

Extensions of truncated discrete valuation rings

Dedicated to Professor Jean-Pierre Serre
on the Occasion of His Eightieth Birthday

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Abstract

An equivalence is established between the category of at most a -ramified finite separable extensions of a complete discrete valuation field K and the category of at most a -ramified finite extensions of the “length- a truncation” $\mathcal{O}_K/\mathfrak{m}_K^a$ of the integer ring of K .

1 Introduction

Let K be a complete discrete valuation field (abbr. cdvf in the following), \mathcal{O}_K its valuation ring, and \mathfrak{m}_K its maximal ideal. Let a be an integer ≥ 1 . In this paper, we prove that the category $\mathcal{FE}_K^{\leq a}$ of finite étale K -algebras with ramification “bounded by a ” (cf. Def. 3.1) depends only on $\mathcal{O}_K/\mathfrak{m}_K^a$. More precisely, let m be any rational number such that $0 < m \leq a$ and put $A = \mathcal{O}_K/\mathfrak{m}_K^a$. We give an equivalence of $\mathcal{FE}_K^{\leq m}$ with a category $\mathcal{FFP}_A^{\leq m}$ of finite flat principal A -algebras² with ramification “bounded by m ” (cf. Def. 3.2). The morphisms in $\mathcal{FFP}_A^{\leq m}$ are defined (cf. Def. 3.3) by using Hattori’s functor ([6]); they are the usual A -algebra homomorphisms modulo a certain equivalence relation.

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²We mean by a *principal* A -algebra an A -algebra of which every ideal is generated by one element. All algebras in this paper are commutative.

For each object L in $\mathcal{FE}_{\bar{K}}^{\leq m}$, let \mathcal{O}_L be the integral closure of \mathcal{O}_K in L . Then the quotient ring $T(L) := \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L$ is an object of $\mathcal{FFP}_A^{\leq m}$ (Cor. 3.5). This correspondence $L \mapsto T(L)$ is functorial, and thus we obtain a functor

$$T : \mathcal{FE}_{\bar{K}}^{\leq m} \rightarrow \mathcal{FFP}_A^{\leq m}.$$

Our main result in this paper is:

Theorem 1.1. *The functor T is an equivalence of categories.*

Remarks. (i) The case of $a = 1$ in the Theorem is well-known (*cf.* [12], Chap. III, §5). Indeed, if $m \leq 1$, the objects of $\mathcal{FE}_{\bar{K}}^{\leq m}$ are direct products of finite unramified extensions of K , and the Theorem implies that the objects of $\mathcal{FFP}_A^{\leq m}$ are étale over A . Thus our main interest is in the case $a > 1$.

(ii) Let $G_K = \text{Gal}(\bar{K}/K)$ denote the absolute Galois group of K , and G_K^a its a th ramification subgroup defined by Abbes and Saito ([2], [3]). The category $\mathcal{FE}_{\bar{K}}^{\leq m}$ is, and hence $\mathcal{FFP}_A^{\leq m}$ is also, a Galois category whose fundamental group is G_K/G_K^m by the very definition of the ramification filtration (*cf.* Sect. 3). Note that $\mathcal{FE}_{\bar{K}}^{\leq m}$ is equivalent also to the category of coverings of $\text{Spec}(\mathcal{O}_K)$ with ramification bounded by \mathfrak{m}_K^m ([7], Def. 2.3); in the terminology of *op. cit.*, we have $\pi_1(\text{Spec}(\mathcal{O}_K), \mathfrak{m}_K^m) = G_K/G_K^m$.

A finite étale K -algebra is the direct product of a finite number of finite separable extension fields of K . Similarly, a finite flat principal A -algebra is the direct product of a finite number of local objects (*cf.* [9], Th. 1.1, Th. 1.2). Since the boundedness of ramification of direct products of K - and A -algebras may be considered componentwise, the above Theorem is equivalent with the following Corollary, in which $\text{FE}_{\bar{K}}^{\leq m}$ (resp. $\text{FFP}_A^{\leq m}$) denotes the full subcategory of $\mathcal{FE}_{\bar{K}}^{\leq m}$ (resp. $\mathcal{FFP}_A^{\leq m}$) consisting of local rings.

