehat{P}^\psi)$ is again standard, we have $\rho_G(w)(\widehat{P}^\psi) = \widehat{P}^\psi$ and $\psi^0(w) \in \widehat{P}^\psi$ for any $w \in W_F$, as claimed.

Since G is quasisplit, we can fix its F-splitting $\mathbf{spl}_G = (\mathbf{B}, \mathbf{T}, \{X\})$. We now define $P^\psi = M^\psi U^\psi$ to be the standard parabolic subgroup of G with respect to $(\mathbf{B}, \mathbf{T})$ having ${}^LP^\psi = \widehat{P}^\psi \rtimes_{\rho_G} W_F$ as its L-group. The above discussion allows us to view ψ as an element of $\Psi(M^\psi)$.

Lemma 3.1. $\psi \in \Psi_0(M^\psi)$. *As a consequence, we have the disjoint decomposition*

$$\Psi(G) = \coprod_{[P]} \Psi_0(M)/W(M).$$

Here, $[P]$ runs over a system of representatives of the associated classes of parabolic subgroups of G, and $W(M) := \mathrm{Norm}(A_M, G)/M$ is the Weyl group of A_M in G.

Proof. It follows from $\widehat{M}^\psi = \mathrm{Cent}(A^\psi, \widehat{G})$ that $A^\psi \subset \widehat{A}_{M^\psi}$. Since A^ψ is a maximal torus in $S_\psi(G)^0$, we have

$$S_\psi(M^\psi)^0 = \mathrm{Cent}(\psi, \mathrm{Cent}(A^\psi, G))^0 = \mathrm{Cent}(A^\psi, S_\psi(G)^0) = A^\psi \subset \widehat{A}_{M^\psi}.$$

□

In the G_4 case, a system of representatives of the associated classes of parabolic subgroups is given by the set of standard parabolic subgroups with respect to $\mathbf{B}_4$. This consists of $P_i = M_iU_i$, $(i = 1, 2)$ with

$$M_1 = \left\{ m_1(a,g) = \left(\begin{array}{c|c|c} a & & \\ \hline & g & \\ \hline & & \theta_1\tilde{\sigma}(a) \end{array}\right) \middle| \begin{array}{c} a \in H_1 \\ g \in G_2 \end{array} \right\},$$

$$U_1 = \left\{ u_1(y,\beta) = \left(\begin{array}{c|cc|c} 1 & y'' & y' & \beta - \langle y, y\rangle/2 \\ \hline & 1 & & -\tilde{\sigma}(y') \\ & & 1 & \tilde{\sigma}(y'') \\ \hline & & & 1 \end{array}\right) \middle| \begin{array}{c} y = (y'', y') \in W_1 \\ \beta \in \mathbb{G}_a \end{array} \right\},$$

$$M_2 = \left\{ m_2(a) = \left(\begin{array}{c|c} a & \\ \hline & \theta_2\tilde{\sigma}(a) \end{array}\right) \middle| a \in H_2 \right\},$$

$$U_2 = \left\{ u_2(b) = \left(\begin{array}{c|c} \mathbf{1}_2 & b \\ \hline & \mathbf{1}_2 \end{array}\right) \middle| \begin{array}{c} b \in (\mathrm{R}_{E/F}\mathbb{M}_2) \\ {}^\iota(\tilde{\sigma}(b)) = -b \end{array} \right\},$$

and G_4, $\mathbf{B}_4$. Notice that our numbering of standard parabolics has been changed from [Kon98], [Kon01]. Here, for an algebraic group G over F, we write $\tilde{\sigma}$ for the F-automorphism on $\mathrm{R}_{E/F}G$ associated to σ by the F-structure of G. $(W_1, \langle\ ,\ \rangle)$ denotes the hyperbolic skew-hermitian space $(\mathrm{R}_{E/F}\mathbb{G}_a)^2$ with the form

$$\langle (x'', x'), (y'', y')\rangle = x''\tilde{\sigma}(y') - x'\tilde{\sigma}(y'').$$

ι denotes the main involution on $\mathbb{M}_2$: ${}^\iota\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

Again as in the global case, we save η and μ for characters of $E^\times$ whose restriction to $F^\times$ are $\mathbf{1}$ and $\omega_{E/F}$, respectively. Here $\omega_{E/F}$ is the sign character of $F^\times/\mathrm{N}_{E/F}(E^\times)$. Any quasi-character ω of $E^\times$ is identified with a character of W_E by the local classfield theory. For a connected reductive group G over F, we write $\Pi(G(F)) \supset \Pi_{\mathrm{unit}}(G(F)) \supset \Pi_{\mathrm{temp}}(G(F)) \supset \Pi_{\mathrm{disc}}(G(F))$ for the set of isomorphism classes of irreducible admissible, unitarizable, tempered and square integrable representations (Harish-Chandra modules if v is archimedean. This abuse of terminology will be adopted throughout the paper. See § 3.4.1.) of $G(F)$, respectively. If F is non-archimedean, we write $\Pi_{\mathrm{cusp}}(G(F))$ for the subset of $\Pi_{\mathrm{disc}}(G(F))$ consisting of supercuspidal elements. Finally we write $\mathcal{A}_F := \mathcal{L}_F \times SL(2, \mathbb{C})$ as in the global case.

Proposition 3.2. *The equivalence classes of A-parameters for G_4 with non-trivial restrictions to $SL(2,\mathbb{C})$ are the followings. The reference to the global situation indicates the numbers in Cor. 2.3.*
(G) $\Psi_0(G)$ consists of the following six types of elements as in the global case.

(i) Stable parameters.

(a) ψ_η defined similarly as in (1.a)

(b) $\psi_{\Pi,\mu}$ defined similarly as in (1.b). In the local case, the stable elliptic L-packet Π has the base change lift $\Pi_E = \xi_{\mathbf{1}}(\Pi) \in \Pi_{\mathrm{disc}}(H_2(F))$ such that $\sigma(\Pi_E) \simeq \Pi_E^\vee$, $\omega_{\Pi_E}|_{F^\times} = \mathbf{1}$ and $L_{\mathrm{Asai}}(s, \Pi_E)$ is holomorphic at $s = 0$.

(ii) Endoscopic parameters.

(a) $\psi_{\underline{\mu}}$ defined similarly as in (2.a), where $\underline{\mu} = (\mu, \mu')$ and μ' can be μ.

(b) $\psi_{\Pi,\eta}$ defined similarly as in (2.b), where Π is the same as in (G.1.b) above.

(c) $\psi_{\underline{\eta}}$ defined similarly as in (2.c), where $\underline{\eta} = (\eta, \eta')$ with $\eta \neq \eta'$.

(d) $\psi_{\eta,\underline{\mu}}$ defined similarly as in (2.d), where $\underline{\mu} = (\mu, \mu')$ with $\mu \neq \mu'$.

(M_1) $\Psi_0(M_1)$ consists of the parameters $\psi^{M_1}_{\omega,\eta}$, ($\omega \in \Pi_{\mathrm{unit}}(E^\times)$):

$$\psi^{M_1}_{\omega,\eta}|_{\mathcal{A}_E} = [(\omega \oplus \sigma(\omega)^{-1}) \oplus (\eta \otimes \rho_2)] \times p_{W_E}, \quad \psi^{M_1}_{\omega,\eta}(w_\sigma) = [(\omega(w_\sigma^2) \oplus \mathbf{1}) \oplus \mathbf{1}_2] \rtimes w_\sigma,$$

(M_2) $\Psi_0(M_2)$ consists of the parameters $\psi^{M_2}_{\omega}$, ($\omega \in \Pi_{\mathrm{unit}}(E^\times)$):

$$\psi^{M_2}_{\omega}|_{\mathcal{A}_E} = [(\omega \otimes \rho_2) \oplus (\sigma(\omega)^{-1} \otimes \rho_2)] \times p_{W_E}, \quad \psi^{M_2}_{\omega}(w_\sigma) = [(\omega(w_\sigma^2) \otimes \mathbf{1}_2) \oplus \mathbf{1}_2] \rtimes w_\sigma.$$

Proof. The elliptic cases are similar to the global case. The only point is that a stable Langlands parameter φ_Π for G_2 is elliptic if and only if $\Pi_E \in \Pi_{\mathrm{disc}}(H_2(F))$ (see for example [GL79]). The characterization of the image of the local base change in terms of the Asai-Oda L-factor is due to Goldberg [Gol93] (at least in the non-archimedean case). The parabolic cases are well known (cf. [Rog90]). □

3.2 Review of the local Arthur conjecture

To obtain the local A-packets associated to the parameters in Prop. 3.2, we postulate some local assertions of Arthur's conjecture.

Let G be a connected reductive quasisplit group over F. We retain the notation of § 3.1. For an F-parabolic subgroup $P = MU$ of G with a Levi component M, we have the real vector spaces $\mathfrak{a}_M = \mathrm{Hom}(X^*(M)_F, \mathbb{R})$ and $\mathfrak{a}_M^* = X^*(M)_F \otimes \mathbb{R}$ dual to each other. Here $X^*(M)_F$ is the lattice of F-rational characters of M. We have the map $H_M : M(F) \to \mathfrak{a}_M$ defined by

$$\exp\langle \chi, H_M(m)\rangle = |\chi(m)|_F, \quad \forall \chi \in X^*(M)_F.$$

Using this, we define the character $e^\lambda : M(F) \to \mathbb{C}^\times$ associated to $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}} = \mathfrak{a}^*_M \otimes_{\mathbb{R}} \mathbb{C}$ by $e^\lambda(m) = m^\lambda := e^{\langle \lambda, H_M(m)\rangle}$. If we write A_M for the maximal F-split torus in the center of M, the set Σ_P of roots of A_M in P can be viewed as a subset of $\mathfrak{a}_M^*$. Similarly the set $\Sigma_P^\vee$ of corresponding coroots is a subset of $\mathfrak{a}_M$. We write Δ_P and $\Delta_P^\vee$ for the subsets of simple roots and coroots in Σ_P and $\Sigma_P^\vee$, respectively. Write

$$\mathfrak{a}_P^{*,+} := \{\lambda \in \mathfrak{a}_M^* \mid \alpha^\vee(\lambda) > 0, \forall \alpha^\vee \in \Delta_P^\vee\}.$$

For $\psi \in \Psi(G)$, we define

$$\mu_\psi : \mathbb{G}_m(\mathbb{C}) \ni t \longmapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in SL(2,\mathbb{C}) \xrightarrow{\psi} \widehat{G}.$$

After a suitable $\widehat{G}$-conjugation of ψ, one may assume that $\mu_\psi \in X_*(\mathcal{T})$ and

$$\alpha^\vee(\mu_\psi) \geq 0, \quad \forall \alpha^\vee \in \Delta(\mathcal{B},\mathcal{T}) = \Delta^\vee(\mathbf{B},\mathbf{T}).$$

Here $\Delta(\mathcal{B},\mathcal{T})$ denotes the set of simple roots of $\mathcal{T}$ in $\mathcal{B}$. Then $\widehat{M_\psi} := \mathrm{Cent}(\mu_\psi, \widehat{G})$ is a standard Levi subgroup of $\widehat{G}$, and we write $\widehat{P}_\psi = \widehat{M_\psi}\widehat{U}_\psi$ for the corresponding standard parabolic subgroup. Since μ_ψ is stable under $\mathrm{Ad}(\psi(W_F)) \subset \mathrm{Ad}(\widehat{M_\psi}) \circ \rho_G(W_F)$, we see that $\widehat{P}_\psi$ is $\rho_G(W_F)$-stable. We obtain a standard parabolic subgroup $P_\psi = M_\psi U_\psi$ of G having ${}^L P_\psi := \widehat{P}_\psi \rtimes_{\rho_G} W_F$ as its L-group. We can view $\psi|_{\mathcal{L}_F}$ as an element of $\Psi(M_\psi)$.

For an admissible representation (τ, V) of $M(F)$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, $(I_P^G(\tau_\lambda), I_P^G(V_\lambda))$ denotes the parabolically induced representation $\mathrm{ind}_{P(F)}^{G(F)}[\tau_\lambda \otimes \mathbf{1}_{U(F)}]$ with $(\tau_\lambda := e^\lambda \otimes \tau, V_\lambda := V)$. If $\tau \in \Pi_{\mathrm{temp}}(M(F))$ and $\lambda \in \mathfrak{a}_P^{*,+}$, we write $J_P^G(\tau_\lambda)$ for the Langlands quotient of $I_P^G(\tau_\lambda)$. For any admissible representation π of finite length of $G(F)$, $\mathrm{JH}(\pi)$ denotes the set of isomorphism classes of its irreducible subquotients.

Let χ be a non-degenerate character of $\mathbf{U}(F)$ in the sense that its stabilizer in $\mathbf{B}(F)$ is $Z(G)(F)\mathbf{U}(F)$. A χ-*Whittaker functional* on an admissible representation (π, V) of $G(F)$ is a (continuous if F is archimedean) linear functional $\Lambda : V \to \mathbb{C}$ satisfying

$$\Lambda(\pi(u)v) = \chi(u)v, \quad \forall u \in \mathbf{U}(F),\, v \in V.$$

We say (π, V) is χ-*generic* if it admits a non-zero χ-Whittaker functional. When π is irreducible, it was shown by Shalika that the space of χ-Whittaker functionals on V is at most one-dimensional [Sha74]. We write $W = W^G$ for the relative Weyl group $\mathrm{Norm}(A_0, G(F))/\mathbf{T}(F)$, where we have written $A_0 := A_{\mathbf{T}}$. It is a Coxeter group generated by the simple reflections r_α associated to the simple roots $\alpha \in \Delta_0$ of A_0 in $\mathbf{B}$. In particular, we have the associated length function $\ell_{\mathbf{B}}$ on W. We write $w_- = w_-^G$ for the longest element in W. In what follows we identify W with its fixed system of representatives in $\mathrm{Norm}(A_0, G(F))$. For a standard parabolic subgroup $P = MU$, we set $w_M = w_M^G := w_-(w_-^M)^{-1} \in G(F)$. We recall the following from [Sha81].

Proposition 3.3 ([Sha81], Prop. 3.1, 3.2). *Let χ and $P = MU$ be as above.*
(a) $\chi_M := \mathrm{Ad}(w_M^{-1})\chi|_{\mathbf{U}^M(F)}$ is a non-degenerate character of $\mathbf{U}^M(F) \subset M(F)$. (Note that this depends on the choice of the representative of w_M !)
(b) Let (τ, V) be an admissible χ_M-generic representation of $M(F)$ and Λ be a χ_M-Whittaker functional on it. Then the integral

$$I_P^G(\Lambda_\lambda, \phi) := \int_{w_M(\bar{U})(F)} \Lambda(\phi(w_M^{-1}v))\overline{\chi(v)}\, dv, \quad \phi \in I_P^G(V_\lambda)$$

converges absolutely for any $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ *and defines a* χ*-Whittaker functional on* $I_P^G(V_\lambda)$*. Moreover* $I_P^G(\Lambda_\lambda, e^\lambda\phi)$ *is holomorphic in lambda for any* $\phi \in I_P^G(V)$*.*

We can now state the local Arthur conjecture which we shall need.

Conjecture 3.4 ([Art89] §§ 6, 7). *(A) For each* $\psi \in \Psi_0(G)$ *there exists a finite subset* $\Pi_\psi(G) \subset \Pi_{\text{unit}}(G(F))$ *called the A-*packet *associated to* ψ*. Let us write* $\Phi_0(G)$ *for the subset of* $\varphi \in \Psi_0(G)$ *such that* $\varphi|_{SL(2,\mathbb{C})}$ *is trivial. Then we should have the disjoint decomposition*

$$\Pi_{\text{disc}}(G(F)) = \coprod_{\varphi \in \Phi_0(G)} \Pi_\varphi(G). \tag{3.1}$$

For general $\psi \in \Psi(G)$*, we have the A-packet* $\Pi_\psi(M^\psi)$ *attached to* $\psi \in \Psi_0(M^\psi)$*. Then we define*

$$\Pi_\psi(G) := \bigcup_{\pi \in \Pi_\psi(M^\psi)} \text{JH}(I_{P^\psi}^G(\pi)).$$

Notice that $I_{P^\psi}^G(\pi)$ *is completely reducible since* π *is unitarizable. Writing* $\Phi_{\text{temp}}(G)$ *for the set of* $\varphi \in \Phi(G)$ *with* $\varphi|_{SL(2,\mathbb{C})} = 1$*, we deduce from* (3.1) *and the Harish-Chandra's classification of tempered representation that*

$$\Pi_{\text{temp}}(G(F)) = \coprod_{\varphi \in \Phi_{\text{temp}}(G)} \Pi_\varphi(G).$$

(B) (1) Part (A) asserts the existence of the packet $\Pi_{\psi|_{\mathcal{L}_F}}(M_\psi) \subset \Pi_{\text{unit}}(M_\psi(F))$ *associated to* $\psi|_{\mathcal{L}_F} \in \Phi_{\text{temp}}(M_\psi)$*. The A-packet* $\Pi_\psi(G)$ *should contain*

$$\Pi'_\psi(G) := \{J_{P_\psi}^G(\tau_{\mu_\psi}) \,|\, \tau \in \Pi_{\psi|_{\mathcal{L}_F}}(M_\psi)\}.$$

(2) Fix a non-degenerate character χ *of* $\mathbf{U}(F)$*. For* $\varphi \in \Phi_0(G)$*,* $\Pi_\varphi(G)$ *should contain a unique* χ*-generic element* δ_χ *(the* generic packet conjecture*). For* $\varphi \in \Phi_{\text{temp}}(G)$*, we have*

$$\Pi_\varphi(G) = \coprod_{\delta \in \Pi_\varphi(M^\varphi)} \text{JH}(I_{P^\varphi}^G(\delta)).$$

Applying Prop. 3.3, $\Pi_\varphi(G)$ *contains a unique* χ*-generic element* $\tau_\chi \in \text{JH}(I_{P^\varphi}^G(\delta_{\chi_{M^\varphi}}))$*. Finally for general* $\psi \in \Psi(G)$*,* $\Pi_{\psi|_{\mathcal{L}_F}}(M_\psi)$ *contains a unique* χ_{M_ψ}*-generic element* $\tau_{\chi_{M_\psi}}$*. The element*

$$\pi_\chi := J_{P_\psi}^G(\tau_{\chi_{M_\psi}, \mu_\psi}) \in \Pi'_\psi(G) \subset \Pi_\psi(G)$$

is called the χ-base point *of* $\Pi_\psi(G)$*.*
(C) (1) There should be a function $\langle\ ,\ |\pi_\chi\rangle_\psi : \mathcal{S}_\psi(G) \times \Pi_\psi(G) \to \mathbb{C}$ *satisfying the following conditions.*

(a) For $\pi \in \Pi_\psi(G)$*,* $\mathcal{S}_\psi(G) \ni \bar{s} \mapsto \langle \bar{s}, \pi|\pi_\chi\rangle_\psi \in \mathbb{C}$ *is a class function such that* $\langle 1, \pi|\pi_\chi\rangle_\psi \in \mathbb{R}_+^\times$*.*

(b) If we write $\bar{s}_\psi$ *for the image of* $\psi(1 \times -\mathbf{1}_2) \in S_\psi(G)$ *in* $\mathcal{S}_\psi(G)$*, then there exists a sign character* $e_\psi(\ ,\pi|\pi_\chi) : \mathcal{S}_\psi(G) \to \{\pm 1\}$ *such that*

$$\langle \bar{s}_\psi \bar{s}, \pi|\pi_\chi\rangle_\psi = e_\psi(\bar{s}_\psi, \pi|\pi_\chi)\langle \bar{s}, \pi|\pi_\chi\rangle_\psi, \quad \forall \bar{s} \in \mathcal{S}_\psi(G).$$

(2) If we define $\langle\ ,\ |\pi_\chi\rangle'_\psi : \mathcal{S}_{\psi|_{\mathcal{L}_F}}(M_\psi) \times \Pi'_\psi(G) \ni (\bar{s}, J^G_{P_\psi}(\tau_{\mu_\psi})) \mapsto \langle \bar{s}, \tau|\tau_{\chi_{M_\psi}}\rangle_{\psi|_{\mathcal{L}_F}} \in \mathbb{C}$, *then the following diagram commutes.*

$$\begin{array}{ccc} \Pi'_\psi(G) \ni \pi & \longrightarrow & \langle\ ,\pi|\pi_\chi\rangle'_\psi \in \Pi(\mathcal{S}_{\psi|_{\mathcal{L}_F}}(M_\psi)) \\ {\scriptstyle inclusion}\downarrow & & \downarrow{\scriptstyle inclusion} \\ \Pi_\psi(G) \ni \pi & \longrightarrow & \langle\ ,\pi|\pi_\chi\rangle_\psi \in \Pi(\mathcal{S}_\psi(G)). \end{array}$$

Here $\Pi(\mathcal{S}_\psi(G))$ *is the set of isomorphism classes of irreducible representations of* $\mathcal{S}_\psi(G)$, *whose elements we identify with their characters.*

Remark 3.5. *Since the generalized Ramanujan conjecture is not yet known, A-packets* $\Pi_\psi(G)$ *for* ψ *with* $\psi(W_F)$ *bounded might not be sufficient. Thus, in this paper, we replace the condition (i) in the definition of local A-parameter by the following.*

- *(i)′* $\psi|_{W_F}$ *is semisimple. Each member of the L-packet* $\Pi_{\psi|_{\mathcal{L}_F}}(M_\psi)$ *appears as a local component of some cuspidal automorphic representation which is not of CAP type. Also the L-packet* $\Pi'_\psi(G)$ *is contained in* $\Pi_{\rm unit}(G(F))$.

3.3 Non-supercuspidal representations of $G_4(F)$

In this subsection, we assume F is non-archimedean and calculate $\Pi'_\psi(G_4)$ for the elliptic ψ in Prop. 3.2. We first review the classification of $\Pi_{\rm unit}(G_4(F))$ from [Kon01].

η defines a character $\eta_u : G_1(F) \ni x\sigma(x)^{-1} \mapsto \eta(x) \in \mathbb{C}^1$. We write η_{G_n} for the composite $G_n(F) \overset{\det}{\to} G_1(F) \overset{\eta_u}{\to} \mathbb{C}^1$. Similarly for a character ω of $E^\times$, we write ω_{H_m} : $H_m(F) \ni h \mapsto \omega(\det h) \in \mathbb{C}^1$. We express irreducible representations of standard Levi subgroups of $G(F) = G_4(F)$ as follows.

$$\begin{aligned} \underline{\omega}[\underline{\lambda}] : \mathbf{T}(F) \ni d(a_1,a_2) &\longmapsto \omega_1(a_1)|a_1|_E^{\lambda_1/2}\omega_2(a_2)|a_2|_E^{\lambda_2/2} \in \mathbb{C}^\times, \\ \omega[\lambda]\otimes\pi : M_1(F) \ni m_1(a,g) &\longmapsto \omega(a)|a|_E^{\lambda/2}\pi(g) \in GL(V_\pi), \\ \pi_E[\lambda] : M_2(F) \ni m_2(a) &\longmapsto |\det a|_E^{\lambda/2}\pi_E(a) \in GL(V_{\Pi_E}). \end{aligned}$$

Here $\underline{\omega} = (\omega_1, \omega_2)$, ω_i, $\omega \in \Pi(E^\times)$, $(\pi_E, V_{\pi_E}) \in \Pi(H_2(F))$, $(\pi, V_\pi) \in \Pi(G_2(F))$, and $|\ |_E$ is the module of E. Also $d(a_1,a_2) = \mathrm{diag}(a_1, a_2, \sigma(a_2)^{-1}, \sigma(a_1)^{-1}) \in \mathbf{T}(F)$.

Recall that the endoscopic liftings in the following three settings were established by Rogawski [Rog90, Ch. 11].

Standard base change for $\mathrm{R}_{E/F}G_2$ This is the endoscopic lifting from the twisted endoscopic data $(G_2, {}^LG_2, 1, \xi_\eta)$ for $(H_2, \theta_2\circ\tilde{\sigma}, \mathbf{1})$ (see [KS99, Ch. II]), where

$$\xi_\eta : {}^LG_2 \ni g \rtimes w \longmapsto \begin{cases} (\eta(w)g, \eta(w)^{-1}\theta_2(g)) \times w & \text{if } w \in W_E, \\ (g, \theta_2(g)) \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^LH_2.$$

Twisted base change for $\mathrm{R}_{E/F}G_2$ This is the same as above except that the twisted endoscopic data $(G_2, {}^LG_2, 1, \xi_\mu)$ is given by

$$\xi_\mu : {}^LG_2 \ni g \rtimes w \longmapsto \begin{cases} (\mu(w)g, \mu(w)^{-1}\theta_2(g)) \times w & \text{if } w \in W_E, \\ (g, -\theta_2(g)) \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^LH_2.$$

Endoscopic lift for G_2 This is the endoscopic lifting from the unique elliptic endoscopic data $(G_1^2, {}^L(G_1^2), s, \lambda_{\mu^{-1}})$ for G_2. Here

$$\lambda_{\mu^{-1}} : {}^L(G_1^2) \ni (z_1, z_2) \rtimes w \longmapsto \begin{cases} \begin{pmatrix} z_1\mu(w) & \\ & z_2\mu(w) \end{pmatrix} \times w & \text{if } w \in W_E, \\ \begin{pmatrix} & -z_1 \\ z_2 & \end{pmatrix} \rtimes w_\sigma & \text{if } w = w_\sigma \end{cases} \in {}^L G_2.$$

It follows from (the local analogue of) Prop. 2.2 that the elliptic endoscopic parameters for $G_2(F)$ are of the form

$$\varphi_{\underline{\mu}}|_{\mathcal{L}_E} = (\mu \oplus \mu') \times \mathrm{pr}_{W_E}, \quad \varphi_{\underline{\mu}}(w_\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rtimes w_\sigma.$$

We recall from [Rog90] that the corresponding L-packet of $G_2(F)$ is

$$\Pi_{\varphi_{\underline{\mu}}}(G_2) = \lambda_{\mu^{-1}}(\mathbf{1} \otimes (\mu^{-1}\mu')_{G_1}).$$

This description is valid even if $\mu = \mu'$.

Lemma 3.6. *The Langlands data $(P_\psi, \Pi_{\psi|_{\mathcal{L}_F}}(M_\psi), \mu_\psi)$ for the elliptic local A-parameters in Prop. 3.2 (G) are given by the following.*

A-parameter	P_ψ	μ_ψ	$\Pi_{\psi\|_{\mathcal{L}_F}}(M_\psi)$
(1.a) ψ_η	$\mathbf{B}$	$(3,1)$	$\eta \otimes \eta$
(1.b) $\psi_{\Pi,\mu}$	P_2	1	$\mu_{H_2}\Pi_E$
(2.a) $\psi_{\underline{\mu}}$	P_1	2	$\mu \otimes \Pi_{\varphi_{\underline{\mu}}}(G_2)$
(2.b) $\psi_{\Pi,\eta}$	P_1	1	$\eta \otimes \pi$
(2.c) $\psi_{\underline{\eta}}$	P_2	1	$I_{\mathbf{B}^{H_2}}^{H_2}(\eta \otimes \eta')$
(2.d) $\psi_{\eta,\underline{\mu}}$	P_1	1	$\eta \otimes \Pi_{\varphi_{\underline{\mu}}}(G_2)$

Here in (2.b), π is the unique member of Π.

This is an immediate consequence of the description of parameters.

Now we recall the results of [Kon01] on the composition series of $I_{P_\psi}^G(\tau_{\mu_\psi})$, $\tau \in \Pi_{\psi|_{\mathcal{L}_F}}(M_\psi)$ in the above lemma. We write δ_0^H for the Steinberg representation of a connected reductive group $H(F)$. We often drop the subscript 4 and write $G = G_4$. We fix a non-trivial character ψ_F of F. This determines a non-degenerate character χ_n^H of $\mathbf{U}_n^H(F)$ such that

$$\chi_n^H\left(\begin{pmatrix} 1 & x_{1,2} & \dots & x_{1,n} \\ & 1 & \ddots & \vdots \\ & & \ddots & x_{n-1,n} \\ & & & 1 \end{pmatrix}\right) = \psi_F\left(\frac{1}{2}\mathrm{Tr}_{E/F}\left(\sum_{i=1}^{n-1} x_{i,i+1}\right)\right),$$

and its restriction to $\mathbf{U}_n(F)$ gives a non-degenerate character χ_n. We also fix a system of representatives of W in $\mathrm{Norm}(A_0, G(F))$ as follows. Write α_i, $(i = 1, 2)$ for the simple roots of A_0 in G given by

$$\alpha_1(d(a_1, a_2)) = a_1 a_2^{-1}, \quad \alpha_2(d(a_1, a_2)) = a_2^2.$$

If we write $r_i \in W$ for the reflection associated to α_i, W is the type C_2 Weyl group generated by them. We choose a system of representatives so that

$$r_1 = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right), \quad r_2 = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & 1 \end{pmatrix}, \quad w_- = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

These choices assure that $\chi_M = \chi|_{\mathbf{U}^M(F)}$ for any standard Levi subgroup $M \subset G$. All the equalities are those in the Grothendieck group of admissible representations of finite length of $G(F)$.

(1.a) For ψ_η we have

$$I^G_{\mathbf{B}}(\eta[3] \otimes \eta[1]) = \eta_G \delta^G_0 + J^G_{P_1}(\eta[3] \otimes \eta_{G_2}\delta^{G_2}_0) + J^G_{P_2}(\eta_{H_2}\delta^{H_2}_0[2]) + \eta_G.$$

$\eta_G\delta^G_0 \in \Pi_{\mathrm{disc}}(G(F))$, $\eta_G \in \Pi_{\mathrm{unit}}(G(F))$ but the other two constituents are not unitarizable.

(1.b) For $\psi_{\Pi,\mu}$ we have the following two possibilities.

(i) If Π is supercuspidal, we have

$$I^G_{P_2}(\mu_{H_2}\Pi_E[1]) = \delta^G_2(\mu_{H_2}\Pi_E) + J^G_{P_2}(\mu_{H_2}\Pi_E[1]),$$

where $\delta^G_2(\mu_{H_2}\Pi_E) \in \Pi_{\mathrm{disc}}(G(F))$ and $J^G_{P_2}(\mu_{H_2}\Pi_E[1]) \in \Pi_{\mathrm{unit}}(G(F))$.

(ii) $\Pi_E = \eta_{H_2}\delta^{H_2}_0$. We have

$$I^G_{P_2}((\mu\eta)_{H_2}\delta^{H_2}_0[1]) = \delta^G_0(\eta\mu)_+ + \delta^G_0(\eta\mu)_- + J^G_{P_2}((\mu\eta)_{H_2}\delta^{H_2}_0[1]),$$

where $\delta^G_0(\mu\eta)_\pm \in \Pi_{\mathrm{disc}}(G(F))$ are labeled in such a way that $\delta^G_0(\eta\mu)_+$ is χ-generic. $J^G_{P_2}((\mu\eta)_{H_2}\delta^{H_2}_0[1]) \in \Pi_{\mathrm{unit}}(G(F))$.

(2.a) For $\psi_{\underline{\mu}}$, again we have two cases.

(i) If $\mu \neq \mu'$, then the L-packet $\Pi_{\varphi_{\underline{\mu}}}(G_2)$ consists of two distinct supercuspidal representations $\pi^{G_2}(\underline{\mu})_\pm$ [Rog90, p. 172, (5)], and only one of them, say, $\pi^{G_2}(\underline{\mu})_+$ is χ_2-generic. We have

$$I^G_{P_1}(\mu[2] \otimes \pi^{G_2}(\underline{\mu})_\pm) = \delta^G_2(\underline{\mu})_\pm + J^G_{P_1}(\mu[2] \otimes \pi^{G_2}(\underline{\mu})_\pm).$$

Here $\delta^G_2(\underline{\mu})_\pm \in \Pi_{\mathrm{disc}}(G(F))$ and $\delta^G_2(\underline{\mu})_+$ is χ-generic. $J^G_{P_1}(\mu[2]\otimes\pi^{G_2}(\underline{\mu})_\pm) \in \Pi_{\mathrm{unit}}(G(F))$.

(ii) If $\mu = \mu'$, then $\Pi_{\varphi_{\underline{\mu}}}(G_2)$ consists of the limit of discrete series representations $\tau^{G_2}(\mu)_\pm$, where only $\tau^{G_2}(\mu)_+$ is χ_2-generic [Rog90, p. 172, (6)]. We have

$$I^G_{P_1}(\mu[2] \otimes \tau^{G_2}(\mu)_\pm) = \delta^G_0(\mu)_\pm + J^G_{P_1}(\mu[2] \otimes \tau^{G_2}(\mu)_\pm) + J^G_{P_2}(\mu_{H_2}\delta^{H_2}_0[1]),$$

where $\delta^G_0(\mu)_\pm \in \Pi_{\mathrm{disc}}(G(F))$ and only $\delta^G_0(\mu)_+$ is χ-generic. The other constituents are also unitarizable.

(2.b) The following three cases occur for $\psi_{\Pi,\eta}$.

(i) If Π_E is supercuspidal, then so is π and

$$I^G_{P_1}(\eta[1]\otimes\pi)=\delta^G_1(\eta,\pi)+J^G_{P_1}(\eta[1]\otimes\pi),$$

where $\delta^G_1(\eta,\pi)\in\Pi_{\mathrm{disc}}(G(F))$ and $J^G_{P_1}(\eta[1]\otimes\pi)\in\Pi_{\mathrm{unit}}(G(F))$.

(ii) If $\Pi_E=\eta'_{H_2}\delta^{H_2}_0$ with $\eta'\neq\eta$, then $\pi=\eta'_{G_2}\delta^{G_2}_0$. We have

$$I^G_{P_1}(\eta[1]\otimes\eta'_{G_2}\delta^{G_2}_0)=\delta^G_0(\underline{\eta})+J^G_{P_1}(\eta[1]\otimes\eta'_{G_2}\delta^{G_2}_0).$$

We have written $\underline{\eta}=(\eta,\eta')$ and $\delta^G_0(\underline{\eta})\in\Pi_{\mathrm{disc}}(G(F))$, $J^G_{P_1}(\eta[1]\otimes\eta'_{G_2}\delta^{G_2}_0)\in\Pi_{\mathrm{unit}}(G(F))$.

(iii) If $\pi=\eta_{G_2}\delta^{G_2}_0$, then

$$I^G_{P_1}(\eta[1]\otimes\eta_{G_2}\delta^{G_2}_0)=\eta_G\tau^G_0(\delta^{G_2}_0)+J^G_{P_1}(\eta[1]\otimes\eta_{G_2}\delta^{G_2}_0),$$

where $\tau^G_0(\delta^{G_2}_0)\in\Pi_{\mathrm{temp}}(G(F))$ and $J^G_{P_1}(\eta[1]\otimes\eta_{G_2}\delta^{G_2}_0)\in\Pi_{\mathrm{unit}}(G(F))$. For later use, we also recall

$$I^G_{P_1}(\eta[1]\otimes\eta_{G_2})=\eta_G\tau^G_0(\mathbf{1}_{G_2})+J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta)[1]),$$

$\tau^G_0(\mathbf{1}_{G_2})\in\Pi_{\mathrm{temp}}(G(F))$, $J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta)[1])\in\Pi_{\mathrm{unit}}(G(F))$.

(2.c) For $\psi_{\underline{\eta}}$, we have

$$\begin{aligned}I^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta')[1])=&\delta^G_0(\underline{\eta})+J^G_{P_1}(\eta[1]\otimes\eta'_{G_2}\delta^{G_2}_0)\\&+J^G_{P_1}(\eta'[1]\otimes\eta_{G_2}\delta^{G_2}_0)+J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta')[1]),\end{aligned}$$

where $\delta^G_0(\underline{\eta})$ is as in (2.b.ii). The other constituents are also unitarizable.
(2.d) For $\psi_{\eta,\underline{\mu}}$, we have

$$I^G_{P_1}(\eta[1]\otimes\pi^{G_2}(\underline{\mu})_\pm)=\delta^G_1(\eta,\underline{\mu})_\pm+J^G_{P_1}(\eta[1]\otimes\pi^{G_2}(\underline{\mu})_\pm),$$

where $\delta^G_1(\eta,\underline{\mu})_\pm\in\Pi_{\mathrm{disc}}(G(F))$ and only $\delta^G_1(\eta,\underline{\mu})_+$ is χ-generic. $J^G_{P_1}(\eta[1]\otimes\pi^{G_2}(\underline{\mu})_\pm)\in\Pi_{\mathrm{unit}}(G(F))$, and $\pi^{G_2}(\underline{\mu})_\pm$ are as in (2.a.i).

We are now ready to determine the packets $\Pi'_\psi(G)$ associated to the parameters in Prop. 3.2.

Proposition 3.7. *The L-packets $\Pi'_\psi(G)$ and their χ-base points for the elliptic ψ in Prop. 3.2 (G) are given by the following.*

A-parameter	$\mathcal{S}_\psi(G)$	$\Pi'_\psi(G)$	*χ-base point*
(1.a) ψ_η	*trivial*	$\{\eta_G\}$	η_G
(1.b) $\psi_{\Pi,\mu}$	*trivial*	$\{J^G_{P_2}(\mu_{H_2}\Pi_E[1])\}$	$J^G_{P_2}(\mu_{H_2}\Pi_E[1])$
(2.a) $\psi_{\underline{\mu}}$	$\mathbb{Z}/2\mathbb{Z}$	$\{J^G_{P_1}(\mu[2]\otimes\pi_\pm)\mid\pi_\pm\in\Pi_{\varphi_{\underline{\mu}}}(G_2)\}$	$J^G_{P_1}(\mu[2]\otimes\pi_+)$
(2.b) $\psi_{\Pi,\eta}$	$\mathbb{Z}/2\mathbb{Z}$	$\{J^G_{P_1}(\eta[1]\otimes\pi)\}$	$J^G_{P_1}(\eta[1]\otimes\pi)$
(2.c) $\psi_{\underline{\eta}}$	$\mathbb{Z}/2\mathbb{Z}$	$\{J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta')[1])\}$	$J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta')[1])$
(2.d) $\psi_{\eta,\underline{\mu}}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\{J^G_{P_1}(\eta[1]\otimes\pi^{G_2}(\underline{\mu})_\pm)\}$	$J^G_{P_1}(\eta[1]\otimes\pi^{G_2}(\underline{\mu})_+)$

Here in (2.a) and (2.d), the elements of $\Pi_{\varphi_{\underline{\mu}}}(G_2)$ *are labeled so that* π_+ *is* χ_2*-generic.*

Proof. The only thing that does not follow from the above list is the description of $\mathcal{S}_\psi(G)$. Since the argument does not change, we consider the global case. Recall the description of $\psi \in \Psi_0(G_n)$ of Prop. 2.2. It follows from Schur's lemma and the form of $\psi(w_\sigma)$ that $S_\psi(G) = (\mathbb{Z}/2\mathbb{Z})^r$. Since $Z(\widehat{G}_n)^\Gamma$ is the diagonal subgroup $\mathbb{Z}/2\mathbb{Z}$ of $S_\psi(G)$, we obtain $\mathcal{S}_\psi(G) \simeq (\mathbb{Z}/2\mathbb{Z})^{r-1}$. □

3.4 Induced representations over the real field

In this subsection, we consider the case $E_v/F_v \simeq \mathbb{C}/\mathbb{R}$. Each of the characters η and μ of $\mathbb{C}^\times$ are written explicitly as

$$\eta^a(z) := (z/\bar{z})^{a/2}, \quad \mu^b(z) := (z/\bar{z})^{b/2}, \quad a \in 2\mathbb{Z},\, b \in 2\mathbb{Z}+1,$$

so that $\eta^{2a} = (\eta^2)^a$ and $\mu^b = (\mu^1)^b$ as notation suggests. Since $\Pi_{\mathrm{disc}}(H_n)$ is $\{\eta^a\}_{a \in 2\mathbb{Z}}$ if $n = 1$ and empty otherwise, the Langlands parameters for G_n are given by the following.

(i) $\Phi_0(G_n)$ consists of the elements $\varphi_{\underline{a}}$, where $\underline{a} = (a_1, \dots, a_n) \in (2\mathbb{Z}+n+1)^n$ satisfying $a_1 > a_2 > \cdots > a_n$:

$$\varphi_{\underline{a}}|_{W_\mathbb{C}} = \begin{cases} \mathrm{diag}(\eta^{a_1}, \dots, \eta^{a_n}) & \text{if } n \text{ is odd,} \\ \mathrm{diag}(\mu^{a_1}, \dots, \mu^{a_n}) & \text{if } n \text{ is even,} \end{cases} \quad \varphi(w_\sigma) = I_n \rtimes w_\sigma. \tag{3.2}$$

(ii) As for general parameters, we have the decomposition (Lem. 3.1)

$$\Phi(G_n) = \coprod_{r=0}^{[n/2]} \Phi_0(M^{(r)})/W(M^{(r)}),$$

where $P^{(r)} = M^{(r)}U^{(r)}$ is the standard parabolic subgroup with $M^{(r)} \simeq H_1^r \times G_{n-2r}$.

(iii) $\Phi_0(M^{(r)})$ consists of the elements $\varphi_{\underline{b},\underline{\nu},\underline{a}}$, ($\underline{b} \in \mathbb{Z}^r$, $\underline{\nu} \in \mathbb{C}^r$, $\underline{a} \in (2\mathbb{Z}+n+1)^{n-2r}$) given by

$$\varphi_{\underline{b},\underline{\nu},\underline{a}}|_{W_\mathbb{C}} = \mathrm{diag}(\omega_{b_1,\nu_1}, \dots, \omega_{b_r,\nu_r}; \varphi_{\underline{a}}|_{W_\mathbb{C}}; \omega_{b_r,-\nu_r}, \dots, \omega_{b_1,-\nu_1}) \times \mathrm{id}_{W_\mathbb{C}}$$
$$\varphi_{\underline{b},\underline{\nu},\underline{a}}(w_\sigma) = \mathrm{diag}(\mathbf{1}_r; I_{n-2r}; (-1)^{b_r}, \dots, (-1)^{b_1}).$$

Here $\omega_{b,\nu}(z) = (z/\bar{z})^{b/2}|z|_\mathbb{C}^\nu$, ($z \in \mathbb{C}^\times$).

3.4.1 Explicit Langlands classification for G_2, G_4

In this section we describe the L-packets associated to the parameters listed above for the unitary groups in two and four variables by means of Vogan's Langlands classification [KV95], [Vog84]. The groups to be considered are $U(1,1)$, $U(2)$ and $U(2,2)$. We first prepare some notation for general $U(p,q)$, an inner form of G_m.

