A SIMPLE PROOF OF THE MODULAR IDENTITY FOR THETA SERIES

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ABSTRACT. We characterize the function spanned by theta series. As an application we derive a simple proof of the modular identity of the theta series.

1. Introduction

The standard proof of the theta modular transformation formula uses the Poisson summation formula. In [1] using the elliptic shift relations a simple proof of the theta modular transformation formula was given for the theta function $\theta(\tau, z) = \sum_{r\in\mathbb{Z}} e^{\pi i r^2\tau + 2\pi i r z}$. In this paper we show that a similar method can be employed to derive the theta modular transformation formula for more general theta series by taking into account the heat kernel property in addition to the elliptic shift relations. These properties characterize the functions spanned by certain theta functions (Section 2). This is applied in Section 3 to derive the modular identity for the theta series.

2. Theta series

Let $\mathcal{H} \subset \mathbb{C}$ be the complex upper half plane. The following theta series has been studied in connection with Jacobi forms ([2], §5); for each $m \in \mathbb{Z}$ $(m \geq 1)$ and each $\mu \pmod{2m}$,

$$\theta_{m,\mu}(\tau, z) = \sum_{r\in\mathbb{Z}, r\equiv\mu \pmod{2m}} q^{\frac{r^2}{4m}} \xi^r, \qquad q = e^{2\pi i\tau}, \xi = e^{2\pi i z}. \tag{2.1}$$

This series converges uniformly in $z \in \mathbb{C}$ and $\tau \in \mathcal{H}$ and hence defines a holomorphic function on $\mathcal{H} \times \mathbb{C}$. One easily checks that $\theta_{m,\mu}(\tau, z+1) =$

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$\theta_{m,\mu}(\tau,z)$ and $\theta_{m,\mu}(\tau,z+\tau)=q^{-m}\xi^{-2m}\theta_{m,\mu}(\tau,z)$. Further one can check that $L_m(\theta_{m,\mu}(\tau,z))=0$, where L_m is the heat operator defined by

$$L_m = 8\pi i m\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial z^2}.$$

Since these conditions are $\mathbb{C}$-linear, any $\mathbb{C}$-linear combination of the $\theta_{m,\mu}(\tau,z)$'s satisfies these three conditions. Conversely, we prove the following result:

Theorem 2.1. *Let $f(\tau,z)$ be a holomorphic function defined over $\mathcal{H}\times\mathbb{C}$ satisfying the following three conditions;*

1. *(***shift***) $f(\tau,z+1)=f(\tau,z)$,*
2. *(***elliptic property***) $f(\tau,z+\tau)=e^{-2\pi i m(\tau+2z)}f(\tau,z)$,*
3. *(***heat kernel***) $L_m(f(\tau,z))=0$.*

Then $f(\tau,z)$ belongs to the vector space spanned by $\{\theta_{m,\mu}(\tau,z)\,|\,0\le\mu\le(2m-1)\}$ over $\mathbb{C}$.

Proof Since f is periodic with respect to z, f has the following Fourier expansion in $\xi=e^{2\pi iz}$:

$$f(\tau,z)=\sum_{r\in\mathbb{Z}}\phi_r(\tau)\xi^r.$$

Secondly, since $L_m(f)=0$, we note that

$$4m\frac{d\phi_r(\tau)}{d\tau}=2\pi i r^2\phi_r(\tau),$$

so that $\phi_r(\tau)=c_r e^{2\pi i\tau\frac{r^2}{4m}}=c_r q^{\frac{r^2}{4m}}$, where c_r is a constant depending on r. So, $f(\tau,z)=\sum_{r\in\mathbb{Z}}c_r q^{\frac{r^2}{4m}}\xi^r$. On the other hand, since

$$\begin{aligned} f(\tau,z+\tau) &= \sum_{r\in\mathbb{Z}}c_r q^{\frac{r^2}{4m}}(\xi q)^r=\sum_{r\in\mathbb{Z}}c_r q^{\frac{(r+2m)^2}{4m}}q^{-m}\xi^{(r+2m)-2m}\\ &= q^{-m}\xi^{-2m}\sum_{r\in\mathbb{Z}}c_{r-2m}q^{\frac{r^2}{4m}}\xi^r,\end{aligned}$$

the elliptic property $f(\tau,z+\tau)=q^{-m}\xi^{-2m}f(\tau,z)$ implies that $c_r=c_{r+2m}$ for every $r\in\mathbb{Z}$. Now it follows that $f(\tau,z)=\sum_{\mu=0}^{2m-1}c_\mu\theta_{m,\mu}(\tau,z)$. □

Remark 2.2. (1) The case of $m = \frac{1}{2}$ was studied in [1]. In that case, only the first two conditions of Theorem2.1 (that is, the shift and elliptic properties) were needed to show that $f(\tau, z)$ is a constant multiple of the theta series $\theta_{1/2,0}(\tau, z) = \sum_{r\in\mathbb{Z}} e^{2\pi i r^2\tau+2\pi i r z}$. This is because such an $f(\tau, z)$ has only one zero, $\frac{\tau+1}{2}$, modulo the period lattice $\mathbb{Z}+\tau\mathbb{Z}$, which is already a strong enough restriction. This doesn't hold anymore in our case.