Corollary 1.2. *The functor T induces an equivalence $\text{FE}_{\bar{K}}^{\leq m} \simeq \text{FFP}_A^{\leq m}$.*

This extends a theorem of Deligne ([4], Th. 2.8) to the imperfect residue field case, except that our construction of the category $\mathcal{FFP}_A^{\leq m}$ for $A = \mathcal{O}_K/\mathfrak{m}_K^a$ depends on the cdvf K and hence our result is somewhat weaker than the “true” generalization of Deligne’s theorem. We expect, however, the category $\mathcal{FFP}_A^{\leq m}$ depends only on the isomorphism class of A as a ring (such a ring as $A = \mathcal{O}_K/\mathfrak{m}_K^a$ is called a *truncated discrete valuation ring*; see Section 2). If this is the case, we may define the Galois group G_A of A to be G_K/G_K^a (or equivalently, to be the fundamental group of the Galois category

$\mathcal{FFP}_A^{\leq a}$) together with the ramification subgroups $G_A^m := G_K^m/G_K^a$, where K is any cdvf such that $A \simeq \mathcal{O}_K/\mathfrak{m}_K^a$. The filtered group G_A should depend (up to inner automorphisms) only on the isomorphism class of A as a ring. It is natural to ask the converse:

Question. If A and A' are two truncated discrete valuation rings of length a and if there is an isomorphism $\gamma : G_A \rightarrow G_{A'}$ of groups such that $\gamma(G_A^m) = G_{A'}^m$ for all $m \leq a$, then is it true that $A \simeq A'$ as a ring?

This problem is a version of the Grothendieck conjecture in anabelian geometry. It will certainly be necessary to assume that the residue fields of A and A' are either finite or of some “anabelian” nature. For the case of local fields (or, the case of “ $a = \infty$ ” and finite residue fields), see [10] and [1].

In Section 2, we study basic properties of truncated discrete valuation rings. After recalling some basics of the ramification theory of Abbes-Saito ([2], [3]) and Hattori ([6]), we construct the category $\mathcal{FFP}_A^{\leq m}$ and prove the Theorem in Section 3.

Throughout this paper, K is a complete discrete valuation field with residual characteristic $p > 0$. We denote by \mathcal{O}_K the valuation ring of K , \mathfrak{m}_K the maximal ideal of \mathcal{O}_K , π_K a uniformizing element of K , and \bar{K} a fixed separable closure of K . For any étale K -algebra L , we denote by \mathcal{O}_L the integral closure of \mathcal{O}_K in L . For A -algebras B and B' , we denote by $\text{Hom}_A(B, B')$ the set of A -algebra homomorphisms $B \rightarrow B'$. We use the following abbreviations:

cdvf := complete discrete valuation field,
 cdvr := complete discrete valuation ring,
 tdvr := truncated discrete valuation ring.

It is our pleasure to dedicate this paper to Professor Jean-Pierre Serre, whose mathematical influence on us has been enormous. In particular, the Book *Corps Locaux* has ever been our main source of inspiration in ramification theory.

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2 Truncated discrete valuation rings

A *tdvr* is an Artinian local ring whose maximal ideal is generated by one element. The *length* of a tdivr A is the length of A as an A -module. It is known that a tdivr A is principal, and any ideal is of the form \mathfrak{m}_A^i for some $i \geq 0$ if \mathfrak{m}_A is the maximal ideal of A . Any generator π_A of \mathfrak{m}_A is said to be a *uniformizer* of A . Any non-zero element x of A can be written as $x = u\pi_A^i$ with $u \in A^\times$, π_A a uniformizer of A , and $0 \leq i < \text{length}(A)$ (with the convention $0^0 = 1$ if $\text{length}(A) = 1$). If $\text{length}(A) > 1$ (resp. $\text{length}(A) = 1$), we mean by an *extension* B/A of tdivr's a local ring homomorphism $A \rightarrow B$ of tdivr's via which B is flat over A (resp. an extension B/A of fields); thus we refrain from calling a homomorphism such as $A \hookrightarrow A[t]/(t^a)$ an extension if A is a field. An extension B/A is said to be *finite* if B is finite as an A -module. If $a > 1$, an A -algebra is a finite extension of A if and only if it is finite, flat, principal and local. In general, the objects of the category $\text{FFP}_A^{\leq m}$ are finite extensions of the tdivr A . The *ramification index* $e_{B/A}$ of a homomorphism $f : A \rightarrow B$ of tdivr's is defined to be the integer e such that $f(\mathfrak{m}_A)B = \mathfrak{m}_B^e$ (with the convention $e_{B/A} = 1$ if $\text{length}(A) = 1$). Note that the homomorphism f is an extension of tdivr's if and only if one has the equality $\text{length}(B) = e_{B/A} \text{length}(A)$ (cf. [4], Sect. 1.4 and [8], Exer. 22.1).