Thus we take an $\mathbb{R}$-form $G_{p,q} = U(p,q)(\mathbb{R})$ of G_m:

$$G_{p,q} = \{g \in H_m(\mathbb{R}) \,|\, {}^t\bar{g} I_{p,q} g = I_{p,q}\}, \quad I_{p,q} = \mathrm{diag}(\mathbf{1}_p, -\mathbf{1}_q).$$

$G_{p,q}$ has the Lie algebra

$$\mathfrak{g}_{p,q} = \mathfrak{u}(p,q)(\mathbb{R}) = \left\{ \left(\begin{array}{c|c} a & b \\ \hline {}^t\bar{b} & d \end{array} \right) \,\middle|\, \begin{array}{l} {}^t\bar{a} = -a \in \mathbb{M}_p(\mathbb{C}) \\ {}^t\bar{d} = -d \in \mathbb{M}_q(\mathbb{C}) \end{array} b \in \mathbb{M}_{p,q}(\mathbb{C}) \right\},$$

whose complexification is denoted by $\mathfrak{g} = \mathfrak{gl}(m,\mathbb{C})$. As usual, we take the Cartan involution $\theta_{p,q} = \mathrm{Ad}(I_{p,q})$ on $\mathfrak{g}_{p,q}$, which determines the Cartan decomposition $\mathfrak{g} = \mathfrak{k}_{p,q} \oplus \mathfrak{p}_{p,q}$:

$$\mathfrak{k}_{p,q} := \left\{ \left(\begin{array}{c|c} a & \\ \hline & d \end{array} \right) \,\middle|\, \begin{array}{l} a \in \mathbb{M}_p(\mathbb{C}) \\ d \in \mathbb{M}_q(\mathbb{C}) \end{array} \right\}, \quad \mathfrak{p}_{p,q} := \left\{ \left(\begin{array}{c|c} \mathbf{0}_p & b \\ \hline c & \mathbf{0}_q \end{array} \right) \,\middle|\, \begin{array}{l} b \in \mathbb{M}_{p,q}(\mathbb{C}) \\ c \in \mathbb{M}_{q,p}(\mathbb{C}) \end{array} \right\}.$$

The corresponding maximal compact subgroup is

$$\mathbf{K}_{p,q} = \left\{ \left(\begin{array}{c|c} k_1 & \\ \hline & k_2 \end{array} \right) \,\middle|\, \begin{array}{l} k_1 \in U(p) \\ k_2 \in U(q) \end{array} \right\}.$$

We write $T \simeq U(1)^m$ for the diagonal fundamental Cartan subgroup of $G_{p,q}$, and $\mathfrak{t}$ for its Lie algebra. Writing $X^*(T) = \sum_{i=1}^m \mathbb{Z} e_i$ with the standard basis

$$e_i : T \ni \mathrm{diag}(t_1, \ldots, t_m) \longmapsto t_i \in \mathbb{G}_m,$$

the set $R(\mathfrak{g},\mathfrak{t})$ of roots of $\mathfrak{t}$ in $\mathfrak{g}$ is given by $R(\mathfrak{g},\mathfrak{t}) = R_{\mathrm{cpt}} \sqcup R_{\mathrm{ncpt}}$ where

$$R_{\mathrm{cpt}} = R(\mathfrak{k}_{p,q},\mathfrak{t}) = \left\{ e_i - e_j \,\middle|\, \begin{array}{c} 1 \leq i \neq j \leq p \text{ or} \\ p+1 \leq i \neq j \leq m \end{array} \right\},$$
$$R_{\mathrm{ncpt}} = R(\mathfrak{p}_{p,q},\mathfrak{t}) = \left\{ \pm e_i \mp e_j \,\middle|\, \begin{array}{c} 1 \leq i \leq p \\ p+1 \leq j \leq m \end{array} \right\}$$

are the sets of compact and non-compact roots, respectively. We use the basis $\{e_i\}_{1 \leq i \leq m}$ to identify $\mathfrak{t}^*$ with $\mathbb{C}^m$.

In what follows, we call an admissible $(\mathfrak{g}_{p,q}, \mathbf{K}_{p,q})$-module as an *admissible representation* of $G_{p,q}$ by abuse of terminology. Similarly, a $(\mathfrak{g}_{p,q}, \mathbf{K}_{p,q})$-module is called a *smooth representation* of $G_{p,q}$. This latter terminology is quite unusual, but it is harmless for our purpose and avails us to unify our descriptions in archimedean and non-archimedean cases.

Discrete L-packets For each elliptic parameter $\varphi_{\underline{a}} \in \Phi_0(G_m)$, the corresponding L-packet $\Pi_{\varphi_{\underline{a}}}(G_{p,q})$ of $G_{p,q}$ contains ${}_mC_p = {}_mC_q$ distinct discrete series representations. In fact, for each permutation

$$(a_{i_1}, \ldots, a_{i_p}; a_{j_1}, \ldots, a_{j_q})$$

of the ordered family $\underline{a}$ satisfying

$$a_{i_1} > a_{i_2} > \cdots > a_{i_p}, \quad a_{j_1} > a_{j_2} > \cdots > a_{j_q},$$

there correspond the discrete series representation $\delta(\lambda)$ with the Harish-Chandra parameter

$$\lambda := \frac{1}{2}(a_{i_1}, \dots, a_{i_p}; a_{j_1}, \dots, a_{j_q}).$$

If we write B for the upper triangular Borel subgroup (over $\mathbb{C}$) of G, then the $B \cap \mathbf{K}_{p,q,\mathbb{C}}$-highest weight of the minimal $\mathbf{K}_{p,q}$-type in $\delta(\lambda)$ is given by

$$\Lambda := \lambda + \frac{1}{2} \sum_{\alpha \in R^+_{\mathrm{ncpt}}(\lambda)} \alpha - \frac{1}{2} \sum_{\alpha \in R^+_{\mathrm{cpt}}(\lambda)} \alpha,$$

where $R^+_{\mathrm{ncpt}}(\lambda) := \{\alpha \in R_{\mathrm{ncpt}} \,|\, \alpha^\vee(\lambda) > 0\}$, $R^+_{\mathrm{cpt}}(\lambda) := \{\alpha \in R_{\mathrm{cpt}} \,|\, \alpha^\vee(\lambda) > 0\}$. We apply these to the groups $G_{2,0}$, $G_{1,1}$ and $G_{2,2}$ and obtain the following.

Lemma 3.8. *The members of the discrete L-packet $\Pi_{\varphi_{\underline{a}}}(G_{p,q})$ for $(p,q) = (2,0)$, $(1,1)$, $(2,2)$ are given by the following table.*

Group	$\mathcal{S}_\varphi(G)$	2λ	Λ	*Comments*
$G_{2,0}$	$\mathbb{Z}/2\mathbb{Z}$	(a_1, a_2)	$(\frac{a_1-1}{2}, \frac{a_2+1}{2})$	
$G_{1,1}$	$\mathbb{Z}/2\mathbb{Z}$	$(a_1; a_2)$	$(\frac{a_1+1}{2}, \frac{a_2-1}{2})$	*holomorphic*
		$(a_2; a_1)$	$(\frac{a_2-1}{2}, \frac{a_1+1}{2})$	*anti-holomorphic*
$G_{2,2}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(a_1, a_2; a_3, a_4)$	$(\frac{a_1+1}{2}, \frac{a_2+3}{2}, \frac{a_3-3}{2}, \frac{a_4-1}{2})$	*holomorphic*
		$(a_1, a_3; a_2, a_4)$	$(\frac{a_1+1}{2}, \frac{a_3+1}{2}, \frac{a_2-1}{2}, \frac{a_4-1}{2})$	
		$(a_1, a_4; a_2, a_3)$	$(\frac{a_1+1}{2}, \frac{a_4-1}{2}, \frac{a_2-1}{2}, \frac{a_3+1}{2})$	
		$(a_2, a_3; a_1, a_4)$	$(\frac{a_2-1}{2}, \frac{a_3+1}{2}, \frac{a_1+1}{2}, \frac{a_4-1}{2})$	
		$(a_2, a_4; a_1, a_3)$	$(\frac{a_2-1}{2}, \frac{a_4-1}{2}, \frac{a_1+1}{2}, \frac{a_3+1}{2})$	
		$(a_3, a_4; a_1, a_2)$	$(\frac{a_3-3}{2}, \frac{a_4-1}{2}, \frac{a_1+1}{2}, \frac{a_2+3}{2})$	*anti-holomorphic*

Limit of discrete series L-packets Here we recall the classification of the limit of discrete series representations from [Vog84, § 2]. The limit of discrete series representations of $G_{p,q}$ are in one-to-one correspondence with the pairs (λ, Ψ) of

- $\lambda = \frac{1}{2}(\underbrace{\overbrace{a_1, \dots, a_1}^{k_1}, \dots, \overbrace{a_r, \dots, a_r}^{k_r}}_{p}; \underbrace{\overbrace{a_1, \dots, a_1}^{\ell_1}, \dots \overbrace{a_r, \dots, a_r}^{\ell_r}}_{q}) \in \frac{1}{2}(2\mathbb{Z} + m + 1)^{p+q}$,
- Ψ is a positive system in $R(\mathfrak{g}, \mathfrak{t})$,

satisfying the following conditions.

(i) $a_1 > a_2 > \cdots > a_r$ (weak dominance for compact roots);

(ii) $|k_i - \ell_i| \leq 1$, $(1 \leq i \leq r)$;

(iii) Ψ contains $R^+_{\mathrm{cpt}} := \{e_i - e_j \,|\, 1 \leq i < j \leq p, \text{ or } p+1 \leq i < j \leq m\}$;

(iv) $\alpha^\vee(\lambda) \geq 0$, $(\forall \alpha \in \Psi)$;

(v) (Condition (F-1) in [Vog84]) Any Ψ-simple root α satisfying $\alpha^\vee(\lambda) = 0$ must be non-compact.

The corresponding limit of discrete series $\tau(\lambda, \Psi)$ has the minimal $\mathbf{K}_{p,q}$-type with the highest weight

$$\Lambda := \lambda + \frac{1}{2} \sum_{\alpha \in \Psi_{\mathrm{ncpt}}} \alpha - \frac{1}{2} \sum_{\alpha \in \Psi_{\mathrm{cpt}}} \alpha,$$

where $\Psi_{\mathrm{cpt}} := \Psi \cap R_{\mathrm{cpt}}$, $\Psi_{\mathrm{ncpt}} := \Psi \cap R_{\mathrm{ncpt}}$. Also this has the Langlands parameter $\varphi_{\underline{a}}$ given as in the discrete series case (3.2) but with the singular parameter

$$\underline{a} := (\overbrace{a_1, \ldots, a_1}^{k_1}, \ldots, \overbrace{a_r, \ldots, a_r}^{k_r}, \overbrace{a_1, \ldots, a_1}^{\ell_1}, \ldots, \overbrace{a_r, \ldots, a_r}^{\ell_r}) \in (2\mathbb{Z} + m + 1)^{p+q}.$$

Specializing these to G_2 and G_4, we have the following lemma. Since $\mathfrak{g} = \mathfrak{gl}(m, \mathbb{C})$, the set of positive systems Ψ are in one-to-one correspondence with that of permutations $(\sigma_1, \ldots, \sigma_m)$ of $(1, \ldots, m)$ by setting:

$$\Psi = \{e_i - e_j \mid 1 \leq i \neq j \leq m, \, \sigma_i < \sigma_j\}.$$

Also notice that $G_{2,0}$ has no limit of discrete series representations by the condition (v).

Lemma 3.9. *The limit of discrete series L-packets for $G_{1,1}$ and $G_{2,2}$ are given as follows. Notice that the L-packet is determined by $\underline{a}$ modulo permutation.*
(i) $G_{1,1}$-case. There is only one type of such packets.

2λ	Ψ	*Representation*	Λ
$(a; a)$	$(1, 2)$	$\tau(\lambda)_+$	$(\frac{a+1}{2}, \frac{a-1}{2})$
	$(2, 1)$	$\tau(\lambda)_-$	$(\frac{a-1}{2}, \frac{a+1}{2})$

(ii) $G_{2,2}$-case. We have the following types of limit of discrete series L-packets.

Case	2λ	Ψ	*Representation*	Λ
(1)	$(a, a; a, a)$	$(1, 3, 2, 4)$	$\tau(\lambda)_+$	$(\frac{a+1}{2}, \frac{a+1}{2}, \frac{a-1}{2}, \frac{a-1}{2})$
		$(3, 1, 4, 2)$	$\tau(\lambda)_-$	$(\frac{a-1}{2}, \frac{a-1}{2}, \frac{a+1}{2}, \frac{a+1}{2})$
(2.i)	$(a_1, a_2; a_1, a_2)$	$(1, 3, 2, 4)$	$\tau(\lambda)_{++}$	$(\frac{a_1+1}{2}, \frac{a_2+1}{2}, \frac{a_1-1}{2}, \frac{a_2-1}{2})$
		$(1, 3, 4, 2)$	$\tau(\lambda)_{+-}$	$(\frac{a_1+1}{2}, \frac{a_2-1}{2}, \frac{a_1-1}{2}, \frac{a_2+1}{2})$
		$(3, 1, 2, 4)$	$\tau(\lambda)_{-+}$	$(\frac{a_1-1}{2}, \frac{a_2+1}{2}, \frac{a_1+1}{2}, \frac{a_2-1}{2})$
		$(3, 1, 4, 2)$	$\tau(\lambda)_{--}$	$(\frac{a_1-1}{2}, \frac{a_2-1}{2}, \frac{a_1+1}{2}, \frac{a_2+1}{2})$
(2.ii)	$(a_1, a_2; a_1, a_1)$	$(3, 1, 4, 2)$	$\tau(\lambda)$	$(\frac{a_1-1}{2}, \frac{a_2-1}{2}, \frac{a_1+1}{2}, \frac{a_1+1}{2})$
	$(a_1, a_1; a_1, a_2)$	$(1, 3, 2, 4)$	$\tau(\lambda)$	$(\frac{a_1+1}{2}, \frac{a_1+1}{2}, \frac{a_1-1}{2}, \frac{a_2-1}{2})$
(2.iii)	$(a_1, a_2; a_2, a_2)$	$(1, 3, 2, 4)$	$\tau(\lambda)$	$(\frac{a_1+1}{2}, \frac{a_2+1}{2}, \frac{a_2-1}{2}, \frac{a_2-1}{2},)$
	$(a_2, a_2; a_1, a_2)$	$(3, 1, 4, 2)$	$\tau(\lambda)$	$(\frac{a_2-1}{2}, \frac{a_2-1}{2}, \frac{a_1+1}{2}, \frac{a_2+1}{2},)$
(3.i)	$(a_1, a_2; a_1, a_3)$	$(1, 3, 2, 4)$	$\tau(\lambda)_+$	$(\frac{a_1+1}{2}, \frac{a_2+1}{2}, \frac{a_1-1}{2}, \frac{a_3-1}{2})$
		$(3, 1, 2, 4)$	$\tau(\lambda)_-$	$(\frac{a_1-1}{2}, \frac{a_2+1}{2}, \frac{a_1+1}{2}, \frac{a_3-1}{2})$
	$(a_1, a_3; a_1, a_2)$	$(1, 3, 4, 2)$	$\tau(\lambda)_+$	$(\frac{a_1+1}{2}, \frac{a_3-1}{2}, \frac{a_1-1}{2}, \frac{a_2+1}{2})$
		$(3, 1, 4, 2)$	$\tau(\lambda)_-$	$(\frac{a_1-1}{2}, \frac{a_3-1}{2}, \frac{a_1+1}{2}, \frac{a_2+1}{2})$

Case	2λ	Ψ	*Representation*	Λ
(3.ii)	$(a_1,a_2;a_2,a_3)$	$(1,2,3,4)$	$\tau(\lambda)_+$	$(\frac{a_1+1}{2},\frac{a_2+3}{2},\frac{a_2-3}{2},\frac{a_3-1}{2})$
		$(1,3,2,4)$	$\tau(\lambda)_-$	$(\frac{a_1+1}{2},\frac{a_2+1}{2},\frac{a_2-1}{2},\frac{a_3-1}{2})$
	$(a_2,a_3;a_1,a_2)$	$(3,1,4,2)$	$\tau(\lambda)_+$	$(\frac{a_2-1}{2},\frac{a_3-1}{2},\frac{a_1+1}{2},\frac{a_2+1}{2})$
		$(3,4,1,2)$	$\tau(\lambda)_-$	$(\frac{a_2-3}{2},\frac{a_3-1}{2},\frac{a_1+1}{2},\frac{a_2+3}{2})$
(3.iii)	$(a_1,a_3;a_2,a_3)$	$(1,3,2,4)$	$\tau(\lambda)_+$	$(\frac{a_1+1}{2},\frac{a_3+1}{2},\frac{a_2-1}{2},\frac{a_3-1}{2})$
		$(1,3,4,2)$	$\tau(\lambda)_-$	$(\frac{a_1+1}{2},\frac{a_3-1}{2},\frac{a_2-1}{2},\frac{a_3+1}{2})$
	$(a_2,a_3;a_1,a_3)$	$(3,1,2,4)$	$\tau(\lambda)_+$	$(\frac{a_2-1}{2},\frac{a_3+1}{2},\frac{a_1+1}{2},\frac{a_3-1}{2})$
		$(3,1,4,2)$	$\tau(\lambda)_-$	$(\frac{a_2-1}{2},\frac{a_3-1}{2},\frac{a_1+1}{2},\frac{a_3+1}{2})$

The other L-packets We now give a list of the rest L-packets for $G_{1,1}$ and $G_{2,2}$. We begin with the $G_{1,1}$-case.

Recall the Borel pair $(\mathbf{B}_2,\mathbf{T}_2)$ of G_2. Each $\varphi\in\Phi_0(\mathbf{T}_2)/W(\mathbf{T}_2)$ is of the form $\varphi_{b,\nu}$, ($b\in\mathbb{Z}$, $\nu\in\mathbb{C}/\{\pm1\}$) in the notation of p. 23. We always choose ν so that $\mathrm{Re}\nu\geq 0$. If b is odd and $\nu=0$, this is of limit of discrete series type treated in Lem. 3.9 (i). In the other cases, we have

$$\Pi_{\varphi_{b,\nu}}=\{J^{G_2}_{\mathbf{B}_2}(\omega_{b,\nu})\},$$

which belongs to $\Pi_{\mathrm{temp}}(G_{1,1})$ if and only if $\mathrm{Re}\nu=0$. Here, for $\mathrm{Re}\nu=0$, we have written $J^{G_2}_{\mathbf{B}_2}(\omega_{b,\nu})=I^{G_2}_{\mathbf{B}_2}(\omega_{b,\nu})$.

Next comes the $G_{2,2}$-case.

(i) Each $\varphi\in\Phi_0(M_1)/W(M_1)$ is of the form $\varphi_{b,\nu,\underline{a}}$ ($b\in\mathbb{Z}$, $\nu\in\mathbb{C}/\{\pm1\}$, $\underline{a}=(a_1,a_2)$, $a_1>a_2\in 2\mathbb{Z}+1$) in the notation of p. 23. We always take ν such that $\mathrm{Re}\nu\geq 0$. The case $\nu=0$, $b\in 2\mathbb{Z}+1$ is treated in Lem. 3.9 (ii). In the other cases, we have

$$\Pi_{\varphi_{b,\nu,\underline{a}}}=\{J^{G_{2,2}}_{P_1}(\omega_{b,\nu}\otimes\delta)\,|\,\delta\in\Pi_{\varphi_{\underline{\mu}}}(G_{1,1})\},\quad \underline{\mu}=(\mu^{a_1},\mu^{a_2}).$$

(ii) Each $\varphi\in\Phi(\mathbf{T}_4)/W^{G_{2,2}}$ is of the form $\varphi_{\underline{b},\underline{\nu}}$, $((\underline{b}=(b_1,b_2),\underline{\nu}=(\nu_1,\nu_2))\in(\mathbb{Z}^2\times\mathbb{C}^2)/W^{G_{2,2}})$. The diagonal action of $W^{G_{2,2}}$ on the $\mathbb{C}^2$-component is as usual, while that on $\mathbb{Z}^2$-component factors through $\mathfrak{S}_2$.

  (a) If $\nu_2=0$ and $b_2\in 2\mathbb{Z}+1$, $\varphi_{\underline{b},\underline{\nu}}$ is a limit of discrete series parameter $\varphi_{b,\nu,a}$ for M_1, with $b=b_1\in\mathbb{Z}$, $\nu=\nu_1\in\mathbb{C}/\{\pm1\}$, $a=b_2\in 2\mathbb{Z}+1$. Further if $\nu_1=0$, $b\in 2\mathbb{Z}+1$, this gives the limit of discrete series packet in Lem. 3.9 (ii). Otherwise, we have

$$\Pi_{\varphi_{b,\nu,a}}(G_{2,2})=\{J^{G_{2,2}}_{P_1}(\omega_{b,\nu}\otimes\tau(a/2;a/2)_\pm)\}.$$

  (b) Otherwise, we may assume $\mathrm{Re}\nu_1\geq\mathrm{Re}\nu_2\geq 0$. $\Pi_{\varphi_{\underline{b},\underline{\nu}}}(G_{2,2})$ consists of the unique irreducible quotient of $I^{G_{2,2}}_{\mathbf{B}_4}(\omega_{b_1,\nu_1}\otimes\omega_{b_2,\nu_2})$.

3.4.2 The packets $\Pi'_\psi(G_{2,2})$

By the same reasoning as in the beginning of this section, the equivalence classes of A-parameters for $G_{2,2}$ are

(1.a) ψ_{η^a}, $a \in 2\mathbb{Z}$;

(2.a) $\psi_{\underline{\mu}^{\underline{a}}}$,($\underline{\mu}^{\underline{a}} = (\mu^{a_1}, \mu^{a_2})$, a_1, $a_2 \in 2\mathbb{Z}+1$);

(2.c) $\psi_{\underline{\eta}^{\underline{a}}}$, ($\underline{\eta}^{\underline{a}} = (\eta^{a_1}, \eta^{a_2})$, $a_1 > a_2 \in 2\mathbb{Z}$);

(2.d) $\psi_{\eta^a, \underline{\mu}^{\underline{a}}}$, ($a \in 2\mathbb{Z}$, $\underline{\mu}^{\underline{a}} = (\mu^{a_1}, \mu^{a_2})$, $a_1 > a_2 \in 2\mathbb{Z}+1$);

(M_1) $\psi^{M_1}_{\omega_{b,\nu}, \eta^a}$, ($b \in \mathbb{Z}$, $\nu \in i\mathbb{R}$, $a \in 2\mathbb{Z}$);

(M_2) $\psi^{M_2}_{\omega_{b,\nu}}$, ($b \in \mathbb{Z}$, $\nu \in i\mathbb{R}$).

Using the lists in § 3.4.1, we have the following archimedean counter part of Prop. 3.7. We take the non-trivial character $\psi_{\mathbb{R}}(x) = e^{2\pi\sqrt{-1}x}$ of $\mathbb{R}$, and use it to construct the non-degenerate character $\chi = \chi_n$ of $\mathbf{U}_n(\mathbb{R})$ as in p. 20.

Lemma 3.10. *The L-packets $\Pi'_{\psi}(G)$ and their χ-base points attached to the elliptic ψ in the above list are given by the following.*

A-parameter	$\mathcal{S}_{\psi}(G)$	$\Pi'_{\psi}(G)$	χ*-base point*
$(1.a)$ ψ_{η^a}	*trivial*	$\{\eta^a_G = \det^{a/2}\}$	η^a_G
$(2.a)$ $\psi_{\underline{\mu}^{\underline{a}}}$, $(a_1 > a_2)$	$\mathbb{Z}/2\mathbb{Z}$	$\left\{\begin{matrix} J^{G_{2,2}}_{P_1}(\mu^{a_1}[2] \otimes \delta(\frac{a_1}{2}; \frac{a_2}{2})) \\ J^{G_{2,2}}_{P_1}(\mu^{a_1}[2] \otimes \delta(\frac{a_2}{2}; \frac{a_1}{2})) \end{matrix}\right\}$	$J^G_{P_1}(\mu^{a_1}[2] \otimes \delta(\frac{a_1}{2}; \frac{a_2}{2}))$
$(2.a)$ $\psi_{\underline{\mu}^{\underline{a}}}$, $(a_1 < a_2)$	$\mathbb{Z}/2\mathbb{Z}$	$\left\{\begin{matrix} J^{G_{2,2}}_{P_1}(\mu^{a_1}[2] \otimes \delta(\frac{a_1}{2}; \frac{a_2}{2})) \\ J^{G_{2,2}}_{P_1}(\mu^{a_1}[2] \otimes \delta(\frac{a_2}{2}; \frac{a_1}{2})) \end{matrix}\right\}$	$J^G_{P_1}(\mu^{a_1}[2] \otimes \delta(\frac{a_2}{2}; \frac{a_1}{2}))$
$(2.a)$ $\psi_{\underline{\mu}^{\underline{a}}}$, $(\underline{a} = (a, a))$	$\mathbb{Z}/2\mathbb{Z}$	$\{J^{G_{2,2}}_{P_1}(\mu^a[2] \otimes \tau(\frac{a}{2}; \frac{a}{2})_{\pm})\}$	$J^{G_{2,2}}_{P_1}(\mu^a[2] \otimes \tau(\frac{a}{2}, \frac{a}{2})_{+})$
$(2.c)$ $\psi_{\underline{\eta}^{\underline{a}}}$	$\mathbb{Z}/2\mathbb{Z}$	$\{J^{G_{2,2}}_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta^{a_1} \otimes \eta^{a_2})[1])\}$	$J^{G_{2,2}}_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta^{a_1} \otimes \eta^{a_2})[1])$
$(2.d)$ $\psi_{\eta^a, \underline{\mu}^{\underline{a}}}$, $(a_1 > a_2)$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\left\{\begin{matrix} J^G_{P_1}(\eta^a[1] \otimes \delta(\frac{a_1}{2}; \frac{a_2}{2})) \\ J^G_{P_1}(\eta^a[1] \otimes \delta(\frac{a_2}{2}; \frac{a_1}{2})) \end{matrix}\right\}$	$J^G_{P_1}(\eta^a[1] \otimes \delta(\frac{a_1}{2}; \frac{a_2}{2}))$

4 Restriction rule from $GU(2)$ to $U(2)$ via base change

In this section we assume F is non-archimedean. We shall extend the base change lift of representations of $G_2(F)$ to $H_2(F)$ [Rog90] to the corresponding unitary similitude group $GU_{E/F}(2)$. Then we compare it with the base change lift for $GL(2)$ [Lan80] and deduce the restriction rule for irreducible representations of $GU_{E/F}(2)$ to $G_2(F) = U_{E/F}(2)$.

4.1 The groups

Throughout this section, we change the notation a little so that we write $G = G_2$, $\widetilde{G} = GU_{E/F}(2)$ the quasisplit unitary similitude group in two variables, and $G' := GL(2)_F$.

More precisely, for any linear algebraic group H over F, we write $\mathrm{R}_{E/F}H$ for its Weil's restriction of scalars from E to F. Thus $\mathrm{R}_{E/F}H(R) = H(R \otimes_F E)$ for any commutative F-algebra R. If further R is an E-algebra, we have from $E \otimes_F E \simeq E \oplus E$ that $\mathrm{R}_{E/F}H(R) =$

$H(R\otimes_E E\otimes_F E)\simeq H(R)\times H(R)$. Thus the scalar extension $(\mathrm{R}_{E/F}H)_E$ is isomorphic to $H_E\times H_E$, where the factors correspond to 1 and $\sigma\in\mathrm{Hom}_F(E,\bar F)$, respectively. There exists an E-automorphism $\tilde\sigma=\tilde\sigma_H$ on $(\mathrm{R}_{E/F}H)_E$ which is transported to $\tilde\sigma(x,y)=(y,x)$ on $H_E\times H_E$. On the other hand, the generator σ of $\Gamma_{E/F}$ acts on each H_E components while it also transposes $\mathrm{Hom}_F(E,\bar F)$. Thus σ on $\mathrm{R}_{E/F}H(R)$ is transported to

$$\sigma(x,y)=(\sigma(y),\sigma(x)),\quad (x,y)\in H(R)\times H(R).$$

In particular $\tilde\sigma$ turns out to be F-rational. If $H=\mathbb{G}_m$, we write $\mathrm{N}_{E/F}:\mathrm{R}_{E/F}\mathbb{G}_m\ni x\mapsto x\tilde\sigma(x)\in\mathbb{G}_m$ for the norm map.

We always write $\theta=\theta_2\circ\tilde\sigma\in\mathrm{Aut}_F(\mathrm{R}_{E/F}GL(2))$. In this section, we adopt the notation

$$\begin{aligned}G=&G_2=\{g\in\mathrm{R}_{E/F}GL(2)\,|\,\theta(g)=g\}\\ \widetilde G=&GU_{E/F}(2):=\{g\in\mathrm{R}_{E/F}GL(2)\,|\,\nu(g)\theta(g)=g,\ \exists\nu(g)\in\mathbb{G}_m\}.\end{aligned}$$

We also write $L:=\mathrm{R}_{E/F}G$, $\widetilde L:=\mathrm{R}_{E/F}\widetilde G$ and $L':=\mathrm{R}_{E/F}G'$. To describe these groups explicitly, we again identify $\mathrm{R}_{E/F}(\mathrm{R}_{E/F}GL(2))\simeq\mathrm{R}_{E/F}GL(2)^2$. Note that $\tilde\sigma\in\mathrm{R}_{E/F}GL(2)$ lifts to $\tilde\sigma(x,y)=(\tilde\sigma(y),\tilde\sigma(x))$, so that θ lifts to $\theta(x,y)=(\theta(y),\theta(x))$. We obtain the identification

$$\begin{aligned}L\simeq&\{(g,\theta(g))\,|\,g\in\mathrm{R}_{E/F}GL(2)\}\simeq\mathrm{R}_{E/F}GL(2)\\ \widetilde L\simeq&\{(g,\tilde\sigma(\nu)\theta(g))\,|\,g\in\mathrm{R}_{E/F}GL(2),\ \nu\in\mathrm{R}_{E/F}\mathbb{G}_m\}\simeq\mathrm{R}_{E/F}(GL(2)\times\mathbb{G}_m).\end{aligned}\tag{4.1}$$

This realization of groups is consistent with [Rog90]. But to compare our results with that of M. Harris [Har93], we also need the following realization. First notice that the condition $\nu(g)\theta(g)=g$ in the definition of $\widetilde G$ amounts to

$$\tilde\sigma(g)g^{-1}=\det(\tilde\sigma(g))\nu(g)^{-1}\in\mathrm{R}_{E/F}\mathbb{G}_m.$$

An argument similar to the proof of Hilbert 90 theorem shows that this $\mathrm{R}_{E/F}\mathbb{G}_m$-valued 1-cocycle on $\langle\sigma\rangle$ splits: $\tilde\sigma(g)g^{-1}=\tilde\sigma(z)z^{-1}$, $\exists z\in\mathrm{R}_{E/F}\mathbb{G}_m$. This implies $\theta_2(g'):=z^{-1}g$ belongs to $(\mathrm{R}_{E/F}GL(2))^{\tilde\sigma}=GL(2)_F$ and $\nu(g)=\nu(z\theta_2(g'))=\mathrm{N}_{E/F}(z)\det g'^{-1}$. Thus we obtain an isomorphism

$$(\mathrm{R}_{E/F}\mathbb{G}_m\times G')/\Delta\mathbb{G}_m\ni(z,g')\longmapsto z\theta_2(g')\in\widetilde G,\tag{4.2}$$

which sends $\nu(z,g'):=\mathrm{N}_{E/F}(z)\det g'^{-1}$ on the left to the similitude norm on the right. Δ stands for the diagonal embedding $z\mapsto(z,z\mathbf{1}_2)$. $G\subset\widetilde G$ consists of (z,g') satisfying $\mathrm{N}_{E/F}(z)=\det g'$. (4.2) lifts to an isomorphism

$$\begin{aligned}\widetilde L\simeq&\Big(\mathrm{R}_{E/F}(\mathrm{R}_{E/F}\mathbb{G}_m)\times\mathrm{R}_{E/F}G'\Big)\Big/\Delta\mathrm{R}_{E/F}\mathbb{G}_m\\ \simeq&(\mathrm{R}_{E/F}\mathbb{G}_m^2\times L')/\Delta\mathrm{R}_{E/F}\mathbb{G}_m\\ \simeq&\mathrm{R}_{E/F}\mathbb{G}_m\times L'.\end{aligned}\tag{4.3}$$

Notice that $\tilde\sigma$ on $\mathrm{R}_{E/F}\mathbb{G}_m\subset\widetilde G$ lifts to $\tilde\sigma(x,y;g')\mapsto(\tilde\sigma(y),\tilde\sigma(x);g')$ on $\mathrm{R}_{E/F}\mathbb{G}_m^2\times L'$. This together with $\mathbb{G}_m=(\mathrm{R}_{E/F}\mathbb{G}_m)^{\tilde\sigma}$ shows that Δ in (4.3) is given by

$$\Delta:\mathrm{R}_{E/F}\mathbb{G}_m\ni z\longmapsto(z,\tilde\sigma(z);z\mathbf{1}_2)\in\mathrm{R}_{E/F}\mathbb{G}_m^2\times L'.$$

Also note that $\tilde{\sigma} = \tilde{\sigma}_{\widetilde{G}} \in \mathrm{Aut}_F(\widetilde{L})$ for $\widetilde{G}$ comes from the outer restriction of scalars: $\tilde{\sigma}(x, y; g') = (y, x; \tilde{\sigma}(g'))$, and

$$\tilde{\sigma}_{\widetilde{G}} : \widetilde{L} \ni (z, g') = (z, 1; g') \longmapsto (\tilde{\sigma}(z)^{-1}, \tilde{\sigma}(z^{-1}g')) \in \widetilde{L}.$$

L is realized as the subgroup $\{(\det g, g) \,|\, g \in L'\}$ in this setting.

The transition between the realizations (4.1) and (4.3) is given by

$$\widetilde{L} \ni \left\{ \begin{array}{ccc} (g, \tilde{\sigma}(\nu)\theta(g)) & \longrightarrow & (\nu^{-1}\det g, \nu^{-1}g) \\ (z\theta_2(g'), \theta(g')) & \longleftarrow & (z, g') \end{array} \right\} \in \mathrm{R}_{E/F}\mathbb{G}_m \times L'. \tag{4.4}$$

This restricts to the injection $L \ni g \mapsto (\det g, g) \in \mathrm{R}_{E/F}\mathbb{G}_m \times L'$.

4.2 Representations and test functions

In general, for a connected reductive p-adic group $G(F)$ and a character ω of its center $Z_G(F)$, we write $\Pi(G(F))_\omega$ for the set of isomorphism classes of irreducible admissible representations of $G(F)$ having the central character ω. Also $\mathcal{H}(G(F), \omega)$ denotes the space of $\mathbb{C}$-valued smooth (*i.e.* locally constant) functions f on $G(F)$ satisfying

- $f(zg) = \omega(z)^{-1}f(g)$, $z \in Z(F)$, $g \in G(F)$;
- f is compactly supported modulo $Z(F)$.

Going back to $G = G_2$, we fix a character η of $E^\times$ satisfying $\eta|_{F^\times} = \mathbf{1}$. This gives a character η_u of $Z_G(F) \simeq G_1(F)$ (§ 3.3). We shall be concerned with the sets $\Pi(G(F))_{\eta_u}$, $\Pi(L(F))_\eta$ of representations of $G(F)$, $L(F)$ and the spaces $\mathcal{H}(G(F), \eta_u)$, $\mathcal{H}(L(F), \eta)$ of test functions. As for $\widetilde{G}$ and $\widetilde{L}$, we take a character ω of $E^\times \simeq Z_{\widetilde{G}}(F)$ whose restriction to $G_1(F)$ is η_u, and set

$$\omega_E : Z_{\widetilde{L}}(F) \ni (z\mathbf{1}_2, z'\mathbf{1}_2) \longmapsto \omega(zz') \in \mathbb{C}^\times$$

in the realization (4.1). This specifies the sets $\Pi(\widetilde{G}(F))_\omega$, $\Pi(\widetilde{L}(F))_{\omega_E}$, and the spaces $\mathcal{H}(\widetilde{G}(F), \omega)$, $\mathcal{H}(\widetilde{L}(F), \omega_E)$.

Since $\nu|_{Z_{\widetilde{G}}(F)} : Z_{\widetilde{G}}(F) \ni z\mathbf{1}_2 \longmapsto z\mathbf{1}_2\theta(z\mathbf{1}_2)^{-1} = \mathrm{N}_{E/F}(z) \in \mathbb{G}_m$, we see that

$$G(F)Z_{\widetilde{G}}(F) = \{g \in \widetilde{G}(F) \,|\, \nu(g) \in \mathrm{N}_{E/F}(E^\times)\}$$

is an index two subgroup of $\widetilde{G}(F)$. This gives a surjection

$$\mathcal{H}(\widetilde{G}(F), \omega) \ni f \longmapsto f_1 := f|_{G(F)} \in \mathcal{H}(G(F), \eta_u). \tag{4.5}$$

On the other hand, $Z_{\widetilde{L}} = \{(z\mathbf{1}_2, \tilde{\sigma}(\nu z^{-1})\mathbf{1}_2) \,|\, z, \nu \in \mathrm{R}_{E/F}\mathbb{G}_m\}$ shows $\widetilde{L}(F) = L(F)Z_{\widetilde{L}}(F)$, and we have an isomorphism

$$\mathcal{H}(\widetilde{L}(F), \omega_E) \ni \phi \longmapsto \phi_1 := \phi|_{L(F)} \in \mathcal{H}(L(F), \eta). \tag{4.6}$$

In the realization (4.2), we have

$$\Pi(\widetilde{G}(F))_\omega = \{\widetilde{\pi} = \omega \otimes \pi' \,|\, \pi' \in \Pi(G'(F))_{\omega^{-1}|_{F^\times}}\}. \tag{4.7}$$

Also if we write an irreducible admissible representation $\widetilde{\pi}_E$ of $\widetilde{L}(F)$ as $\widetilde{\pi}_E = \chi \otimes \pi'_E$, $\chi \in \Pi(E^\times)$, $\pi'_E \in \Pi(L'(F))$ in the realization (4.3), then its central character is given by

$$\omega_{\widetilde{\pi}_E} : Z_{\widetilde{L}}(F) \ni (z\mathbf{1}_2, z'\mathbf{1}_2) \overset{(4.4)}{\longmapsto} (z/\sigma(z'), \sigma(z')^{-1}\mathbf{1}_2) \overset{\widetilde{\pi}_E}{\longmapsto} \chi(z)\sigma(\chi\omega_{\pi'_E})^{-1}(z') \in \mathbb{C}^\times.$$

Thus $\omega_{\widetilde{\pi}_E} = \omega_E$ if and only if $\chi = \omega$ and $\sigma(\omega_{\pi'_E}) = \omega^{-1} \circ \mathrm{N}_{E/F} = \omega^{-2}\eta$. Thus we obtain

$$\begin{aligned}\Pi(\widetilde{L}(F))_{\omega_E} =& \{\widetilde{\pi}_E = \omega \otimes \pi'_E \,|\, \pi'_E \in \Pi(L'(F))_{\omega^{-1}\circ\mathrm{N}_{E/F}}\} \\ =& \{\widetilde{\pi}_E = \omega \otimes \omega(\det)^{-1}\pi_E \,|\, \pi_E \in \Pi(L(F))_\eta\}.\end{aligned} \tag{4.8}$$

In this notation, the restriction of $\widetilde{\pi}_E$ to $L(F)$ is just π_E.

4.3 Norm map in the theory of base change

Here we review the general construction of the norm map for the base change lifting from [Kot82], [KS99] and [Lab99].

Let E/F be a cyclic extension of degree ℓ of local fields of characteristic zero. We fix a generator σ of the Galois group $\Gamma_{E/F}$ of this extension. For the moment, we write G for a quasisplit connected reductive group over F whose derived group is simply connected, and put $L := \mathrm{R}_{E/F}G$. We have the E-isomorphism $L_E \xrightarrow{\sim} G_E^{|\Gamma_{E/F}|}$ as in § 4.1, and each $\tau \in \Gamma_{E/F}$ gives an F-automorphism $\tilde{\tau}$ of L which is transported to

$$\tilde{\tau} : G_E^{|\Gamma_{E/F}|} \ni (g_\gamma)_{\gamma\in\Gamma_{E/F}} \longmapsto (g_{\tau^{-1}\gamma})_{\gamma\in\Gamma_{E/F}} \in G_E^{|\Gamma_{E/F}|}.$$

In particular we have $\tilde{\sigma} \in \mathrm{Aut}_F(L)$.

We fix an algebraic closure $\bar{F}$ of F containing E. To define the norm map, we start with a construction at the level of $\bar{F}$-varieties. L acts on itself by the *σ-conjugation*:

$$\mathrm{Ad}_\sigma(g)x := gx\tilde{\sigma}(g)^{-1}, \quad g,\, x \in L.$$

We say δ, $\delta' \in L$ are *σ-conjugate* if they belong to a same $\mathrm{Ad}_\sigma(L)$-orbit. We write $Cl_\sigma(L)$ for the set of σ-conjugacy classes in L. The set of ordinary conjugacy classes in L is denoted by $Cl(L)$. Define the *concrete norm* by

$$\mathrm{N}_{E/F} : L \ni g \longmapsto g\tilde{\sigma}(g)\cdots\tilde{\sigma}^{\ell-1}(g) \in L.$$

If $\delta' = g^{-1}\delta\tilde{\sigma}(g) \in L$ we have $\mathrm{N}_{E/F}(\delta') = g^{-1}\mathrm{N}_{E/F}(\delta)g$. Also we have $\tilde{\sigma}(\mathrm{N}_{E/F}(\delta)) = \mathrm{Ad}(\delta^{-1})\mathrm{N}_{E/F}(\delta)$. These show that $\mathrm{N}_{E/F}$ gives a map

$$\mathrm{N}_{E/F} : Cl_\sigma(L) \longrightarrow Cl(L)^{\tilde{\sigma}}.$$

Now we consider the F-valued points. The set of F-valued points $C_\sigma(F) = C_\sigma(\bar{F}) \cap L(F)$ of $C_\sigma \in Cl_\sigma(L)$ is a *stable σ-conjugacy class* in $L(F)$. We write $\mathfrak{O}_\sigma^{st}(L(F))$ for the set of stable σ-conjugacy classes in $L(F)$. In general, the stable σ-conjugacy class $\mathfrak{O}^{st}_\sigma(\delta)$ of $\delta \in L(F)$ is larger than the rational σ-conjugacy class $\mathfrak{O}_\sigma(\delta) = \mathrm{Ad}_\sigma(L(F))\delta$. Replacing $(L, \tilde\sigma)$ by (G, id_G) we have the set of stable conjugacy classes $\mathfrak{O}^{st}(G(F))$ and that of rational conjugacy classes $\mathfrak{O}(G(F))$ in $G(F)$.