Lemma 2.1. *Let B and C be extensions of A . Then any A -algebra homomorphism $f : B \rightarrow C$ is an extension.*

Proof. We have to show that $\text{length}(C) = e_{C/B} \text{length}(B)$. We may assume that $\text{length}(A) > 1$. Let \mathfrak{m}_A , \mathfrak{m}_B and \mathfrak{m}_C be respectively the maximal ideals of A , B and C . By the definition of ramification index, we have $\mathfrak{m}_A B = \mathfrak{m}_B^{e_{B/A}}$, $\mathfrak{m}_A C = \mathfrak{m}_C^{e_{C/A}}$, and $f(\mathfrak{m}_B)C = \mathfrak{m}_C^{e_{C/B}}$. The equality $\mathfrak{m}_C^{e_{C/A}} = f(\mathfrak{m}_B^{e_{B/A}})C$ (= the ideal generated by \mathfrak{m}_A) implies that $e_{C/A} = e_{C/B} e_{B/A}$. Since B and C are extensions of A , we have $\text{length}(C) = e_{C/A} \text{length}(A) = e_{C/B} e_{B/A} \text{length}(A) = e_{C/B} \text{length}(B)$. \square

If K is a cdvf, then $\mathcal{O}_K/\mathfrak{m}_K^a$ is a tdivr for any integer $a \geq 1$. If L/K is a finite extension of cdvf's, then $B = \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L$ is a finite extension of $A = \mathcal{O}_K/\mathfrak{m}_K^a$. Conversely, it is known that any tdivr is a quotient of a cdvf ([9], Th. 3.3). More precisely, we have:

Proposition 2.2. (i) *Let A be a tdivr with residue field k of characteristic $p \geq 0$, and let a be the length of A . Then there exists a cdvf \mathcal{O} such that A*

is isomorphic to $\mathcal{O}/\mathfrak{m}^a$, where \mathfrak{m} is the maximal ideal of \mathcal{O} . If $pA = 0$, then this \mathcal{O} can be taken to be the power series ring $k[[\pi]]$; if $pA \neq 0$, then \mathcal{O} as above must be finite over a Cohen p -ring ([5], 0_{IV}, 19.8) with residue field k . (If $pA = 0$ and $p \neq 0$, then both types of \mathcal{O} are possible.)

(ii) Let K be a cdvf and let $A = \mathcal{O}_K/\mathfrak{m}_K^a$ with $a \geq 1$. For any finite extension B/A of tdrv's, there exist a finite separable extension L/K and an isomorphism $\psi : \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L \rightarrow B$ such that the diagram

$$(1) \quad \begin{array}{ccc} \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L & \xrightarrow{\psi} & B \\ \uparrow & & \uparrow \\ \mathcal{O}_K/\mathfrak{m}_K^a & \xlongequal{\quad} & A \end{array}$$

is commutative, where the left vertical arrow is the one induced by $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$.

Proof. (i) Let W be a Cohen p -ring with residue field k . The reduction map $W \rightarrow k$ lifts by the formal smoothness of W to a local ring homomorphism $W \rightarrow A$ ([5], 0_{IV}, 19.8.6).

If $pA = 0$, the map $W \rightarrow A$ factors through the residue field k , which makes A a k -algebra. Then there exists a surjective A -algebra homomorphism $k[[\pi]] \rightarrow A$ which maps π to π_A , where π_A is a uniformizer of A . Hence A is isomorphic to $k[[\pi]]/(\pi^a)$ (cf. [9], Th. 3.1).

In the general case, we can write A as a quotient of the polynomial ring $W[X]$ by sending X to π_A . Then we obtain a surjection onto A from a cdvf \mathcal{O} which is finite over W by the same procedure as in the proof of (ii) below.

(ii) Since B is finite over $A = \mathcal{O}_K/\mathfrak{m}_K^a$, there exists a surjective \mathcal{O}_K -algebra homomorphism $\phi : R \rightarrow B$ from a polynomial ring $R = \mathcal{O}_K[X_1, \dots, X_n]$ onto B . Let $\mathfrak{m} = \phi^{-1}(\mathfrak{m}_B)$ and $R_{\mathfrak{m}}$ the localization of R at the maximal ideal \mathfrak{m} . Then $R_{\mathfrak{m}}$ is a regular local ring of Krull dimension $n+1$ ([5], 0_{IV}, 17.3.7), and ϕ extends to a surjective \mathcal{O}_K -algebra homomorphism $\varphi : R_{\mathfrak{m}} \rightarrow B$. By abuse of notation, we denote also by \mathfrak{m} the maximal ideal of $R_{\mathfrak{m}}$. Put $\mathfrak{n} = \text{Ker}(\varphi)$. We identify the residue field k' of $R_{\mathfrak{m}}$ with that of B via φ . Since $\varphi(\mathfrak{m}^2) = \mathfrak{m}_B^2$, the map φ induces a surjective k' -linear map $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ and its kernel is $(\mathfrak{n} + \mathfrak{m}^2)/\mathfrak{m}^2 \simeq \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$. Thus we have an exact sequence

$$0 \rightarrow \mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2) \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2 \rightarrow 0.$$

Assume $a \geq 2$, as the case $a = 1$ can be treated similarly and more easily. Then $\dim_{k'}(\mathfrak{m}_B/\mathfrak{m}_B^2) = 1$ and $\dim_{k'}(\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)) = n$. Choose a regular

system of parameters (w, f_1, \dots, f_n) of $R_{\mathfrak{m}}$ such that $\varphi(w)$ gives a basis of $\mathfrak{m}_B/\mathfrak{m}_B^2$ and $f_1, \dots, f_n \in \mathfrak{n}$ give a basis of $\mathfrak{n}/(\mathfrak{n} \cap \mathfrak{m}^2)$. Let \mathfrak{p} be the ideal of $R_{\mathfrak{m}}$ generated by f_1, \dots, f_n . Then by [5], 0IV, Prop. 17.1.7, the quotient ring $\mathcal{O} = R_{\mathfrak{m}}/\mathfrak{p}$ is a regular local ring of dimension 1 and hence a discrete valuation ring. It contains \mathcal{O}_K since φ maps π_K to a non-zero non-unit in B , and is finite over \mathcal{O}_K . Hence it is a cdvr. Since $\mathfrak{n} \supset \mathfrak{p}$, the map φ factors through \mathcal{O} . Thus we see the diagram (1) commutes (with \mathcal{O} in place of \mathcal{O}_L). Since B is flat over A , the induced homomorphism ψ is bijective.

To make the fraction field L of \mathcal{O} separable over K , we “deform” the prime ideal \mathfrak{p} if necessary. By multiplying the f_i with some $u \in R \setminus \mathfrak{m}$, we may assume that all f_i are in the polynomial ring R . Note that the composite map $R \hookrightarrow R_{\mathfrak{m}} \rightarrow R_{\mathfrak{m}}/\mathfrak{p} = \mathcal{O}$ is surjective by Nakayama’s lemma, since its image generates $B = \mathcal{O}/\mathfrak{m}_K^a \mathcal{O}$. Let \mathfrak{q} be its kernel, so that $\mathcal{O} = R/\mathfrak{q}$. We have $\mathfrak{q}R_{\mathfrak{m}} = \mathfrak{p}$, *i.e.*, \mathfrak{q} is generated by f_1, \dots, f_n locally at \mathfrak{m} . By the Jacobian criterion ([11], V, Sect. 2, Th. 5), the K -algebra L is separable (*i.e.*, the \mathcal{O}_K -algebra \mathcal{O} is étale at the generic point of $\text{Spec}(\mathcal{O})$) if and only if the Jacobian $\det \left(\frac{\partial f_i}{\partial X_j} \right)_{1 \leq i, j \leq n} \not\equiv 0 \pmod{\mathfrak{q}}$. Let $g_i := f_i + xX_i$ with $x \in \mathfrak{m}_K^a$. Then, since $g_i \in \mathfrak{n}$ and $g_i \equiv f_i \pmod{\mathfrak{n} \cap \mathfrak{m}^2}$, the ideal $\mathfrak{p}' = (g_1, \dots, g_n)$ of $R_{\mathfrak{m}}$ has similar properties as \mathfrak{p} so that the quotient ring $\mathcal{O}' := R_{\mathfrak{m}}/\mathfrak{p}'$ is a cdvr which contains \mathcal{O}_K and surjects onto B . Moreover, if $J := \left(\frac{\partial f_i}{\partial X_j} \right)_{1 \leq i, j \leq n}$, we have