$C \in Cl(G)$ is F-rational if it is fixed under the Γ-action. Clearly the conjugacy class of a $\gamma \in G(F)$ is F-rational. Conversely, it was shown by Kottwitz [Kot82, Th. 4.1] that each F-rational conjugacy class C has an F-rational point: $C(\bar F) \cap G(F) \neq \emptyset$. Take $\delta \in L(F) = G(E)$ and write $C_\sigma := \mathrm{Ad}_\sigma(L)\delta$. Then $\mathrm{N}_{E/F}(C_\sigma)$ is F-rational in L, so that one can take $x \in \mathrm{N}_{E/F}(C_\sigma)(\bar F) \cap G(E)$. But since $\mathrm{N}_{E/F}(C_\sigma)$ is $\tilde\sigma$-stable, $C = \mathrm{Ad}(G)x$ is F-rational and contains an element $\gamma \in G(F)$. One can easily check that $\mathfrak{O}^{st}(\gamma)$ depends only on $\mathfrak{O}^{st}_\sigma(\delta)$. Thus we obtain the *norm map*

$$\mathcal{N}_{E/F} : \mathfrak{O}^{st}_\sigma(L(F)) \longrightarrow \mathfrak{O}^{st}(G(F)).$$

Take a maximal torus $T_G \subset G$ and set $T := \mathrm{Cent}(T_G, L)$. T is a $\tilde\sigma$-stable maximal torus in L and $T_G = T^{\tilde\sigma}$. Write $T(\tilde\sigma) := \{t\tilde\sigma(t)^{-1} \,|\, g \in T\}$ and $T_{\tilde\sigma} := T/T(\tilde\sigma)$. One can easily verify the following lemma, which shows our concrete norm coincides with the *abstract norm* in [KS99].

Lemma 4.1. $\mathrm{N}_{E/F} : T \to T_G$ *gives an isomorphism* $\mathrm{N}_{E/F} : T_{\tilde\sigma} \xrightarrow{\sim} T_G$.

4.4 Classification of σ-conjugacy classes

We still continue the general construction. We use $\mathcal{N}_{E/F}$ to describe σ-conjugacy classes in $L(F)$. We say $\delta \in L(F)$ is *σ-semisimple* if $\mathcal{N}_{E/F}(\delta)$ consists of semisimple elements. We write $\mathfrak{O}^{st}_\sigma(L(F))_{ss}$ for the subset of σ-semisimple elements in $\mathfrak{O}^{st}_\sigma(L(F))$.

Lemma 4.2. $\mathcal{N}_{E/F} : \mathfrak{O}^{st}_\sigma(L(F))_{ss} \to \mathfrak{O}^{st}(G(F))_{ss}$ *is an injection.*

Proof. Suppose σ-semisimple δ and $\delta' \in L(F)$ share the same norm $\mathfrak{O}^{st}(\gamma)$, $\gamma \in G(F)$. We choose a maximal F-torus $T_G \subset G$ containing γ, and set $T := \mathrm{Cent}(T_G, L)$. We write $\Omega(L, T)$ for the absolute Weyl group of T in L. In the base change case, one can easily check that $\Omega(L, T)^{\tilde\sigma} = \Omega(G, T_G)$ which acts on $T_{\tilde\sigma}$. Thanks to [KS99, Lem. 3.2.A], the images of $\mathrm{Ad}_\sigma(L)\delta \cap T$, $\mathrm{Ad}_\sigma(L)\delta' \cap T$ in $T_{\tilde\sigma}$ are single $\Omega(G, T_G)$-orbits. Then by Lem. 4.1, their images under $\mathrm{N}_{E/F} : T_{\tilde\sigma} \xrightarrow{\sim} T_G$ must be the $\Omega(G, T_G)$-orbit of γ and hence coincide. □

$\delta \in L(F)$ is *σ-regular* if $\mathcal{N}_{E/F}(\delta)$ is a regular semisimple stable class in $G(F)$. We write $\mathfrak{O}^{st}_\sigma(L(F))_{\mathrm{reg}}$ for the subset of σ-regular elements in $\mathfrak{O}^{st}_\sigma(L(F))$. Take a σ-regular $\delta \in L(F)$ and $\gamma \in \mathcal{N}_{E/F}(\delta)$. Then the centralizer $T_G := G_\gamma$ of γ in G is a maximal F-torus, and we have the $\tilde\sigma$-stable maximal torus $T := \mathrm{Cent}(T_G, L)$ of L. By the proof of Lem. 4.2, we can take $\delta^* \in T(\bar F)$, $g \in L(\bar F)$ such that

- $\delta = g^{-1}\delta^*\tilde\sigma(g)$,
- $\mathrm{N}_{E/F}(\delta^*) = \gamma$.

We write $L_{\delta,\sigma}$ for the fixed part of L under $\mathrm{Ad}(\delta)\circ\tilde{\sigma}$ and call it the *σ-centralizer* of δ. $\mathrm{Ad}(g^{-1}\delta^*\tilde{\sigma}(g))\circ\tilde{\sigma} = \mathrm{Ad}(g^{-1})\circ\mathrm{Ad}(\delta^*)\circ\tilde{\sigma}\circ\mathrm{Ad}(g)$ shows $\mathrm{Ad}(g)L_{\delta,\sigma} = L_{\delta^*,\sigma}$. Then $\tilde{\sigma}(\delta^*)\in T(\bar{F})$ gives

$$L_{\delta^*,\sigma} = (L_{\mathrm{N}_{E/F}(\delta^*)})^{\mathrm{Ad}(\delta^*)\circ\tilde{\sigma}} = T^{\tilde{\sigma}\circ\mathrm{Ad}(\tilde{\sigma}(\delta^*))} = T^{\tilde{\sigma}} = G_\gamma.$$

In general δ^* may not be F-rational, but it does not matter. We cite the following lemma.

Lemma 4.3 (Lem. 4.4.A in [KS99]). *For any $\tau\in\Gamma$, we have $g\tau(g)^{-1}\in T(\bar{F})$, so that $\mathrm{Ad}(g) : L_{\delta,\sigma}\xrightarrow{\sim} G_\gamma$ is defined over F.*

Finally we define $\mathfrak{D}(L_{\delta,\sigma}) := \mathrm{Ker}[\mathrm{H}^1(F, L_{\delta,\sigma})\to\mathrm{H}^1(F, L)]$. The map which associates to $g^{-1}\delta\tilde{\sigma}(g)\in\mathfrak{O}^{st}_\sigma(\delta)$ the class of the 1-cocycle $\{g\tau(g)^{-1}\}_{\tau\in\Gamma}$ in $\mathrm{H}^1(F, L_{\delta,\sigma})$ gives a bijection

$$\mathfrak{O}^{st}_\sigma(\delta)/\mathrm{Ad}_\sigma(L(F)) \xrightarrow{\ \sim\ } \mathfrak{D}(L_{\delta,\sigma}). \tag{4.9}$$

For $\alpha\in\mathfrak{D}(L_{\delta,\sigma})$, we write δ^α for the corresponding element in $\mathfrak{O}^{st}_\sigma(\delta)/\mathrm{Ad}_\sigma(L(F))$. Combining Lem. 4.2 with (4.9), we can describe the σ-regular σ-conjugacy classes in $L(F)$ in terms of $\mathcal{N}_{E/F}$ and $\mathfrak{D}(L_{\delta,\sigma})$.

4.5 σ-conjugacy classes in L and $\widetilde{L}$

We now apply the above construction to our L and $\widetilde{L}$.

We first describe the maximal tori in the related groups. For any maximal torus $T_G\subset G$, $T := \mathrm{Cent}(T_G, L)$ is a $\tilde{\sigma}$-stable maximal torus in L as above. Conversely for any $\tilde{\sigma}$-stable maximal torus T in L, $T_G := T^{\tilde{\sigma}}$ is a maximal torus in G. Similar relation holds for $\widetilde{G}$ and $\widetilde{L}$. Since G and $\widetilde{G}$ share the derived group $SL(2)$, we have the following relationships between their maximal tori:

$$\begin{array}{ccc}
\widetilde{G}\supset\widetilde{T}_G & \overset{\mathrm{Cent}(\widetilde{T}_G,\widetilde{L})}{\underset{\widetilde{T}^{\tilde{\sigma}}}{\rightleftarrows}} & \widetilde{T}\subset\widetilde{L} \\
{\scriptstyle\mathrm{Cent}(T_G,\widetilde{G})}\uparrow\ \downarrow{\scriptstyle\widetilde{T}_G\cap G} & & {\scriptstyle\widetilde{T}\cap L}\downarrow\ \uparrow{\scriptstyle\mathrm{Cent}(T,\widetilde{L})} \\
G\supset T_G & \overset{\mathrm{Cent}(T_G,L)}{\underset{T^{\tilde{\sigma}}}{\rightleftarrows}} & T\subset L
\end{array}$$

Here $\widetilde{T}$ and T are $\tilde{\sigma}$-stable. Thus to classify the ($\tilde{\sigma}$-stable for L and $\widetilde{L}$) maximal tori in these groups, it suffices to describe those in G. We review the classification from [Rog90, 3.4].

Write (B,ϵ) for the central simple algebra $\mathbb{M}_2(E)$ together with the involution of the second kind $\epsilon(a) = \mathrm{Ad}(I_2)^t\sigma(a)$, so that $G(R) = \{g\in(B\otimes_F R)^\times \mid g\epsilon(g) = 1\}$ for any commutative F-algebra R. For a maximal torus $T_G\subset G$, its centralizer B_T in B and the restriction of ϵ to B_T gives rise to a pair (B_T,ϵ_T) of a two-dimensional abelian semisimple algebra over E and an involution of the second kind on it. Of course T_G is recovered as $T_G(R) = \{t\in(B_T\otimes_F R)^\times \mid t\epsilon_T(t) = 1\}$. Such pairs are easily classified as follows.

(i) $(B_T = E^2, \epsilon_T(x,y) = ({}^\sigma y, {}^\sigma x))$, $T_G \simeq \mathrm{R}_{E/F}\mathbb{G}_m$.

(ii) $(B_T = E^2, \epsilon_T(x,y) = ({}^\sigma x, {}^\sigma y))$, $T_G \simeq G_1 \times G_1$.

(iii) $(B_T = K := E.E', \epsilon_T(z) = {}^\sigma z)$ for a quadratic extension E' of F other than E, where σ denotes $\sigma \otimes \mathrm{id}$ on $E \otimes_F E' \simeq K$. $T_G \simeq \mathrm{R}_{E'/F}(U_{K/E'}(1))$.

Next we review the classification of the $\tilde\sigma$-regular $\tilde\sigma$-conjugacy classes in $L(F)$ [Rog90, 3.12]. The stable $\tilde\sigma$-regular classes are parameterized by their images under $\mathcal{N}_{E/F}$, the regular semisimple stable classes in $G(F)$. Thus let $\delta_1 \in L(F)$ be a $\tilde\sigma$-regular element, and take $\gamma_1 \in \mathcal{N}_{E/F}(\delta_1)$, δ_1^* and $g_1 \in L(\bar F)$ as in § 4.4. We need to calculate $\mathfrak{O}^{st}_{\tilde\sigma}(\delta_1)/\mathrm{Ad}_\sigma(L(F))$ or equivalently $\mathfrak{D}(L_{\delta_1,\sigma})$. Since $\mathrm{H}^1(F,L) \simeq \mathrm{H}^1(E, GL(2)) = \{1\}$ by Shapiro's lemma and Hilbert 90 theorem, we see that

$$\mathfrak{D}(L_{\delta_1,\sigma}) = \mathrm{H}^1(F, L_{\delta_1,\sigma}) \xrightarrow[\sim]{\mathrm{Ad}(g)} \mathrm{H}^1(F, G_{\gamma_1}) \xrightarrow{\sim} \mathrm{H}^1(E/F, G_{\gamma_1}).$$

For a 1-cocycle $\{t_\sigma\} \subset G_{\gamma_1}(E)$ representing the image of $\alpha \in \mathrm{H}^1(E/F, G_{\gamma_1})$, $\delta_1^\alpha \in \mathfrak{O}^{st}_\sigma(\delta_1)/\mathrm{Ad}_\sigma(L(F))$ can be explicitly given as follows. We identify the cocycle with $t_\sigma \in G_{\gamma_1}(E)$ satisfying $t_\sigma \sigma(t_\sigma) = 1$. Here, identifying $G(E) = L(F)$ with $GL(2,E)$, the σ-action on $G(E)$ is given by $\sigma(g) = \theta_2({}^\sigma g)$ where $g \mapsto {}^\sigma g$ is the σ-action on $GL(2,E)$. Hence the image of t_σ under the isomorphism $L(E) \xrightarrow{\sim} GL(2,E)^2$ is $(t_\sigma, \theta_2(t_\sigma)) = (t_\sigma, {}^\sigma t_\sigma^{-1})$. Taking $a \in L(E)$ which is transported to $(t_\sigma, 1) \in GL(2,E)^2$, t_σ splits in $L(E)$ as

$$(t_\sigma, {}^\sigma t_\sigma^{-1}) = (t_\sigma, 1)(1, {}^\sigma t_\sigma^{-1}) = a^\sigma a^{-1}.$$

Correspondingly, α equals the class of $\{\mathrm{Ad}(g^{-1})(a^\sigma a^{-1})\}_{\sigma \in \Gamma_{E/F}}$. It follows from the definition of the bijection (4.9) that

$$\begin{aligned}(\delta_1^\alpha, {}^\sigma\delta_1^\alpha) &= (\mathrm{Ad}(g^{-1})a)^{-1}(\delta_1, {}^\sigma\delta_1)\theta_2({}^\sigma(\mathrm{Ad}(g^{-1})a)) \\ &= (\mathrm{Ad}(g^{-1})t_\sigma^{-1} \cdot \delta_1, {}^\sigma(\mathrm{Ad}(g^{-1})t_\sigma)^{-1} \cdot \delta_1).\end{aligned}$$

Thus $\delta_1^\alpha = \mathrm{Ad}(g^{-1})t_\sigma^{-1} \cdot \delta_1$.

This completes the classification of the σ-regular σ-conjugacy classes in $L(F)$. We recall the following exact sequence [Rog90, Prop. 3.12.1]:

$$\mathrm{H}^1(E/F, Z_G) \longrightarrow \mathrm{H}^1(E/F, G_{\gamma_1}) \xrightarrow{\det} \mathrm{H}^1(E/F, G_{\mathrm{ab}}). \tag{4.10}$$

Here $G_{\mathrm{ab}} := G/G_{\mathrm{der}} \simeq G_1$ is the abelianization of G. Notice that $\mathrm{H}^1(E/F, G_1)$ is identified with $F^\times/\mathrm{N}_{E/F}(E^\times)$. Moreover, under the bijection (4.9), the image of $z \in F^\times/\mathrm{N}_{E/F}(E^\times) = \mathrm{H}^1(E/F, Z_G)$ in $\mathfrak{D}(L_{\delta_1,\sigma}) \simeq \mathrm{H}^1(E/F, G_{\gamma_1})$ corresponds to $z^{-1}\delta_1 \in \mathfrak{O}^{st}_\sigma(\delta_1)$. Following Rogawski, we define

$${}_\sigma\mathfrak{D}(L_{\delta_1,\sigma}) := \mathrm{cok}[\mathrm{H}^1(E/F, Z_G) \to \mathrm{H}^1(E/F, G_{\gamma_1})].$$

We now proceed to $\widetilde{L}$. We take σ-regular $\delta \in \widetilde{L}(F)$, $\gamma \in \mathcal{N}_{E/F}(\delta)$, δ^* and $\tilde g \in \widetilde{L}(\bar F)$ satisfying the conditions in § 4.4. Here again, we need to describe $\mathfrak{D}(\widetilde{L}_{\delta,\sigma})$. Since $\widetilde{L}(F) = Z_{\widetilde{L}}(F)L(F)$, we may write

$$\delta = \zeta\delta_1, \quad \zeta \in Z_{\widetilde{L}}(F),\ \delta_1 \in L(F),$$

where δ_1 is obviously σ-regular. Take $\gamma_1 \in \mathcal{N}_{E/F}(\delta_1)$, δ_1^*, $g \in L(\bar F)$ as above. Then $\mathrm{Ad}_\sigma(g)\delta = \zeta\delta_1^*$ and $\mathrm{N}_{E/F}(\zeta\delta_1^*) = \mathrm{N}_{E/F}(\zeta)\gamma_1$ imply

(i) $\mathcal{N}_{E/F}(\delta) = \mathfrak{O}^{st}(\mathrm{N}_{E/F}(\zeta)\gamma_1)$. We may choose $\gamma := \mathrm{N}_{E/F}(\zeta)\gamma_1$.

(ii) We can choose $(\delta^* = \zeta\delta_1^*, g)$ as (δ^*, g) of § 4.4 for δ.

Thus in the relation of maximal tori above, we have $T_G = G_{\gamma_1}$, $\widetilde{T}_G = \widetilde{G}_\gamma$, $T = L_{\gamma_1}$, $\widetilde{T} = \widetilde{L}_\gamma$. Again we have the isomorphism

$$\mathfrak{D}(\widetilde{L}_{\delta,\sigma}) = \mathrm{H}^1(F, \widetilde{L}_{\delta,\sigma}) \xrightarrow[\sim]{\mathrm{Ad}(g)} \mathrm{H}^1(F, \widetilde{G}_\gamma).$$

Lemma 4.4. *For a maximal torus $T_G \subset G$, the following is exact.*

$$\mathrm{H}^1(F, Z_G) \longrightarrow \mathrm{H}^1(F, T_G) \longrightarrow \mathrm{H}^1(F, \widetilde{T}_G) \longrightarrow \{1\}.$$

Proof. Recall the pair (B_T, ϵ_T) associated to T_G. Then $\widetilde{T}_G$ is given by

$$\widetilde{T}_G(R) = \{t \in (B_T \otimes_F R)^\times \,|\, t\epsilon_T(t) \in R^\times\},$$

while $Z_G(R) = Z(B \otimes_F R) \cap T_G(R) = (E \otimes_F R)^\times \cap T_G(R)$. We examine the three types of maximal tori in turn.

(1) If $T_G \simeq \mathrm{R}_{E/F}\mathbb{G}_m$, $(x, y) \in B_T \otimes R$ belongs to $\widetilde{T}_G$ if and only if

$$(x, y)\epsilon_T(x, y) = (x\,{}^\sigma y, {}^\sigma x y) = (a, a), \quad \exists a \in R^\times.$$

Thus $\widetilde{T}_G = \{(t, a\tilde{\sigma}(t)^{-1}) \,|\, t \in \mathrm{R}_{E/F}\mathbb{G}_m,\ a \in \mathbb{G}_m\} \simeq \mathrm{R}_{E/F}\mathbb{G}_m \times \mathbb{G}_m$, so that $\mathrm{H}^1(F, T_G) = \mathrm{H}^1(F, \widetilde{T}_G) = \{1\}$ and the exactness is trivial.

(2) If $T_G \simeq G_1^2$, we have $\widetilde{T}_G = \{(x, xt) \,|\, x \in \mathrm{R}_{E/F}\mathbb{G}_m,\ t \in G_1\} \simeq \mathrm{R}_{E/F}\mathbb{G}_m \times G_1$. Thus

$$1 \longrightarrow Z_G \longrightarrow T_G \longrightarrow \widetilde{T}_G/\mathrm{R}_{E/F}\mathbb{G}_m \longrightarrow 1$$

is exact. The result follows from this together with the injectiveness of $\mathrm{H}^1(F, \widetilde{T}_G) \to \mathrm{H}^1(F, \widetilde{T}_G/\mathrm{R}_{E/F}\mathbb{G}_m)$.

(3) If $T_G \simeq \mathrm{R}_{E'/F}(U_{K/E'}(1))$, we look at the long exact sequence of Galois cohomology associated to

$$1 \longrightarrow T_G \longrightarrow \widetilde{T}_G \xrightarrow{\nu} \mathbb{G}_m \longrightarrow 1.$$

Noting $\mathrm{N}_{K/E'}(K^\times) \supset F^\times$, we can check the surjectivity of $\nu : \widetilde{T}_G(F) \to F^\times$. Hence the long sequence gives $\mathrm{H}^1(F, T_G) \xrightarrow{\sim} \mathrm{H}^1(F, \widetilde{T}_G)$. This combined with the triviality of the image of $\mathrm{H}^1(F, Z_G) = F^\times/\mathrm{N}_{E/F}(E^\times)$ in $\mathrm{H}^1(F, T_G) \simeq \mathrm{H}^1(E', U_{K/E'}(1)) \simeq E'^\times/\mathrm{N}_{K/E'}(K^\times)$ yields the desired exactness. □

This gives an isomorphism ${}_\sigma\mathfrak{D}(L_{\delta_1,\sigma}) \xrightarrow{\sim} \mathrm{H}^1(F, \widetilde{G}_\gamma) \xrightarrow{\sim} \mathfrak{D}(\widetilde{L}_{\delta,\sigma})$. In particular for $\alpha \in \mathfrak{D}(\widetilde{L}_{\delta,\sigma})$, we have $\delta^\alpha = \mathrm{Ad}(g^{-1})t_\sigma^{-1} \cdot \delta$, where $\{t_\sigma\}$ is a $G_{\gamma_1}(E)$-valued 1-cocycle on $\Gamma_{E/F}$ representing an inverse image of α in $\mathrm{H}^1(E/F, G_{\gamma_1})$.

4.6 Orbital integral transfers

To define the base change lift, one needs the relevant orbital integral transfers. We review this for (G, L) from [Rog90, 4.11] and then deduce an analogous result for $(\widetilde{G}, \widetilde{L})$.

Let us define the twisted orbital integrals on $L(F)$. Write $p : L \twoheadrightarrow L_{\mathrm{ad}} := L/Z_L$ for the natural projection. Take a σ-regular $\delta_1 \in L(F)$ and $\gamma_1 \in \mathcal{N}_{E/F}(\delta_1)$. Define

$$\begin{aligned} I_{\delta_1,\sigma} :=& p^{-1}(p(L_{\delta_1,\sigma})) = \{g \in L \,|\, (gz)^{-1}\delta_1\tilde{\sigma}(gz) = \delta_1, \exists z \in Z_L\} \\ =& \{g \in L \,|\, g^{-1}\delta_1\tilde{\sigma}(g) \in \delta_1 Z_L(\tilde{\sigma})\}, \end{aligned}$$

where $Z_L(\tilde{\sigma}) := (1-\tilde{\sigma})Z_L$. Under the identification $Z_L \xrightarrow{\sim} \mathrm{R}_{E/F}\mathbb{G}_m$, $\tilde{\sigma}$ on Z_L corresponds to $z \mapsto \tilde{\sigma}(z)^{-1}$. Thus $(1-\tilde{\sigma})$ is just $\mathrm{N}_{E/F} : \mathrm{R}_{E/F}\mathbb{G}_m \to \mathbb{G}_m$ and $Z_L(\tilde{\sigma}) = \mathbb{G}_m$. Notice that

$$I_{\delta_1,\sigma}(F) = \{g \in L(F) \,|\, gz \in L_{\delta_1,\sigma}(\bar{F}), \exists z \in Z_L(\bar{F})\}$$

is strictly larger than $L_{\delta_1,\sigma}(F)Z_L(F)$. Since the central character η restricted to $Z_L(\tilde{\sigma}, F) = F^\times$ is trivial, the *σ-orbital integral* of $\phi_1 \in \mathcal{H}(L(F), \eta)$ at δ_1

$$O_{\sigma,\delta_1}(\phi_1) := \int_{I_{\delta_1,\sigma}(F)\backslash L(F)} \phi_1(g^{-1}\delta_1\sigma(g))\,\frac{dg}{dt}$$

is well-defined. Here we have fixed an invariant measures dg and dt on $L(F)$ and $I_{\delta_1,\sigma}(F)$, respectively.

Next consider $\widetilde{L}$. We take σ-regular δ and $\gamma \in \mathcal{N}_{E/F}(\delta)$. We again write $p : \widetilde{L} \twoheadrightarrow \widetilde{L}_{\mathrm{ad}}$ for the natural projection and define

$$\widetilde{I}_{\delta,\sigma} := p^{-1}(p(\widetilde{L}_{\delta,\sigma})) = \{g \in \widetilde{L} \,|\, gz \in \widetilde{L}_{\delta,\sigma}, \exists z \in Z_{\widetilde{L}}\}.$$

In this case $Z_{\widetilde{L}}$ is identified with $\mathrm{R}_{E/F}(\mathrm{R}_{E/F}\mathbb{G}_m) \simeq \mathrm{R}_{E/F}\mathbb{G}_m^2$ and $\tilde{\sigma}$ is just the transposition of the two components. Since $Z_{\widetilde{L}}(\tilde{\sigma}) = \{(x, x^{-1}) \,|\, x \in \mathrm{R}_{E/F}\mathbb{G}_m\}$ and $1-\tilde{\sigma} : Z_{\widetilde{L}}(F) \to Z_{\widetilde{L}}(\tilde{\sigma}, F)$ is surjective, we have

$$\widetilde{I}_{\delta_1,\sigma}(F) = \widetilde{L}_{\delta_1,\sigma}(F)Z_{\widetilde{L}}(F).$$

Since $\omega_E : Z_L(\tilde{\sigma}, F) \ni (z, z^{-1}) \mapsto \omega(zz^{-1}) = 1 \in \mathbb{C}^\times$, the σ-orbital integral

$$O_{\sigma,\delta}(\phi) := \int_{\widetilde{I}_{\delta,\sigma}(F)\backslash\widetilde{L}(F)} \phi(g^{-1}\delta\sigma(g))\,\frac{dg}{dt} = \int_{\widetilde{L}_{\delta,\sigma}(F)Z_{\widetilde{L}}(F)\backslash\widetilde{L}(F)} \phi(g^{-1}\delta\sigma(g))\,\frac{dg}{dt}$$

of $\phi \in \mathcal{H}(\widetilde{L}(F), \omega_E)$ is well-defined. Let us write $\delta = \zeta\delta_1$, $\zeta \in Z_{\widetilde{L}}(F)$, $\delta_1 \in L(F)$, and adopt the notation of § 4.5. It follows from $L_{\mathrm{ad}} = \widetilde{L}_{\mathrm{ad}}$ that $\widetilde{I}_{\delta,\sigma} \cap L = I_{\delta_1,\sigma}$. We may and do choose the invariant measures dg and dt on $\widetilde{L}(F)$ and $\widetilde{I}_{\delta,\sigma}(F)$, respectively, in such a way that the isomorphism (4.6) gives

$$\begin{aligned} O_{\sigma,\delta}(\phi) =& \int_{I_{\delta_1,\sigma}(F)Z_{\widetilde{L}}(F)\backslash L(F)Z_{\widetilde{L}}(F)} \phi(\zeta g^{-1}\delta_1\sigma(g))\,\frac{dg}{dt} \\ =& \omega_E(\zeta)^{-1}\int_{I_{\delta_1,\sigma}(F)\backslash L(F)} \phi_1(g^{-1}\delta_1\sigma(g))\,\frac{dg}{dt} \\ =& \omega_E(\zeta)^{-1}O_{\sigma,\delta_1}(\phi_1). \end{aligned} \tag{4.11}$$

For $f_1 \in \mathcal{H}(G(F), \eta_u)$ and regular semisimple $\gamma_1 \in G(F)$, we have the usual orbital integral

$$O_{\gamma_1}(f_1) := \int_{G_{\gamma_1}(F)\backslash G(F)} f_1(x^{-1}\gamma_1 x)\,\frac{dx}{dt}.$$

As in the σ-twisted case, we define $\mathfrak{D}(G_{\gamma_1}) := \mathrm{Ker}[\mathrm{H}^1(F, G_{\gamma_1}) \to \mathrm{H}^1(F, G)]$. There is a bijection $\mathfrak{D}(G_{\gamma_1}) \ni \alpha \overset{\sim}{\mapsto} \gamma_1^\alpha \in \mathfrak{O}^{st}(\gamma_1)/\mathrm{Ad}(G(F))$. In fact, this is the composite of the isomorphism

$$(G_{\gamma_1}\backslash G)(F) \ni x \overset{\sim}{\longmapsto} x^{-1}\gamma_1 x \in \mathfrak{O}^{st}(\gamma_1)/\mathrm{Ad}(G(F))$$

and the bijection

$$(G_{\gamma_1}\backslash G)(F) \ni x \overset{\sim}{\longmapsto} \text{class of } \{x\tau(x)^{-1}\}_{\tau\in\Gamma} \in \mathfrak{D}(G_{\gamma_1}).$$

Using the former, we can take compatible measures on the rational conjugacy classes in $\mathfrak{O}^{st}(\gamma_1)$ as was explained in [LS87, (1.4)]. The *stable orbital integral* of f_1 at γ_1 is defined by

$$SO_{\gamma_1}(f_1) := \sum_{\alpha\in\mathfrak{D}(G_{\gamma_1})} O_{\gamma_1^\alpha}(f_1),$$

where the orbital integrals in the right hand side are taken with respect to the compatible measures. As for $\widetilde{G}$ we have the following.

Lemma 4.5. *For any maximal torus $\widetilde{T}_G$ of $\widetilde{G}$, $\mathfrak{D}(\widetilde{T}_G) = \mathrm{Ker}[\mathrm{H}^1(F, \widetilde{T}_G) \to \mathrm{H}^1(F, \widetilde{G})]$ is trivial.*

Proof. The realization (4.2) gives the long exact sequence

$$\begin{aligned} 1 &\longrightarrow F^\times \longrightarrow E^\times \times GL(2, F) \longrightarrow \widetilde{G}(F) \\ &\longrightarrow \mathrm{H}^1(F, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(F, \mathrm{R}_{E/F}\mathbb{G}_m \times GL(2)) \longrightarrow \mathrm{H}^1(F, \widetilde{G}) \xrightarrow{\iota_G} \mathrm{Br}(F), \end{aligned}$$

and hence ι_G becomes an injection of $\mathrm{H}^1(F, \widetilde{G})$ into the Brauer group $\mathrm{Br}(F)$ of F by Hilbert 90. Let $\widetilde{T}_G \simeq (\mathrm{R}_{E/F}\mathbb{G}_m \times \mathrm{R}_{E'/F}\mathbb{G}_m)/\Delta\mathbb{G}_m$ be any maximal torus in $\widetilde{G}$, where E' is a two dimensional abelian semisimple algebra over F. We have the similar long exact sequence

$$\begin{aligned} 1 &\longrightarrow F^\times \longrightarrow E^\times \times E'^\times \longrightarrow \widetilde{T}_G(F) \\ &\longrightarrow \mathrm{H}^1(F, \mathbb{G}_m) \longrightarrow \mathrm{H}^1(F, \mathrm{R}_{E/F}\mathbb{G}_m \times \mathrm{R}_{E'/F}\mathbb{G}_m) \longrightarrow \mathrm{H}^1(F, \widetilde{T}_G) \xrightarrow{\iota_T} \mathrm{Br}(F), \end{aligned}$$

and an injection $\iota_T : \mathrm{H}^1(F, \widetilde{T}_G) \hookrightarrow \mathrm{Br}(F)$. It follows from the functoriality of the Galois cohomology that

$$\begin{array}{ccc} \mathrm{H}^1(F, \widetilde{T}_G) & \longrightarrow & \mathrm{H}^1(F, \widetilde{G}) \\ {\scriptstyle \iota_T}\downarrow & & \downarrow{\scriptstyle \iota_G} \\ \mathrm{Br}(F) & = & \mathrm{Br}(F) \end{array}$$

is commutative. Hence the lemma follows from the injectivity of ι_G and ι_T. □

Thus each stable conjugacy class consists of a single rational class, so that the stable orbital integral of $f \in \mathcal{H}(\widetilde{G}(F), \omega)$ at a regular semisimple $\gamma \in \widetilde{G}(F)$ is just the orbital integral

$$O_\gamma(f) := \int_{\widetilde{G}_\gamma(F)\backslash \widetilde{G}(F)} f(x^{-1}\gamma x)\, \frac{dx}{dt}.$$

Further suppose that $\nu(\gamma) \in \mathrm{N}_{E/F}(E^\times)$. Thanks to p. 35 (i), any regular semisimple norm $\gamma \in \mathcal{N}_{E/F}(\delta)$ satisfies this condition. Then we can write $\gamma = z\gamma_1$, $(z \in Z_{\widetilde{G}}(F),\ \gamma_1 \in G(F))$ and have $(G_{\gamma_1}\backslash G)(F) \simeq (\widetilde{G}_\gamma\backslash \widetilde{G})(F) \simeq \widetilde{G}_\gamma(F)\backslash \widetilde{G}(F)$. We choose the invariant measures dx and dt on $\widetilde{G}(F)$ and $\widetilde{G}_\gamma(F)$, respectively, so that we have

$$\begin{aligned} O_\gamma(f) = &\int_{\widetilde{G}_\gamma(F)\backslash \widetilde{G}(F)} f(zx^{-1}\gamma_1 x)\, \frac{dx}{dt} = \omega(z)^{-1} \int_{(\widetilde{G}_\gamma\backslash \widetilde{G})(F)} f(x^{-1}\gamma_1 x)\, \frac{dx}{dt} \\ =&\omega(z)^{-1} \int_{(G_{\gamma_1}\backslash G)(F)} f_1(x^{-1}\gamma_1 x)\, \frac{dx}{dt} \\ =&\omega(z)^{-1} SO_{\gamma_1}(f_1) \end{aligned} \tag{4.12}$$

for $f \in \mathcal{H}(\widetilde{G}(F), \omega)$ and its image $f_1 \in \mathcal{H}(G(F), \eta_u)$ under (4.5).

We also need the transfer factors. First we consider the case of G and L. We are concerned with the twisted endoscopy problem for $(L, \tilde{\sigma}, \mathbf{1})$ in the sense of [KS99], which is the case (A) with $n = 2$ in [Kon02, Appendix]. Th. A.6 therein asserts that we have two isomorphism classes of principal endoscopic data $\mathcal{E}_2 = (G, {}^LG, (\mathbf{1}_2, \mathbf{1}_2), \xi_{\mathbf{1}})$ and $\mathcal{E}_0 = (G, {}^LG, (\mathbf{1}_2, -\mathbf{1}_2), \xi_\mu)$. The choice of a representative for the latter class specifies a character μ of $E^\times$ such that $\mu|_{F^\times} = \omega_{E/F}$. Since $L \simeq L' = \mathrm{R}_{E/F}GL(2)$, we have $\det : L \to \mathrm{R}_{E/F}\mathbb{G}_m$. Moreover $\det(g\delta_1\theta(g)^{-1}) = \det\delta_1 \cdot \mathrm{N}_{E/F}(\det g)$ shows that the class of $\det\delta$ in $E^\times/\mathrm{N}_{E/F}(E^\times)$ depends only on the σ-conjugacy class of δ in $L(F)$. (Recall that $\tilde{\sigma} \in \mathrm{Aut}_F(L)$ corresponds to $\theta \in \mathrm{Aut}_F(L')$.) Hence we can define the transfer factor $\Delta_\mu : \mathfrak{O}^{st}(G(F))_{\mathrm{reg}} \times \mathfrak{O}_\theta(L(F))_{\mathrm{reg}} \to \mathbb{C}$ for $\mathcal{E}_0$ by

$$\Delta_\mu(\gamma_1, \delta_1) := \begin{cases} \mu(\det\delta_1) & \text{if } \gamma_1 \in \mathcal{N}_{E/F}(\delta_1), \\ 0 & \text{otherwise.} \end{cases}$$

The transfer factor $\Delta_{\mathbf{1}}$ for $\mathcal{E}_2$ is just

$$\Delta_{\mathbf{1}}(\gamma_1, \delta_1) := \begin{cases} 1 & \text{if } \gamma_1 \in \mathcal{N}_{E/F}(\delta_1), \\ 0 & \text{otherwise.} \end{cases}$$

Extending these, we define the transfer factor Δ_μ and $\Delta_{\mathbf{1}}$ on $\mathfrak{O}(\widetilde{G}(F))_{\mathrm{reg}} \times \mathfrak{O}_\sigma(\widetilde{L}(F))_{\mathrm{reg}}$ by

$$\Delta_\mu(\gamma, \delta) = \begin{cases} \mu(z) & \text{if } \gamma \in \mathcal{N}_{E/F}(\delta) \text{ and } \delta = (z, g'), \\ 0 & \text{otherwise,} \end{cases}$$

$$\Delta_{\mathbf{1}}(\gamma, \delta) = \begin{cases} 1 & \text{if } \gamma \in \mathcal{N}_{E/F}(\delta), \\ 0 & \text{otherwise.} \end{cases}$$

Here in the former, we have used the realization (4.3). In this case we have

$$\mathrm{Ad}_\sigma((x,h))(z,g') = (x,h)(z,g')(\sigma(x),\sigma(x)\cdot h^{-1}) = (z\mathrm{N}_{E/F}(x),\sigma(x)\mathrm{Ad}(h)g'),$$

so that Δ_μ is well-defined.

We now recall the following from [Rog90, Prop. 4.11.1]. For an F-maximal torus $T_G \subset G$, $T_G(F)_{\mathrm{reg}}$ denotes the set of $\gamma \in T_G(F)$ which is regular semisimple in G.

Proposition 4.6 (Orbital integral transfer for (G,L)). *(i) For any $\phi_1 \in \mathcal{H}(L(F),\eta)$, there exist $f_1 \in \mathcal{H}(G(F),\eta_u)$ and $f_1^\mu \in \mathcal{H}(G(F),\eta_u\mu^{-1})$ satisfying*

$$\begin{aligned}\sum_{\alpha\in{}_\sigma\mathfrak{D}(L_{\delta_1,\sigma})} \Delta_{\mathbf{1}}(\gamma_1,\delta_1^\alpha)O_{\sigma,\delta_1}(\phi_1) =& SO_{\gamma_1}(f_1),\\ \sum_{\alpha\in{}_\sigma\mathfrak{D}(L_{\delta_1,\sigma})} \Delta_\mu(\gamma_1,\delta_1^\alpha)O_{\sigma,\delta_1}(\phi_1) =& SO_{\gamma_1}(f_1^\mu),\end{aligned} \tag{4.13}$$

for any σ-regular $\delta_1 \in L(F)$ and $\gamma_1 \in \mathcal{N}_{E/F}(\delta_1)$.
(ii) Conversely, for any $f_1 \in \mathcal{H}(G(F),\eta_u)$ and $f_1^\mu \in \mathcal{H}(G(F),\eta_u\mu^{-1})$ satisfying

$$O_{\gamma_1}(f_1^\mu) = \mu(\gamma_1)O_{\gamma_1}(f_1), \quad \forall\gamma_1 \in \mathbf{T}_G(F)_{\mathrm{reg}},$$

there exists $\phi_1 \in \mathcal{H}(L(F),\eta)$ such that (4.13) holds. Here we view μ as the character $\mathbf{T}_G(F) \ni \left(\begin{smallmatrix} x & \\ & \sigma(x)^{-1}\end{smallmatrix}\right) \mapsto \mu(x) \in \mathbb{C}^\times$.

We can deduce the transfer for $\widetilde{G}$ and $\widetilde{L}$ from this. We regard μ as the character

$$\mu : \widetilde{\mathbf{T}}_G(F) \ni \begin{pmatrix} z & \\ & \mathrm{N}_{E/F}(\nu)\sigma(z)^{-1}\end{pmatrix} \longmapsto \mu(z) \in \mathbb{C}^\times.$$

Proposition 4.7. *(i) For $\phi \in \mathcal{H}(\widetilde{L}(F),\omega_E)$, there exist $f \in \mathcal{H}(\widetilde{G}(F),\omega)$, $f^\mu \in \mathcal{H}(\widetilde{G}(F),\omega\mu^{-1})$ supported on the index two subgroup $Z_{\widetilde{G}}(F)G(F)$ such that*

$$\begin{aligned}\sum_{\delta'\in\mathfrak{D}_\sigma^{st}(\delta)/\mathrm{Ad}_\sigma(L(F))} \Delta_{\mathbf{1}}(\gamma,\delta')O_{\sigma,\delta'}(\phi) &= O_\gamma(f),\\ \sum_{\delta'\in\mathfrak{D}_\sigma^{st}(\delta)/\mathrm{Ad}_\sigma(L(F))} \Delta_\mu(\gamma,\delta')O_{\sigma,\delta'}(\phi) &= O_\gamma(f^\mu)\end{aligned} \tag{4.14}$$

hold for any σ-regular $\delta \in \widetilde{L}(F)$ and $\gamma \in \mathcal{N}_{E/F}(\delta)$.
(ii) Conversely, for any $f \in \mathcal{H}(\widetilde{G}(F),\omega)$, $f^\mu \in \mathcal{H}(\widetilde{G}(F),\omega\mu^{-1})$ satisfying

$$O_\gamma(f^\mu) = \mu(\gamma)O_\gamma(f), \quad \forall\gamma \in \widetilde{\mathbf{T}}_G(F)_{\mathrm{reg}},$$

there exists $\phi \in \mathcal{H}(\widetilde{L}(F),\omega_E)$ such that (4.14) holds.

Proof. We prove only (i). (ii) can be proved in a similar way using Prop. 4.6 (ii) and reversing the argument. Take $\phi \in \mathcal{H}(\widetilde{L}(F),\omega_E)$ and write $\phi_1 \in \mathcal{H}(L(F),\eta)$ for its image under the surjection (4.6). Then ϕ_1 transfers to $f_1 \in \mathcal{H}(G(F),\eta_u)$ and $f_1^\mu \in \mathcal{H}(G(F),\eta_u\mu^{-1})$

by Prop. 4.6 (i). Write $f \in \mathcal{H}(\widetilde{G}(F),\omega)$ and $f^\mu \in \mathcal{H}(\widetilde{G}(F),\omega\mu^{-1})$ for any elements in the inverse images of f_1 and f_1^μ under the surjections (4.5), respectively. Take σ-regular $\delta = \zeta\delta_1 \in \widetilde{L}(F)$, $(\zeta \in Z_{\widetilde{L}}(F),\ \delta_1 \in L(F))$, and $\gamma = \mathrm{N}_{E/F}(\zeta)\gamma_1 \in \mathcal{N}_{E/F}(\delta)$, $(\gamma_1 \in \mathcal{N}_{E/F}(\delta_1))$. We write $\xi = \mathbf{1}$ or μ, and $\widetilde{\xi}(z,g') := \xi(z)$ inthe realization (4.3). Then one has

$$\begin{aligned}\sum_{\delta'\in\mathfrak{O}_\sigma^{st}(\delta)/\mathrm{Ad}_\sigma(\widetilde{L}(F))}\Delta_\xi(\gamma,\delta')O_{\sigma,\delta'}(\phi) &= \sum_{\alpha\in\mathfrak{D}(\widetilde{L}_{\delta,\sigma})}\Delta_\xi(\gamma,\delta^\alpha)O_{\sigma,\delta^\alpha}(\phi)\\ &= \sum_{\alpha\in_\sigma\mathfrak{D}(L_{\delta_1,\sigma})}\Delta_\xi(\gamma,\zeta\delta_1^\alpha)O_{\sigma,\zeta\delta_1^\alpha}(\phi)\\ &\overset{(4.11)}{=}\widetilde{\xi}\omega_E^{-1}(\zeta)\sum_{\alpha\in_\sigma\mathfrak{D}(L_{\delta_1,\sigma})}\Delta_\xi(\gamma_1,\delta_1^\alpha)O_{\sigma,\delta_1^\alpha}(\phi_1)\\ &\overset{(4.13)}{=}\widetilde{\xi}\omega_E^{-1}(\zeta)SO_{\gamma_1}(f_1^\xi) = \omega^{-1}\xi(\mathrm{N}_{E/F}(\zeta))SO_{\gamma_1}(f_1^\xi)\\ &\overset{(4.12)}{=}O_\gamma(f^\xi),\end{aligned}$$

as desired. Notice that if $\zeta = (z\mathbf{1}_2, z'\mathbf{1}_2) = (z\mathbf{1}_2, \sigma(z\sigma(z'))\theta(z\mathbf{1}_2))$ in the realization (4.1), then

$$\widetilde{\xi}(\zeta) = \xi((z\sigma(z'))^{-1}\det(z\mathbf{1}_2)) = \xi(z\sigma(z')^{-1}) = \xi(zz') = \xi(\mathrm{N}_{E/F}(\zeta)).$$

□

4.7 Definition of the base change for $\widetilde{G}$

We are now able to extend the definition of the base change lift for $G = U_{E/F}(2)$ to $\widetilde{G} = GU_{E/F}(2)$. First we review the definition for G from [Rog90, 11.4].

Recall that $\Pi(G(F))_{\eta_u}$ is partitioned into a disjoint union of the finite sets called *L-packets* [Rog90, 11.1]:

$$\Pi(G(F))_{\eta_G} = \coprod_{\Pi\in\Phi(G)_{\eta_u}}\Pi.$$

Here we have written $\Phi(G)_{\eta_u}$ for the set of L-packets of $G(F)$ with the central character η_u. For $\Pi \in \Phi(G)_{\eta_u}$, we define

$$\mathrm{tr}\Pi(f) := \sum_{\pi\in\Pi}\mathrm{tr}\pi(f),\quad f \in \mathcal{H}(G(F),\eta_u).$$

Rogawski proved that this is a stable distribution so that this depends only on the stable orbital integrals $SO_\bullet(f)$ of f.

$\pi_E \in \Pi(L(F))_\eta$ is *σ-stable* if $\sigma(\pi_E) := \pi_E\circ\tilde{\sigma}^{-1}$ is isomorphic to π_E. We write $\Pi(L(F))_\eta^\sigma$ for the subset of σ-stable elements in $\Pi(L(F))_\eta$. For $\pi_E \in \Pi(L(F))_\eta^\sigma$, we take an $L(F)$-module isomorphism $\pi_E(\sigma) : \sigma(\pi_E) \xrightarrow{\sim} \pi_E$ such that $\pi_E(\sigma)^2 = \mathrm{id}$. This extends π_E to an irreducible representation of the non-connected group $L(F) \rtimes \langle\sigma\rangle$. We define the *$\sigma$-twisted character* of π_E by

$$\mathrm{tr}\pi_E(\phi_1)\pi_E(\sigma) = \mathrm{tr}\int_{Z_L(F)\backslash L(F)}\phi_1(g)\pi_E(g)\pi_E(\sigma)\,dg,\quad \phi_1 \in \mathcal{H}(L(F),\eta).$$

This is well-defined only up to $\{\pm 1\}$ due to the possible choices of $\pi_E(\sigma)$.

Definition 4.8. *$\pi_E \in \Pi(L(F))^\sigma_\eta$ is a* stable base change lift *of $\Pi \in \Phi(G)_{\eta_u}$ and $\Pi_\mu \in \Phi(G)_{\eta_u\mu^{-1}}$ (written $\pi_E \simeq \xi_{\mathbf{1}}(\Pi) \simeq \xi_{\mathbf{1}}(\Pi_\mu)$ in § 3.3) if*

$$\mathrm{tr}\pi_E(\phi_1)\pi_E(\sigma) = \mathrm{tr}\Pi(f_1), \quad \mathrm{tr}(\mu(\det)^{-1}\pi_E)(\mu(\det)\phi_1)\pi_E(\sigma) = \mathrm{tr}\Pi_\mu(f_1^\mu) \tag{4.15}$$

hold for any $\phi_1 \in \mathcal{H}(L(F),\eta)$ and its transfer $f_1 \in \mathcal{H}(G(F),\eta_G)$, $f_1^\mu \in \mathcal{H}(G(F),\eta_G\mu^{-1})$ as in Prop. 4.6.

Notice that (4.15) determines the linear span of $\mathrm{tr}\pi_E(\bullet)\pi_E(\sigma)$. Since the irreducible σ-twisted characters are linearly independent, this definition is independent of the choice of $\pi_E(\sigma)$. We have the following trivial extension to $\widetilde{G}$.

Definition 4.9. *$\widetilde{\pi}_E \in \Pi(\widetilde{L}(F))^\sigma_{\omega_E}$ is the* base change lift *of $\widetilde{\pi} \in \Pi(\widetilde{G}(F))_\omega$, $\widetilde{\pi}_\mu \in \Pi(\widetilde{G}(F))_{\omega\mu^{-1}}$ if*

$$\mathrm{tr}\widetilde{\pi}_E(\phi)\widetilde{\pi}_E(\sigma) = \mathrm{tr}\widetilde{\pi}(f), \quad \mathrm{tr}(\widetilde{\mu}^{-1}\widetilde{\pi}_E)(\widetilde{\mu}\phi)\widetilde{\pi}_E(\sigma) = \mathrm{tr}\widetilde{\pi}_\mu(f^\mu) \tag{4.16}$$

hold for any $\phi \in \mathcal{H}(\widetilde{L}(F),\omega_E)$ and its transfer $f \in \mathcal{H}(\widetilde{G}(F),\omega)$, $f^\mu \in \mathcal{H}(\widetilde{G}(F),\omega\mu^{-1})$ as in Prop. 4.7. Here, we have written $\widetilde{\mu}(z,g') := \mu(z)$ as in the proof of Prop. 4.7.

4.8 Restriction from $GU_{E/F}(2)$ to $GL(2)$

In what follows, we calculate the base change lift for $GU_{E/F}(2)$ defined above in the second realization in § 4.1, and compare it with that for $GL(2)$ [Lan80], [AC89]. We start with the restriction of test functions, orbital integrals and representations from $GU_{E/F}(2)$ to $GL(2)$.

We can embed $G' = GL(2)$ and $L' = \mathrm{R}_{E/F}GL(2)$ as the subgroups of $\widetilde{G}$ and $\widetilde{L}$, respectively, by

$$G' \ni g' \longmapsto (1,g')\Delta\mathbb{G}_m \in \mathrm{R}_{E/F}\mathbb{G}_m \times G'/\Delta\mathbb{G}_m \xrightarrow[\sim]{(4.2)} \widetilde{G},$$
$$L' \ni g' \longmapsto (1,g') \in \mathrm{R}_{E/F}\mathbb{G}_m \times L' \xrightarrow[\sim]{(4.4)} \widetilde{L}.$$

In this setting, the characters ω and ω_E become

$$\omega : Z_{\widetilde{G}}(F) \ni (z, z'\mathbf{1}_2)\Delta(F^\times) \overset{(4.2)}{\longmapsto} zz'^{-1}\mathbf{1}_2 \longmapsto \omega(zz'^{-1}) \in \mathbb{C}^\times,$$
$$\omega_E : Z_{\widetilde{L}}(F) \ni (z, z'\mathbf{1}_2) \overset{(4.4)}{\longmapsto} (zz'^{-1}\mathbf{1}_2, \sigma(z')^{-1}\mathbf{1}_2) \longmapsto \omega(z)\omega(\mathrm{N}_{E/F}(z'))^{-1} \in \mathbb{C}^\times.$$

Since $\mathrm{H}^1(F,Z_{G'}) = \mathrm{H}^1(F,Z_{\widetilde{G}}) = \{1\}$ by Hilbert 90 theorem, we have $\widetilde{G}(F)/Z_{\widetilde{G}}(F) \simeq \widetilde{G}_{\mathrm{ad}}(F) \simeq G'_{\mathrm{ad}}(F) \simeq G'(F)/Z_{G'}(F)$, and similarly $\widetilde{L}(F)/Z_{\widetilde{L}}(F) \simeq L'(F)/Z_{L'}(F)$ as p-adic manifolds. These together with the above give the isomorphisms

$$\begin{aligned} \mathcal{H}(\widetilde{G}(F),\omega) \ni & f \overset{\sim}{\longmapsto} \bar{f}' := f|_{G'(F)} \in \mathcal{H}(G'(F),\omega^{-1}|_{F^\times}) \\ \mathcal{H}(\widetilde{L}(F),\omega_E) \ni & \phi \overset{\sim}{\longmapsto} \bar{\phi}' := \phi|_{L'(F)} \in \mathcal{H}(L'(F),\omega^{-1}\circ \mathrm{N}_{E/F}). \end{aligned} \tag{4.17}$$

As for representations, we have the bijections

$$\begin{aligned}\Pi(\widetilde{G}(F))_\omega \ni \widetilde{\pi} = \omega \otimes \pi' \longmapsto \pi' \in \Pi(G'(F))_{\omega^{-1}|_{F^\times}},\\ \Pi(\widetilde{L}(F))_{\omega_E} \ni \omega \otimes \pi'_E \longmapsto \pi'_E \in \Pi(L'(F))_{\omega^{-1}\circ \mathrm{N}_{E/F}},\end{aligned} \tag{4.18}$$

from (4.7), (4.8), respectively.

Let us calculate the relation between orbital integrals corresponding to (4.17). For a maximal torus $\widetilde{T}_G \subset \widetilde{G}$, $T_{G'} := \widetilde{T}_G \cap G'$ is a maximal torus in G', and $\widetilde{T}_G = \mathrm{Cent}(T_{G'}, \widetilde{G})$. Moreover if $\widetilde{T}_G$ is defined over F, we have $\widetilde{T}_G(F) = Z_{\widetilde{G}}(F)T_{G'}(F)$ from $\widetilde{G}(F) = Z_{\widetilde{G}}(F)G'(F)$. $\gamma = (z, \gamma')\Delta F^\times \in \widetilde{G}(F)$, $((z, 1)\Delta F^\times \in Z_{\widetilde{G}}(F)$, $\gamma' \in G'(F))$ is regular semisimple if and only if so is γ'. Suppose that this is the case. Then $\widetilde{G}_\gamma = Z_{\widetilde{G}}G'_{\gamma'}$. Since $\mathrm{H}^1(F, G'_{\gamma'})$ is trivial, we have an isomorphism $\widetilde{G}_\gamma(F)\backslash\widetilde{G}(F) \simeq (\widetilde{G}_\gamma\backslash\widetilde{G})(F) \simeq (G'_{\gamma'}\backslash G')(F) \simeq G'_{\gamma'}(F)\backslash G'(F)$ given by the second projection. Hence we have

$$\begin{aligned}O_\gamma(f) =& \int_{\widetilde{G}_\gamma(F)\backslash\widetilde{G}(F)} \omega(z)^{-1} f(x^{-1}(1, \gamma')x)\, \frac{dx}{dt}\\ =&\omega(z)^{-1} O_{\gamma'}(\bar{f}')\end{aligned} \tag{4.19}$$

in the notation of (4.17). Here the invariant measures dx, dt on $G'(F)$, $G'_{\gamma'}(F)$ are the transports of those on the second components of $\widetilde{G}(F)$, $\widetilde{G}_\gamma(F)$, respectively.

Similarly $\delta = \zeta(1, \delta') \in \widetilde{L}(F)$, $(\zeta \in Z_{\widetilde{L}}(F)$, $\delta' \in L'(F))$ is σ-regular if and only if so is δ'. If this is the case, we define the σ-orbital integral of $\bar{\phi}' \in \mathcal{H}(L'(F), \omega^{-1} \circ \mathrm{N}_{E/F})$ at such δ' by

$$O_{\sigma,\delta'}(\bar{\phi}') := \int_{I'_{\delta',\sigma}(F)\backslash L'(F)} \bar{\phi}'(g^{-1}\delta'\sigma(g))\, \frac{dg}{dt},$$

where $I'_{\delta',\sigma} = L'_{\delta',\sigma}Z_{L'}$. The measures are the second components of the measures on corresponding subgroups of $\widetilde{L}(F)$ in § 4.6. Then we have

$$\begin{aligned}O_{\sigma,\delta}(\phi) =& \int_{I'_{\delta',\sigma}(F)Z_{\widetilde{L}}(F)\backslash L'(F)Z_{\widetilde{L}}(F)} \omega_E(\zeta)^{-1}\phi(g^{-1}(1, \delta')\sigma(g))\, \frac{dg}{dt}\\ =&\omega_E(\zeta)^{-1} \int_{I'_{\delta',\sigma}(F)\backslash L'(F)} \bar{\phi}'(g^{-1}\delta'\sigma(g))\, \frac{dg}{dt}\\ =&\omega_E(\zeta)^{-1} O_{\sigma,\delta'}(\bar{\phi}').\end{aligned} \tag{4.20}$$

4.9 Relation between the transfers for $GU_{E/F}(2)$ and $GL(2)$

We use the above relations to recapitulate the orbital integral transfer Prop. 4.7. We say that a σ-regular $\delta \in \widetilde{L}(F)$ is *σ-elliptic* if $\mathcal{N}_{E/F}(\delta)$ consists of elliptic elements in $\widetilde{G}(F)$. One can easily check that (see the proof of Lem. 4.4)

$$\mathfrak{D}(\widetilde{L}_{\delta,\sigma}) = \begin{cases}\mathbb{Z}/2\mathbb{Z} & \text{if } \delta \text{ is } \sigma\text{-elliptic},\\ \{1\} & \text{otherwise.}\end{cases}$$

When δ is σ-elliptic, we take a $G_{\gamma_1}(E)$-valued 1-cocycle t_σ on $\Gamma_{E/F}$ whose class $\alpha \in \mathrm{H}^1(F,\widetilde{G}_\gamma) \simeq \mathfrak{D}(\widetilde{L}_{\delta,\sigma})$ generates $\mathfrak{D}(\widetilde{L}_{\delta,\sigma})$. As was explained at the end of § 4.5, we have

$$\mathfrak{O}_\sigma^{st}(\delta) = \mathfrak{O}_\sigma(\delta) \sqcup \mathfrak{O}_\sigma(\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta)$$

in this case. Note that $\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta \overset{(4.4)}{\mapsto} \zeta\cdot(\det t_\sigma^{-1}, \mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta')$ in the present realization. Writing $\zeta = (z, z'\mathbf{1}_2)$, we have $\delta = (z, z'\delta')$, $\gamma = (z, \mathrm{N}_{E/F}(z')\gamma')\Delta F^\times$, and the formulae (4.14) read

$$\begin{aligned}
&\xi(z)\omega_E(\zeta)^{-1}\big(O_{\sigma,\delta'}(\bar{\phi}') + \omega\xi^{-1}(\det t_\sigma)O_{\sigma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta'}(\bar{\phi}')\big)\\
&\qquad=\xi(z)\omega_E(\zeta)^{-1}O_{\sigma,\delta'}(\bar{\phi}') + \xi(z\det t_\sigma^{-1})\omega_E(\zeta(\det t_\sigma^{-1},1))^{-1}O_{\sigma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta'}(\bar{\phi}')\\
&\qquad\overset{(4.20)}{=}\Delta_\xi(\gamma,\delta)O_{\sigma,\delta}(\phi) + \Delta_\xi(\gamma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta)O_{\sigma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta}(\phi)\\
&\qquad=O_\gamma(f^\xi) = O_{(z\mathrm{N}_{E/F}(z')^{-1},\gamma')}(f^\xi) \overset{(4.19)}{=} \omega\xi^{-1}(z^{-1}\mathrm{N}_{E/F}(z'))O_{\gamma'}(\bar{f}^{\xi\prime})\\
&\qquad=\xi(z)\omega_E(\zeta)^{-1}O_{\gamma'}(\bar{f}^{\xi\prime}),
\end{aligned}$$

where we have written $\xi = \mathbf{1}$ or μ as in the proof of Prop. 4.7. Since $\det t_\sigma \in F^\times \setminus \mathrm{N}_{E/F}(E^\times)$, this becomes

$$\begin{aligned}
O_{\sigma,\delta'}(\bar{\phi}') + \omega(\det t_\sigma)O_{\sigma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta'}(\bar{\phi}') &= O_{\gamma'}(\bar{f}'),\\
O_{\sigma,\delta'}(\bar{\phi}') - \omega(\det t_\sigma)O_{\sigma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta'}(\bar{\phi}') &= O_{\gamma'}(\bar{f}^{\mu\prime})
\end{aligned}\tag{4.21}$$

To compare the base change lift of $\widetilde{G}$ and G', we first compare the orbital integral transfer for $\widetilde{G}$ (4.21) with that for G'. The transfer for G' and L' can be stated as follows.

Proposition 4.10 (Prop. I.3.1 in [AC89]). *(i) For $\phi' \in \mathcal{H}(L'(F))$, there exists $f' \in \mathcal{H}(G'(F))$ such that*

$$O_{\gamma'}(f') = \begin{cases} O_{\sigma,\delta'}(\phi') & \text{if } \gamma' \in \mathcal{N}_{E/F}(\delta'),\\ 0 & \text{otherwise}\end{cases}$$

holds for any regular semisimple $\gamma' \in G'(F)$.
(ii) Conversely for any $f' \in \mathcal{H}(G'(F))$, there is $\phi' \in \mathcal{H}(L'(F))$ satisfying $O_{\sigma,\delta'}(\phi') = O_{\mathcal{N}_{E/F}(\delta')}(f')$ for any σ-regular $\delta \in L'(F)$.

Notice that this is stated for $\mathcal{H}(G'(F))$ while the statements for $\widetilde{G}$ is always for $\mathcal{H}(G'(F),\omega^{-1}|_{F^\times})$. These two are related by the surjections

$$\mathcal{H}(G'(F)) \ni f' \longmapsto \bar{f}'(x) := \int_{Z_{G'}(F)} f'(zg)\omega^{-1}(z)\,dz \in \mathcal{H}(G'(F),\omega^{-1}|_{F^\times}),$$

$$\mathcal{H}(L'(F)) \ni \phi' \longmapsto \bar{\phi}'(g) := \int_{Z_{L'}(F)} \phi'(zg)\omega^{-1}(\mathrm{N}_{E/F}(z))\,dz \in \mathcal{H}(L'(F),\omega^{-1}\circ\mathrm{N}_{E/F}).$$

Thus Fubini's theorem gives a simple passage for the ordinary orbital integrals:

$$\begin{aligned}
O_{\gamma'}(\bar f') &= \int_{G'_{\gamma'}(F)\backslash G'(F)} \int_{Z_{G'}(F)} f'(zx^{-1}\gamma' x)\omega^{-1}(z)\,dz\,\frac{dx}{dt} \\
&= \int_{Z_{G'}(F)} \int_{G'_{\gamma'}(F)\backslash G'(F)} f'(x^{-1}z\gamma' x)\omega^{-1}(z)\,\frac{dx}{dt}\,dz \\
&= \int_{Z_{G'}(F)} O_{\gamma'}(f')\omega^{-1}(z)\,dz.
\end{aligned} \tag{4.22}$$

The counterpart for twisted orbital integrals is a little more complicated. We have from $L'_{\delta',\sigma}\cap Z_{L'} = Z_{L'}^{\mathrm{Ad}(\delta')\circ\tilde\sigma} = Z_{G'}$ the bijection

$$L'_{\delta',\sigma}(F)Z_{L'}(F)\backslash L'(F)\times Z_{L'}(F)/Z_{G'}(F)\ni (g,z)\overset{\sim}{\longmapsto} gz^{-1}\in L'_{\delta',\sigma}(F)\backslash L'(F).$$

Using this we have

$$\begin{aligned}
O_{\sigma,\delta'}(\bar\phi') &= \int_{L'_{\delta',\sigma}(F)Z_{L'}(F)\backslash L'(F)} \int_{Z_{L'}(F)} \phi'(zg^{-1}\delta'^{\sigma} g)\omega^{-1}(\mathrm{N}_{E/F}(z))\,dz\,\frac{dg}{dt} \\
&= \int_{Z_{L'}(F)/Z_{L'}(\tilde\sigma,F)} \int_{L'_{\delta',\sigma}(F)Z_{L'}(F)\backslash L'(F)} \\
&\qquad\qquad \int_{Z_{L'}(\tilde\sigma,F)} \phi'(g^{-1}\bar z z(\sigma)\delta'^{\sigma} g)\,dz(\sigma)\,\frac{dg}{dt}\omega^{-1}(\mathrm{N}_{E/F}(\bar z))\,d\bar z \\
&= \int_{Z_{L'}(F)/Z_{L'}(\tilde\sigma,F)} \int_{L'_{\delta',\sigma}(F)Z_{L'}(F)\backslash L'(F)} \\
&\qquad \int_{Z_{L'}(F)/Z_{G'}(F)} \phi'(zg^{-1}\bar z\delta'^{\sigma}(z^{-1}g))\,dz\,\frac{dg}{dt}\,\omega^{-1}(\mathrm{N}_{E/F}(\bar z))\,d\bar z \\
&= \int_{Z_{L'}(F)/Z_{L'}(\tilde\sigma,F)} \int_{L'_{\delta',\sigma}(F)\backslash L'(F)} \phi'(g^{-1}\bar z\delta'^{\sigma} g)\,\frac{dg}{dt}\,\omega^{-1}(\mathrm{N}_{E/F}(\bar z))\,d\bar z \\
&= \int_{Z_{L'}(F)/Z_{L'}(\tilde\sigma,F)} O_{\sigma,\bar z\delta'}(\phi')\omega^{-1}(\mathrm{N}_{E/F}(\bar z))\,d\bar z
\end{aligned}$$

Here we have used the isomorphism $Z_{L'}(F)/Z_{G'}(F)\ni z\overset{\sim}{\mapsto} z^{\sigma}z^{-1}\in Z_{L'}(\tilde\sigma,F)$. Suppose $\phi'\in\mathcal{H}(L'(F))$ and $f'\in\mathcal{H}(G'(F))$ are as in Prop. 4.10. Then this becomes

$$\begin{aligned}
O_{\sigma,\delta'}(\bar\phi') &= \int_{Z_{L'}(F)/Z_{L'}(\tilde\sigma,F)} O_{\mathrm{N}_{E/F}(\bar z)\mathcal{N}_{E/F}(\delta')}(f')\,\omega^{-1}(\mathrm{N}_{E/F}(\bar z))\,d\bar z \\
&= \int_{\mathrm{N}_{E/F}(E^\times)} O_{z\mathcal{N}_{E/F}(\delta')}(f')\omega^{-1}(z)\,dz.
\end{aligned}$$

This combined with (4.22) yields

$$\begin{aligned}
\frac{1}{2}(O_{\gamma'}(\bar f') + O_{\gamma'}(\bar f^{\mu\prime})) &= \int_{F^\times} O_{z\gamma'}(f')\omega^{-1}(z)\frac{1+\omega_{E/F}(z)}{2}\,dz \\
&= O_{\sigma,\delta'}(\bar\phi')
\end{aligned} \tag{4.23}$$

for $\gamma' \in \mathcal{N}_{E/F}(\delta')$. Also we note that $\mathcal{N}_{E/F}(\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta')$ contains

$$\begin{aligned}\mathrm{N}_{E/F}(t_\sigma^{-1}\delta'^*) =&(t_\sigma{}^\sigma t_\sigma)^{-1}\gamma' = (t_\sigma\theta_2(t_\sigma)^{-1})^{-1}\gamma' = (t_\sigma(\det t_\sigma^{-1}\cdot t_\sigma)^{-1})^{-1}\gamma'\\ =&\det t_\sigma^{-1}\cdot\gamma'\end{aligned}$$

(recall the cocycle condition $t_\sigma\theta(t_\sigma) = 1$). Hence we have

$$\begin{aligned}\omega(\det t_\sigma)O_{\sigma,\mathrm{Ad}(g^{-1})t_\sigma^{-1}\cdot\delta'}(\bar\phi') = \omega(\det t_\sigma)&\int_{\mathrm{N}_{E/F}(E^\times)} O_{z\det t_\sigma\cdot\gamma'}(f')\omega^{-1}(z)\,dz\\ =&\omega(\det t_\sigma)\int_{F^\times\backslash\mathrm{N}_{E/F}(E^\times)} O_{z\gamma'}(f')\omega^{-1}(z\det t_\sigma)\,dz\\ =&\int_{F^\times\backslash\mathrm{N}_{E/F}(E^\times)} O_{z\gamma'}(f')\omega^{-1}(z)\,dz\\ =&\frac{1}{2}(O_{\gamma'}(\bar f') - O_{\gamma'}(\bar f^{\mu\prime})).\end{aligned}\tag{4.24}$$

We conclude that (4.23) plus (4.24) and (4.23) minus (4.24) yield the first and second formulae in (4.21), respectively.

4.10 Comparison of the base changes for $GU_{E/F}(2)$ and $GL(2)$

We write $\Pi(L'(F))^\sigma$ for the set of isomorphism classes of irreducible admissible σ-stable representations of $L'(F)$. For $\pi'_E \in \Pi(L'(F))^\sigma$, its σ-twisted character $\mathrm{tr}\pi'_E(\phi')\pi'_E(\sigma)$ is defined modulo a factor ±1. Recall that $\pi'_E \in \Pi(L'(F))^\sigma$ is the *base change lift* of $\pi' \in \Pi(G'(F))$ if

$$\mathrm{tr}\pi'_E(\phi')\pi'_E(\sigma) = \mathrm{tr}\pi'(f')$$

holds for any $\phi' \in \mathcal{H}(L'(F))$, $f' \in \mathcal{H}(G'(F))$ as in Prop. 4.10 (see [AC89, I.6]). Again this is well-defined because of the linear independence of the irreducible twisted characters [*loc.cit.* Lem. I.6.3]. Then we know from [*loc.cit.* Th. I.6.2] that $\omega_{\pi'_E} = \omega_{\pi'}\circ\mathrm{N}_{E/F}$. This allows us to restrict the lifting to the one between $\Pi(G'(F))_{\omega^{-1}|_{F^\times}}$ and $\Pi(L'(F))_{\omega^{-1}\circ\mathrm{N}_{E/F}}$. As for characters, the passage to the modulo center situation is just

$$\mathrm{tr}\pi'_E(\phi')\pi'_E(\sigma) = \mathrm{tr}\pi'_E(\bar\phi')\pi'_E(\sigma), \quad \mathrm{tr}\pi'(f') = \mathrm{tr}\pi'(\bar f').$$

Hence, taking the consideration in § 4.9 into account, we can restate the definition as follows.

Definition 4.11. $\pi'_E \in \Pi(L'(F))_{\omega^{-1}\circ\mathrm{N}_{E/F}}$ *is the* base change lift *of* $\pi' \in \Pi(G'(F))_{\omega^{-1}|_{F^\times}}$ *if*

$$\mathrm{tr}\pi'_E(\bar\phi')\pi'_E(\sigma) = \mathrm{tr}\pi'(\bar f')$$

holds for any $\phi \in \mathcal{H}(\widetilde{L}(F),\omega_E)$ *and* $f \in \mathcal{H}(\widetilde{G}(F),\omega)$ *as in Prop. 4.7.*

Now we can state the first result of this section. Recall that π' and $\omega_{E/F}(\det)\pi'$ share the same base change lift [AC89, Prop. 6.8].

Theorem 4.12. *Suppose $\pi'_E \in \Pi(L'(F))_{\omega^{-1}\circ \mathrm{N}_{E/F}}$ is the base change lift of π', $\omega_{E/F}(\det)\pi' \in \Pi(G'(F))_{\omega^{-1}|_{F^\times}}$. Then the (stable) base change of $\widetilde{\pi} = \omega \otimes \pi' \in \Pi(\widetilde{G}(F))_\omega$ and $\widetilde{\pi}_\mu = \omega\mu^{-1} \otimes \omega_{E/F}(\det)\pi' \in \Pi(\widetilde{G}(F))_{\omega\mu^{-1}}$ is $\widetilde{\pi}_E := \omega \otimes \pi'_E \in \Pi(\widetilde{L}(F))_{\omega_E}$.*

Proof. As before, we write $\xi = \mathbf{1}_{E^\times}$ or μ. We have from Def. 4.11 that

$$\begin{aligned}\mathrm{tr}\widetilde{\pi}_E(\phi)\widetilde{\pi}(\sigma) =&\mathrm{tr}\int_{Z_{\widetilde{L}}(F)\backslash \widetilde{L}(F)} \phi(g)\widetilde{\pi}_E(g)\widetilde{\pi}_E(\sigma)\,dg \\ =&\mathrm{tr}\int_{Z_{L'}(F)\backslash L'(F)} \bar{\phi}'(g)\pi'_E(g)\pi'_E(\sigma)\,dg\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}(\widetilde{\mu}^{-1}\widetilde{\pi}_E)(\widetilde{\mu}\phi)\widetilde{\pi}_E(\sigma) =&\mathrm{tr}\int_{Z_{\widetilde{L}}(F)\backslash \widetilde{L}(F)} (\widetilde{\mu}\phi)(g)\widetilde{\mu}^{-1}\widetilde{\pi}_E(g)\widetilde{\pi}_E(\sigma)\,dg \\ =&\mathrm{tr}\int_{Z_{L'}(F)\backslash L'(F)} \bar{\phi}'(g)\pi'_E(g)\pi'_E(\sigma)\,dg\end{aligned}$$

equal

$$\begin{aligned}\mathrm{tr}\pi'(\bar{f}') =&\mathrm{tr}\int_{Z_{G'}(F)\backslash G'(F)} \bar{f}'(g)\pi'(g)\,dg \\ =&\mathrm{tr}\int_{Z_{\widetilde{G}}(F)\backslash \widetilde{G}(F)} f(g)\widetilde{\pi}(g)\,dg = \mathrm{tr}\widetilde{\pi}(f),\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\omega_{E/F}(\det)\pi'((\bar{f}^{\mu})') =&\mathrm{tr}\int_{Z_{G'}(F)\backslash G'(F)} (\bar{f}^{\mu})'(g)\omega_{E/F}(\det)\pi'(g)\,dg \\ =&\mathrm{tr}\int_{Z_{\widetilde{G}}(F)\backslash \widetilde{G}(F)} f^{\mu}(g)\widetilde{\pi}_\mu(g)\,dg = \mathrm{tr}\widetilde{\pi}_\mu(f^\mu),\end{aligned}$$

for $\phi \in \mathcal{H}(\widetilde{L}(F), \omega_E)$ and $f \in \mathcal{H}(\widetilde{G}(F), \omega)$, $f^\mu \in \mathcal{H}(\widetilde{G}(F), \omega\mu^{-1})$ as in Prop. 4.7. The theorem follows. □

4.11 Restriction from $GU_{E/F}(2)$ to $U_{E/F}(2)$

Here we deduce the restriction rule for $GU_{E/F}(2)$ to $U_{E/F}(2)$ from Th. 4.12. It follows from the definition of L-packets of $G(F)$ [Rog90, 11.1] that, for $\widetilde{\pi} \in \Pi(\widetilde{G}(F))_\omega$, there exists a unique L-packet $\Pi(\widetilde{\pi}) \in \Phi(G)_{\eta_u}$ such that

$$\widetilde{\pi}|_{G(F)} \simeq \bigoplus_{\pi\in\Pi(\widetilde{\pi})} \pi.$$

Also if we fix ω satisfying $\omega|_{G_1(F)} = \eta_u$, $\widetilde{\pi} \in \Pi(\widetilde{G}(F))_\omega$ is uniquely determined by $\pi' = \widetilde{\pi}|_{G'(F)} \in \Pi(G'(F))_{\omega^{-1}|_{F^\times}}$. The following lemma describes $\Pi(\widetilde{\pi})$ in terms of ω and π'.

Lemma 4.13. *Suppose* $\pi_E \in \Pi(L(F))_\eta$ *is the stable base change lift of* $\Pi \in \Phi(G)_{\eta_u}$ *and* $\Pi_\mu \in \Phi(G)_{\eta_u\mu^{-1}}$. *If we write* $\Pi = \Pi(\widetilde{\pi})$ *for some* $\widetilde{\pi} \in \Pi(\widetilde{G}(F))_\omega$ *as above, then the base change lift of* $\widetilde{\pi}$ *is* $\widetilde{\pi}_E := \omega \otimes \omega(\det)^{-1}\pi_E$ *and* $\Pi_\mu = \Pi(\widetilde{\pi}_\mu)$ *with* $\widetilde{\pi}_\mu := \omega\mu^{-1} \otimes \omega\omega_{E/F}(\det)^{-1}\pi_E \in \Pi(\widetilde{G}(F))_{\omega\mu^{-1}}$.

Proof. It suffices to deduce the base change identity (4.16) for $\widetilde{\pi}$, $\widetilde{\pi}_\mu$, $\widetilde{\pi}_E$ from (4.15) for Π, Π_μ, π_E. We write $\xi = \mathbf{1}_{E^\times}$ or μ as before. We know from (4.8) that $\widetilde{\pi}_E$ and

$$\widetilde{\mu}^{-1}\widetilde{\pi}_E = \omega\mu^{-1} \otimes \omega(\det)^{-1}\pi_E = \omega\mu^{-1} \otimes (\omega\mu^{-1})(\det)^{-1}\mu^{-1}(\det)\pi_E$$

have the restriction π_E and $\mu^{-1}(\det)\pi_E$ to $L(F)$, respectively. Thus we have

$$\begin{aligned}
\mathrm{tr}(\widetilde{\xi}^{-1}\widetilde{\pi}_E)(\widetilde{\xi}\phi)\widetilde{\pi}_E(\sigma) = \mathrm{tr}\int_{Z_{\widetilde{L}}(F)\backslash\widetilde{L}(F)} (\widetilde{\xi}\phi)(g)(\widetilde{\xi}^{-1}\widetilde{\pi}_E)(g)\widetilde{\pi}_E(\sigma)\,dg \\
=&\mathrm{tr}\int_{Z_L(F)\backslash L(F)} (\xi(\det)\phi_1)(g)(\xi(\det)^{-1}\pi_E)(g)\pi_E(\sigma)\,dg \\
=&\mathrm{tr}(\xi(\det)^{-1}\pi_E)(\xi(\det)\phi_1)\pi_E(\sigma) \\
\overset{(4.15)}{=}&\mathrm{tr}\Pi_\xi(f_1^\xi) = \sum_{\pi_\xi\subset\widetilde{\pi}_\xi|_{G(F)}} \mathrm{tr}\pi_\xi(f_1^\xi) = \mathrm{tr}\bigoplus_{\pi_\xi\subset\widetilde{\pi}_\xi|_{G(F)}} \pi_\xi(f_1^\xi) \\
=&\mathrm{tr}\int_{Z_G(F)\backslash G(F)} f_1^\xi(g)\widetilde{\pi}_\xi(g)\,dg
\end{aligned}$$

using the support condition for f^ξ in Prop. 4.7 (i)

$$=\mathrm{tr}\int_{Z_{\widetilde{G}}(F)\backslash\widetilde{G}(F)} f^\xi(x)\widetilde{\pi}_\xi(x)\,dx = \mathrm{tr}\widetilde{\pi}_\xi(f^\xi),$$

as desired. □

From this, we obtain the restriction result.

Corollary 4.14. *The L-packet* $\Pi(\widetilde{\pi})$ *obtained by restricting* $\widetilde{\pi} = \omega \otimes \pi' \in \Pi(\widetilde{G}(F))_\omega$ *to* $G(F)$ *is characterized by the property that its stable base change lift* $\xi_{\mathbf{1}}(\Pi(\widetilde{\pi}))$ *equals* $\omega(\det)\pi'_E$. *Here* π'_E *is the base change lift of* $\pi' \in \Pi(G'(F))_{\omega^{-1}|_{F^\times}}$ *to* $L'(F)$.

Proof. This is immediate from the above lemma and Th. 4.12. We remark that the notation $\xi_{\mathbf{1}}(\Pi)$ makes sense, because Π determines $\widetilde{\pi}_\mu$ and hence Π_μ uniquely. □

5 Local theta correspondence for unitary groups in two variables

To construct the missing members of $\Pi_\psi(G)$ for ψ of the types (2.b), (2.c), (2.d) in Prop. 3.7 and Lem. 3.10, we use the local theta correspondence.

5.1 Weil representations of unitary dual pairs

We begin with a brief review of the construction of Weil representations for unitary dual pairs [Kud94], (see also [HKS96] for the non-archimedean case).

Fix a generator δ of E over F such that $\Delta := \delta^2 \in F^\times$. Let $(W, \langle\, , \,\rangle) = (W_n, \langle\, , \,\rangle_n)$ be the hyperbolic skew-hermitian space

$$W_n = E^{2n}, \quad \langle (x, x'), (y, y')\rangle_n = x^t\sigma(y') - x'^t\sigma(y).$$

The unitary group $G(W) = G(W_n)$ of this space is isomorphic to $G_{2n}(F)$. We shall be concerned with the cases $n = 1$ and 2. If F is p-adic, we have only two isometry classes of 2-dimensional hermitian spaces over E. We take representatives $(V_\pm, (\, , \,)_\pm)$ of these isometry classes to be $V_\pm = E^2$ with

$$(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix})_+ := \delta(\sigma(x_1)y_2 - \sigma(x_2)y_1), \quad (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix})_- := -\sigma(x_1)y_1 + \gamma\sigma(x_2)y_2.$$

Here we have fixed $\gamma \in F^\times \setminus \mathrm{N}_{E/F}(E^\times)$. The unitary group $G(V_+)$ for V_+ is our $G_2(F)$, while $G(V_-)$ is its anisotropic inner form. If $E/F \simeq \mathbb{C}/\mathbb{R}$, we write $(V_{p,q}, (\, , \,)_{p,q}) := (\mathbb{C}^2, I_{p,q})$, ($p + q = 2$, $I_{p,q}$ is as in § 3.4.1) so that $G(V_{p,q}) = G_{p,q}$.

For such $(V, (\, , \,))$ and W, let us introduce an $8n$-dimensional symplectic space

$$\mathbb{W} := V \otimes_E W, \quad \langle\!\langle v \otimes w, v' \otimes w' \rangle\!\rangle := \frac{1}{2}\mathrm{Tr}_{E/F}[(v, v')\sigma(\langle w, w'\rangle)]$$

over F. We have a homomorphism

$$\iota_{V,W} : G(V) \times G_{2n}(F) \ni (h, g) \longmapsto h \otimes g \in Sp(\mathbb{W}).$$

5.1.1 Splitting of the metaplectic 2-cocycle

In §§ 3.3, 3.4, we have fixed a non-trivial additive character ψ_F of F. Choose two maximal isotropic subspaces

$$Y := \{(0, \dots, 0, y_1, \dots, y_n) \in W_n\}, \quad Y' := \{(y'_1, \dots, y'_n, 0, \dots, 0) \in W_n\},$$

dual to each other. These give the Lagrangians $\mathbb{Y} := V \otimes_E Y$, $\mathbb{Y}' := V \otimes_E Y'$ of $\mathbb{W}$. Let $P_n = M_n U_n$ be the Siegel parabolic subgroup of G_{2n} associated to Y:

$$P_n := \mathrm{Stab}(Y, G_{2n}), \quad M_n := \mathrm{Stab}(Y', P_n), \quad U_n := \{g \in P_n \,|\, (g|_Y) = \mathrm{id}_Y\},$$

or explicitly,

$$M_n = \left\{ m_n(a) := \begin{pmatrix} a & \\ & {}^t\tilde{\sigma}(a)^{-1} \end{pmatrix} \,\middle|\, a \in H_n \right\},$$
$$U_n = \left\{ u_n(b) := \begin{pmatrix} \mathbf{1}_n & b \\ & \mathbf{1}_n \end{pmatrix} \,\middle|\, b = {}^t\tilde{\sigma}(b) \in \mathrm{R}_{E/F}\mathbb{M}_n \right\}$$

Recall the metaplectic group $Mp(\mathbb{W})$ of $Sp(\mathbb{W})$:

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow Mp(\mathbb{W}) \longrightarrow Sp(\mathbb{W}) \longrightarrow 1.$$

The Lagrangian $\mathbb{Y}$ specifies a homeomorphism (not a group homomorphism) $Sp(\mathbb{W}) \times \mathbb{C}^1 \xrightarrow{\sim} Mp(\mathbb{W})$ so that the multiplication of $Mp(\mathbb{W})$ is given by

$$(g_1, \varepsilon_1)(g_2, \varepsilon_2) = (g_1 g_2, \varepsilon_1 \varepsilon_2 c_{\mathbb{Y}}(g_1, g_2)), \quad c_{\mathbb{Y}}(g_1, g_2) = \gamma_{\psi_F}(L(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)).$$

Here $L(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_2)$ is the Leray invariant of $(\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_2)$ [RR93, Def. 2.10] and $\gamma_{\psi_F}(L)$ denotes the Weil constant of a quadratic space L over F. Thus we have

$$\int_F \phi(x)\psi_F\big(\frac{ax^2}{2}\big)\, dx = \gamma_{\psi_F}(a)|a|_F^{-1/2} \int_F \widehat{\phi}(x)\psi_F\big(-\frac{x^2}{2a}\big)\, dx, \quad \phi \in \mathcal{S}(F)$$

for the invariant measure dx on F self-dual with respect to ψ_F.