$$\det \left(\frac{\partial g_i}{\partial X_j} \right)_{1 \leq i, j \leq n} = \det(xI_n + J) = x^n + \text{Tr}(J)x^{n-1} + \dots + \det(J).$$

Considering this modulo \mathfrak{q} and noticing that $\mathcal{O}_K \subset \mathcal{O} = R/\mathfrak{q}$, we find an $x \in \mathfrak{m}_K^a$ such that $\det \left(\frac{\partial g_i}{\partial X_j} \right) \not\equiv 0 \pmod{\mathfrak{q}}$. Then the fraction field of \mathcal{O}' is separable over K . \square

3 Ramification

Let G_K be the absolute Galois group of K . A. Abbes and T. Saito ([2], [3]) defined a decreasing filtration $(G_K^m)_{m \geq 0}$ by closed normal subgroups G_K^m of G_K indexed with rational numbers $m \geq 0$, in such a way that $\bigcap_{m \geq 0} G_K^m = 1$, $G_K^0 = G_K$ and G_K^1 is the inertia subgroup of G_K . The filtration coincides

with the classical upper numbering ramification filtration shifted by one if the residue field of K is perfect (see [12], Chap. IV, §3, for the classical case). It is defined by using certain functors F and F^m from the category \mathcal{FE}_K of finite étale K -algebras to the category \mathcal{S}_K of finite G_K -sets. We recall here the definition of F and F^m assuming for simplicity that m is a positive integer. Let L be a finite étale K -algebra, and let \mathcal{O}_L be the integral closure of \mathcal{O}_K in L . We define $F(L) := \text{Hom}_K(L, \bar{K}) = \text{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{\bar{K}})$. The functor F gives an anti-equivalence of \mathcal{FE}_K with \mathcal{S}_K , thereby making \mathcal{FE}_K a Galois category. To define F^m , we proceed as follows: An *embedding* of \mathcal{O}_L is a pair $(\mathbb{B}, \mathbb{B} \rightarrow \mathcal{O}_L)$ consisting of an \mathcal{O}_K -algebra \mathbb{B} which is formally of finite type and formally smooth over \mathcal{O}_K and a surjection $\mathbb{B} \rightarrow \mathcal{O}_L$ of \mathcal{O}_K -algebras which induces an isomorphism $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \rightarrow \mathcal{O}_L/\mathfrak{m}_L$, where $\mathfrak{m}_{\mathbb{B}}$ and \mathfrak{m}_L are respectively the radicals of \mathbb{B} and \mathcal{O}_L (cf. [3], Def. 1.1). Let I be the kernel of the surjection $\mathbb{B} \rightarrow \mathcal{O}_L$. Define an affinoid algebra \mathbf{B}^m over K by $\mathbf{B}^m = \mathbb{B}[I/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$, where \wedge means the π_K -adic completion. Let $X^m(\mathbb{B} \rightarrow \mathcal{O}_L)$ be the affinoid variety $\text{Sp}(\mathbf{B}^m)$ associated with \mathbf{B}^m . For any affinoid variety X over K , let $\pi_0(X_{\bar{K}})$ denote the set $\varprojlim_{K'} \pi_0(X \otimes_K K')$ of geometric connected components, where K' runs through the finite separable extensions of K . Then we define the functor F^m by

$$F^m(L) := \varprojlim_{(\mathbb{B} \rightarrow \mathcal{O}_L)} \pi_0(X^m(\mathbb{B} \rightarrow \mathcal{O}_L)_{\bar{K}}),$$

where $(\mathbb{B} \rightarrow \mathcal{O}_L)$ runs through the category of embeddings of \mathcal{O}_L . The projective system in the right-hand side is constant. The finite set $F(L)$ can be identified with a subset of $X^m(\mathbb{B} \rightarrow \mathcal{O}_L)_{\bar{K}}$, and this causes a natural surjective map $F(L) \rightarrow F^m(L)$. The m th ramification subgroup G_K^m is characterized by the property that $F(L)/G_K^m = F^m(L)$ for all L .