Using the Bruhat decomposition $G_{2n} = \coprod_{r=0}^n P_n w_r P_n$ with

$$w_r = \left(\begin{array}{cc|cc} \mathbf{0}_r & & -\mathbf{1}_r & \\ & \mathbf{1}_{n-r} & & \\ \hline \mathbf{1}_r & & \mathbf{0}_r & \\ & & & \mathbf{1}_{n-r} \end{array}\right),$$

write $g \in G_{2n}(F)$ as

$$g = \begin{pmatrix} a_1 & * \\ & {}^t\sigma(a_1)^{-1} \end{pmatrix} w_r \begin{pmatrix} a_2 & * \\ & {}^t\sigma(a_2)^{-1} \end{pmatrix}.$$

Define $r(g) := r$ and $d(g) := \det(a_1 a_2) \in E^\times/\mathrm{N}_{E/F}(E^\times)$. As before η denotes a character of $E^\times/F^\times$. Also we need Langlands' λ-factor $\lambda(E/F, \psi_F) = \gamma_{\psi_F}(1)/\gamma_{\psi_F}(\Delta)$ for E/F. Now if we set

$$\begin{aligned} \beta_V(g) :&= (\lambda(E/F,\psi_F)^2 \omega_{E/F}(\det V))^{-r(g)} \eta(d(g)) \\ &= \begin{cases} \eta(d(g)) & \text{if } F \text{ is non-archimedean and } V = V_+ \\ (-1)^{r(g)}\eta(d(g)) & \text{if } F \text{ is non-archimedean and } V = V_- \\ i^{(q-p)r(g)}\eta(d(g)) & \text{if } E/F \simeq \mathbb{C}/\mathbb{R}, \end{cases} \end{aligned}$$

then

$$\widetilde{\iota}_{V,W,\eta} : G(V) \times G_{2n}(F) \ni (h, g) \longmapsto (\iota_{V,W}(h, g), \beta_V(g)) \in Mp(\mathbb{W})$$

is a continuous homomorphism which makes the following diagram commute [Kud94, Th. 3.1].

$$\begin{array}{ccc} G(V) \times G_{2n}(F) & \xrightarrow{\widetilde{\iota}_{V,W,\eta}} & Mp(\mathbb{W}) \\ \| & & \downarrow \text{proj} \\ G(V) \times G_{2n}(F) & \xrightarrow{\iota_{V,W}} & Sp(\mathbb{W}) \end{array}$$

5.1.2 Weil representations

The Heisenberg group $\mathcal{H}(\mathbb{W})$ associated to $\mathbb{W}$ is $\mathbb{W} \oplus F$ with the multiplication

$$(w; z)(w'; z') = (w + w'; z + z' + \frac{\langle\langle w, w' \rangle\rangle}{2}).$$

By the Stone-von Neumann theorem, there is a unique irreducible unitary representation ρ_{ψ_F} of $\mathcal{H}(\mathbb{W})$ on which the center F acts by ψ_F. Its underlying smooth representation is realized on $\mathcal{S}(\mathbb{Y}') = \mathcal{S}(V^n)$:

$$\rho_{\psi_F}(y', y; z)\phi(x') = \psi_F\Big(z + \frac{\langle\!\langle 2x' + y', y\rangle\!\rangle}{2}\Big)\phi(x' + y'), \quad \phi \in \mathcal{S}(\mathbb{Y}').$$

This extends uniquely to an irreducible admissible representation ω_{ψ_F} of the metaplectic Jacobi group $Mp(\mathbb{W}) \ltimes \mathcal{H}(\mathbb{W})$. Here the action of $Mp(\mathbb{W})$ on $\mathcal{H}(\mathbb{W})$ is through the $Sp(\mathbb{W})$-action on $\mathbb{W}$. The composite

$$\omega_{V,W,\eta} : G(V) \times G_{2n}(F) \overset{\widetilde{\iota}_{V,W,\eta}}{\longrightarrow} Mp(\mathbb{W}) \overset{\omega_{\psi_F}}{\longrightarrow} U(\mathcal{S}(V^n))$$

is the *Weil representation* of $G(V) \times G_{2n}(F)$ associated to η. It is characterized by the explicit formulae [Kud94, § 5]:

$$\omega_{V,W,\eta}(m_n(a))\phi(v) = \eta(\det a)|\det a|_E\phi(v.a), \quad a \in GL(n, E), \tag{5.1}$$

$$\omega_{V,W,\eta}(u_n(b))\phi(v) = \psi_F(\frac{\mathrm{tr}(v, v)b}{2})\phi(v), \quad b = {}^t\sigma(b) \in \mathbb{M}_n(E), \tag{5.2}$$

$$\omega_{V,W,\eta}(w_r)\phi(v) = (\pm 1)^r\mathcal{F}_{V_\pm,r}\phi(-v_1, v_2), \tag{5.3}$$

$$\omega_{V,W,\eta}(h)\phi(v) = \phi(h^{-1}v), \quad h \in G(V), \tag{5.4}$$

where

$$\mathcal{F}_{V,r}\phi(v_1, v_2) := \int_{V^r} \phi(v', v_2)\psi_E\big(\frac{\mathrm{tr}(v_1, v')}{2}\big)\, dv', \quad \psi_E := \psi_F \circ \mathrm{Tr}_{E/F}.$$

If the base field F is archimedean, we fix a Cartan involution $\theta_{\mathbb{W}}$ on $\mathfrak{sp}(\mathbb{W})$ which induces Cartan involutions $\theta_{p,q}$ and $\theta_{n,n}$ on $\mathfrak{g}_{p,q}$ and $\mathfrak{g}_{n,n}$, respectively. In particular, this determines maximal compact subgroups $\mathbf{K}_{\mathbb{W}} \subset Sp(\mathbb{W})$, $\mathbf{K}_{p,q} \subset G_{p,q} = G(V)$, $\mathbf{K}_{n,n} \subset G_{n,n}$, which are compatible with $\iota_{V,W}$. We write $(\omega_{V,W,\eta}, \mathcal{S}_0(V^n))$ for the *Fock subspace* of $(\omega_{V,W,\eta}, \mathcal{S}(V^n))$, which is the underlying $(\mathfrak{sp}(\mathbb{W}), \widetilde{\mathbf{K}}_{\mathbb{W}})$-module or $(\mathfrak{g}_{p,q} \times \mathfrak{g}_{n,n}, \mathbf{K}_{p,q} \times \mathbf{K}_{n,n})$-module of the Weil representation. When F is non-archimedean, we just put $\mathcal{S}_0(V_n) := \mathcal{S}(V_n)$.

We write $\mathscr{R}(G_{2n}(F), \omega_{V,W,\eta})$ (resp. $\mathscr{R}(G(V), \omega_{V,W,\eta})$) for the set of isomorphism classes of the irreducible admissible representations of $G_{2n}(F)$ (resp. $G(V)$) which appear as quotients of $\omega_{V,W,\eta}$. For $\pi_W \in \mathscr{R}(G_{2n}(F), \omega_{V,W,\eta})$ (resp. $\pi_V \in \mathscr{R}(G(V), \omega_{V,W,\eta})$), we write $\mathcal{S}_0(V^n, \pi_W)$ (resp. $\mathcal{S}_0(V^n, \pi_V)$) for the maximal quotient of $\mathcal{S}_0(V^n)$ on which $G_{2n}(F)$ (resp. $G(V)$) acts by some copy of π_W (resp. π_V). Thus we have an smooth representation $\Theta_\eta(\pi_W, V)$ of $G(V)$ (resp. $\Theta_\eta(\pi_V, W)$ of $G_{2n}(F)$) such that

$$\mathcal{S}_0(V^n, \pi_W) \simeq \Theta_\eta(\pi_W, V) \otimes \pi_W, \quad (\text{resp. } \mathcal{S}_0(V^n, \pi_V) \simeq \pi_V \otimes \Theta_\eta(\pi_V, W)).$$

The following conjecture is proved by R. Howe if F is archimedean [How89] and by Waldspurger if F is non-archimedean and the residual characteristic of F is odd [Wal90].

Conjecture 5.1 (Local Howe duality conjecture). *(i) $\Theta_\eta(\pi_W, V)$ (resp. $\Theta_\eta(\pi_V, W)$) is an admissible (quasisimple if F is archimedean) representation of finite length.*
(ii) $\Theta_\eta(\pi_W, V)$ (resp. $\Theta_\eta(\pi_V, W)$) admits a unique maximal submodule and hence a unique irreducible quotient $\theta_\eta(\pi_W, V)$ (resp. $\theta_\eta(\pi_V, W)$).
(iii) $\pi_W \mapsto \theta_\eta(\pi_W, V)$ and $\pi_V \mapsto \theta_\eta(\pi_V, W)$ are bijections converse to each other between $\mathscr{R}(G_{2n}(F), \omega_{V,W,\eta})$ and $\mathscr{R}(G(V), \omega_{V,W,\eta})$.

In § 5.2, we shall obtain a precise description of this correspondence in the case $n = 2$ and non-archimedean F. Along the way, we also affirm the conjecture in this special case over F of residual characteristic two. We remark that (i) in the conjecture is always valid [MVW87, Ch.3, IV, Th.4] so that $\Theta_\eta(\pi_W, V)$ or $\Theta_\eta(\pi_V, W)$ admits an irreducible quotient. Our tools are the local theta correspondence between the unitary similitude groups studied by M. Harris [Har93] and the restriction rule Cor. 4.14. The corresponding result for $F = \mathbb{R}$ is reviewed in § 5.3.

5.2 Local theta correspondence for non-archimedean $U(2)$

From now on, we assume F is non-archimedean. We first review the definition of the local theta correspondence for the similitude groups from [Har93, § 3].

5.2.1 The group $GU({}_EB)$

Let B be a quaternion algebra over F, so that it is isomorphic to either $\mathbb{M}_2(F)$ or the unique quaternion division algebra D over F. Using the quadratic extension E of F, we fix an isomorphism $D \otimes_F E \simeq \mathbb{M}_2(E)$ so that

$$D = \left\{ X \in \mathbb{M}_2(E) \,\middle|\, \mathrm{Ad}\left(\begin{pmatrix} & 1 \\ \gamma & \end{pmatrix}\right) \circ {}^\sigma X = X \right\} = \left\{ \begin{pmatrix} x & y \\ \gamma^\sigma y & {}^\sigma x \end{pmatrix} \,\middle|\, x,\, y \in E \right\}.$$

Thus $D \simeq_F E^2$ but this is not an E-algebra isomorphism. Using the main involution ι on B, we define a quadratic form $(x, y)_B := \tau_{B/F}(x^\iota \cdot y)/2$ on B, where $\tau_{B/F}$ is the reduced trace. Thus

$$\begin{cases} \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right)_B = \dfrac{ad' + a'd - bc' - b'c}{2} & \text{if } B \text{ splits,} \\ \left(\begin{pmatrix} x & y \\ \gamma^\sigma y & {}^\sigma x \end{pmatrix}, \begin{pmatrix} x' & y' \\ \gamma^\sigma y' & {}^\sigma x' \end{pmatrix}\right)_B = \dfrac{1}{2}\mathrm{Tr}_{E/F}({}^\sigma x x' - \gamma^\sigma y y') & \text{otherwise.} \end{cases}$$

By abuse of notation, we write B and $B^\times$ for the algebraic group which associate to a commutative F-algebra R $B \otimes_F R$ and $(B \otimes_F R)^\times$, respectively. We write $O(B)$ for the orthogonal group of $(B, (\ ,\ )_B)$, and $GO(B)$ for its similitude group:

$$GO(B, R) := \{g \in GL_F(B \otimes_F R) \,|\, (gx, gy)_B = \nu_B(g)(x, y)_B,\ \exists \nu_B(g) \in R^\times\}.$$

We have an isomorphism $\rho : (B^\times \times B^\times)/\Delta\mathbb{G}_m \xrightarrow{\sim} GO(B)^0$ given by

$$\rho(g, g') : B \ni x \longmapsto g x g'^{-1} \in B.$$

This extends to an isomorphism $\rho : (B^\times \times B^\times)/\Delta\mathbb{G}_m \rtimes \langle \epsilon \rangle \xrightarrow{\sim} GO(B)$, where $\epsilon(g, g') := ((g'^\iota)^{-1}, (g^\iota)^{-1})$. Then, writing $\nu_{B/F}$ for the reduced norm on B, we have $\nu_B(\rho(g, g')) = \nu_{B/F}(g g'^{-1})$.

We view E as a subalgebra of B by

$$i : E \ni \begin{cases} x + y\delta & \longmapsto & \begin{pmatrix} x & -y \\ -y\Delta & x \end{pmatrix} & \in \mathbb{M}_2(F) & \text{if } B \text{ splits,} \\ z & \longmapsto & \begin{pmatrix} {}^\sigma z & \\ & z \end{pmatrix} & \in D & \text{otherwise.} \end{cases}$$

If we take the basis

$$(e_1, e_2) := \begin{cases} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right) & \text{if } B \simeq \mathbb{M}_2(F), \\ \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \gamma & 0 \end{pmatrix}\right) & \text{if } B \simeq D \end{cases}$$

of B over E, then writing $v = z_1 e_1 + z_2 e_2$, $v' = z'_1 e_1 + z'_2 e_2$ with $z_i = x_i + y_i\delta$, $z'_i = x'_i + y'_i\delta$, we have

$$\begin{aligned}(v, v')_B =& \left(\begin{pmatrix} y_2 & -y_1 \\ -x_2 & x_1 \end{pmatrix}, \begin{pmatrix} y'_2 & -y'_1 \\ -x'_2 & x'_1 \end{pmatrix}\right)_B = \frac{x_1 y'_2 - y_1 x'_2 - x_2 y'_1 + y_2 x'_1}{2} \\ =& \frac{1}{2}\mathrm{Tr}_{E/F}\Big(\frac{1}{2\delta}\big((x_1 - y_1\delta)(x'_2 + y'_2\delta) - (x_2 - y_2\delta)(x'_1 + y'_1\delta)\big)\Big) \\ =& \frac{1}{2}\mathrm{Tr}_{E/F}\left(({}^\sigma z_1, {}^\sigma z_2)\begin{pmatrix} 0 & (2\delta)^{-1} \\ -(2\delta)^{-1} & 0 \end{pmatrix}\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}\right)\end{aligned}$$

if B splits, and

$$\begin{aligned}(v, v')_B =& \left(\begin{pmatrix} {}^\sigma z_1 & {}^\sigma z_2 \\ \gamma z_2 & z_1 \end{pmatrix}, \begin{pmatrix} {}^\sigma z'_1 & {}^\sigma z'_2 \\ \gamma z'_2 & z'_1 \end{pmatrix}\right)_B = \frac{1}{2}\mathrm{Tr}_{E/F}({}^\sigma z_1 z'_1 - \gamma {}^\sigma z_2 z'_2) \\ =& \frac{1}{2}\mathrm{Tr}_{E/F}\left(({}^\sigma z_1, {}^\sigma z_2)\begin{pmatrix} 1 & 0 \\ 0 & -\gamma \end{pmatrix}\begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}\right)\end{aligned}$$

otherwise. Thus if we write

$$({}_E B, (\ ,\ )_{{}_E B}) := \begin{cases} \left(E^2, \begin{pmatrix} 0 & (2\delta)^{-1} \\ -(2\delta)^{-1} & 0 \end{pmatrix}\right) & \text{if } B = \mathbb{M}_2(F), \\ (V_-, (\ ,\ )_-) & \text{if } B = D, \end{cases}$$

then $(B, (\ ,\ )_B)$ is the quadratic space associated to this hermitian space over E. In particular, the centralizer of $i(E^\times)$ in $GO(B)$ is the unitary similitude group $GU({}_E B)$:

$$\begin{aligned}GU({}_E B) :=& \mathrm{Cent}(i, GO(B)) = \mathrm{Cent}(i, GO(B)^0) \\ =& \{\rho(g, g') \mid g i(z) x g'^{-1} = i(z) g x g'^{-1},\ \forall z \in E^\times,\ x \in B\} \\ =& \{\rho(z, g') \mid z \in \mathrm{Cent}(i, B^\times) = i(E^\times),\ g' \in B^\times\}.\end{aligned}$$

Moreover the above ρ restricts to the analogue for $GU({}_E B)$ of the isomorphism (4.2):

$$GU({}_E B) = (E^\times \times B^\times)/\Delta F^\times \ni (z, g') \overset{\sim}{\longmapsto} z\nu_{B/F}(g')^{-1} g' \in \widetilde{G}(B),$$

where $\widetilde{G}(B)$ is the unitary similitude group of V_+ if $B = \mathbb{M}_2(F)$ and V_- if $B = D$:

$$z(z_1 e_1 + z_2 e_2) g'^{-1} = z'_1 e_1 + z'_2 e_2 \quad \text{if} \quad \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} = z\nu_{B/F}(g')^{-1} g' \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The similitude norm ν composed with this isomorphism is given by $\nu(z, g') = \mathrm{N}_{E/F}(z)\nu_{B/F}(g')^{-1}$.

5.2.2 Dual pairs and Weil representations for similitude groups

Suppose for the moment that E is a quadratic extension of F or F itself, and write σ for the generator of $\Gamma_{E/F}$. Let $(W, \langle\ ,\ \rangle)$ and $(V, (\ ,\ ))$ be skew-hermitian and hermitian space over E, respectively. We have the corresponding similitude groups

$$GU(W) := \{g \in GL_E(W) \,|\, \langle wg, w'g\rangle = \nu(g)\langle w, w'\rangle,\ w,\ w' \in W,\ \exists\nu(g) \in \mathbb{G}_m\},$$
$$GU(V) := \{h \in GL_E(V) \,|\, (hv, hv') = \nu(h)(v, v'),\ v,\ v' \in V,\ \exists\nu(h) \in \mathbb{G}_m\}.$$

Define $R(V, W) := \{(h, g) \in GU(V) \times GU(W) \,|\, \nu(h) = \nu(g)\}$. The first and the second projections $\mathrm{pr}_1 : R(V, W) \to GU(V)$, $\mathrm{pr}_2 : R(V, W) \to GU(W)$ have images of finite index in $GU(V)$ and $GU(W)$, respectively. We have the symplectic space

$$(\mathbb{W}, \langle\!\langle\ ,\ \rangle\!\rangle) := (V \otimes_E W, [E : F]^{-1}\mathrm{Tr}_{E/F}((\ ,\ ) \otimes_E {}^{\sigma}\langle\ ,\ \rangle)).$$

Then $\iota : GU(V) \times GU(W) \ni (h, g) \mapsto h \otimes g \in GL(\mathbb{W})$ sends $R(V, W)$ into $Sp(\mathbb{W})$.

Our first example is the case $E = F$, $(W, \langle\ ,\ \rangle) := (F^2, \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right))$, $(V, (\ ,\ )) = (B, (\ ,\ )_B)$. In this non-archimedean case, pr_i projects $R(B) := R(V, W)$ onto $GO(B)$ and $GL(2, F) = GSp(1, F)$. In what follows, we write $\mathbb{W}$ for the symplectic space associated to this (V, W) unless otherwise stated. As before, we consider the metaplectic group $Mp(\mathbb{W})$ and its Weil representation $(\omega_{\psi_F}, \mathcal{S}(B))$. Put

$$\beta_B(g) := \epsilon_B(\det B, d(g))_F \left(\frac{\gamma_{\psi_F}(1)}{\gamma_{\psi_F}(\det B)}\right)^{r(g)} = \epsilon_B^{r(g)}, \quad g \in SL(2, F)$$

where

$$\epsilon_B := \begin{cases} (-1, -1)_F & \text{if } B \text{ splits}, \\ -(-1, -1)_F & \text{otherwise.} \end{cases}$$

Then [Kud94, Th. 3.1] asserts that

$$\widetilde{\iota} : O(B, F) \times SL(2, F) \ni (h, g) \longmapsto (\iota(h, g), \beta_B(g)) \in Mp(\mathbb{W})$$

is an well-defined homomorphism lifting ι. This yields the Weil representation $(\omega_B := \omega_{\psi_F} \circ \widetilde{\iota}, \mathcal{S}(B))$ of $O(B, F) \times SL(2, F)$. Noting that scalar multiplications do not change the isometry class of $(B, (\ ,\ )_B)$, we deduce the following explicit formulae which characterize $(\omega_B, \mathcal{S}(B))$.

$$\omega_B(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix})\phi(v) = |a|_F^2 \phi(v.a), \quad a \in F^\times \tag{5.5}$$

$$\omega_B(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix})\phi(v) = \psi_F(\frac{b(v, v)_B}{2})\phi(v), \quad b \in F \tag{5.6}$$

$$\omega_B(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix})\phi(v) = \frac{1}{\gamma_{\psi_F}(B)} \int_B \phi(u)\psi_F((u, -v)_B)\, du \tag{5.7}$$

$$\omega_B(h)\phi(v) = \phi(h^{-1}v), \quad h \in O(B, F). \tag{5.8}$$

Here $\gamma_{\psi_F}(B)$ is 1 if B splits and -1 otherwise.

As in [Shi72], we extend this to a representation of $R(V, W)$. Using the embeddings

$$GO(B) \ni h \longmapsto (h, \begin{pmatrix} 1 & \\ & \nu(h) \end{pmatrix}) \in R(V, W)$$

and $SL(2) \ni g \mapsto (1, g) \in R(V, W)$, we have $R(V, W) \simeq GO(B) \ltimes SL(2)$ where h acts on $SL(2)$ by $\mathrm{Ad}((\begin{smallmatrix} 1 & \\ & \nu(h) \end{smallmatrix}))$. Thus we can extend $(\omega_B, \mathcal{S}(B))$ by extending (5.8) to

$$\omega_B(h)\phi(v) := |\nu(h)|_F^{-1}\phi(h^{-1}v), \quad h \in GO(B, F).$$

Also we shall consider the case where E is a quadratic extension of F and W, V are $(W_1, \langle \, , \, \rangle_1)$, $({}_EB, (\, , \,)_{{}_EB})$, respectively. Since we are considering non-archimedean F, $\nu_{B/F}(B^\times) = F^\times$ and $R({}_EB) = R({}_EB, W_1)$ projects onto $\widetilde{G}(B)$ and $\widetilde{G}(F)$. If we write $(\mathbb{W}_{/E}, \langle\!\langle \, , \, \rangle\!\rangle_{/E})$ for the symplectic space associated to $({}_EB, W_1)$, then we have the isomorphism $\mathbb{W}_{/E} = B \otimes_{i,E} E^2 \xrightarrow{\sim} B \otimes_F F^2 = \mathbb{W}$. We know from § 5.2.1 that this is in fact an isometry of symplectic space. We already constructed the Weil representation $(\omega_{B,\eta} = \omega_{{}_EB,W_1,\eta}, \mathcal{S}(B))$ of $G({}_EB) \times G_2(F)$ in § 5.1.2. As in the first example, we have $R({}_EB) \simeq GU({}_EB) \ltimes G_2(F)$ with respect to the action

$$\mathrm{Ad}(h)g = \mathrm{Ad}(\rho(1, \begin{pmatrix} 1 & \\ & \nu(h)^{-1} \end{pmatrix}))g, \quad h \in GU({}_EB), \, g \in G_2(F).$$

We can extend $(\omega_{B,\mathbf{1}}, \mathcal{S}(B))$ to $R({}_EB)$ by setting

$$\omega_{B,\mathbf{1}}(h)\phi(v) := |\nu(h)|_F^{-1}\phi(h^{-1}v).$$

5.2.3 Review of the Shimizu-Jacquet-Langlands correspondence

Our input to deduce sharp informations on local theta correspondences is the Shimizu-Jacquet-Langlands correspondence [JL70], [Shi72]. Jacquet-Langlands showed that there exists a bijection

$$\Pi_{\mathrm{disc}}(GL(2, F)) \ni \pi \longleftrightarrow \pi^B \in \Pi(B^\times),$$

which is characterized by the character formula

$$\mathrm{tr}\pi^B(g') = \begin{cases} \mathrm{tr}\pi(g) & \text{if } B \text{ is split,} \\ -\mathrm{tr}\pi(g) & \text{otherwise,} \end{cases}$$

for any regular semisimple $g \in GL(2, F)$ and $g' \in B^\times$ sharing the eigen values.

Moreover they also constructed the global correspondence. Let k be a number field and write $\mathbb{A}$ for its ring of adeles. For a quaternion algebra B over k, we write S_B for the set of places of k where B is ramified. We write $\mathcal{A}_{\mathrm{cusp}}(GL(2))^B$ for the set of irreducible cuspidal automorphic representations π of $GL(2, \mathbb{A})$ whose local components π_v at $v \in S_B$ are square integrable. Also write $\mathcal{A}_{\mathrm{cusp}}(B^\times)^*$ for the set of irreducible automorphic representations of $B_{\mathbb{A}}^\times := (B \otimes_k \mathbb{A})^\times$ which are not 1-dimensional. Then Jacquet-Langlands proved that

$$\mathcal{A}_{\mathrm{cusp}}(GL(2))^B \ni \pi = \bigotimes_v \pi_v \longleftrightarrow \pi^B = \bigotimes_v \pi_v^B \in \mathcal{A}_{\mathrm{cusp}}(B^\times)^*$$

is an well-defined bijection. Notice that $B_v^\times \simeq GL(2,k_v)$ and $\pi_v^B \simeq \pi_v$ at almost every v.

By taking the restricted tensor product of the local Weil representations $(\omega_{B_v}, \mathcal{S}(B_v))$ (§ 5.2.2), we have the Weil representation $(\omega_B, \mathcal{S}(B_\mathbb{A}))$ of $O(B,\mathbb{A}) \times SL(2,\mathbb{A})$. We extend this to $R(B_\mathbb{A}) = \{(h,g) \in GO(B,\mathbb{A}) \times GL(2,\mathbb{A}) \,|\, \nu(h) = \det g\}$ as in the local case. For each $\phi \in \mathcal{S}(B_\mathbb{A})$, we have the theta kernel

$$\theta_\phi(h,g) := \sum_{\xi \in B} \omega_B(h,g)\phi(\xi), \quad h \in GO(B_\mathbb{A}),\, g \in GL(2,\mathbb{A}).$$

For each irreducible cuspidal representation π of $GL(2,\mathbb{A})$, we write $\mathcal{A}(\pi)$ for its unique realization in the space of automorphic forms on $GL(2,k)\backslash GL(2,\mathbb{A})$ with the central character ω_π. Let $\Theta(\pi, B)$ be the space of automorphic forms

$$GO(B)\backslash GO(B_\mathbb{A}) \ni h \longmapsto \int_{GL(2,k)\mathbb{A}^\times \backslash GL(2,\mathbb{A})} \theta(h,g) f(g)\, dg, \quad \phi \in \mathcal{S}_0(B_\mathbb{A}),\, f \in \mathcal{A}(\pi^\vee)$$

on $GO(B)\backslash GO(B_\mathbb{A})$, where $\mathcal{S}_0(B_\mathbb{A})$ is the $\mathbf{K}$-finite part of $\mathcal{S}(B_\mathbb{A})$ with respect to a maximal compat subgroup $\mathbf{K} \subset GO(B_\mathbb{A})$. Shimizu showed that $\Theta(\pi, B)$ is an irreducible cuspidal representation of $GO(B_\mathbb{A})$ and, at almost all v where $B_v \simeq \mathbb{M}_2(k_v)$, its local component restricted to $GO(B_v)^0$ is isomorphic to $\pi_v \otimes \pi_v^\vee$. Here we use ρ to identify the representations of $GO(B_v)^0$ with those of $B_v \times B_v^\times$ whose restriction to $\Delta(k_v^\times)$ is trivial.

Now we go back to the local (non-archimedean) situation. Each $(\pi, V_\pi) \in \Pi(GL(2,F))$ can be regarded as a representation $(\pi \circ \mathrm{pr}_2, V_\pi)$ of $R(B)$. But since $GL(2,F)$ is not a Howe subgroup of $R(V,W)$, we cannot define $\Theta(\pi, B)$ as in § 5.1.2. Instead we consider the $SL(2,F)$-coinvariant space

$$\Theta(\pi, B) := (\mathcal{S}(B) \otimes V_\pi^\vee)_{SL(2,F)}.$$

In this notation, the v-component of the global $\Theta(\pi, B)$ is apparently $\Theta(\pi_v, B)$. Since both the cusp forms on $GL(2)$ and those on $B^\times$ satisfy the strong multiplicity one theorem, we deduce the following result.

Proposition 5.2 (Shimizu, Harris). *(i) If $B = D$ and $\pi \in \Pi(GL(2,F))$ is not square integrable, $\Theta(\pi, B) = 0$. Otherwise $\Theta(\pi, B)$ is an irreducible admissible representation of $GO(B)$.*
(ii) In the latter case, we have

- *(a) $\Theta(\pi, B)|_{GO(B)^0} \simeq \pi^B \otimes \pi^{B,\vee}$.*
- *(b) If we write Q for the linear form $V_{\pi^B} \otimes V_{\pi^B}^\vee \ni (v, v^\vee) \mapsto \langle v, v^\vee \rangle \in \mathbb{C}$, then ϵ acts as ± 1 and it preserves Q.*

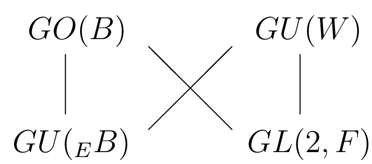
5.2.4 Local theta correspondence for $GU(2)$

Now we deduce the local Howe duality correspondence for $GU({}_EB) \times \widetilde{G}(F)$. As in § 4.2, we write each irreducible representation of $GU({}_EB)$ as $\widetilde{\pi}^B = \omega \otimes \pi^B$. Here, $\omega \in \Pi(E^\times)$, $\pi \in$

$\Pi(GL(2,F))$ with $(\omega|_{F^\times})\omega_\pi = \mathbf{1}$, and $\pi^B \in \Pi(B^\times)$ is the Jacquet-Langlands correspondent of π. Using the Weil representation $(\omega_{B,\mathbf{1}}, \mathcal{S}(B))$, we define

$$\Theta_{\mathbf{1}}(\widetilde{\pi}^B, W_1) := (\mathcal{S}(B) \otimes V^\vee_{\widetilde{\pi}^B})_{U({}_EB)},$$

where $U({}_EB) := \{(z, g') \in GU({}_EB) \,|\, \nu_{B/F}(g') = \mathrm{N}_{E/F}(z)\}$ is the unitary group of $({}_EB, (\ ,\ )_{{}_EB})$. Let us consider the "seesaw dual pair"



Here the vertical lines assign the inclusions while the diagonal ones stand for the dual pairs. For $\widetilde{\pi}^B \in \Pi(GU({}_EB))$ and $\pi' \in \Pi(GL(2,F))$, we have the seesaw duality

$$\mathrm{Hom}_{GL(2,F)}(\Theta_{\mathbf{1}}(\widetilde{\pi}^B, W_1) \otimes \pi'^\vee, \mathbb{C}) \simeq \mathrm{Hom}_{GU({}_EB)}(\widetilde{\pi}^{B,\vee} \otimes \Theta(\pi', B), \mathbb{C}).$$

Thanks to Prop. 5.2, the right hand side becomes

$$\begin{aligned}&\mathrm{Hom}_{GU({}_EB)}((\omega^{-1} \otimes \pi^{B,\vee}) \otimes (\pi'^B \circ i \otimes \pi'^{B,\vee}), \mathbb{C})\\ &\qquad\simeq \mathrm{Hom}_{E^\times}(\omega, \pi'^B \circ i) \otimes \mathrm{Hom}_{GL(2,F)}(\pi^B, \pi'^{B,\vee}).\end{aligned}$$

Thus we have:

(i) $\Theta_{\mathbf{1}}(\widetilde{\pi}^B, W_1)$ is non-trivial if and only if $\pi^{B,\vee} \circ i$ contains ω as a submodule.

(ii) In that case, $\Theta_{\mathbf{1}}(\widetilde{\pi}^B, W_1)$ is isomorphic to $\widetilde{\pi}^\vee = \omega^{-1} \otimes \pi^\vee$.

The condition $\omega \hookrightarrow \pi^{B,\vee} \circ i$ is well understood. That is, the following result was proved by Waldspurger, Tunnel, H. Saito [Sai93], [Tun83], [Wal84].

(iii) $\omega \hookrightarrow \pi^{B,\vee} \circ i$ if and only if

$$\varepsilon(\frac{1}{2}, \pi_E^\vee \otimes \omega^{-1}, \psi_E)\omega_\pi(-1) = \epsilon(B), \quad \epsilon(B) := \begin{cases} 1 & \text{if } B \text{ splits,} \\ -1 & \text{otherwise.} \end{cases}$$

Here $\varepsilon(s, \pi_E \times \omega, \psi_E)$ is the standard ε-factor for the base change lift $\pi_E \in \Pi(GL(2,E))$ of $\pi \in \Pi(GL(2,F))$ twisted by ω [JL70]. Its value at $s = 1/2$ is independent of the choice of a non-trivial character $\psi_E = \psi_F \circ \mathrm{Tr}_{E/F}$ of E.

In the above, we have taken $\eta = \mathbf{1}$ to make the splittings $\widetilde{\iota}_\eta$ for $GU({}_EB) \times \widetilde{G}(F)$ and $\widetilde{\iota}$ for $GO(B) \times GL(2,F)$ compatible. Now let us calculate the effect of η. We know from [HKS96, (1.8)] that $\widetilde{\iota}_\eta|_{G_2(F)} = \eta_{G_2} \cdot \widetilde{\iota}_{\mathbf{1}}|_{G_2(F)}$, where η_{G_2} is multiplied to $\mathbb{C}^1 \subset Mp(\mathbb{W})$. In the realization (4.2), we have

$$\eta_G : G_2(F) \ni (z, g') \longmapsto \eta_u(\det(z\theta_2(g'))) = \eta_u(z\sigma(z)^{-1}) = \eta(z) \in \mathbb{C}^\times$$

so that we can extend this to the character $\eta : \widetilde{G}(F) \ni (z, g') \mapsto \eta(z) \in \mathbb{C}^\times$. Let us define the Weil representation $(\omega_{B,\eta}, \mathcal{S}(B))$ of $R({}_E B)$ to be the twist of $(\omega_{B,\mathbf{1}}, \mathcal{S}(B))$ by $\eta \circ \mathrm{pr}_2 : R({}_E B) \to \mathbb{C}^\times$. This restricts to the Weil representation $(\omega_{B,\eta}, \mathcal{S}(B))$ defined in § 5.1. Since

$$\Theta_\eta(\widetilde{\pi}^B, W_1) = (\eta\omega_{B,\mathbf{1}} \otimes \widetilde{\pi}^{B,\vee})_{U(B)} = \eta\Theta_{\mathbf{1}}(\widetilde{\pi}^B, W_1),$$

we have proved the following.

Proposition 5.3 (Lem. 4.3.3 in [Har93]). *(i) For* $\widetilde{\pi} = \omega \otimes \pi'$, $(\omega \in \Pi(E^\times)$, $\pi' \in \Pi(GL(2,F))$, $\omega_\pi(\omega|_{F^\times}) = \mathbf{1})$, $\Theta_\eta(\widetilde{\pi}, {}_E B) \neq 0$ *if and only if*

$$\varepsilon(1/2, \pi'_E \times \omega\eta^{-1}, \psi_E)\omega_{\pi'}(-1) = \epsilon(B).$$

Here $\epsilon(B)$ *is as in* (3) *above.*
(ii) If this is the case, then $\Theta_\eta(\widetilde{\pi}, {}_E B) \simeq \eta\omega^{-1} \otimes {\pi'}^{B,\vee}$.

5.2.5 Correspondence for unitary groups in two variables

Now we can describe the local theta correspondence for $G(V_\pm) \times G_2(F)$. Recall that each L-packet Π' for $G(V_-)$ consists of the irreducible components of some $\widetilde{\pi}^D = \omega \otimes {\pi'}^D \in \Pi(\widetilde{G}(D))$ [LL79, § 4]. We write this packet $\Pi(\widetilde{\pi}^D)$. As in the $GL(2)$-case, we have the bijection between the set of the discrete L-packets of $G_2(F)$ and that of the L-packets of $G(V_-)$ given by $\Pi(\widetilde{\pi}) \leftrightarrow \Pi(\widetilde{\pi}^D)$. This is independent of the choice of $\widetilde{\pi}$ so that we denote this correspondence by $\Pi \mapsto \mathrm{JL}(\Pi)$. The cardinality of $\Pi(\widetilde{\pi}^D)$ is at most two and it is so if and only if the same is true for $\Pi(\widetilde{\pi})$ [*loc.cit.* Lem. 7.1]. Note that the problematic case treated in [*loc.cit.* Prop. 7.4] does not occur for $G(V_-)$.

Theorem 5.4 (ε-dichotomy for $U(2)$). *We write* $\epsilon(V_\pm) := \pm 1$.
(i) Let Π *be an* L*-packet of* $G_2(F)$ *and* $\pi \in \Pi$. *For a 2-dimensional hermitian space* V *over* E, $\Theta_\eta(\pi, V) \neq 0$ *if and only if*

$$\varepsilon(1/2, \Pi \times \eta^{-1}, \psi_F)\omega_\Pi(-1)\lambda(E/F, \psi_F)^{-2} = \epsilon(V). \tag{5.9}$$

Here $\varepsilon(s, \Pi \times \omega, \psi_F)$ *denotes the standard* ε*-factor for* Π *and* $\omega \in \Pi(E^\times)$ *defined by the Langlands-Shahidi theory [Sha90], and* $\lambda(E/F, \psi_F)$ *is Langlands'* λ*-factor* *(see p. 49)**.*
(ii) For such V *and* $\pi \in \Pi$, $\Theta_\eta(\pi, V)$ *is irreducible, and we have a bijection*

$$\Pi \ni \pi \longmapsto \Theta_\eta(\pi, V) \in \begin{cases} \eta_G \Pi^\vee & \text{if } \epsilon(V) = 1, \\ \eta_G \mathrm{JL}(\Pi)^\vee & \text{otherwise.} \end{cases}$$

Remark 5.5. *This is the* ε-dichotomy property *which was proved by Harris-Kudla-Sweet [HKS96, Th. 6.1] for general unitary dual pairs at least for supercuspidal* π. *But they used the* ε*-factors defined by the doubling method of Piatetskii-Shapiro-Rallis [PSR86]. The comparison conjecture between their* ε*-factor and that defined by the Langlands-Shahidi method (see the introduction of [HKS96]) is not yet established in the present case. This is the reason why we rely on the Shimizu-Jacquet-Langlands correspondence together with the restriction rule for* $GU(2)$ *to* $U(2)$.

Proof. Take $\widetilde{\pi} = \omega \otimes \pi' \in \Pi(\widetilde{G}(F))$ such that $\Pi = \Pi(\widetilde{\pi})$ in the notation of § 4.11. Let B be the quaternion algebra over F associated to the hermitian space V in § 5.2.1. We first remark that the condition (5.9) is equivalent to the condition in Prop. 5.3 (i). In fact, we know from Cor. 4.14 that $\xi_{\mathbf{1}}(\Pi) = \omega(\det)\pi'_E$, so that [Kon01, Prop. 3.2] shows

$$\begin{aligned}\varepsilon(s, \Pi \times \eta^{-1}, \psi_F) =& \lambda(E/F, \psi_F)^2 \varepsilon(s, \xi_{\mathbf{1}}(\Pi) \times \eta^{-1}, \psi_E)\\ =& \lambda(E/F, \psi_F)^2 \varepsilon(s, \pi'_E \otimes \omega\eta^{-1}, \psi_E).\end{aligned}$$

Also (4.2) gives $\omega_{\pi'}(-1) = \omega_{\Pi}(-1)$, hence the conditions are equivalent. If $\Pi(\widetilde{\pi})$ consists of a single π, that is, $\widetilde{\pi}|_{G_2(F)} = \pi$, the theorem is just the restriction of Prop. 5.3.

Otherwise $\Pi(\widetilde{\pi})$ consists of two elements $\pi_\pm$, where we may assume π_+ is χ_2-generic and π_- is not. Notice that $\widetilde{G}(F)/G(F)Z_{\widetilde{G}}(F) \simeq \mathbb{Z}/2\mathbb{Z}$, which is generated by the image of any $g \in \widetilde{G}(F)$ with $\nu(g) \notin \mathrm{N}_{E/F}(E^\times)$. Writing $V_{\widetilde{\pi}}$ for a realization of $\widetilde{\pi}$ and so on, we know from definition that

$$\Theta_\eta(\widetilde{\pi}, {}_EB) \simeq (\mathcal{S}(V) \otimes V_{\pi_+}^\vee)_{G(F)} \oplus (\mathcal{S}(V) \otimes V_{\pi_-}^\vee)_{G(F)}. \tag{5.10}$$

Moreover g transposes the two terms in the right hand side.

Now (5.9) is equivalent to $\Theta_\eta(\widetilde{\pi}, {}_EB) \neq \{0\}$, which in turn amounts to the non-vanishing of at least one of the terms in the right hand side of (5.10). But since g transposes these, this is equivalent to the non-vanishing of both $\Theta_\eta(\pi_+, V)$ and $\Theta_\eta(\pi_-, V)$. We have proved (i). Furthermore, in this case, (5.10) combined with Prop. 5.3 (ii) yields

$$\eta\widetilde{\pi}^{B,\vee}|_{G(V)} \simeq \Theta_\eta(\pi_+, V) \oplus \Theta_\eta(\pi_-, V).$$

Now (ii) follows from $\eta\widetilde{\pi}^{B,\vee}|_{G(V)} \simeq \bigoplus_{\pi_V \in \Pi(\widetilde{\pi}^B)} \eta_G \pi_V^\vee$. □

5.3 The case $F = \mathbb{R}$

If $E/F \simeq \mathbb{C}/\mathbb{R}$, the result analogous to Th. 5.4 was proved by A. Paul for general unitary groups in n-variables [Pau98]. Using the description of the representations in § 3.4, [Pau98, Th. 6.1] specializes to the following.

Proposition 5.6. *For a 2-dimensional hermitian space $(V, (\ ,\ ))$ over $\mathbb{C}$, we set $\epsilon(V) := -\det V$.*
(1) For any $\pi \in \Pi(G_{1,1})$, $\theta_{\eta^n}(\pi, V) \neq \{0\}$ only if

$$\varepsilon(1/2, \pi \times \eta^{-n}, \psi_{\mathbb{R}})\omega_\pi(-1)\lambda(\mathbb{C}/\mathbb{R}, \psi_{\mathbb{R}})^{-2} = \epsilon(V).$$

(2) More precisely, for each $\pi \in \Pi(G_{1,1})$, there exists a unique signature (p, q), $(p+q = 2)$ satisfying the above condition such that $\theta_{\eta^n}(\pi, V_{p,q})$ does not vanish. Such (p, q) and the Howe correspondent $\theta_{\eta^n}(\pi, V_{p,q})$ is explicitly given as follows.

π	*Conditions*	(p,q)	$\theta_{\eta^n}(\pi, V_{p,q})$
$\delta(\frac{a_1}{2};\frac{a_2}{2})$ $\delta(\frac{a_2}{2};\frac{a_1}{2})$	$a_1 > a_2 > n$	$(1,1)$	$\delta(\frac{n-a_1}{2};\frac{n-a_2}{2})$ $\delta(\frac{n-a_2}{2};\frac{n-a_1}{2})$
$\delta(\frac{a_1}{2};\frac{a_2}{2})$ $\delta(\frac{a_2}{2};\frac{a_1}{2})$	$a_1 > n > a_2$	$(2,0)$ $(0,2)$	$\delta(\frac{n-a_2}{2},\frac{n-a_1}{2})$ $\delta(\frac{n-a_2}{2},\frac{n-a_1}{2})$
$\delta(\frac{a_1}{2};\frac{a_2}{2})$ $\delta(\frac{a_2}{2};\frac{a_1}{2})$	$n > a_1 > a_2$	$(1,1)$	$\delta(\frac{n-a_2}{2};\frac{n-a_1}{2})$ $\delta(\frac{n-a_1}{2};\frac{n-a_2}{2})$
$\tau(\frac{a}{2};\frac{a}{2})_\pm$	$a > n$	$(1,1)$	$\tau(\frac{n-a}{2};\frac{n-a}{2})_\pm$
$\tau(\frac{a}{2};\frac{a}{2})_\pm$	$n > a$	$(1,1)$	$\tau(\frac{n-a}{2};\frac{n-a}{2})_\mp$
$J^{G_2}_{\mathbf{B}_2}(\omega_{a,\nu})$	—	$(1,1)$	$J^{G_2}_{\mathbf{B}_2}(\omega_{n-a,\bar\nu})$

Proof. As for (ii), there are two points to translate the reuslts of Paul into our setting. Firstly, the Howe duality correspondence in [Pau98] is the contragredient of ours. Secondly, we use the Weil representation ω_{V,W,η^n} while $\omega_{V,W,\mathbf{1}}$ was used in [Pau98]. According to [HKS96, (1,8)], we have $\omega_{V,W,\eta^n} = \eta^n_{G_2}\omega_{V,W,\mathbf{1}}$ so that $\theta_{\eta^n}(\pi, V) \simeq \theta_{\mathbf{1}}(\eta^{-n}_{G_2}\pi, V)$. Now the table follows from [Pau98, Th. 6.1].

We recall the definition of Artin L and ε-factors at archimedean places. As usual we write $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. We need the following two cases.

(i) If $F = \mathbb{C}$, a quasi-character of $W_{\mathbb{C}} = \mathbb{C}^\times$ is of the form $\omega_{b,\nu}(z) = (z/\bar z)^{b/2}|z|^\nu_{\mathbb{C}}$, ($b \in \mathbb{Z}$, $\nu \in \mathbb{C}$). We have $L(s, \omega_{b,\nu}) := \Gamma_{\mathbb{C}}(s + \nu + |b|/2)$, $\varepsilon(s, \omega_{b,\nu}, \psi_{\mathbb{C}}) = \sqrt{-1}^{|b|}$.

(ii) Each two dimensional irreducible representation ρ of $W_{\mathbb{R}}$ is an induced moudule $\mathrm{ind}^{W_{\mathbb{R}}}_{W_{\mathbb{C}}}(\omega_{b,\nu})$. Then we have $L(s,\rho) = L(s,\omega_{b,\nu})$, $\varepsilon(s,\rho,\psi_{\mathbb{R}}) = \lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})\varepsilon(s,\omega_{b,\nu},\psi_{\mathbb{C}})$, where $\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}}) = \sqrt{-1}$.

Suppose first $\pi = \delta(a_1/2; a_2/2)$. Looking at the Langlands parameter (3.2), we have

$$\begin{aligned}\varepsilon(1/2, \pi\times\eta^{-n}, \psi_{\mathbb{R}})\omega_\pi(-1)\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}})^{-2} &= \sqrt{-1}^{|a_1-n|+|a_2-n|}(-1)^{(a_1+a_2)/2} \\ &= \begin{cases}(-1)^{a_1+a_2-2n} = 1 & \text{if } a_1 > a_2 > n; \\ (-1)^{a_1-n} = -1 & \text{if } a_1 > n > a_2; \\ (-1)^0 = 1 & \text{if } n > a_1 > a_2,\end{cases}\end{aligned}$$

which always equals $\epsilon(V)$ for V indicated by the table. For the other π, we have only to note that

$$\varepsilon(1/2, \pi\times\eta^{-n}, \psi_{\mathbb{R}})\omega_\pi(-1)\lambda(\mathbb{C}/\mathbb{R},\psi_{\mathbb{R}}) = \sqrt{-1}^{2|a-n|}(-1)^a = 1.$$

□

6 Candidates for local A-packets of $U(4)$

In this section, we combine the induction principle in the local theta correspondence [AB95], [Kud86], [MVW87, Ch.3] with the results of § 5, and construct the candidates for the A-packets in the cases in Prop. 3.7 (2.b, c, d), and Lem. 3.10 (2.c, d). We first treat the non-archimedean case.

6.1 Non-archimedean A-packets

Throughout this subsection, we take F to be non-archimedean.

6.1.1 Induction principle in the non-archimedean case

In order to have the induction principle, we need explicit formulae for the Jacquet modules of the Weil representation. This was done by Kudla [Kud86, Th. 2.8] in the symplectic orthogonal dual pair case. We begin with the analogous formula for our unitary dual pairs.

We use the following temporal notation. Let $(\mathbb{H}, (\ ,\ )_{\mathbb{H}})$ be the hyperbolic hermitian plane. We fix an anisotropic hermitian space $(V_0, (\ ,\ )_0)$ of dimension m_0, and write $(V, (\ ,\ )) = (V_r := V_0 \oplus \mathbb{H}^r, (\ ,\ )_r := (\ ,\ )_0 \oplus (\ ,\ )_{\mathbb{H}}^r)$. Thus $(\ ,\ )_r$ is realized as the hermitian matrix

$$q_r := \left(\begin{array}{c|c|c} & \mathbf{1}_r & \\ \hline \mathbf{1}_r & & \\ \hline & & q_0 \end{array}\right)$$

where q_0 is a $m_0 \times m_0$ hermitian matrix representing $(\ ,\ )_0$. Its unitary group is given by

$$G(V_r) := \{g \in GL_E(V_r) \mid \sigma({}^t g) q_r g = q_r\}.$$

For $0 \leq k \leq r$, we have the maximal standard parabolic subgroup $P_k(V) = M_k(V)U_k(V) \subset G(V)$ given by

$$M_k(V) = \left\{ m_k^V(a, g) = \left(\begin{array}{cc|cc|c} a & & & & \\ & A & & B & x \\ \hline & & \sigma({}^t a)^{-1} & & \\ & C & & D & y \\ \hline & z & & w & g_0 \end{array}\right) \;\middle|\; \begin{array}{c} a \in GL(k, E) \\ g = \begin{pmatrix} A & B & x \\ C & D & y \\ z & w & g_0 \end{pmatrix} \in G(V_{r-k}) \end{array} \right\},$$

$$U_k(V) = \left\{ u_k^V(y, z) = \left(\begin{array}{cc|cc|c} \mathbf{1}_k & & & & \\ y'' & \mathbf{1}_{r-k} & & & \\ \hline z - \frac{(y,y)_{r-k}}{2} & -{}^\tau y' & \mathbf{1}_k & -\sigma({}^t y'') & -\sigma({}^t y_0) q_0 \\ y' & & & \mathbf{1}_{r-k} & \\ \hline y_0 & & & & \mathbf{1}_{m_0} \end{array}\right) \;\middle|\; \begin{array}{c} y = \begin{pmatrix} y'' \\ y' \\ y_0 \end{pmatrix} \in V_{r-k}^k, \\ z = -\sigma({}^t z) \in \mathbb{M}_k(E) \end{array} \right\}.$$

The standard parabolic subgroups $P_{\mathbf{p}}$ of $GL(k, E)$ are in one to one correspondence with the partitions $\mathbf{p}$ of k. For each partition $\mathbf{p}$ of k, we write $P_{\mathbf{p}}(V) = M_{\mathbf{p}}(V)U_{\mathbf{p}}(V)$ for the standard parabolic subgroup $m_k^V(\bar{P}_{\mathbf{p}} \times G(V_{r-k}))U_k(V)$ of $P_k(V)$, where $\bar{P}_{\mathbf{p}}$ is the transpose of $P_{\mathbf{p}}$. Similarly, G_{2n} has the maximal standard parabolic subgroups $P_k(W) = M_k(W)U_k(W)$ given by

$$M_k(W) = \left\{ m_k(a, g) := \left(\begin{array}{cc|cc} a & & & \\ & A & & B \\ \hline & & \sigma({}^t a)^{-1} & \\ & C & & D \end{array}\right) \;\middle|\; \begin{array}{c} a \in GL(k, E) \\ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{2(n-k)} \end{array} \right\},$$

$$U_k(W) = \left\{ u_k(y,z) := \left(\begin{array}{cc|cc} \mathbf{1}_k & y'' & z + \frac{\langle y,y\rangle_{n-k}}{2} & y' \\ & \mathbf{1}_{n-k} & \sigma({}^t y') & \\ \hline & & \mathbf{1}_k & \\ & & -\sigma({}^t y'') & \mathbf{1}_{n-k} \end{array} \right) \middle| \begin{array}{c} y = (y', y'') \in W_{n-k}^k \\ z = \sigma({}^t z) \in \mathbb{M}_k(E) \end{array} \right\}.$$

Again for each partition $\mathbf{p}$ of k, we have the standard parabolic subgroup $P_{\mathbf{p}}(W) := m_k(P_{\mathbf{p}} \times G_{n-k})U_k(W)$ of $P_k(W)$.

As in § 5.1.2, we have the Weil representation $\omega_{r,n} := \omega_{V_r,W_n,\eta}$ of $G(V_r) \times G_{2n}(F)$. It suffices to calculate the Jacquet modules along the maximal parabolic subgroups $P_k(V)$ and $P_k(W)$. For a smooth representation (π, V_π) of $G_{2n}(F)$, we write $(\pi_{P_k(W)}, V_{\pi,P_k(W)})$ for its Jacquet module along $P_k(W)$. We also use similar notation for $G(V)$.

Proposition 6.1. *(1) G_{2n}-side. For $0 \le k \le n$, we write $s := \min(k, r)$. There exists a $G(V_r) \times P_k(W)$-invariant filtration*

$$(\omega_{r,n})_{P_k(W)} = S^0 \supset S^1 \supset S^2 \supset \cdots \supset S^s \supset \{0\}$$

such that

$$S_j := S^j/S^{j+1} \simeq I^{G(V)\times M_k(W)}_{P_j(V)\times(P_{(k-j,j)}\times G(W_{n-k}))}(\sigma^j_{r,n} \otimes \omega_{r-j,n-k}).$$

Here $\sigma^j_{r,n}$ is the representation of $M_j(V) \times M_{(k-j,j)}(F)$ on $\mathcal{S}(GL(j,E))$ given by

$$\begin{aligned}\sigma^j_{r,n}\big(m^V_j(a,g), \begin{pmatrix} A' & \\ & a' \end{pmatrix}\big)\phi(x)& \\ =&\eta(\det A' \det a')|\det A'|_E^{\frac{m-j+k-2n}{2}} |\det a' a|_E^{\frac{m-j}{2}} \phi(\sigma({}^t a')xa).\end{aligned}$$

(2) $G(V_r)$-case. For $0 \le k \le r$, we put $t := \min(k, n)$. There is a $P_k(V) \times G_{2n}(F)$-invariant filtration

$$(\omega_{r,n})_{P_k(V)} = T^0 \supset T^1 \supset T^2 \supset \cdots \supset T^t \supset \{0\}$$

such that

$$T_j := T^j/T^{j+1} \simeq I^{M_k(V)\times G_{2n}(F)}_{(\bar{P}_{(k-j,j)}\times G(V_{r-k}))\times P_j(W)}(\tau^j_{r,n} \otimes \omega_{r-k,n-j}).$$

Here $\tau^j_{r,n}$ is the representation of $M_{(k-j,j)}(V) \times M_j(V)$ on $\mathcal{S}(GL(j,E))$ given by

$$\begin{aligned}\tau^j_{r,n}(\begin{pmatrix} A & \\ & a \end{pmatrix}, g), m_j(a', g'))\phi(x)& \\ =&|\det A|_E^{\frac{m-k+j-2n}{2}} \eta(\det a)|\det aa'|_E^{\frac{j-2n}{2}} \phi(\sigma({}^t a)xa').\end{aligned}$$

From this, we can deduce the following special case of the induction principle [Kud94], [MVW87, Ch.3, IV].

Corollary 6.2. *Let $V = V_r$ with $r \le 1$. For $\pi \in \Pi(G(V))$, there exists a unique $1 \le n(\pi) \le \dim_E V$ such that the followings hold.*
(1) If π is supercuspidal, then $\Theta_\eta(\pi, W_n)$ is an irreducible admissible representation of $G_{2n}(F)$ if $n \ge n(\pi)$ and $\Theta_\eta(\pi, W_n) = \{0\}$ otherwise. Further, we have

(i) $\Theta_\eta(\pi, W_{n(\pi)}) \in \Pi_{\mathrm{cusp}}(G_{2n(\pi)}(F))$.

(ii) When $n = n(\pi) + t$, $(t \geq 1)$, $\Theta_\eta(\pi, W_n)_{P_t(W_n)} \simeq \eta(\det)|\det|_E^{\frac{m+t}{2}-n} \otimes \Theta_\eta(\pi, W_{n(\pi)})$.

(2) Otherwise, we have $r = 1$ *and* $\pi_{P_1(V)} \neq \{0\}$. *Again* $\Theta_\eta(\pi, W_n) = \{0\}$ *if* $n < n(\pi)$, *and it is an admissible representation of* $G_{2n}(F)$ *of finite length if* $n \geq n(\pi)$. *In particular it admits an irreducible quotient.*

(i) If $\mathrm{JH}(\pi_{P_1(V)})$ *consists of representations of the form* $|\,|_E^{\frac{m-1}{2}-n} \otimes \pi_0$, *then such* $\pi_0 \in \Pi_{\mathrm{cusp}}(G(V_0))$ *is uniquely determined by* π *and* $\Theta_\eta(\pi_0, W_n)$ *is the unique irreducible quotient* $\theta_\eta(\pi, W_n)$ *of* $\Theta_\eta(\pi, W_n)$.

(ii) Otherwise, for each irreducible quotient π_W *of* $\Theta_\eta(\pi, W_n)$, *there exists* $\omega \otimes \pi_0 \in \mathrm{JH}(\pi_{P_1(V)})$ *such that* π_W *is a quotient of* $I^{G_{2n}}_{P_1(W)}(\eta\sigma(\omega) \otimes \Theta_\eta(\pi_0, W_{n-1}))$.

6.1.2 Local non-archimedean A-packets

Now we are ready to complete the A-packets in the cases Prop. 3.7 (2.b, c, d). For $\pi \in \Pi(G_2(F))$, define

$$\begin{aligned}\epsilon(\pi, \eta) :=& \varepsilon(1/2, \pi \times \eta^{-1}, \psi_F)\omega_\pi(-1)\lambda(E/F, \psi_F)^{-2} \\ =& \varepsilon(1/2, \Pi_E \times \eta^{-1}, \psi_E)\omega_\pi(-1),\end{aligned}$$

and let $(V, (\ ,\ ))$ be a representative of the isometry class of 2-dimensional hermitian spaces over E with $\epsilon(V) = \epsilon(\pi, \eta)$ (cf. Th. 5.4). We also take a representative $(V', (\ ,\ )')$ of the other isomety class of 2-dimensional hermitian spaces.

(2.b) $\psi_{\Pi,\eta}$ We need to find the partner of $J^G_{P_1}(\eta[1] \otimes \pi)$, $\pi \in \Pi$.

(2.b.i) $\pi \in \Pi_{\mathrm{cusp}}(G_2(F))$. Writing

$$\pi_V := \begin{cases} \pi^\vee & \text{if } V \simeq V_+, \\ \mathrm{JL}(\pi)^\vee & \text{otherwise,} \end{cases}$$

we have $\Theta_\eta(\pi, V) \simeq \eta_{G(V)}\pi_V$ (Th. 5.4). Cor. 6.2 (1.ii) implies $\Theta_\eta(\eta_{G(V)}\pi_V, W_2) \simeq J^G_{P_1}(\eta[1] \otimes \pi)$. On the other hand, Th. 5.4 asserts that the Jacquet-Langalnds correspondent $\eta_{G(V')}\pi_{V'} := \eta_{G(V')}\mathrm{JL}(\pi_V)$ satisfies $n(\eta_{G(V')}\pi_{V'}) > 1$ so that Cor. 6.2 (1.i) gives $\Theta_\eta(\eta_{G(V')}\pi_{V'}, W_2) \in \Pi_{\mathrm{cusp}}(G(F))$ (the early lift). We define

$$\Pi_{\psi_{\Pi,\eta}}(G) := \{\pi^+ := J^G_{P_1}(\eta[1] \otimes \pi), \pi^- := \Theta_\eta(\eta_{G(V')}\pi_{V'}, W_2)\}.$$

(2.b.ii) $\pi \simeq \eta'_{G_2}\delta_0^{G_2}$, with $\eta' \neq \eta$. In this case, $\epsilon(\eta'_{G_2}\delta_0^{G_2}, \eta)$ is equal to the value of

$$\varepsilon(s, \eta'\bar{\eta}(\det)\delta_0^{H_2} \times, \psi_E) = \varepsilon(s + 1/2, \eta'\bar{\eta}, \psi_E)\varepsilon(s - 1/2, \eta'\bar{\eta}, \psi_E)\frac{L(1/2 - s, \bar{\eta}'\eta)}{L(s - 1/2, \eta'\bar{\eta})}. \tag{6.1}$$

at $s = 1/2$. If $\eta'\bar{\eta}$ is ramified, this equals (see for example [Tat79])

$$\varepsilon(s + 1/2, \eta'\bar{\eta}, \psi_E)\varepsilon(s - 1/2, \eta'\bar{\eta}, \psi_E) = 1.$$

Otherwise, we have $\eta'\bar{\eta} = |\,|_E^{\pi i/\log q_F}$ (q_F is the cardinality of the residue field of $\mathcal{O}_F$), and

$$\varepsilon(s, \eta'\bar{\eta}(\det)\delta_0^{H_2}\times, \psi_E) = \frac{1+q_F^{1/2-s}}{1+q_F^{s-1/2}} = q_F^{1/2-s}.$$

In any case we have $\epsilon(\eta'_{G_2}\delta_0^{G_2}, \eta) = 1$, so that $V \simeq V_+$ and Th. 5.4 gives

$$\pi_{V_+} := \theta_\eta(\eta'_{G_2}\delta_0^{G_2}, V_+) \simeq (\eta\bar{\eta}')_{G_2}\delta_0^{G_2}.$$

Since $(\pi_{V_+})_{P_1(V_+)} \simeq \eta\bar{\eta}'[-1]$ (note that $P_1(V_+)$ is opposite to $\mathbf{B}_2$, cf. § 6.1.1), we can apply Cor. 6.2 (2.ii) to see that any irreducible quotient π_W of $\Theta_\eta(\pi_{V_+}, W_2)$ is a quotient of $I_{P_1}^G(\eta'[-1] \otimes \eta_{G_2})$. The hermitian dual of this is $I_{P_1}^G(\eta'[1] \otimes \eta_{G_2})$ which has the unique submodule $J_{P_1}^G(\eta[1] \otimes \eta'_{G_2}\delta_0^{G_2})$ [Kon01, Prop.5.6]. Since this last representation is unitarizable [Kon01, Th.6.2], it turns out that this is also the unique irreducible quotient of $I_{P_1}^G(\eta'[-1] \otimes \eta_{G_2})$. Hence

$$\theta_\eta(\pi_{V_+}, W_2) \simeq J_{P_1}^G(\eta[1] \otimes \eta'_{G_2}\delta_0^{G_2}).$$

Now the Jacquet-Langlands correspondent $\pi_{V_-} := \mathrm{JL}(\pi_{V_+})$ is $(\eta\bar{\eta}')_{G(V_-)}$. This is supercuspidal since $G(V_-)$ is anisotropic, and Cor. 6.2 (1.i) shows $\Theta_\eta(\pi_{V_-}, W_2) \in \Pi_{\mathrm{cusp}}(G_4(F))$. We set

$$\Pi_{\psi_{\Pi,\eta}}(G) := \{\pi^+ := J_{P_1}^G(\eta[1] \otimes \eta'_{G_2}\delta_0^{G_2}), \pi^- := \Theta_\eta((\eta\bar{\eta}')_{G(V_-)}, W_2)\}.$$

(2.b.iii) $\pi \simeq \eta_{G_2}\delta_0^{G_2}$. Again we have from (6.1)

$$\epsilon(\pi, \eta) = \lim_{s\to 1/2} \frac{\zeta_E(1/2-s)}{\zeta_E(s-1/2)} = -1,$$

so that $V = V_-$ and $\pi_{V_-} := \theta_\eta(\eta_G\delta_0^{G_2}, V_-) = \mathbf{1}_{G(V_-)}$. Cor. 6.2 (1.ii) tells us $\Theta_\eta(\pi_{V_-}, W_2)_{P_2} \simeq \eta(\det)$, and we can deduce from the proof of [Kon01, Prop.5.8] that

$$\Theta_\eta(\pi_{V_-}, W_2) \simeq J_{P_1}^G(\eta[1] \otimes \eta_{G_2}\delta_0^{G_2}).$$

In this case, the Jacquet-Langlands correspondent of π_{V_-} is $\pi_{V_+} = \delta_0^{G(V_+)}$ (compare this with the case (2.c) below). It follows from Cor. 6.2 that any irreducible quotient π_W of$\Theta_\eta(\pi_{V_+}, W_2)$ is a quotient of $I_{P_1}^G(\eta[-1] \otimes \eta_{G_2})$. Its hermitian dual $I_{P_1}^G(\eta[1] \otimes \eta_{G_2})$ admits a unique irreducible submodule $\eta_{G_4}\tau(\mathbf{1}_{G_2})$ [Kon01, Prop.5.8], which is tempered. Thus this is also the unique irreducible quotient of $I_{P_1}^G(\eta[-1] \otimes \eta_{G_2})$:

$$\theta_\eta(\delta_0^{G(V_+)}, W_2) \simeq \eta_{G_4}\tau(\mathbf{1}_{G_2}).$$

We define

$$\Pi_{\psi_{\Pi,\eta}}(G) := \{\pi^+ := J_{P_1}^G(\eta[1] \otimes \eta_{G_2}\delta_0^{G_2}), \pi^- := \eta_G\tau(\mathbf{1}_{G_2})\}.$$

(2.c) $\psi_{\underline{\eta}}$ We start with η'_{G_2}. As above, we have

$$\epsilon(\eta'_{G_2}, \psi_F) = \varepsilon(1, \eta'\bar{\eta}, \psi_E)\varepsilon(0, \eta'\bar{\eta}, \psi_E) = 1,$$

so that $V = V_+$ and $\pi_{V_+} := \theta_\eta(\eta'_{G_2}, V_+) \simeq (\eta\bar{\eta}')_{G(V_+)}$. Cor. 6.2 (2.ii) combined with [Kon01, Prop.5.6] shows

$$\theta_\eta(\pi_{V_+}, W_2) \simeq J^G_{P_2}(I^{H_2}_{B_2^H}(\eta \otimes \eta')[1]).$$

The Jacquet-Langlands correspondent of π_{V_+} is again $\pi_{V_-} := (\eta\bar{\eta}')_{G(V_-)}$, and we define

$$\Pi_{\psi_{\underline{\eta}}}(G) := \{\pi^+ := J^G_{P_2}(I^{H_2}_{B_2^H}(\eta \otimes \eta')[1]), \pi^- := \Theta_\eta((\eta\bar{\eta}')_{G(V_-)}, W_2)\}.$$

Notice that $\Theta_\eta((\eta\bar{\eta}')_{G(V_-)}, W_2) \simeq \Theta_{\eta'}((\eta'\bar{\eta})_{G(V_-)}, W_2)$.

(2.d) $\psi_{\eta,\underline{\mu}}$**,** $\mu \neq \mu'$**.** This case is completely similar to the case (2.b.i). That is,

- $\pi_{V,\pm} := \Theta_\eta(\pi^{G_2}(\underline{\mu})_\pm, V)$ are exactly the members of the L-packet $\eta_G\Pi_{\varphi_{\underline{\mu}}}(G(V))^\vee = \Pi_{\varphi_{\eta\underline{\mu}^{-1}}}(G(V))$.
- We have $\Theta_\eta(\pi_{V,\pm}, W_2) \simeq J^G_{P_1}(\eta[1] \otimes \pi^{G_2}(\underline{\mu})_\pm)$.
- Writing $\pi_{V',\pm}$ for the members of $\Pi_{\varphi_{\eta\underline{\mu}^{-1}}}(G(V'))$, $\Theta_\eta(\pi_{V',\pm}, W_2)$ are two distinct elements of $\Pi_{\mathrm{cusp}}(G_4(F))$.

Thus we set

$$\Pi_{\psi_{\eta,\underline{\mu}}}(G) := \{\pi^{+,\pm} := J^G_{P_1}(\eta[1] \otimes \pi^{G_2}(\underline{\mu})_\pm), \pi^{-,\pm} := \Theta_\eta(\pi_{V',\pm}, W_2)\}.$$

6.2 Relation with ZASS involution

Here we verify that our definition of A-packets for $G_4(F)$ is consistent with Hiraga's conjecture on ZASS duality.

For the moment let G be a connected reductive group over a p-adic field F and we adopt the notation of § 3.2. We write $\mathcal{R}(G(F))$ for the category of admissible representations of finite length of $G(F)$ and $\mathcal{K}(G(F))$ for its Grothendieck group. For a parabolic subgroup $P = MU$ of G, we have the parabolic induction functor

$$\mathcal{R}(M(F)) \ni \pi \longmapsto I^G_P(\pi) \in \mathcal{R}(G(F)),$$

and the Jacquet functor

$$\mathcal{R}(G(F)) \ni \pi \longmapsto \pi_P \in \mathcal{R}(M(F)).$$

Both of these are exact and we write $i^G_P : \mathcal{K}(M(F)) \to \mathcal{K}(G(F))$ and $r^G_P : \mathcal{K}(G(F)) \to \mathcal{K}(M(F))$ for the homomorphisms defined by them.

In [Zel80, 9.16], Zelevinsky introduced an involution $\mathcal{D}_G$ on $\mathcal{K}(GL(n, F))$. Aubert [Aub95] extended this to general reductive groups by setting

$$\mathcal{D}_G(\pi) := \sum_P (-1)^{\dim(A_M/A_G)} i_P^G \circ r_P^G(\pi).$$

Extending the result of Zelevinsky for $GL(n)$, Aubert [Aub95], [Aub96] and Schneider-Stühler [SS97] proved that this sends irreducible representations to irreducible ones up to signs. (Precisely speaking, Schneider and Stühler adopted a different definition of $\mathcal{D}_G$, and we do not know whether their definition coincides with Aubert's.) We call $\mathcal{D}_G$ *ZASS involution*.

By analogy with the role played by the Curtis-Kawanaka-Alvis duality in the representation theory of reductive groups over finite fields, we expect some relationships between $\mathcal{D}_G$ and the A-packets. In fact, some of these relationships have already been used in [Jan95] and [Mœgl]. The following conjecture is communicated to us by K. Hiraga.

Conjecture 6.3 (Conj. 1 in [Hir]). *$\mathcal{D}_G$ sends A-packets to A-packets. More precisely, if we write an A-parameter $\psi \in \Psi(G)$ in the form:*

$$\psi : W_F \times SU(2) \times SL(2, \mathbb{C}) \ni (w, h, g) \longmapsto \rho(w)\lambda(h)\tau(g) \in {}^L G,$$

then $\mathcal{D}_G$ should send the A-packet $\Pi_\psi(G)$ to $\Pi_{\mathcal{D}_G(\psi)}(G)$, where

$$\mathcal{D}_G(\psi) : W_F \times SU(2) \times SL(2, \mathbb{C}) \ni (w, h, g) \longmapsto \rho(w)\tau(h)\lambda(g) \in {}^L G.$$

Here rational representations of $SL_2(\mathbb{C})$ are identified with those of $SU(2)$ by restriction.

We now examine this for the group $G = G_4$. By using the calculation of Jacquet modules in [Kon01], we can check the following.

Corollary 6.4. *Conj. 6.3 is consistent with our definition of the A-packets for $G_4(F)$.*

Since the proof of this is a lengthy case by case verification, we omit it. Instead we explain two examples.

(1) Cases (2.b.ii) and (2.c) We have constructed $\pi^- \in \Pi_{\text{cusp}}(G(F))$ such that

$$\Pi_\psi(G) = \{J_{P_1}^G(\eta[1] \otimes \eta'_{G_2}\delta_0^{G_2}), \pi^-\}, \quad \Pi_{\psi'}(G) = \{J_{P_1}^G(\eta'[1] \otimes \eta_{G_2}\delta_0^{G_2}), \pi^-\},$$
$$\Pi_{\psi''}(G) = \{J_{P_2}^G(I_{B_2^H}^{H_2}(\eta \otimes \eta')[1]), \pi^-\},$$

where the A-parameters have the restrictions to $\mathcal{A}_E = W_E \times SU(2) \times SL(2, \mathbb{C})$ given by

$$\psi|_{\mathcal{A}_E} = (\eta \otimes \rho_{2,SL(2)}) \oplus (\eta' \otimes \rho_{2,SU(2)}), \quad \psi'|_{\mathcal{A}_E} = (\eta' \otimes \rho_{2,SL(2)}) \oplus (\eta \otimes \rho_{2,SU(2)}),$$
$$\psi''|_{\mathcal{A}_E} = (\eta \otimes \rho_{2,SL(2)}) \oplus (\eta' \otimes \rho_{2,SL(2)}).$$

We can verify $\mathcal{D}_G(J_{P_1}^G(\eta[1] \otimes \eta'_{G_2}\delta_0^{G_2})) = J_{P_1}^G(\eta'[1] \otimes \eta_{G_2}\delta_0^{G_2})$ and $\mathcal{D}_G(\psi) = \psi'$, and hence Conj. 6.3 is valid in this case. On the other hand, since $\mathcal{D}_G(J_{P_2}^G(I_{B_2^H}^{H_2}(\eta \otimes \eta')[1])) = \delta_0^G(\underline{\eta})$

(see § 3.3), Conj. 6.3 asserts that the L-packet associated to the Langlands parameter $\varphi'' := \mathcal{D}_G(\psi'')$ with

$$\varphi''|_{\mathcal{L}_E} = (\eta \otimes \rho_{2,SU(2)}) \oplus (\eta' \otimes \rho_{2,SU(2)})$$

should be $\Pi_{\varphi''}(G) := \{\delta_0^G(\underline{\eta}), \pi^-\}$. In this way, it will be possible to determine certain discrete L-packets of general classical groups by applying the ZASS involution to certain A-packets. This strategy was already taken by Moeglin [Mœgl, 5.6], where she applied $\mathcal{D}_G$ to her quadratic unipotent A-packets. Notice that $\Pi_{\psi''}(G)$ is precisely an analogue of quadratic unipotent A-packet for G.

(2) Case (2.b.iii) $\mathcal{D}_G$ transposes $\eta_G\tau_0^G(\delta_0^{G_2})$, $\eta_G\tau_0^G(\mathbf{1}_{G_2})$ and $J_{P_2}^G(I_{\mathbf{B}_2^H}^{H_2}(\eta\otimes\eta)[1])$, $J_{P_1}^G(\eta[1]\otimes\eta_{G_2}\delta_0^{G_2})$, respectively. The tempered Langlands parameter $\varphi_{\underline{\eta}}$ in this case corresponds to the tempered L-packet $\Pi_{\varphi_{\underline{\eta}}} = \{\eta_G\tau_0^G(\delta_0^{G_2}), \eta_G\tau_0^G(\mathbf{1}_{G_2})\}$. As is conjectured, the A-packet corresponding to $\mathcal{D}_G(\varphi_{\underline{\eta}}) = \psi_{\underline{\eta}}$ is $\{J_{P_2}^G(I_{\mathbf{B}_2^H}^{H_2}(\eta\otimes\eta)[1]), J_{P_1}^G(\eta[1]\otimes\eta_{G_2}\delta_0^{G_2})\}$. (These are not treated in § 6.1 because $\psi_{\underline{\eta}}$ is not elliptic. See Prop. 6.11.) On the other hand, $\psi_{\eta_{G_2}\delta_0^{G_2},\eta}$ is unchanged under $\mathcal{D}_G$ while the two members of the corresponding A-packet are transposed with each other. Also note that the elliptic tempered (but not square-integrable) representation $\eta_G\tau_0^G(\mathbf{1}_{G_2})$ appears in the elliptic A-packet.

6.3 A-packets in the archimedean case

Here we determine the candidates for the A-packets of $G_4(\mathbb{R}) = U(2,2)$ following Adams' conjecture (see the introduction). We first review some general results on the local theta correspondence over $\mathbb{R}$.

6.3.1 Some tools in the Howe duality over $\mathbb{R}$

Adams-Barbasch's induction principle The key tool in our calculation is the induction principle of [AB95]. To state this, we need Weil representations of twisted type II dual pairs. Let X' and Y' be $\mathbb{C}$-vector spaces of dimension r and r', respectively, and write $X := \mathrm{Hom}_{\mathbb{C},\sigma}(X', \mathbb{C})$, $Y := \mathrm{Hom}_{\mathbb{C},\sigma}(Y', \mathbb{C})$. Then we have a symplectic space

$$\mathbb{W}_{X,Y} := \mathbb{Y}'_{X,Y} \oplus \mathbb{Y}_{X,Y},$$
$$\mathbb{Y}'_{X,Y} := X \otimes_{\mathbb{C}} Y', \quad \mathbb{Y}_{X,Y} := X' \otimes_{\mathbb{C}} Y$$

with the symplectic form

$$\langle\!\langle (y_1', y_1), (y_2', y_2) \rangle\!\rangle_{X,Y} := \frac{1}{2}\mathrm{Tr}_{E/F}\mathrm{tr}(y_2 \circ y_1' - y_1 \circ y_2'), \quad y_i \in \mathbb{Y}_{X,Y},\ y_i' \in \mathbb{Y}'_{X,Y}.$$

In particular $GL_{\mathbb{C}}(X')$ and $GL_{\mathbb{C}}(Y')$ form a twisted type II dual pair in $Sp(\mathbb{W}_{X,Y})$:

$$\iota_{X,Y} : GL_{\mathbb{C}}(X') \times GL_{\mathbb{C}}(Y') \ni (a, a') \longmapsto \begin{pmatrix} \sigma({}^t a)^{-1} \otimes a' & \\ & a \otimes \sigma({}^t a')^{-1} \end{pmatrix} \in Sp(\mathbb{W}_{X,Y}).$$

Let $\underline{\xi} = (\xi, \xi')$ be a pair of characters of $E^\times$ trivial on $\mathrm{N}_{E/F}(E^\times)$. Clearly $\iota_{X,Y}$ lifts to a continuous homomorphism

$$\iota_{X,Y,\underline{\xi}} : GL_{\mathbb{C}}(X') \times GL_{\mathbb{C}}(Y') \ni (a, a') \longmapsto (\iota_{X,Y}(a, a'), \xi(\det a)\xi'(\det a')) \in Mp(\mathbb{W}_{X,Y}).$$

The composite $\omega_{X,Y,\underline{\xi}} := \omega_{\psi_{\mathbb{R}}} \circ \iota_{X,Y,\underline{\xi}}$ of the Weil representation $\omega_{\psi_{\mathbb{R}}}$ of $Mp(\mathbb{W}_{X,Y})$ with this is the *Weil representation* of the twisted type II pair $GL_{\mathbb{C}}(X') \times GL_{\mathbb{C}}(Y')$. An explicit formula in the Schrödinger model $\mathcal{S}(\mathbb{Y}'_{X,Y})$ is given by

$$\omega_{X,Y,\underline{\xi}}(a, a')\phi(y) = \xi(\det a)|\det a|_{\mathbb{C}}^{-r'/2}\xi'(\det a')|\det a'|_{\mathbb{C}}^{r/2}\phi(\sigma({}^t a).y.a')$$

for $a \in GL_{\mathbb{C}}(X')$, $a' \in GL_{\mathbb{C}}(Y')$.

Now we can state the induction principle. Recall the setting of § 5.1.

Theorem 6.5 ([Pau98] Th.4.5.5). *We adopt the notation analogous to § 6.1.1. Suppose that admissible representations π_V, π_W of $G(V_{r-k})$, $G_{2(n-k')}(\mathbb{R})$ satisfy*

$$\mathrm{Hom}_{G(V_{r-k})\times G(W_{n-k'})}(\omega_{V_{r-k},W_{n-k'},\underline{\xi}}, \pi_V \otimes \pi_W) \neq 0,$$

and admissible representations π, π' of $GL(k,\mathbb{C})$, $GL(k,\mathbb{C}')$ satisfy

$$\mathrm{Hom}_{GL(k,\mathbb{C})\times GL(k',\mathbb{C})}(\omega_{X,Y,\underline{\xi}}, \pi \otimes \pi') \neq 0.$$

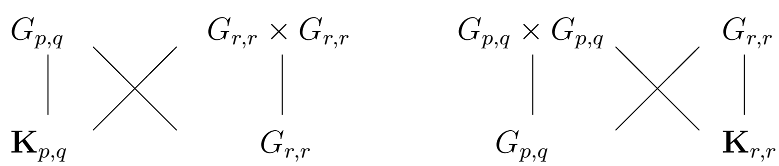
Then we have a non-trivial $G(V_r) \times G_{2n}(\mathbb{R})$-homomorphism

$$\omega_{V,W,\underline{\xi}} \longrightarrow I_{P_k(V)}^{G(V)}(|\det|_{\mathbb{C}}^{t/2}\pi \otimes \pi_V) \otimes I_{P_{k'}(W)}^{G_{2n}}(|\det|_{\mathbb{C}}^{t/2}\pi' \otimes \pi_W).$$

Here, by abuse of notation, we write $\omega_{V,W,\underline{\xi}}$ for the Harish-Chandra module (the Fock model) of the Weil representation $\omega_{V,W,\underline{\xi}}$ and $t := m - k + k' - 2n$.

θ-corresondence for K-types We adopt the notation of 3.4.1. We also need the correspondence of **K**-types [How89], [Li90].

Consider the unitary dual pair $G_{p,q} \times G_{r,r}$. We always assume $m := p + q$ is even. Temporally we use the abbreviation $V = V_{p,q}$, $W = W_r$. We have the seasaw diagrams



Notice that $K_{p,q} \simeq G_{p,0} \times G_{0,q}$ and so on. The (Harish-Chandra modules of) Weil representations of $\mathbf{K}_{p,q} \times (G_{r,r} \times G_{r,r})$ and $(G_{p,q} \times G_{p,q}) \times \mathbf{K}_{r,r}$ are realized on the same Fock model $\mathscr{P}_{V,W,\eta^n}$ of ω_{V,W,η^n}. Recall that $\mathfrak{p}_{p,q} \subset \mathfrak{g}_{p,q}$ is equipped with the decomposition $\mathfrak{p}_{p,q} = \mathfrak{p}^+_{p,q} \oplus \mathfrak{p}^-_{p,q}$. By definition, the spaces of $G_{r,r}$-harmonics and $G_{p,q}$-harmonics are given by

$$\mathscr{H}(G_{r,r}) := \{\phi \in \mathscr{P}_{V,W,\eta^n} \,|\, X.\phi = 0, \forall X \in \mathfrak{p}^-_{r,r} \times \mathfrak{p}^-_{r,r}\},$$
$$\mathscr{H}(G_{p,q}) := \{\phi \in \mathscr{P}_{V,W,\eta^n} \,|\, X.\phi = 0, \forall X \in \mathfrak{p}^-_{p,q} \times \mathfrak{p}^-_{p,q}\},$$

respectively. The intersection $\mathscr{J}_{V,W,\eta^n} := \mathscr{H}(G_{r,r}) \cap \mathscr{H}(G_{p,q})$ is called the space of *joint harmonics*. We recall from [How89, § 3] the followings.

- $\mathscr{J}_{V,W,\eta^n}$ generates $\mathscr{P}_{V,W,\eta^n}$ as $(\mathfrak{g}_{p,q} \times \mathfrak{g}_{r,r}, \mathbf{K}_{p,q} \times \mathbf{K}_{r,r})$-module.
- $\mathscr{J}_{V,W,\eta^n}$ is multiplicity free as $\mathbf{K}_{p,q} \times \mathbf{K}_{r,r}$-module:

$$\mathscr{J}_{V,W,\eta^n} \simeq \bigoplus_i \tau_i \otimes \tau_i', \quad \tau_i \in \Pi(\mathbf{K}_{p,q}),\ \tau_i' \in \Pi(\mathbf{K}_{r,r}). \tag{6.2}$$

  Thus τ_i and τ_i' determine each other. We refer them as *θ-correspondents*. Moreover the $\tau_i \times \tau_i'$-isotypic part of $\mathscr{J}_{V,W,\eta^n}$ consists of homogeneous polynomials, whose degree we denote by $\deg \tau_i = \deg \tau_i'$.

For a $\mathbf{K}_{p,q}$-type τ appearing in $\mathscr{P}_{V,W,\eta^n}$, we call the minimum degree of polynomials in the τ-isotypic subspace of $\mathscr{P}_{V,W,\eta^n}$ its *W-degree*. Such τ is said to be *W-harmonic* if it occurs in the decomposition (6.2). Similarly, V-degree of a $\mathbf{K}_{r,r}$-type and V-harmonic $\mathbf{K}_{r,r}$-types are defined.

Fact 6.6 ([How89] § 4). *For $\pi_V \in \mathscr{R}(G_{p,q}, \omega_{V,W,\eta^n})$, take a $\mathbf{K}_{p,q}$-type τ of the minimum W-degree in it. Then τ is W-harmonic and its θ-correspondent τ' is of minimum V-degree in $\theta_{\eta^n}(\pi_V, W_r)$.*

The θ-corresopndence for $\mathbf{K}$-types is explicitly described as follows. Notice that our correspondence is the contragredient of the one in [Li90] twisted by $\eta_G^n := \det^n$.