Definition 3.1 ([2], Def. 6.3). Let L be a finite étale K -algebra. We say that the *ramification of L is bounded by m* if $F(L) \rightarrow F^m(L)$ is bijective.

Thus the category $\mathcal{FE}_{\bar{K}}^{\leq m}$ of finite étale K -algebras with ramification bounded by m forms a Galois full-subcategory of \mathcal{FE}_K whose fundamental group is G_K/G_K^m ([2], Prop. 2.1) as noted in the Introduction. Note that the above definition of “ramification bounded by m ” coincides with Deligne’s one in [4] when L is a field and \mathcal{O}_L is monogenic over \mathcal{O}_K (cf. [2], Prop. 6.7).

Let a be an integer ≥ 1 , and put $A = \mathcal{O}_K/\mathfrak{m}_K^a$. For each rational number $0 < m \leq a$, Hattori ([6]) defined another functor \mathcal{F}^m from the category of

finite flat A -algebras to the category \mathcal{S}_K of finite G_K -sets. We next recall the definition of \mathcal{F}^m assuming for simplicity that m is a positive integer. Let B be a finite flat A -algebra. An *embedding* of B is a pair $(\mathbb{B}, \mathbb{B} \rightarrow B)$ consisting of an \mathcal{O}_K -algebra \mathbb{B} which is formally of finite type and formally smooth over \mathcal{O}_K and a surjection $\mathbb{B} \rightarrow B$ of \mathcal{O}_K -algebras which induces an isomorphism $\mathbb{B}/\mathfrak{m}_{\mathbb{B}} \rightarrow B/\mathfrak{m}_B$, where $\mathfrak{m}_{\mathbb{B}}$ and \mathfrak{m}_B are respectively the radicals of \mathbb{B} and B . Let \mathcal{I} be the kernel of the surjection $\mathbb{B} \rightarrow B$. Define an affinoid algebra \mathcal{B}^m over K by $\mathcal{B}^m = \mathbb{B}[\mathcal{I}/\pi_K^m]^\wedge \otimes_{\mathcal{O}_K} K$. Let $\mathcal{X}^m(\mathbb{B} \rightarrow B)$ be the affinoid variety $\mathrm{Sp}(\mathcal{B}^m)$ associated with \mathcal{B}^m . Then we define the functor \mathcal{F}^m by

$$\mathcal{F}^m(B) := \varprojlim_{(\mathbb{B} \rightarrow B)} \pi_0(\mathcal{X}^m(\mathbb{B} \rightarrow B)_{\bar{K}}),$$

where $(\mathbb{B} \rightarrow B)$ runs through the category of embeddings of B . In general, we have $\sharp \mathcal{F}^m(B) \leq \mathrm{rank}_A(B)$. Two key definitions in this paper are the following:

Definition 3.2. Let B be a finite flat A -algebra. We say that the *ramification of B is bounded by m* if $\sharp \mathcal{F}^m(B) = \mathrm{rank}_A(B)$.

Definition 3.3. For any rational number m with $0 < m \leq a$, we define $\mathcal{FFP}_A^{\leq m}$ to be the category whose objects are finite flat principal A -algebras with ramification bounded by m and whose morphisms are defined as follows: For any B and B' in $\mathcal{FFP}_A^{\leq m}$, set

$$(2) \quad \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B') := \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)).$$

We also define $\mathrm{FFP}_A^{\leq m}$ to be the full-subcategory of $\mathcal{FFP}_A^{\leq m}$ consisting of local objects.

To prove Theorem 1.1, we recall the following lemma due to Hattori ([6], Lemma 1):

Lemma 3.4. *Let L be a finite étale K -algebra, and a an integer ≥ 1 . If $B = \mathcal{O}_L/\mathfrak{m}_K^a \mathcal{O}_L$, then we have $\mathcal{F}^m(B) = F^m(L)$ as an object of \mathcal{S}_K for any rational number $0 < m \leq a$.*

This is because one may choose a common \mathbb{B} in the embeddings $(\mathbb{B}, \mathbb{B} \rightarrow \mathcal{O}_L)$ and $(\mathbb{B}, \mathbb{B} \rightarrow B)$, so that, if $m \leq a$, we have $X^m(\mathbb{B} \rightarrow \mathcal{O}_L) = \mathcal{X}^m(\mathbb{B} \rightarrow B)$.