Lemma 6.7 ([Li90] § 5, [Pau98]). *Any $\mathbf{K}_{p,q}$-type τ has the unique highest weight of the form*

$$\Lambda_\tau := (\underbrace{a_1, \dots, a_k, 0, \dots, 0, -b_\ell, \dots, -b_1}_{p}; \underbrace{c_1, \dots, c_{k'}, 0, \dots, 0, -d_{\ell'}, \dots, -d_1}_{q}),$$

where $\{a_i\}_{i=1}^k$, $\{b_i\}_{i=1}^\ell$, $\{c_i\}_{i=1}^{k'}$, $\{d_i\}_{i=1}^{\ell'}$ are strictly decreasing sequence of positive integers. Then τ is W_r-harmonic if and only if $k+\ell'$, $k'+\ell \leq r$. Further, its θ-correspondent τ' has the highest weight

$$\begin{aligned}\Lambda_{\tau'} :=& \frac{1}{2}(\overbrace{n+p-q, \dots, n+p-q}^{r}; \overbrace{n+q-p, \dots, n+q-p}^{r}) \\ &+ (\underbrace{b_1, \dots, b_\ell, 0, \dots, 0, -c_{k'}, \dots, -c_1}_{r}; \underbrace{d_1, \dots, d_{\ell'}, 0, \dots, 0, -a_k, \dots, -a_1}_{r}).\end{aligned}$$

θ-correspondence for infinitesimal characters Finally we review the correspondence between infinitesimal characters [Li90, § 5]. We still consider the Weil representation ω_{V,W,η^n} of the dual pair $G_{p,q} \times G_{r,r}$, and assume $2r \geq m$. As usual, we identify an infinitesimal character with an element of $\mathfrak{t}^*$ invariant under the Weyl group.

Lemma 6.8 ([Li90] p. 926). *If $\pi_V \in \Pi(G_{p,q})$ has the infinitesimal character $\lambda = (\lambda_1, \dots, \lambda_m)$, then that of $\theta_{\eta^n}(\pi_V, W_r)$ is given by*

$$\begin{aligned}\lambda' =& \left(\frac{n}{2}, \dots, \frac{n}{2}\right) \\ &- \left(\lambda_1, \dots, \lambda_m, r - \frac{m+1}{2}, \frac{m+1}{2} - r, r - \frac{m-1}{2}, \frac{m-1}{2} - r, \dots, \frac{1}{2}, -\frac{1}{2}\right).\end{aligned}$$

6.3.2 Candidates for real A-packets

Now we calculate the candidates for the A-packets of $G_4(\mathbb{R}) = G_{2,2}$ associated to the parameters of the types (2.c, d) in § 3.4.2.

(2.d) $\psi_{\eta^a,\underline{\mu}^b}$**,** $a \in 2\mathbb{Z}$**,** $a_1 > a_2 \in 2\mathbb{Z}+1$ Following Adams' conjecture, we define

$$\Pi_{\psi_{\eta^a,\underline{\mu}^a}}(G_{2,2}) := \left\{ \begin{array}{cc} \theta_{\eta^a}(\delta(\frac{a-a_1}{2};\frac{a-a_2}{2}),W_2), & \theta_{\eta^a}(\delta(\frac{a-a_2}{2};\frac{a-a_1}{2}),W_2), \\ \theta_{\eta^a}(\delta(\frac{a-a_1}{2},\frac{a-a_2}{2})_{2,0},W_2), & \theta_{\eta^a}(\delta(\frac{a-a_1}{2},\frac{a-a_2}{2})_{0,2},W_2) \end{array} \right\}$$

Here $\delta(\bullet)_{2,0}$, $\delta(\bullet)_{0,2}$ are just $\delta(\bullet)$ viewed as a representation of $G_{2,0}$, $G_{0,2}$, respectively, although $G_{2,0} \simeq G_{0,2}$. Also notice that each of these 4 lifts is non-trivial, since we are in the stable range. Our task is to calculate these local θ-lifts explicitly. We first prove the following.

Lemma 6.9. *Recall the θ-correspondence for $U(2)$ given in Prop. 5.6. We have*

$$J_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(\tfrac{a_1}{2};\tfrac{a_2}{2})) \simeq \begin{cases} \theta_{\eta^a}(\delta(\frac{a-a_1}{2};\frac{a-a_2}{2}),W_2) & \text{if } a_1 > a_2 > a; \\ \theta_{\eta^a}(\delta(\frac{a-a_2}{2},\frac{a-a_1}{2})_{2,0},W_2) & \text{if } a_1 > a > a_2; \\ \theta_{\eta^a}(\delta(\frac{a-a_2}{2};\frac{a-a_1}{2}),W_2) & \text{if } a > a_1 > a_2, \end{cases}$$

$$J_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(\tfrac{a_2}{2};\tfrac{a_1}{2})) \simeq \begin{cases} \theta_{\eta^a}(\delta(\frac{a-a_2}{2};\frac{a-a_1}{2}),W_2) & \text{if } a_1 > a_2 > a; \\ \theta_{\eta^a}(\delta(\frac{a-a_2}{2},\frac{a-a_1}{2})_{0,2},W_2) & \text{if } a_1 > a > a_2; \\ \theta_{\eta^a}(\delta(\frac{a-a_1}{2};\frac{a-a_2}{2}),W_2) & \text{if } a > a_1 > a_2. \end{cases}$$

Proof. We prove the case $a_1 > a_2 > a$ and $J_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(a_1/2;a_2/2))$. The other cases can be treated similarly. Prop. 5.6 implies the existence of a non-zero homomorphism $\omega_{V_{1,1},W_1,\eta^a} \twoheadrightarrow \delta((a-a_1)/2;(a-a_2)/2)\otimes\delta(a_1/2;a_2/2)$. Thus Th. 6.5 gives a non-trivial homomorphism

$$\omega_{V_{1,1},W_2,\eta^a} \longrightarrow \delta\big(\frac{a-a_1}{2};\frac{a-a_2}{2}\big)\otimes I_{P_1}^{G_{2,2}}(\eta^a[-1]\otimes\delta\big(\frac{a_1}{2};\frac{a_2}{2}\big)).$$

This combined with the uniqueness of the Howe quotient shows that $\theta_{\eta^a}(\delta(\frac{a-a_1}{2};\frac{a-a_2}{2}),W_2)$ is an irreducible constituent of $I_{P_1}^{G_{2,2}}(\eta^a[-1]\otimes\delta(\frac{a_1}{2};\frac{a_2}{2}))$, or equivalently, $I_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(\frac{a_1}{2};\frac{a_2}{2}))$. We know from Lem. 6.7 that $\theta_{\eta^a}(\delta(\frac{a-a_1}{2};\frac{a-a_2}{2}),W_2)$ has the minimal $\mathbf{K}_{2,2}$-type with the highest weight

$$\Big(\frac{a_1+1}{2},\frac{a}{2};\frac{a_2-1}{2},\frac{a}{2}\Big).$$

On the other hand, the standard module $I_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(a_1/2;a_2/2))$ corresponds to the "limit character"

$$\lambda+\mu = \Big(\frac{a_1}{2},\frac{a}{2};\frac{a_2}{2},\frac{a}{2}\Big),\quad \nu = \Big(0,\frac{1}{2};0,\frac{1}{2}\Big)$$

in the notation of [Pau98, § 3]. The $\theta_{2,2}$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ associated to $\lambda+\mu$ in [Pau98, Prop. 3.2.7] is given by

$$\mathfrak{l} = \left\{ \left(\begin{array}{cc|cc} * & & & \\ & * & & * \\ \hline & & * & \\ & * & & * \end{array} \right) \in \mathfrak{g} \right\},\quad \mathfrak{u} = \left\{ \left(\begin{array}{cc|cc} & * & * & * \\ & & & \\ \hline & * & & * \\ & & & \end{array} \right) \in \mathfrak{g} \right\}.$$

Hence [*loc.cit.*] asserts that $J_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(a_1/2;a_2/2))$ is the unique constituent of $I_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(a_1/2;a_2/2))$ having the minimal $\mathbf{K}_{2,2}$-type with the highest weight

$$\lambda+\mu+\rho_{\mathfrak{v}\cap\mathfrak{p}_{2,2}}-\rho_{\mathfrak{v}\cap\mathfrak{k}_{2,2}}=\Big(\frac{a_1+1}{2},\frac{a}{2};\frac{a_2-1}{2},\frac{a}{2}\Big).$$

The assertion is proved. □

This assures that our $\Pi_{\psi_{\eta^a,\underline{\mu}^a}}(G_{2,2})$ contains the corresponding L-packet $\Pi'_{\psi_{\eta^a,\underline{\mu}^a}}(G_{2,2})$.

Next we calculate the rest θ-correspondents in the sufficiently regular cases: $a_1>a_2\geq a+3$ or $a_1\geq a+3$, $a-3\geq a_2$ or $a-3\geq a_1>a_2$. This was done for the general dual pair in [Li90]. Thus we only write out the results taking modifications caused by our convention on Weil representation into account.

(i) When $a_1>a_2\geq a+3$, we set

$$\lambda_{1,1}^+:=\frac{1}{2}(a_1-3,a+2;a_2-1,a+2),\quad \lambda_{1,1}^-:=\frac{1}{2}(a_2-1,a+2;a_1-3,a+2),$$
$$\lambda_{2,0}:=\frac{1}{2}(a_1-3,a_2-1;a+2,a+2),\quad \lambda_{0,2}:=\frac{1}{2}(a+2,a+2;a_1-3,a_2-1).$$

(ii) When $a_1\geq a+3$ and $a-3\geq a_2$, we set

$$\lambda_{2,0}:=\frac{1}{2}(a_1-3,a;a,a_2+3),\quad \lambda_{0,2}:=\frac{1}{2}(a,a_2+3;a_1-3,a),$$
$$\lambda_{1,1}^+:=\frac{1}{2}(a_1-3,a_2+3;a,a),\quad \lambda_{1,1}^-:=\frac{1}{2}(a,a;a_1-3,a_2+3).$$

(iii) When $a-3\geq a_1>a_2$, we set

$$\lambda_{1,1}^+:=\frac{1}{2}(a-2,a_1+1;a-2,a_2+3),\quad \lambda_{1,1}^-:=\frac{1}{2}(a-2,a_2+3;a-2,a_1+1),$$
$$\lambda_{2,0}:=\frac{1}{2}(a-2,a-2;a_1+1,a_2+3),\quad \lambda_{0,2}:=\frac{1}{2}(a_1+1,a_2+3;a-2,a-2).$$

In each case, $\lambda_{1,1}^\pm$, $\lambda_{2,0}$ and $\lambda_{0,2}$ determine θ-stable parabolic subalgebras $\mathfrak{q}_{1,1}^+$, $\mathfrak{q}_{2,0}$ and $\mathfrak{q}_{0,2}\subset\mathfrak{g}$ containing $\mathfrak{t}$, respectively, such that

- $\mathfrak{q}$ admits a Levi factor $\mathfrak{l}:=\mathrm{Cent}(\lambda,\mathfrak{g})$;
- $\alpha^\vee(\lambda)\geq 0$, $\forall\alpha\in R(\mathfrak{q},\mathfrak{t})$.

For each of these $(\mathfrak{q},\lambda)$, we have an irreducible unitarizable $(\mathfrak{g},\mathbf{K}_{2,2})$-module $A_{\mathfrak{q}}(\lambda)$ with non-trivial $(\mathfrak{g},\mathbf{K}_{2,2})$-cohomology [KV95], [Vog81], [VZ84].

(i) When $a_1>a_2\geq a+3$, we have

$$\theta_{\eta^a}\Big(\delta\Big(\frac{a-a_1}{2};\frac{a-a_2}{2}\Big),W_2\Big)\simeq A_{\mathfrak{q}_{1,1}^+}(\lambda_{1,1}^+)\simeq J_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(a_1/2;a_2/2)),$$
$$\theta_{\eta^a}\Big(\delta\Big(\frac{a-a_2}{2};\frac{a-a_1}{2}\Big),W_2\Big)\simeq A_{\mathfrak{q}_{1,1}^-}(\lambda_{1,1}^-)\simeq J_{P_1}^{G_{2,2}}(\eta^a[1]\otimes\delta(a_2/2;a_1/2)),$$
$$\theta_{\eta^a}\Big(\delta\Big(\frac{a-a_2}{2},\frac{a-a_1}{2}\Big)_{2,0},W_2\Big)\simeq A_{\mathfrak{q}_{2,0}}(\lambda_{2,0})\simeq\delta\Big(\frac{a_1}{2},\frac{a_2}{2};\frac{a+1}{2},\frac{a-1}{2}\Big),$$
$$\theta_{\eta^a}\Big(\delta\Big(\frac{a-a_2}{2},\frac{a-a_1}{2}\Big)_{0,2},W_2\Big)\simeq A_{\mathfrak{q}_{0,2}}(\lambda_{0,2})\simeq\delta\Big(\frac{a+1}{2},\frac{a-1}{2};\frac{a_1}{2},\frac{a_2}{2}\Big)$$

(ii) When $a_1 \geq a+3$ and $a-3 \geq a_2$, we have

$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}, \frac{a-a_1}{2}\Big)_{2,0}, W_2\big) \simeq A_{\mathfrak{q}_{2,0}}(\lambda_{2,0}) \simeq J_{P_1}^{G_{2,2}}(\eta^a[1] \otimes \delta(a_1/2; a_2/2)),$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}, \frac{a-a_1}{2}\Big)_{0,2}, W_2\big) \simeq A_{\mathfrak{q}_{0,2}}(\lambda_{0,2}) \simeq J_{P_1}^{G_{2,2}}(\eta^a[1] \otimes \delta(a_2/2; a_1/2)),$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_1}{2}; \frac{a-a_2}{2}\Big), W_2\big) \simeq A_{\mathfrak{q}_{1,1}^+}(\lambda_{1,1}^+) \simeq \delta\Big(\frac{a_1}{2}, \frac{a_2}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big),$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}; \frac{a-a_1}{2}\Big), W_2\big) \simeq A_{\mathfrak{q}_{1,1}^-}(\lambda_{1,1}^-) \simeq \delta\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a_1}{2}, \frac{a_2}{2}\Big).$$

(iii) When $a-3 \geq a_1 > a_2$, we have

$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}; \frac{a-a_1}{2}\Big), W_2\big) \simeq A_{\mathfrak{q}_{1,1}^+}(\lambda_{1,1}^+) \simeq J_{P_1}^{G_{2,2}}(\eta^a[1] \otimes \delta(a_1/2; a_2/2)),$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_1}{2}; \frac{a-a_2}{2}\Big), W_2\big) \simeq A_{\mathfrak{q}_{1,1}^-}(\lambda_{1,1}^-) \simeq J_{P_1}^{G_{2,2}}(\eta^a[1] \otimes \delta(a_2/2; a_1/2)),$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}, \frac{a-a_1}{2}\Big)_{2,0}, W_2\big) \simeq A_{\mathfrak{q}_{2,0}}(\lambda_{2,0}) \simeq \delta\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a_1}{2}, \frac{a_2}{2}\Big)$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}, \frac{a-a_1}{2}\Big)_{0,2}, W_2\big) \simeq A_{\mathfrak{q}_{0,2}}(\lambda_{0,2}) \simeq \delta\Big(\frac{a_1}{2}, \frac{a_2}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big).$$

Remark 6.10. *These are precisely the non-tempered cohomological A-packets of Adams-Johnson [AJ87].*

Finally we calculate the Howe correspondents in the singular cases.

(i) When $a_1 > a_2 = a+1$, we need to calculate $\theta_{\eta^a}(\delta(-1/2, (a-a_1)/2)_{2,0}, W_2)$, $\theta_{\eta^a}(\delta(-1/2, (a-a_1)/2)_{0,2}, W_2)$. We deduce from Lemmas 6.7, 6.8 that these have the infinitesimal character $2^{-1}(a_1, n+1, n+1, n-1)$ and the minimal $\mathbf{K}_{2,2}$-types with the highest weights

$$\frac{1}{2}(a_1+1, a+4; a-2, a-2), \quad \frac{1}{2}(a-2, a-2; a_1+1, n+4).$$

Granting the table in Lem. 3.9, we obtain

$$\theta_{\eta^a}\big(\delta\Big(-\frac{1}{2}, \frac{a-a_1}{2}\Big)_{2,0}, W_2\big) \simeq \tau\Big(\frac{a_1}{2}, \frac{a+1}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big)_+$$
$$\theta_{\eta^a}\big(\delta\Big(-\frac{1}{2}, \frac{a-a_1}{2}\Big)_{0,2}, W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a_1}{2}, \frac{a+1}{2}\Big)_-.$$

(ii) There are the following 3 singular cases in the case $a_1 > a > a_2$.

(a) When $a_1 = a+1$, $a-3 \geq a_2$, we have

$$\theta_{\eta^a}\big(\delta\Big(-\frac{1}{2}; \frac{a-a_2}{2}\Big), W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a_2}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big)_+,$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2}; -\frac{1}{2}\Big), W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a+1}{2}, \frac{a_2}{2}\Big)_-.$$

(b) When $a_1 \geq a+3$, $a_2 = a-1$, we have

$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_1}{2};\frac{1}{2}\Big), W_2\big) \simeq \tau\Big(\frac{a_1}{2}, \frac{a-1}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big)_-,$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{1}{2};\frac{a-a_1}{2}\Big), W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a_1}{2}, \frac{a-1}{2}\Big)_+.$$

(c) When $a_1 = a+1$, $a_2 = a-1$, we have

$$\theta_{\eta^a}\big(\delta\Big(-\frac{1}{2};\frac{1}{2}\Big), W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big)_{+-},$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{1}{2};-\frac{1}{2}\Big), W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big)_{-+}.$$

These can be calculated in the same way as in (i).

(iii) When $a_1 = a-1 > a_2$, we have

$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2},\frac{1}{2}\Big)_{2,0}, W_2\big) \simeq \tau\Big(\frac{a+1}{2}, \frac{a-1}{2}; \frac{a-1}{2}, \frac{a_2}{2}\Big)_+,$$
$$\theta_{\eta^a}\big(\delta\Big(\frac{a-a_2}{2},\frac{1}{2}\Big)_{0,2}, W_2\big) \simeq \tau\Big(\frac{a-1}{2}, \frac{a_2}{2}; \frac{a+1}{2}, \frac{a-1}{2}\Big)_-.$$

Again the calculation is similar to the first case.

6.4 Non-elliptic parameters

Here we describe the A-packets associated to non-elliptic parameters listed in Prop. 3.2, § 3.4.2, and study their relation to the local θ-correspondence. There are the following two types of non-elliptic A-parameters for G_4:

(M_1) $\psi^{M_1}_{\omega,\eta}$ whose restriction to $\mathcal{A}_E$ is $\omega \oplus \sigma(\omega)^{-1} \oplus \eta \otimes \rho_2$.

(M_2) $\psi^{M_2}_{\omega}$ whose restriction to $\mathcal{A}_E$ is $\omega \otimes \rho_2 \oplus \sigma(\omega)^{-1} \otimes \rho_2$.

As was remarked in Rem. 3.5, we cannot assume that ω is unitary. Instead we impose the condition (i)$'$ in the remark, which can be stated in the present case as follows.

(M_1) ω in $\psi^{M_1}_{\omega,\eta}$ is either:

- (a) a unitary character with $\omega \neq \sigma(\omega)^{-1}$, or
- (b) $\eta'[s]$, $(0 \leq s < 1)$, where η' can be η, or
- (c) $\omega = \mu$.

(M_2) ω in $\psi^{M_2}_{\omega}$ is either

- (a) a unitary character with $\omega \neq \sigma(\omega)^{-1}$, or
- (b) $\omega = \mu[s]$, $(0 \leq s < 1)$, or
- (c) $\omega = \eta$.

In the case (M_2), $\omega = \eta[s]$, $(0 < s < 1)$ is excluded because the packet $\Pi'_\psi(G)$ contains the representation $J^G_{\mathbf{B}}(\eta[1+s] \otimes \eta[1-s])$ which is not unitarizable. (See [Kon01, Th. 6.2 (1.a)] for the non-archimedean case. The archimedean case can be proved by the same argument.)

The A-packets We first give the list of A-packets.

Proposition 6.11. *(i) The A-packets and other data associated to the A-parameters of type (M_1) are described as follows.*

A-parameter	$S_\psi(G)$	$\Pi_\psi(G)$	*χ-base point*
$\psi^{M_1}_{\omega,\eta}$	$\{\pm 1\}\times\mathbb{C}^\times$	$\{J^G_{P_1}(\eta[1]\otimes I^{G_2}_{\mathbf{B}_2}(\omega))\}$	*unique*
$\psi^{M_1}_{\eta',\eta}$	$\{\pm 1\}\times SL(2,\mathbb{C})$	$\{J^G_{P_1}(\eta[1]\otimes I^{G_2}_{\mathbf{B}_2}(\eta'))\}$	*unique*
$\psi^{M_1}_{\eta'[s],\eta}$, $(0<s<1)$	$\{\pm 1\}\times\mathbb{C}^\times$	$\{J^G_{\mathbf{B}}(\eta[1]\otimes\eta'[s])\}$	*unique*
$\psi^{M_1}_{\mu,\eta}$ *(non-arch.)*	$\{\pm 1\}\times O(2,\mathbb{C})$	$\{J^G_{P_1}(\eta[1]\otimes\tau^{G_2}(\mu)_\pm)\}$	$J^G_{P_1}(\eta[1]\otimes\tau^{G_2}(\mu)_+)$
$\psi^{M_1}_{\mu^b,\eta^a}$ $(E/F\simeq\mathbb{C}/\mathbb{R})$	$\{\pm 1\}\times O(2,\mathbb{C})$	$\{J^G_{P_1}(\eta^a[1]\otimes\tau(\frac{b}{2};\frac{b}{2})_\pm)\}$	$J^G_{P_1}(\eta^a[1]\otimes\tau(\frac{b}{2};\frac{b}{2})_+)$

Here in the first case, ω is a unitary character of $E^\times$ such that $\omega\neq\sigma(\omega)^{-1}$.
(ii) The A-packets and other data associated to the A-parameters of type (M_2) are described as follows.

A-parameter	$S_\psi(G)$	$\Pi_\psi(G)$	*χ-base point*
$\psi^{M_2}_\omega$	$\mathbb{C}^\times$	$\{J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\omega\otimes\sigma(\omega)^{-1})[1])\}$	*unique*
$\psi^{M_2}_\mu$	$SL(2,\mathbb{C})$	$\{J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\mu\otimes\mu)[1])\}$	*unique*
$\psi^{M_2}_\eta$ *(non-arch.)*	$O(2,\mathbb{C})$	$\left\{\begin{array}{c} J^G_{P_1}(\eta[1]\otimes\eta_{G_2}\delta^{G_2}_0), \\ J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta)[1]) \end{array}\right\}$	$J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta\otimes\eta)[1])$
$\psi^{M_2}_{\eta^a}$ $(E/F\simeq\mathbb{C}/\mathbb{R})$	$O(2,\mathbb{C})$	$\left\{\begin{array}{c} J^G_{P_1}(\eta^a[1]\otimes\delta(\frac{a+1}{2};\frac{a-1}{2})), \\ J^G_{P_1}(\eta^a[1]\otimes\delta(\frac{a-1}{2};\frac{a+1}{2})), \\ J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta^a\otimes\eta^a)[1]) \end{array}\right\}$	$J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta^a\otimes\eta^a)[1])$

Here in the first case, ω is either a unitary character of $E^\times$ with $\omega\neq\sigma(\omega)^{-1}$ or of the form $\mu[s]$ with $0<s<1$.

Proof. If F is non-archimedean, this follows immediately from the definition of non-elliptic A-packets Conj. 3.4 (B.2) and results of [Kon01, §§ 5, 6]. Notice that all the representations appeared in the packets are unitarizable.

Next consider the case $E/F=\mathbb{C}/\mathbb{R}$. Everything but the A-packets can be obtained in the same way as in the non-archimedean case. To determine the packets we need the intertwining operator.

We temporally goes back to the notation of § 3.2. Let $P=MU\subset G$ be a standard parabolic subgroup. For $w\in W$ such that $w(M)$ is again a standard Levi subgroup, we write $P(w)=M(w)U(w)$ for the standard parabolic subgroup with the Levi component $M(w):=w(M)$. Take an admissible representation π of finite length of $M(\mathbb{R})$ and $\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$. If $\alpha^\vee(\mathrm{Re}\lambda)>>0$ for any $\alpha^\vee\in\Delta^\vee_P$, the intertwining integral

$$M(w,\pi_\lambda)\phi(g):=\int_{(w(U)\cap U(w))(\mathbb{R})\backslash U(w,\mathbb{R})}\phi(w^{-1}ug)\,du,\quad \phi\in I^G_P(\pi_\lambda).$$

converges absolutely. This extends meromorphically in λ to whole $\mathfrak{a}^*_{M,\mathbb{C}}$, and besides its poles, it defines an intertwining operator $M(w,\pi_\lambda):I^G_P(\pi_\lambda)\to I^G_{P(w)}(w(\pi_\lambda))$. Moreover, if $\pi \in \Pi_{\mathrm{temp}}(M(\mathbb{R}))$ and λ satisfies $\alpha^\vee(\mathrm{Re}\lambda) > 0$, $\forall \alpha^\vee \in \Delta_P^\vee$, $M(w_M, \pi_\lambda)$ is holomorphic and $J_P^G(\pi_\lambda) = \mathrm{Im} M(w_M, \pi_\lambda)$ [Lan89, p. 147], [BW00, IV.4]. Thus, at such λ, $M(w_M, \pi_\lambda)$ is an isomorphism if and only if $I_P^G(\pi_\lambda)$ is irreducible.

Now we return to the proof.

(i) $\psi_{\omega,\eta}^{M_1}$ case. We write $\eta = \eta^a$. Recall [JL70, Th. 6.2] that, writing $\omega_i = \omega_{b_i,\nu_i}$, $(i = 1, 2)$ with $\mathrm{Re}\nu_1 \geq \mathrm{Re}\nu_2$, $I_{\mathbf{B}_2^H}^{H_2}(\omega_1 \otimes \omega_2)$ is irreducible unless $\nu_1 - \nu_2 \pm (b_1 - b_2)/2 \in \mathbb{N}$. We first consider the cases (a), (b) above. One can write $\omega = \omega_{b,\nu}$ with $\nu \in i\mathbb{R} \setminus \{0\}$ in the case (a), and $\omega = \eta^b$ in the case (b) with $s = 0$. In both cases, we have

$$\begin{aligned} J_{P_1}^G(\eta^a[1] \otimes \omega) &= \mathrm{Im} M(w_{M_1}, \eta^a[1] \otimes I_{\mathbf{B}_2^H}^{H_2}(\omega)) \\ &= \mathrm{Im}\big[M(r_1, \omega \otimes \eta^a[-1]) M(r_2, \omega \otimes \eta^a[1]) M(r_1, \eta^a[1] \otimes \omega)\big]. \end{aligned} \tag{6.3}$$

In the case (b) with $s > 0$, we can write $\omega = \eta^b[s]$ so that

$$\begin{aligned} J_{\mathbf{B}}^G(\eta^a[1] \otimes \eta^b[s]) &= \mathrm{Im} M(w_-, \eta^a[1] \otimes \eta^b[s]) \\ &= \mathrm{Im}\big[M(r_1, \eta^b[-s] \otimes \eta^a[-1]) M(r_2, \eta^b[-s] \otimes \eta^a[1]) M(r_1, \eta^a[1] \otimes \eta^b[-s]) \\ &\qquad M(r_2, \eta^a[1] \otimes \eta^b[s])\big]. \end{aligned} \tag{6.4}$$

Since $I_{\mathbf{B}_2^H}^{H_2}(\eta^a[1] \otimes \omega)$, $I_{\mathbf{B}_2^H}^{H_2}(\omega \otimes \eta^a[-1])$ are irreducible, the operators $M(r_1, \eta^a[1] \otimes \omega)$, $M(r_1, \omega \otimes \eta^a[-1])$ are isomorphisms. Similarly, the operators $M(r_2, \eta^a[1] \otimes \eta^b[s])$, $M(r_1, \eta^a[1] \otimes \eta^b[-s])$, $M(r_1, \eta^b[-s] \otimes \eta^a[-1])$ are isomorphisms. Thus we have

$$\begin{aligned} J_{P_1}^G(\eta^a[1] \otimes \omega) &= \mathrm{Im} M(r_2, \omega \otimes \eta^a[1]) = I_{P_1}^G(\omega \otimes \mathrm{Im} M(w_-^{G_2}, \eta^a[1])) \\ &= I_{P_1}^G(\omega \otimes \eta_{G_2}^a), \\ J_{\mathbf{B}}^G(\eta^a[1] \otimes \eta^b[s]) &= \mathrm{Im} M(r_2, \eta^b[-s] \otimes \eta^a[1]) = I_{P_1}^G(\eta^b[-s] \otimes \mathrm{Im} M(w_-^{G_2}, \eta^a[1])) \\ &= I_{P_1}^G(\eta^b[s] \otimes \eta_{G_2}^a), \end{aligned}$$

and this together with Conj. 3.4 (B.2) yields the result. Here in the last line, we can replace $\eta^b[-s]$ by $\eta^b[s]$ by applying w_{M_1} to the inducing representation. Next consider the case (c) $\omega = \mu^b$, $b \in 2\mathbb{Z} + 1$. We know

$$I_{\mathbf{B}}^G(\eta^a[1] \otimes \mu^b) = I_{P_1}^G\big(\eta^a[1] \otimes \tau\big(\frac{b}{2}; \frac{b}{2}\big)_+\big) \oplus I_{P_1}^G\big(\eta^a[1] \otimes \tau\big(\frac{b}{2}; \frac{b}{2}\big)_-\big),$$

so that we have as in the previous cases

$$\begin{aligned} & J_{P_1}^G\big(\eta^a[1] \otimes \tau\big(\frac{b}{2}; \frac{b}{2}\big)_+\big) \oplus J_{P_1}^G\big(\eta^a[1] \otimes \tau\big(\frac{b}{2}; \frac{b}{2}\big)_+\big) \\ &\quad = \mathrm{Im} M\big(w_{M_1}, \eta^a[1] \otimes \tau\big(\frac{b}{2}; \frac{b}{2}\big)_+\big) \oplus \mathrm{Im} M\big(w_{M_1}, \eta^a[1] \otimes \tau\big(\frac{b}{2}; \frac{b}{2}\big)_-\big) \\ &\quad = \mathrm{Im} M\big(w_{M_1}, \eta^a[1] \otimes \mu^b\big) \simeq I_{P_1}^G(\mu^b, \eta_{G_2}^a). \end{aligned}$$

The assertion follows.

(ii) $\psi_\omega^{M_2}$ case. We first note that:

(1) $M(w_-^{G_2}, \omega[s])$ has zero at $s = 1$ if and only if $\omega = \eta^a$ for some $a \in 2\mathbb{Z}$.

(2) In that case, $\mathrm{Im}M(w_-^{G_2},\eta^a[1])=\eta^a_{G_2}$ and

$$\mathrm{Ker}M(w_-^{G_2},\eta^a[1])\simeq\delta\big(\frac{a+1}{2};\frac{a-1}{2}\big)\oplus\delta\big(\frac{a-1}{2};\frac{a+1}{2}\big).$$

Again we begin with the simpler cases (a), (b). Similar argument as in the proof of (i) gives

$$\begin{aligned}&J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\omega\otimes\sigma(\omega)^{-1})[1])\\&=\mathrm{Im}[M(r_2,\omega[-1]\otimes\omega[1])M(r_1,\omega[1]\otimes\omega[-1])M(r_2,\omega[1]\otimes\sigma(\omega)^{-1})].\end{aligned}$$

(1) assures that the first and the last operator in the right hand side are isomorphisms, so that this equals $\mathrm{Im}M(r_1,\omega[1]\otimes\omega[-1])=I^G_{P_2}(\omega(\det))$ and the assertion follows. Finally we consider the case (c) $\omega=\eta^a$. (2) above gives

$$\begin{aligned}&J^G_{P_1}\Big(\eta^a[1]\otimes\delta\Big(\frac{a+1}{2};\frac{a-1}{2}\Big)\Big)\oplus J^G_{P_1}\Big(\eta^a[1]\otimes\delta\Big(\frac{a-1}{2};\frac{a+1}{2}\Big)\Big)\\&\quad=\mathrm{Im}M\Big(w_{M_1},\eta^a[1]\otimes\delta\Big(\frac{a+1}{2};\frac{a-1}{2}\Big)\Big)\oplus\mathrm{Im}M\Big(w_{M_1},\eta^a[1]\otimes\delta\Big(\frac{a-1}{2};\frac{a+1}{2}\Big)\Big)\\&\quad=M(w_{M_1},\eta^a[1]\otimes\eta^a[1])\mathrm{Ker}M(r_2,\eta^a[1]\otimes\eta^a[1]),\end{aligned}$$

while it follows from definition that

$$\begin{aligned}J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta^a\otimes\eta^a)[1])=&\mathrm{Im}M(w_{M_2},\eta^a[1]\otimes\eta^a[1])\\=&\mathrm{Im}[M(w_{M_2},\eta^a[1]\otimes\eta^a[1])M(r_1,\eta^a[1]\otimes\eta^a[1])]\\=&\mathrm{Im}[M(w_{M_1},\eta^a[1]\otimes\eta^a[-1])M(r_2,\eta^a[1]\otimes\eta^a[1])].\end{aligned}$$

By the functional equation [KS80], the following diagram commutes.

$$\begin{array}{ccc}I^G_{\mathbf{B}}(\eta^a[1]\otimes\eta^a[1]) & \xrightarrow{M(w_{M_1},\eta^a[1]\otimes\eta^a[1])} & I^G_{\mathbf{B}}(\eta^a[-1]\otimes\eta^a[1])\\ {\scriptstyle M(r_2,\eta^a[1]\otimes\eta^a[1])}\downarrow & & \downarrow{\scriptstyle M(r_2,\eta^a[-1]\otimes\eta^a[1])}\\ I^G_{\mathbf{B}}(\eta^a[1]\otimes\eta^a[-1]) & \xrightarrow{M(w_{M_1},\eta^a[1]\otimes\eta^a[-1])} & I^G_{\mathbf{B}}(\eta^a[-1]\otimes\eta^a[-1])\end{array}$$

In other words, the homomorphism induced on $I^G_{\mathbf{B}}(\eta^a[1]\otimes\eta^a[1])/\mathrm{Ker}M(r_2,\eta^a[1]\otimes\eta^a[1])$ by $M(w_{M_1},\eta^a[1]\otimes\eta^a[1])$ is the same as the restriction of $M(w_{M_1},\eta^a[1]\otimes\eta^a[-1])$ to $\mathrm{Im}M(r_2,\eta^a[1]\otimes\eta^a[1])$. Noting $\mathrm{Im}M(w_{M_1},\eta^a[1]\otimes\eta^a[1])=I^G_{P_2}(\eta^a(\det))$, we conclude that

$$\begin{aligned}0\longrightarrow J^G_{P_1}\Big(\eta^a[1]\otimes\delta\Big(\frac{a+1}{2};\frac{a-1}{2}\Big)\Big)\oplus J^G_{P_1}\Big(\eta^a[1]\otimes\delta\Big(\frac{a-1}{2};\frac{a+1}{2}\Big)\Big)&\\ \longrightarrow I^G_{P_2}(\eta^a(\det))\longrightarrow J^G_{P_2}(I^{H_2}_{\mathbf{B}^H_2}(\eta^a\otimes\eta^a)[1])\longrightarrow 0.&\end{aligned}$$

This completes the proof. □

Relation with the local θ-correspondence For later use, we describe these A-packets by means of the local θ-correspondence. The cases (M_2.a, b) are not treated, because they appear only as local components of the global parameters of the form Cor. 2.3 (1.b), which contribute only to the residual spectrum.

Lemma 6.12. *Suppose E/F is a quadratic extension of non-archimedean local fields of characteristic zero. We use the notation of § 6.1.*
(i) For ω in (M_1.a, b) above, we have $\epsilon(I_{\mathbf{B}_2}^{G_2}(\omega),\eta)=1$ and $\theta_\eta(I_{\mathbf{B}_2}^{G_2}(\omega),V_+)\simeq I_{\mathbf{B}_2}^{G_2}(\eta\omega^{-1})$ (Th. 5.4). Moreover,

$$\theta_\eta(I_{\mathbf{B}_2}^{G_2}(\eta\omega^{-1}),W_2)=\begin{cases}J_{P_1}^G(\eta[1]\otimes I_{\mathbf{B}_2}^{G_2}(\omega)) & \text{in the case (a),}\\ J_{\mathbf{B}}^G(\eta[1]\otimes\eta'[s]) & \text{in the case (b).}\end{cases}$$

(ii) In the case (M_1.c), we consider the L-packet $\Pi_{\varphi_\mu}(G_2)=\{\tau^{G_2}(\mu)_\pm\}$. We have $\epsilon(\tau^{G_2}(\mu)_\pm,\eta)=1$, and hence $\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon}:=\theta_\eta(\tau^{G_2}(\mu)_\pm,V_+)$ for some $\epsilon\in\{\pm1\}$. Although we cannot specify this ϵ, we have $\theta_\eta(\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon},W_2)=J_{P_1}^G(\eta[1]\otimes\tau^{G_2}(\mu)_\pm)$.
(iii) As for the case (M_2.c), we have $\theta_\eta(\mathbf{1}_{G(V_-)},W_2)=J_{P_1}^G(\eta[1]\otimes\eta_{G_2}\delta_0^{G_2})$, and $\theta_\eta(\mathbf{1}_{G(V_+)},W_2)=J_{P_2}^G(I_{\mathbf{B}_2^H}^{H_2}(\eta\otimes\eta)[1])$.

Proof. (i) Writing $\omega_1:=\omega\eta^{-1}$, we have

$$\begin{aligned}\epsilon(I_{\mathbf{B}_2}^{G_2}(\omega),\eta)=&\epsilon(1/2,\omega',\psi_E)\epsilon(1/2,\sigma(\omega')^{-1},\psi_E)\omega'(-1)\\ =&\epsilon(1/2,\omega',\psi_E)\epsilon(1/2,\omega'^{-1},\psi_E)\omega'(-1),\end{aligned}$$

so that $\epsilon(I_{\mathbf{B}_2}^{G_2}(\omega),\eta))=1$ follows for unitary ω. The case of $\omega=\eta'[s]$ was already treated in § 6.1.2 (2.b.ii). The rest assertion follows from the induction principle Cor. 6.2 (2.ii) combined with [Kon01, Lem.5.3].

(ii) $\epsilon(\tau^{G_2}(\mu)_\pm,\eta)$ can be calculated in the same manner as above. The local θ-correspondence sends $\Pi_{\varphi_\mu}(G_2)$ bijectively to $\Pi_{\varphi_{\eta\mu^{-1}}}(G_2)$ (Th. 5.4), and we name the correspondents as in the lemma. Again the induction principle asserts that $\Theta_\eta(\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon},W_2)$ are non-zero, so that we can take irreducible quotients $\pi_\pm$ of $\Theta_\eta(\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon},W_2)$, respectively. We know (Cor. 6.2 (2.ii)) that they are quotients of

$$I_{P_1}^G(\mu\otimes\eta_{G_2})\simeq J_{P_1}^G(\eta[1]\otimes\tau^{G_2}(\mu)_+)\oplus J_{P_1}^G(\eta[1]\otimes\tau^{G_2}(\mu)_-).$$

Let us introduce the characters

$$\chi_\pm^{M_1}:\mathbf{U}(F)\longrightarrow\mathbf{U}^{M_1}(F)\simeq\mathbf{U}_2(F)\xrightarrow{\chi_2^\pm}\mathbb{C}^1,$$

of $\mathbf{U}(F)=\mathbf{U}_4(F)$ and consider the space $(\pi_\pm)_{\mathbf{U},\chi_\pm^{M_1}}$ of degenerate Whittaker models [BZ77]. Here $\chi_2^+:=\chi_2$ and χ_2^- is the character χ_2 defined with $\psi^\gamma(x):=\psi(\gamma\cdot x)$, $\gamma\in F^\times\setminus\mathrm{N}_{E/F}(E^\times)$ in place of ψ. Also we have written $\mathbf{U}^{M_1}=\mathbf{U}\cap M_1$. One can deduce from [Kon01, Cor. 5.2, Prop. 5.7] that

$$J_{P_1}^G(\eta[1]\otimes\tau^{G_2}(\mu)_\pm)_{P_1}=\mu\otimes\eta_{G_2}+\eta[-1]\otimes\tau^{G_2}(\mu)_\pm \tag{6.5}$$

(in the Grothendieck group), so that for $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ we have

$$\begin{aligned} J^G_{P_1}(\eta[1] \otimes \tau^{G_2}(\mu)_{\epsilon_1})_{\mathbf{U}, \chi^{M_1}_{\epsilon_2}} &= \big(J^G_{P_1}(\eta[1] \otimes \tau^{G_2}(\mu)_{\epsilon_1})_{P_1}\big)_{\mathbf{U}^{M_1}, \chi^{\epsilon_2}_2} \\ &= \begin{cases} \mathbb{C} & \text{if } \epsilon_1 = \epsilon_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.6}$$

On the other hand, Prop. 6.1 asserts that there exists a $G(V_+) \times M_1(F)$-stable filtration $(\omega_{V_+, W_2, \eta})_{P_1} \supset S_1 \supset \{0\}$ such that

$$S_0 := (\omega_{V_+, W_2, \eta})_{P_1}/S_1 \simeq \eta[-1] \otimes \omega_{V_+, W_1, \eta}, \quad S_1 \simeq I^{G_2 \times M_1}_{\mathbf{B}_2 \times M_1}(\sigma^1 \otimes \eta_{G_2}),$$

where σ_1 is a certain representation of $\mathbf{T}_2(F) \times E^\times$, $(E^\times \subset M_1(F))$ on $\mathcal{S}(E^\times)$. Note $\omega_{0, W_1, \eta} = \eta_{G_2}$ by our convention. Obviously $(S_1)_{\mathbf{U}^{M_1}, \chi^{M_1}_\pm} = \{0\}$, so that we have

$$(\omega_{V_+, W_2, \eta})_{\mathbf{U}, \chi^{M_1}_\pm} = (S_0)_{\mathbf{U}^{M_1}, \chi^{M_1}_\pm} = \eta[-1] \otimes (\omega_{V_+, W_1, \eta})_{\mathbf{U}_2, \chi^\pm_2}. \tag{6.7}$$

Let $v_\pm \in V_+$ be any vector satisfying $(v_+, v_+)_+ = 2$, $(v_-, v_-)_- = 2\gamma$, and write $\Omega_\pm$ for the $G(V_+)$-orbits of $v_\pm$. Granting (5.2), we see that the projections $\omega_{V_+, W_1, \eta} \twoheadrightarrow (\omega_{V_+, W_1, \eta})_{\mathbf{U}_2, \chi^\pm_2}$ are, respectively, given by the restriction maps $\mathcal{S}(V_+) \twoheadrightarrow \mathcal{S}(\Omega_\pm)$. We apply the twisted coinvariant functor for $(\mathbf{U}_2, \chi^\pm_2)$ to the Howe duality correspondence $\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon} \leftrightarrow \tau^{G_2}(\mu)_\pm$ to see that

$$\begin{aligned} &\mathrm{Hom}_{G(V_+)}(\mathcal{S}(\Omega_{\epsilon_1}), \tau^{G_2}(\eta\mu^{-1})_{\epsilon\epsilon_2}) \\ &\quad \simeq \mathrm{Hom}_{G(V_+) \times Z_{G_2}(F)}((\omega_{V_+, W_1, \eta})_{\mathbf{U}_2, \chi^{\epsilon_1}_2}, \tau^{G_2}(\eta\mu^{-1})_{\epsilon\epsilon_2} \otimes (\tau^{G_2}(\mu)_{\epsilon_2})_{\mathbf{U}_2, \chi^{\epsilon_1}_2}) \\ &\quad \simeq \begin{cases} \mathbb{C} & \text{if } \epsilon_1 = \epsilon_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.8}$$

Now we go back to $\pi_\pm$. We know from (6.6) that $(\pi_{\epsilon_2})_{\mathbf{U}, \chi^{M_1}_{\epsilon_1}} \simeq \mathbb{C}$ for some pair $(\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$. We apply the twisted Jacquet functor for $(\mathbf{U}, \chi^{M_1}_{\epsilon_1})$ to the Howe duality correspondence $\tau^{G_2}(\eta\mu^{-1})_{\epsilon\epsilon_2} \leftrightarrow \pi_{\epsilon_2}$:

$$\mathrm{Hom}_{G(V_+) \times Z_{M_1}(F)}((\omega_{V_+, W_2, \eta})_{\mathbf{U}, \chi^{M_1}_{\epsilon_1}}, \tau^{G_2}(\eta\mu^{-1})_{\epsilon\epsilon_2} \otimes (\pi_{\epsilon_2})_{\mathbf{U}, \chi^{M_1}_{\epsilon_1}}) \neq \{0\}.$$

The left hand side is equal to that of (6.8) by (6.7), (6.5), so that this is possible only if $\epsilon_1 = \epsilon_2$. It follows from (6.6) that the irreducible quotient $\theta_\eta(\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon}, W_2)$ of $\Theta_\eta(\tau^{G_2}(\eta\mu^{-1})_{\pm\epsilon}, W_2)$ is unique and is isomorphic to $J^G_{P_1}(\eta[1] \otimes \tau^{G_2}(\mu)_\pm)$.

(iii) has already been proved in § 6.1.2 (2.b.iii). □

Lemma 6.13. *Next we consider the case $E/F = \mathbb{C}/\mathbb{R}$.*
(i) In the cases (M_1.a, b), writing $\omega = \omega_{b,\nu}$, we have $\theta_{\eta^a}(I^{G_2}_{\mathbf{B}_2}(\omega_{b,\nu}), V_{1,1}) = I^{G_2}_{\mathbf{B}_2}(\omega_{a-b,-\nu})$ (Prop. 5.6), and

$$\theta_{\eta^a}(I^{G_2}_{\mathbf{B}_2}(\omega_{a-b,-\nu}), W_2) = \begin{cases} J^G_{P_1}(\eta^a[1] \otimes I^{G_2}_{\mathbf{B}_2}(\omega_{b,\nu})) & \textit{in the case (a),} \\ J^G_{\mathbf{B}}(\eta^a[1] \otimes \eta^b[s]) & \textit{in the case (b).} \end{cases}$$

(ii) In the case (M_1.c)*, we write* $\mu = \mu^b$*. Prop. 5.6 gives*

$$\theta_{\eta^a}\big(\tau\Big(\frac{b}{2};\frac{b}{2}\Big)_{\pm}, V_{1,1}\big) = \tau\Big(\frac{a-b}{2};\frac{a-b}{2}\Big)_{\pm\epsilon}, \quad \epsilon = \operatorname{sgn}(b-a),$$

and we have

$$\theta_{\eta^a}\big(\tau\Big(\frac{a-b}{2};\frac{a-b}{2}\Big)_{\pm\epsilon}, W_2\big) = J^G_{P_1}\big(\eta^a[1]\otimes\tau\Big(\frac{b}{2};\frac{b}{2}\Big)_{\pm}\big).$$

(iii) As for the case (M_2.c)*, we have*

$$\theta_{\eta^a}(\mathbf{1}_{G_{2,0}}, W_2) = J^G_{P_1}\big(\eta^a[1]\otimes\delta\Big(\frac{a+1}{2};\frac{a-1}{2}\Big)\big),$$
$$\theta_{\eta^a}(\mathbf{1}_{G_{0,2}}, W_2) = J^G_{P_1}\big(\eta^a[1]\otimes\delta\Big(\frac{a-1}{2};\frac{a+1}{2}\Big)\big),$$
$$\theta_{\eta^a}(\mathbf{1}_{G_{1,1}}, W_2) = J^G_{P_2}(I^{H_2}_{\mathbf{B}_2^H}(\eta^a\otimes\eta^a)[1]).$$

Proof. (i), (ii) can be proved in the same manner as in § 6.3.2. The details are omitted. Note that, if b is odd, $I^{G_2}_{\mathbf{B}_2}(\omega_{a-b,-\nu})$ has the minimal $\mathbf{K}_{1,1}$-type $2^{-1}(a-b\pm 1; a-b\mp 1)$ which corresponds to the $\mathbf{K}_{2,2}$-type $2^{-1}(a, b\pm 1; a, b\mp 1)$ by the local θ-correspondence (Lem. 6.7). The 1/2-shifts are caused by the so-called *fine weight* [Pau98, 3.2].

The first assertion in (iii) is merely a restatement of Lem. 6.9. The last assertion again can be shown as in § 6.3.2. □

6.5 Split case

Finally we treat the case where the quadratic extension E/F is trivial: $E = F\oplus F$. We still write σ for the unique non-trivial element of $\mathrm{Aut}_F(E)$: $\sigma(x,y) = (y,x)$. $H_n = \mathrm{R}_{E/F}GL(n)$ is just the double copy of $GL(n)$ and

$$G_n = \{(g, \theta_n(g)) \in H_n\} \simeq GL(n)_F.$$

In the dual setting, we have the "base change map"

$${}^LG_n = \widehat{G}_n \times W_F \ni g\times w \longmapsto (g,\theta_n(g))\times w \in {}^LH_n$$

so that the base change lift in this case is given by

$$\Pi(G_n(F)) \ni \pi \longmapsto \pi\otimes\pi^\vee \in \Pi(H_n(F)).$$

6.5.1 A-parameters and representations

An A-parameter ψ for G_n in this case is simply a completely reducible representation

$$\psi = \bigoplus_i \varphi_{\pi_i}\otimes\rho_{d_i}$$

of $\mathcal{L}_F\times SL(2,\mathbb{C})$. Here φ_{π_i} is the Langlands parameter for some $\pi_i \in \Pi(G_{m_i}(F))$ and $n = \sum_i d_i m_i$. Recall (Rem. 3.5) that we cannot assume the temperedness of π_i, and we impose some unitarizability condition instead. To make this explicit, we review the classification of the unitary dual of $G_n(F) = GL(n,F)$ from [Tad86, Th.A], [Vog86]. We write $P_{\mathbf{n}} = M_{\mathbf{n}}U_{\mathbf{n}}$ for the standard parabolic subgroup of G_n associated to a partition $\mathbf{n}$ of n.

(i) Speh modules. Let $n=dm$, d, $m\in\mathbb{N}$, and take $\delta\in\Pi_{\rm disc}(G_m(F))$.

$$J(\delta,d):=J_{P_{(m^d)}}^{G_n}(\delta|\det|_F^{(d-1)/2}\otimes\delta|\det|_F^{(d-3)/2}\otimes\cdots\otimes\delta|\det|_F^{(1-d)/2})$$

is known to be unitarizable. This is called the *Speh module* associated to (δ,d).

(ii) (Stein's) complementary series. For $J(\delta,d)$ as above and $0<s<1$,

$$J(\delta,d)\langle s\rangle:=I_{P_{(n,n)}}^{G_{2n}}\big(J(\delta,d)|\det|_F^{s/2}\otimes J(\delta,d)|\det|_F^{-s/2}\big)$$

is irreducible and unitarizable.

(iii) For each finite family of representations $\{\pi_i\}_i$ where

- π_i is either a Speh module or a complementary series representation of $G_{m_i}(F)$;
- $\sum_i m_i=n$,

the induced representation

$$\boxplus_i\,\pi_i:=I_{P_{(m_i)_i}}^{G_n}(\bigotimes_i\pi_i)$$

is irreducible and unitarizable. Any element of $\Pi_{\rm unit}(G_n(F))$ is of this form, and the set $\{\pi_i\}$ is uniquely determined by $\boxplus_i\,\pi_i$ (of course, up to permutations).

Now the A-parameters with non-trivial $SL(2,\mathbb{C})$-components and the associated A-packets, or simply irreducible representations for $G_4(F)$, are given as follows. For a quasi-character ω of $F^\times$, we write ω_{G_n} for the quasi-character $\omega\circ\det$ of $G_n(F)$.
(1) Elliptic cases. Ellipticity of an A-parameter ψ in the $GL(n)$ case is equivalent to its irreducibility.

(a) $\psi=\omega\otimes\rho_4$ with $\omega\in\Pi_{\rm unit}(F^\times)$. $\Pi_\psi(G_4)=\{\omega_{G_4}\}$.

(b) $\psi=\varphi_\delta\otimes\rho_2$, where φ_δ is the Langlands parameter of some $\delta\in\Pi_{\rm disc}(G_2(F))$. $\Pi_\psi(G_4)=\{J(\delta,2)\}$.

(2) Non-elliptic cases. These are reducible parameters. For later global use, we divide them according to if their images are contained in a globally defined parabolic subgroups of G_4 or not.
Globally elliptic (endoscopic) cases.

(a) $\psi=(\omega\otimes\rho_3)\oplus\omega'$ with ω, $\omega'\in\Pi_{\rm unit}(F^\times)$. $\Pi_\psi(G_4)=\{\omega_{G_3}\boxplus\omega'\}$.

(b) $\psi=(\omega\otimes\rho_2)\oplus\varphi_\delta$, $\omega\in\Pi_{\rm unit}(F^\times)$, $\delta\in\Pi_{\rm disc}(G_2(F))$. $\Pi_\psi(G_4)=\{\omega_{G_2}\boxplus\delta\}$.

Globally non-elliptic cases.

(M_1) $\psi^{M_1}=(\omega\otimes\rho_2)\oplus\omega_1\oplus\omega_2^{-1}$ with $\omega\in\Pi_{\rm unit}(F^\times)$. Rem. 3.5 (i)$'$ restricts the possibility for $\omega_1\otimes\omega_2\in\Pi(E^\times)$ to:

(i) $\omega_1,\, \omega_2 \in \Pi_{\mathrm{unit}}(F^\times)$. $\Pi_{\psi^{M_1}}(G_4) = \{\omega_{G_2} \boxplus \omega_1 \boxplus \omega_2^{-1}\}$.

(ii) $\omega_1 = \omega' |\,|_F^{s/2}$, $\omega_2 = \omega'^{-1} |\,|_F^{s/2}$ for some $\omega' \in \Pi_{\mathrm{unit}}(F^\times)$ and $0 < s < 1$. $\Pi_{\psi^{M_1}}(G_4) = \{\omega_{G_2} \boxplus \omega' \langle s \rangle\}$.

(M_2) $\psi^{M_2} = (\omega_1 \otimes \rho_2) \oplus (\omega_2^{-1} \otimes \rho_2)$. Here again Rem. 3.5 (i)$'$ implies that $\omega_1 \otimes \omega_2 \in \Pi(E^\times)$ is either:

(i) $\omega_1,\, \omega_2 \in \Pi_{\mathrm{unit}}(F^\times)$. $\Pi_{\psi^{M_2}}(G_4) = \{\omega_{1,G_2} \boxplus \omega_{2,G_2}\}$.

(ii) $\omega_1 = \omega |\,|_F^{s/2}$, $\omega_2 = \omega^{-1} |\,|_F^{s/2}$ for some $\omega \in \Pi_{\mathrm{unit}}(F^\times)$ and $0 < s < 1$. $\Pi_{\psi^{M_2}}(G_4) = \{J(\omega, 2)\langle s \rangle\}$.

6.5.2 Local θ-correspondence

We shall also need realizations of some of these representations as local θ-lifting from $G_2(F)$.

A rank m *hermitian space* over $E = F \oplus F$ is a pair $(V, (\,,\,))$ consisting of

- A scalar extension $V := X \otimes_F E = X \oplus X$ of some m-dimensional F-vector space X. Note that σ acts on V as the transposition of the first and second X.
- An F-bilinear form $(\,,\,) : V \otimes_F V \to E$ satisfying $(\lambda v, \lambda' v') = \sigma(\lambda)\lambda' \cdot \sigma((v', v))$, $\forall \lambda$, $\lambda' \in E$, v, $v' \in V$.

Writing this hermitian condition in coordinates, it turns out that we can write $V = X' \oplus X$, X' being the dual space of X, and

$$((x_1', x_1), (x_2', x_2)) = (\langle x_1, x_2' \rangle, \langle x_1', x_2 \rangle), \quad x_i' \in X',\, x_i \in X.$$

Here $\langle\,,\,\rangle$ is the duality between X and X'. In fact, this gives the identification

$$\begin{aligned} G(V) :=& \{g \in GL_E(V) \,|\, (g.v, g'.v') = (v, v'),\, \forall v,\, v' \in V\} \\ =& \{g \oplus {}^t g^{-1} \,|\, g \in GL(X')\} \simeq GL(X'), \end{aligned}$$

where ${}^t g$ is the adjoint of g with respect to the duality $\langle\,,\,\rangle$. Also a rank n *skew-hermitian space* over E is a pair $(W, \langle\,,\,\rangle)$ of

- $W = Y \otimes_F E = Y \oplus Y$ for some n-dimensional vector space Y over F.
- An F-bilinear form $\langle\,,\,\rangle : W \otimes_F W \to E$ satisfying $\langle \lambda w, \lambda' w' \rangle = -\lambda \sigma(\lambda') \cdot \sigma(\langle w', w \rangle)$, for λ, $\lambda' \in E$, w, $w' \in W$.

Again we may write $W = Y' \oplus Y$ with

$$\langle (y_1', y_1), (y_2', y_2) \rangle = (\langle y_1', y_2 \rangle, -\langle y_1, y_2' \rangle), \quad y_i' \in Y',\, y_i \in Y,$$

where $\langle\,,\,\rangle$ in the right hand side stands for the duality between Y and its dual Y'. The unitary group $G(W)$ of W is identified with $GL(Y')$.

As in § 5.1, we define a $2nm$-dimensional symplectic space $(\mathbb{W} := V \otimes_E W, \langle\!\langle\,,\,\rangle\!\rangle := 2^{-1}\mathrm{Tr}_{E/F}((\,,\,) \otimes_E \sigma(\langle\,,\,\rangle)))$. Note that $\mathrm{Tr}_{E/F} : E \to F$ is just the summation of the first and second components. If we identify $\mathbb{W} = \mathbb{Y}' \oplus \mathbb{Y}$, with $\mathbb{Y}' := X \otimes Y' = \mathrm{Hom}_F(Y, X)$, $\mathbb{Y} = X' \otimes Y = \mathrm{Hom}_F(X, Y)$, then we have

$$\langle\langle (x', x), (y', y) \rangle\rangle = \frac{1}{2}\mathrm{tr}(x' \circ y - y' \circ x), \quad x',\, y' \in \mathbb{Y}',\, x,\, y \in \mathbb{Y}.$$

Thus in the present case, the dual pair $(G(V), G(W))$ in $Sp(\mathbb{W})$ reduces to the type II dual pair:

$$\iota_{X,Y} : GL(X') \times GL(Y') \ni (g, g') \longmapsto ({}^t g^{-1} \otimes g') \oplus (g \otimes {}^t g'^{-1}) \in Sp(\mathbb{W}).$$

Take a character ω of $F^\times$, which we identify with the character $E^\times \ni (x, y) \mapsto \omega(xy^{-1}) \in \mathbb{C}^1$ trivial on the diagonal subgroup $F^\times$. We adopt the identification $Mp(\mathbb{W}) = Sp(\mathbb{W}) \times \mathbb{C}^1$ in which the metaplectic 2-cocycle is given by $\gamma_\psi(\mathbb{Y}, \mathbb{Y}g_2^{-1}, \mathbb{Y}g_1)$ for $(g_1, g_2) \in Sp(\mathbb{W})^2$. Since the image of $\iota_{X,Y}$ is contained in the Siegel parabolic subgroup $P_{\mathbb{Y}}$ stabilizing $\mathbb{Y}$, this obviously lifts to a continuous homomorphism

$$\iota_{X,Y,\omega} : GL(X') \times GL(Y') \ni (g, g') \longmapsto (\iota_{X,Y}(g, g'), \omega(\det g')) \in Mp(\mathbb{W}).$$

The composite $\omega_{X,Y,\omega} := \omega_\psi \circ \iota_{X,Y,\omega}$ is the Weil representation, which we need below. In the Schrödinger model $\mathcal{S}(\mathbb{Y}')$, we have the explicit formula

$$\omega_{X,Y,\omega}(g, g')\phi(x) = |\det g|_F^{-n/2}\omega(\det g')|\det g'|_F^{m/2}\phi(g^{-1}.x.g), \qquad g \in GL(X'),\, g' \in GL(Y').$$

When F is archimedean, we fix suitable maximal compact subgroups $\mathbf{K}_X \subset GL(X')$, $\mathbf{K}_Y \subset GL(Y')$ and consider the Fock subspace $\mathcal{S}_0(\mathbb{Y}') \subset \mathcal{S}(\mathbb{Y}')$ with respect to them.

As in the inert case § 5.1.2, we have $\mathscr{R}(GL(X'), \omega_{X,Y,\omega}) \subset \Pi(GL(X'))$ and $\Theta_\omega(\pi_X, Y')$ for each $\pi_X \in \mathscr{R}(GL(X'), \omega_{X,Y,\omega})$. By the local Howe duality conjecture, $\Theta_\eta(\pi_X, Y)$ is an admissible finitely generated representation. It admits a unique irreducible quotient $\theta_\omega(\pi_X, Y')$. Similar construction works for $\pi_Y \in \mathscr{R}(GL(Y'), \omega_{X,Y,\omega})$ and $\pi_X \mapsto \theta_\omega(\pi_X, Y')$, $\pi_Y \mapsto \theta_\omega(\pi_Y, X')$ are bijections between $\mathscr{R}(GL(X'), \omega_{X,Y,\omega})$ and $\mathscr{R}(GL(Y'), \omega_{X,Y,\omega})$ converse to each other. Besides these generalities, the following result is well-known, although we cannot find a suitable reference ([KS97] ???).

Proposition 6.14. *(i) Suppose $n = m = 2$. Then for each $\pi \in \Pi(GL(Y'))$, $\theta_\omega(\pi, X') \simeq \omega(\det)\pi^\vee$. Here we identify $\Pi(GL(X')) = \Pi(GL(Y'))$.*
(ii) When $n = 4$, $m = 2$ and $\pi \in \Pi_{\mathrm{unit}}(GL(X'))$, we have $\theta_\omega(\pi, Y') \simeq \omega(\det)\pi^\vee \boxplus \omega_{G_2}$.

7 An incomplete multiplicity formula

In this final section, we give an example of the half inequality of the expected multiplicity formula for A-paramters of type (2.b) in Prop. 2.3 in order to motivate the reader.

7.1 Multiplicity pairing

We still assume that E is a 2-dimensional semisimple algebra over a local field F of characteristic zero. Our task here is to define a local pairing $\langle\,,\,|\pi_\chi\rangle_\psi : \mathcal{S}_\psi(G)\times\Pi_\psi(G)\to\mathbb{C}$ in Conj. 3.4 (C). Since $\mathcal{S}_\psi(G)=\{1\}$ and $\Pi_\psi(G)$ is a singleton for $G=GL(n)_F$, one may exclude the trivial case $E\simeq F\oplus F$.

First we consider the non-archimedean case. The global A-parameters of type (2.b) have the following three types of non-archimedean local components.

(2.b) $\psi_{\Pi,\eta}$ with $\Pi_E\in\Pi_{\mathrm{disc}}(H_2(F))$. In this case, $\Pi_\psi(G)$ consists of two elements $\pi^\pm$ (§ 6.1.2 (2.b)), the χ-base point π_χ is π^+ and $\mathcal{S}_\psi(G)\simeq\mathbb{Z}/2\mathbb{Z}$ (Prop. 3.7). Define $\langle\,,\pi^+|\pi^+\rangle_\psi:=\mathbf{1}$, $\langle\,,\pi^-|\pi^+\rangle_\psi:=\mathrm{sgn}_{\mathcal{S}_\psi(G)}$.

(2.d) $\psi_{\eta,\underline{\mu}}$ with $\underline{\mu}=(\mu,\mu')$, $\mu\neq\mu'$. $\Pi_\psi(G)$ contains 4 representations $\pi^{\pm,\pm}$. $\pi_\chi=\pi^{+,+}$ and $\mathcal{S}_\psi(G)=\mathbb{Z}/2\mathbb{Z}\times\mathcal{S}_{\psi|\mathcal{L}_F}(M_1)\simeq(\mathbb{Z}/2\mathbb{Z})^2$ (Prop. 3.7). We define

- $\langle\bar{s},\pi_{\bullet,\pm}|\pi^{+,+}\rangle_\psi:=\langle\bar{s},\pi^{G_2}(\underline{\mu})_\pm|\pi^{G_2}(\underline{\mu})_+\rangle_{\psi|\mathcal{L}_F}$ on $\bar{s}\in\mathcal{S}_{\psi|\mathcal{L}_F}(M_1)$;
- $\langle\,,\pi_{\pm,\bullet}|\pi^{+,+}\rangle_\psi$ on $\mathbb{Z}/2\mathbb{Z}$ equals the sign character if $\pi_{-,\bullet}$ and the trivial character otherwise.

Notice the ambiguity in the definition of $\langle\bar{s},\pi^{-,\bullet}|\pi^{+,+}\rangle_\psi$, since the labeling of $\pi^{-,\bullet}$ depends on an arbitrary parametrization of $\pi_{V',\pm}\in\Pi_{\varphi_{\eta\underline{\mu}^{-1}}}(G(V'))$. A precise labeling will be given in [Kon]. At present, we only remark that this ambiguity has no effect to the global consequence for the global parameters of type (2.b).

(M_1) $\psi^{M_1}_{\omega,\eta}$, where ω is a quasi-character of $E^\times$ of the type specified in § 6.4. Unless $\omega|_{F^\times}=\omega_{E/F}$, $\Pi_\psi(G)$ consists of single element and $\mathcal{S}_\psi(G)=\{1\}$ (Prop. 6.11 (i)). In the case $\omega=\mu$, $\Pi_\psi(G)=\{J^G_{P_1}(\eta[1]\otimes\tau^{G_2}(\mu)_\pm)\}$, $\pi_\chi=J^G_{P_1}(\eta[1]\otimes\tau^{G_2}(\mu)_+)$, and $\mathcal{S}_\psi(G)=\mathcal{S}_{\psi|\mathcal{L}_F}(M_1)\simeq\mathbb{Z}/2\mathbb{Z}$. Following [Art89, § 7], we define

$$\langle\bar{s},J^G_{P_1}(\eta[1]\otimes\tau^{G_2}(\mu)_\pm)|J^G_{P_1}(\eta[1]\otimes\tau^{G_2}(\mu)_+)\rangle_\psi:=\langle\bar{s},\tau^{G_2}(\mu)_\pm|\tau^{G_2}(\mu)_+\rangle_{\psi|\mathcal{L}_F}.$$

Notice that the R-group R_ψ [*loc. cit.*] for ψ is trivial in this example.

Next we move to the case $E/F=\mathbb{C}/\mathbb{R}$. Since there are no parameters of type (2.b), we have only to consider the A-parameters of type (2.d) and (M_1).

(2.d) $\psi_{\eta^a,\underline{\mu}^{\underline{a}}}$, ($a\in2\mathbb{Z}$, $\underline{\mu}^{\underline{a}}=(\mu^{a_1},\mu^{a_2})$, $a_1>a_2\in2\mathbb{Z}+1$). $\Pi_\psi(G)$ consists of

$$\pi^{+,+}:=J^{G_{2,2}}_{P_1}(\eta^a[1]\otimes\delta(a_1/2;a_2/2)),\quad \pi^{+,-}:=J^{G_{2,2}}_{P_1}(\eta^a[1]\otimes\delta(a_2/2;a_1/2)),$$

and two discrete series or limit of discrete series representations which we temporally label as $\pi^{-,\pm}$. As is remarked in the non-archimedean case, this labeling is arbitrary but does not affect the multiplicity formula for the global parameters of type (2.b). π_χ for χ with respect to $\psi_\mathbb{R}$ is $\pi^{+,+}$, and $\mathcal{S}_\psi(G)=\mathbb{Z}/2\mathbb{Z}\times\mathcal{S}_{\psi|\mathcal{L}_F}(M_1)\simeq(\mathbb{Z}/2\mathbb{Z})^2$ (Lem. 3.10). The pairing $\langle\,,\,|\pi^{+,+}\rangle_\psi$ is defined as in the non-archimedean case.

(M_1) $\psi^{M_1}_{\omega,\eta^a}$, where ω is specified in § 6.4. Only the case $\omega=\mu^b$, ($b\in2\mathbb{Z}+1$) is non-trivial. In that case $\Pi_\psi(G)=\{J^G_{P_1}(\eta^a[1]\otimes\tau(b/2,b/2)_\pm)\}$, $\pi_\chi=J^G_{P_1}(\eta^a[1]\otimes\tau(b/2,b/2)_+)$, and $\mathcal{S}_\psi(G)=\mathcal{S}_{\psi|\mathcal{L}_F}(M_1)\simeq\mathbb{Z}/2\mathbb{Z}$. As in the non-archimedean case $\langle\,,\,|\pi_\chi\rangle_\psi$ is the one induced from $\langle\,,\,|\tau(b/2,b/2)_+\rangle_{\psi|\mathcal{L}_F}$ [Art89, § 7].

7.2 The multiplicity inequality

We go back to the global situation of § 2 where E/F is a quadratic extension of number fields. At each place v of F, we write $E_v := E \otimes_F F_v$. If further v is non-archimedean, we write $\mathcal{O}_v$ for the maximal compact subring in F_v and $\mathcal{O}_{E_v}$ for its integral closure in E_v. We fix a non-trivial character $\psi_F = \bigotimes_v \psi_{F_v}$ of $\mathbb{A}/F$. This determines a non-degenerate character $(\chi = \chi_4) = \bigotimes_v \chi_v$ of $\mathbf{U}(\mathbb{A})$ trivial on $\mathbf{U}(F)$ as in the local case (see p. 20). We also fix a maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v \subset G(\mathbb{A})$, which is in good position with respect to the maximal F-split subtorus in $\mathbf{T} = \mathbf{T}_4$.

Take an A-parameter $\psi = \psi_{\Pi,\eta}$ of type Cor. 2.3 (2.b). $\Pi_E = \bigotimes_v \Pi_{E_v}$ is an irreducible cuspidal representation of $H_2(\mathbb{A}) = GL(2, \mathbb{A}_E)$ such that $\sigma(\Pi_E) \simeq \Pi_E^\vee$, $\omega_{\Pi_E}|_{\mathbb{A}^\times}$ is trivial and $L_{\mathrm{Asai}}(s, \Pi_E)$ is holomorphic at $s = 1$. $\eta = \bigotimes_v \eta_v$ is an idele class character of E trivial on $\mathbb{A}^\times$.

We have the local A-packet $\Pi_{\psi_v}(G_v) \subset \Pi_{\mathrm{unit}}(G(F_v))$ associated to its local components ψ_v at each place v of F. Here we have written $G_v := G \otimes_F F_v$. At all but finite number of non-archimedean v, we have either

- E_v/F_v is an unramified quadratic extension and $\psi_v = \psi_{\omega,\mathbf{1}}^{M_1}$ with some unramified quasi-character ω of type listed at the beginning of § 6.4. Moreover ψ_v is of order zero and our splitting $\mathbf{spl}_G$ is chosen in such a way that $(\mathbf{B}_v, \mathbf{T}_v)$ is defined over $\mathcal{O}_{E_v}$ and $\{X\} \subset \mathfrak{g}_v(\mathcal{O}_{E_v})$. Here, of course, we have used the smooth flat model over $\mathcal{O}_v$ of G_v associated to a hyperspecial point defining $\mathbf{K}_v$. Under these conditions, the χ_v-base point $\pi_{\chi_v} \in \Pi_{\psi_v}(G_v)$ is unramified, and contains a distinguished $\mathbf{K}_v$-spherical vector ϕ_v^0.

- $E_v \simeq F_v \oplus F_v$, $G_v \simeq GL(4)_{F_v}$ and $\psi_v = \psi^{M_1} = \omega \otimes \rho_2 \oplus \omega_1 \oplus \omega_2^{-1}$ for some unramified $\omega \in \Pi_{\mathrm{unit}}(F^\times)$, $\omega_1 \otimes \omega_2 \in \Pi(E_v^\times)$ as in § 6.5.1 (M_1). Again the unique element $\omega_{G_{2,v}} \boxplus \omega_1 \boxplus \omega_2^{-1}$ of $\Pi_{\psi_v}(G_v)$ is unramified and possesses a distinguished unramified vector ϕ_v^0.

We can define the global A-packet $\Pi_\psi(G)$ to be the set of irreducible representations $\bigotimes_v \pi_v$ of $G(\mathbb{A})$, where $\pi_v \in \Pi_{\psi_v}(G_v)$ at each v and $\pi_v = \pi_{\chi_v}$ at all but finite number of v. The restricted tensor product is taken with respect to ϕ_v^0. Note that we have a canonical homomorphism $\mathcal{S}_\psi(G) \ni \bar{s} \mapsto \bar{s}(v) \in \mathcal{S}_{\psi_v}(G_v)$ at each v. The restrictions of $\langle\, , \pi_v | \pi_{\chi_v} \rangle_{\psi_v}$, $\pi_v \in \Pi_{\psi_v}(G_v)$ to the image of this homomorphism is not affected by the ambiguity referred in § 7.1.

Theorem 7.1. *We write $m(\pi)$ for the multiplicity of an irreducible representation π of $G(\mathbb{A})$ in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$. Then for $\pi = \bigotimes_v \pi_v \in \Pi_\psi(G)$, we have*

$$m(\pi) \geq \frac{1}{2} \sum_{\bar{s} \in \mathcal{S}_\psi(G)} \epsilon_\psi(\bar{s}) \prod_v \langle \bar{s}(v), \pi_v | \pi_{\chi_v} \rangle_{\psi_v}.$$

Here

$$\epsilon_\psi(\bar{s}) = \begin{cases} \mathrm{sgn}_{\mathcal{S}_\psi(G)} & \text{if } \varepsilon(1/2, \pi \times \eta^{-1}) = -1, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Here $\varepsilon(s, \pi \times \eta^{-1})$ is the global root number of the standard L-function for $\pi \times \eta^{-1}$ defined by means of the Langlands-Shahidi theory.

Remark 7.2. *The formula is interpreted in terms of [Art89, Conj. 8.1] as follows. Writing $\widehat{\mathfrak{g}} = \mathfrak{gl}(4, \mathbb{C})$ for the Lie algebra of $\widehat{G}$, we consider the representation*

$$\tau_\psi : S_\psi(G) \times \mathcal{L}_F \times SL(2, \mathbb{C}) \ni (s, w, g) \longmapsto \mathrm{Ad}(s\psi(w, g)) \in GL(\widehat{\mathfrak{g}}).$$

It follows immediately from the realization of $\psi = \psi_{\Pi,\eta}$ in Cor. 2.3 (2) that this admits an irreducible decomposition $\tau_\psi = \bigoplus_{i=1}^{5} \tau_i$, where (τ_i, V_i) are the restrictions of τ_ψ to the invariant subspaces

$$V_1 := \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \middle| X \in \mathfrak{sl}(2, \mathbb{C}) \right\}, \quad V_2 := \left\{ \begin{pmatrix} x\mathbf{1}_2 & 0 \\ 0 & 0 \end{pmatrix} \middle| x \in \mathbb{C} \right\},$$
$$V_3 := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix} \middle| X \in \mathfrak{sl}(2, \mathbb{C}) \right\}, \quad V_4 := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x\mathbf{1}_2 \end{pmatrix} \middle| x \in \mathbb{C} \right\},$$
$$V_5 := \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \middle| B, C \in \mathbb{M}_2(\mathbb{C}) \right\},$$

so that

$$\tau_1 = \mathbf{1}_{S_\psi(G)} \otimes \mathbf{1}_{\mathcal{L}_F} \otimes \rho_3, \quad \tau_2 = \mathbf{1}_{S_\psi(G)} \otimes \omega_{E/F} \otimes \rho_1, \quad \tau_3 = \mathbf{1}_{S_\psi(G)} \otimes (\mathrm{Ad} \circ \varphi_\Pi)|_{\mathfrak{sl}(2,\mathbb{C})} \otimes \rho_1,$$
$$\tau_4 = \mathbf{1}_{S_\psi(G)} \otimes \omega_{E/F} \otimes \rho_1, \quad \tau_5 = \mathrm{sgn}_{\mathcal{S}_\psi(G)} \otimes \mathrm{ind}_{\mathcal{L}_E}^{\mathcal{L}_F}(\eta^{-1}\varphi_\Pi|_{\mathcal{L}_E}) \otimes \rho_2.$$

Noting $L(s, \mathrm{ind}_{\mathcal{L}_E}^{\mathcal{L}_F}(\eta^{-1}\varphi_\Pi|_{\mathcal{L}_E})) = L(s, \Pi \times \eta^{-1})$, we see that the sign character ϵ_ψ of $\mathcal{S}_\psi(G)$ defined in [Art89, (8.4)] is exactly the one in the theorem.

Proof. Only some simple arguments in the theory of θ-correspondence is sufficient for our purpose. Let $(V, (\ ,\ ))$ be a two-dimensional hermitian space over E and write $G(V)$ for its unitary group. We fix a Lagrangian subspace $\mathbb{Y}'$ of the symplectic space $\mathbb{W} := V \otimes_E W_2$ defined as in the local case § 5.1. At each place v of F, we have the local Weil representation $(\omega_{V_v,W_2,\eta_v}, \mathcal{S}(\mathbb{Y}'_v))$ of $G(V_v) \times G(F_v)$ associated to η_v and ψ_{F_v}, where $V_v := V \otimes_F F_v$, $G(V_v) = G(V, F_v)$ is its unitary group, and $\mathbb{Y}'_v := \mathbb{Y}' \otimes_F F_v \subset \mathbb{W}_v := \mathbb{W} \otimes_F F_v$. Observe that, at all but finite number of (non-archimedean) places, we have

- E_v/F_v and $G(V_v)$, G are unramified (*i.e.* V_v is hyperbolic), so that we have hyperspecial maximal compact subgroups $\mathbf{K}_{V_v} \subset G(V_v)$ and $\mathbf{K}_v \subset G(F_v)$.
- $\iota_{V_v,W_2} : G(V_v) \times G(F_v) \to Sp(\mathbb{W}_v)$ sends $\mathbf{K}_{V_v} \times \mathbf{K}_v$ to a hyperspecial maximal compact subgroup $\mathbf{K}_{\mathbb{W},v} \subset Sp(\mathbb{W}_v)$. Here we have written $\mathbb{W}_v := V_v \otimes_{E_v} W_2$.
- ψ_{F_v} is of order 0.
- η_v is unramified. (Note, when v is inert, this is equivalent to the triviality of η_v.)
- The residual characteristic of F_v is odd.

At such v, $\widetilde{\iota}_{V_v,W_2,\eta_v} : G(V_v) \times G(F_v) \to Mp(\mathbb{W}_v)$ restricts to a continuous homomorphism $\mathbf{K}_{V_v} \times \mathbf{K}_v \to \mathbf{K}_{\mathbb{W},v} \hookrightarrow Mp(\mathbb{W}_v)$. $\mathbf{K}_{\mathbb{W}}$ is the stabilizer of some self-dual $\mathcal{O}_v$-lattice $\widetilde{L}_v \subset \mathbb{W}_v$.

Writing $L_v := \mathbb{Y}'_v \cap \widetilde{L}_v$, its characteristic function ϕ_{L_v} is fixed under $\mathbf{K}_{\mathbb{W}_v}$. The restricted tensor product

$$(\omega_{V,W_2,\eta}, \mathcal{S}(\mathbb{Y}'_{\mathbb{A}})) = \bigotimes_v (\omega_{V_v,W_2,\eta_v}, \mathcal{S}(\mathbb{Y}'_v))$$

with respect to these $\phi_{L_v} \in \mathcal{S}(\mathbb{Y}'_v)$ is the global Weil representation of $G(V_{\mathbb{A}}) \times G(\mathbb{A})$ associated to η and ψ_F. We have written $V_{\mathbb{A}} := V \otimes_F \mathbb{A}$ and $G(V_{\mathbb{A}}) := G(V, \mathbb{A})$ for brevity.

For archimedean v, let $\mathcal{S}_0(\mathbb{Y}_v)$ be the Fock subspace of $\mathcal{S}(\mathbb{Y}_v)$ (pull-back of the Fock model, cf. § 6.3.1). We set $\mathcal{S}_0(\mathbb{Y}'_{\mathbb{A}}) := \bigotimes_{v|\infty} \mathcal{S}_0(\mathbb{Y}_v) \otimes \bigotimes_{v\nmid\infty} \mathcal{S}(\mathbb{Y}'_v)$. For each $\phi \in \mathcal{S}_0(\mathbb{Y}'_{\mathbb{A}})$, we define the corresponding θ-kernel by

$$\theta_\phi(g, g') := \sum_{\xi \in \mathbb{Y}'} \omega_{V,W_2,\eta}(g, g')\phi(\xi), \quad g \in G(V_{\mathbb{A}}),\ g' \in G(\mathbb{A}).$$

This turns out to be an automorphic form on $(G(V) \times G(F)) \backslash (G(V_{\mathbb{A}}) \times G(\mathbb{A}))$. For an irreducible cuspidal representation π_V of $G(V_{\mathbb{A}})$, we write $\mathcal{A}(\pi_V)$ for its space (precisely speaking, its underlying Hecke algebra module) in the space of cusp forms $\mathcal{A}_0(G(V)\backslash G(V_{\mathbb{A}}))$. We write $\Theta_\eta(\pi_V, W_2)$ for the span of

$$G(\mathbb{A}) \ni g' \longmapsto \int_{G(V)\backslash G(V_{\mathbb{A}})} f(g)\theta_\phi(g, g')\, dg \in \mathbb{C},$$

with $f \in \mathcal{A}(\pi_V^\vee)$, $\phi \in \mathcal{S}_0(\mathbb{Y}'_{\mathbb{A}})$. Since f is rapidly decreasing and θ_ϕ is slowly increasing, the integral always converges absolutely.

Next let $\{V_v\}_v$ be a family of two-dimensional hermitian spaces V_v over E_v at each place v of F. Using the signature $\epsilon(V_v) := \omega_{E_v/F_v}(-\det V_v)$, the classical Hasse principle for $\{V_v\}_v$ can be stated as follows. There exists a hermitian space V over E such that $V \otimes_F F_v \simeq V_v$, at every v if and only if

$$\prod_v \epsilon(V_v) = 1.$$

Now take $\pi = \bigotimes_v \pi_v \in \Pi_\psi(G)$ such that the right hand side of the inequality in the theorem is not zero. If we write S for the set of places v where $\bar{s} \mapsto \langle \bar{s}(v), \pi_v | \pi_{\chi_v} \rangle_{\psi_v}$ is $\mathrm{sgn}_{\mathcal{S}_\psi(G)}$, then this amounts to

$$(-1)^{|S|}\varepsilon(1/2, \Pi \times \eta^{-1}) = 1.$$

We know from the local construction in § 6 that there exists a two-dimensional hermitian space V_v and $\pi_{V_v} \in \eta_{G(V_v)}\Pi_{\varphi_{\Pi,v}}(G(V_v))^\vee$ such that

$$\epsilon(V_v) = \begin{cases} -\varepsilon(1/2, \Pi_v \times \eta_v^{-1}, \psi_{F_v})\omega_{\Pi_v}(-1)\lambda(E_v/F_v, \psi_{F_v})^{-2} & \text{if } v \in S, \\ \varepsilon(1/2, \Pi_v \times \eta_v^{-1}, \psi_{F_v})\omega_{\Pi_v}(-1)\lambda(E_v/F_v, \psi_{F_v})^{-2} & \text{otherwise,} \end{cases} \tag{7.1}$$

$$\pi_v \simeq \theta_{\eta_v}(\pi_{V_v}, W_2). \tag{7.2}$$

Applying the product formula for the Langlands λ-factor, (7.1) gives

$$\prod_v \epsilon(V_v) = (-1)^{|S|}\varepsilon(1/2, \Pi \times \eta^{-1}) = 1,$$

so that there exists a two-dimensional hermitian space $(V, (\ ,\ ))$ over E such that $V_v \simeq V \otimes_F F_v$ at every v. Since the global L-packet $\Pi_{\varphi_\Pi}(G(V))$ is stable, the irreducible representation $\pi_V := \bigotimes_v \pi_{V_v}$ occurs in $\mathcal{A}_0(G(V)\backslash G(V_{\mathbb{A}}))$ [LL79, Prop. 7.2]. We take a realization $\mathcal{A}(\pi_V^\vee)$ of $\pi_V^\vee$ in $\mathcal{A}_0(G(V)\backslash G(V_{\mathbb{A}}))$ and use it to construct $\Theta_\eta(\pi_V, W_2)$. Since we are in the stable range, this is a non-trivial subspace of $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$, the space of automorphic forms on $G(\mathbb{A})$.

We apply the induction principle for global θ-correspondence [Ral84]. First suppose that S is empty. If $L(1/2, \Pi \times \eta^{-1}) \neq 0$, then π appears in the residual discrete spectrum [Kon98, Th. 1.1 (5)] and the theorem follows. Otherwise, [Har93, Th. 4.5] asserts that $\Theta_\eta(\pi_V, W_1) = \{0\}$. In other words $\Theta_\eta(\pi_V, W_2)$ is the early lift and hence belongs to the cuspidal spectrum. If S is non-empty, $\Theta_\eta(\pi_V, W_1) = \{0\}$ because $\Theta_{\eta_v}(\pi_{V_v}, W_1) = \{0\}$ at $v \in S$. Thus $\Theta_\eta(\pi_V, W_2)$ belongs to the cuspidal spectrum by the same reason as above.

We have to complete the proof only in the cases $S \neq \emptyset$ or $L(1/2, \Pi \times \eta^{-1}) = 0$. Since $\theta_{\eta_v}(\pi_{V_v}, W_2) = \pi_v$ by construction, $\Theta_\eta(\pi_V, W_2)$ must contain an irreducible component isomorphic to π. □

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