By Definitions 3.1 and 3.2, we have:

Corollary 3.5. *For any rational number $0 < m \leq a$, the ramification of B is bounded by m if and only if the ramification of L is bounded by m .*

Now we can prove Theorem 1.1. The essential surjectivity of the functor $T : \mathcal{FE}_{\bar{K}}^{\leq m} \rightarrow \mathcal{FFP}_A^{\leq m}$ follows from (ii) of Proposition 2.2 and Corollary 3.5, since any object of $\mathcal{FFP}_A^{\leq m}$ is a direct product of finite extensions of A . To prove the full-faithfulness of T , let L and L' be two objects in $\mathcal{FE}_{\bar{K}}^{\leq m}$, and let $B = T(L)$ and $B' = T(L')$. Since the functor F^m gives an anti-equivalence of the Galois category $\mathcal{FE}_{\bar{K}}^{\leq m}$ with a full-subcategory of \mathcal{S}_K , we have

$$\mathrm{Hom}_{\mathcal{FE}_{\bar{K}}^{\leq m}}(L, L') \simeq \mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)).$$

By Lemma 3.4, we have

$$\mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) = \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)).$$

It follows from our definition (2) of Hom in $\mathcal{FFP}_A^{\leq m}$ that

$$\mathrm{Hom}_{\mathcal{FE}_{\bar{K}}^{\leq m}}(L, L') = \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B').$$

This completes the proof of the Theorem.

Remark. The relation of $\mathrm{Hom}_A(B, B')$ to the Hom sets appearing in the above proof is summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_K(L, L') & \xrightarrow[\mathcal{F}^m]{\simeq} & \mathrm{Hom}_{\mathcal{S}_K}(F^m(L'), F^m(L)) \\ \downarrow & & \parallel \\ \mathrm{Hom}_A(B, B') & \xrightarrow[\mathcal{F}^m]{} & \mathrm{Hom}_{\mathcal{S}_K}(\mathcal{F}^m(B'), \mathcal{F}^m(B)), \end{array}$$

where the left vertical arrow is the reduction mod \mathfrak{m}_K^a of $\mathrm{Hom}_K(L, L') = \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_{L'})$. This shows that the map $\mathcal{F}^m : \mathrm{Hom}_A(B, B') \rightarrow \mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B')$ is surjective and compatible with the composition of morphisms. It can be shown that this map identifies the set $\mathrm{Hom}_{\mathcal{FFP}_A^{\leq m}}(B, B')$ with the quotient of $\mathrm{Hom}_A(B, B')$ by an equivalence relation \simeq^A defined as follows: Put $\bar{A} = \mathcal{O}_{\bar{K}}/\mathfrak{m}_K^a \mathcal{O}_{\bar{K}}$ and let \mathcal{X}^m be the affinoid variety associated with an embedding of B . Recall that there exists a natural surjective map $\mathcal{X}^m(\bar{K}) \rightarrow \mathrm{Hom}_A(B, \bar{A})$ with connected fibers ([2], Lem. 3.2), so that its inverse yields a well-defined map $\xi : \mathrm{Hom}_A(B, \bar{A}) \rightarrow \pi_0(\mathcal{X}_{\bar{K}}^m)$. Then we have a map

$$\mathrm{Hom}_A(B, B') \times \mathrm{Hom}_A(B', \bar{A}) \rightarrow \pi_0(\mathcal{X}_{\bar{K}}^m)$$

which maps (f, α) to $\xi(\alpha \circ f)$. For f and f' in $\text{Hom}_A(B, B')$, define

$$f \simeq f' \iff \xi(\alpha \circ f) = \xi(\alpha \circ f') \quad \text{for all } \alpha \in \text{Hom}_A(B', \bar{A}).$$

It can also be shown that, if B' is local, then for given f and f' , the equality $\xi(\alpha \circ f) = \xi(\alpha \circ f')$ holds for all $\alpha \in \text{Hom}_A(B', \bar{A})$ if it holds for some α .

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