(2) The referee pointed out that characterization of theta series $\theta(\tau, z) = \sum_{r\in\mathbb{Z}} e^{\pi i r^2\tau+2\pi i z}$, i.e. the case when $m = \frac{1}{2}$, has been also studied in [3](see Proposition11.1 in page 53).

Theorem 2.3. *For each fixed τ, let $f(\tau, z)$ be a holomorphic function satisfying the two relations* (1) *and* (2) *in Theorem 2.1. Then*

(1) f vanishes identically or it has $2m$ roots (counting with multiplicity) modulo the lattice $\Lambda(\tau)$ generated by τ and 1, i.e., $\Lambda(\tau) = \mathbb{Z} + \tau\mathbb{Z}$.

(2) The sum of all the roots z_j modulo $\Lambda(\tau)$ of f satisfies

$$z_0 + z_1 + ... + z_{2m-1} \equiv m(\tau + 1) \pmod{\Lambda(\tau)}.$$

Proof Suppose that f does not vanish identically. The shift and elliptic relations imply that, for each τ,

$$\frac{\partial_z f(\tau, z+1)}{f(\tau, z+1)} = \frac{\partial_z f(\tau, z)}{f(\tau, z)}, \qquad \frac{\partial_z f(\tau, z+\tau)}{f(\tau, z+\tau)} = \frac{\partial_z f(\tau, z)}{f(\tau, z)} - 4\pi i m.$$

Let $\mathcal{F} = \{c + a + b\tau \mid a, b \in [0,\ 1]\}$, for $c \in \mathbb{C}$, be a closed fundamental domain of the lattice $\Lambda(\tau)$. By choosing c properly, we may assume that f has no roots on $\partial\mathcal{F}$. Then the number and the sum of roots of f on $\mathcal{F}$ can be computed respectively by the integrals

$$\frac{1}{2\pi i}\int_{\partial\mathcal{F}} \frac{\partial_z f(\tau, z)}{f(\tau, z)} dz \qquad \text{and} \qquad \frac{1}{2\pi i}\int_{\partial\mathcal{F}} \frac{z\partial_z f(\tau, z)}{f(\tau, z)} dz.$$

From the shift relations for f one can check that the first integral evaluates to $2m$, so f has exactly $2m$ roots on $\mathcal{F}$. The second integral evaluates

$$z_0 + z_1 + ... + z_{2m-1} \equiv m(\tau + 1) \pmod{\Lambda(\tau)}.$$

Now the proof is complete. □

Remark 2.4. The referee pointed out that the related result of Theorem2.3 is also contained in Lemma4.1 in [3].

3. The modular identity

Lemma 3.1. *Let*

$$F(\tau,z):=\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau}).$$

Then $F(\tau,z)$ *belongs to the vector space spanned by* $\{\theta_{m,\nu}(\tau,z)\,|\,0\le\nu\le(2m-1)\}$, *so there exist complex numbers* $c_{0,\mu},...,c_{2m-1,\mu}$ *such that*

$$\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau})=c_{0,\mu}\theta_{m,0}(\tau,z)+c_{1,\mu}\theta_{m,1}(\tau,z)+...+c_{2m-1,\mu}\theta_{m,2m-1}(\tau,z).$$

Proof We check the three conditions of Theorem 2.1 for $F(\tau,z)$. This can be done by straightforward calculations, and we omit the details. For example, the condition (1) can be checked as follows by looking at the Fourier expansion:

$$\begin{aligned}F(\tau,z+1) &= \sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{(z+1)^2}{\tau}}\sum_{r\in\mathbb{Z},r\equiv\mu \pmod{2m}}e^{-\frac{2\pi ir^2\tau}{4m}}e^{2\pi ir\frac{z+1}{\tau}}\\ &= \sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}\sum_{r\in\mathbb{Z},r\equiv\mu \pmod{2m}}e^{-\frac{2\pi i(r-2m)^2\tau}{4m}}e^{2\pi i(r-2m)\frac{z}{\tau}}\\ &= F(\tau,z).\end{aligned}$$

□

In what follows, we put $\zeta_{2m}=e^{\frac{2\pi i}{2m}}$.

Lemma 3.2. *(1)* $\theta_{m,\mu}(\tau,z+\frac{\tau}{2m})=q^{-\frac{1}{4m}}\xi^{-1}\theta_{m,\mu+1}(\tau,z)$ *for each* μ, $0\le\mu\le(2m-1)$. *Here we put* $\theta_{m,2m}(\tau,z)=\theta_{m,0}(\tau,z)$.

(2) $\theta_{m,\mu}(\tau,z+\frac{1}{2m})=\zeta_{2m}^{\mu}\theta_{m,\mu}(\tau,z)$.

Proof (1) We have

$$\begin{aligned}\theta_{m,\mu}(\tau,z+\frac{\tau}{2m}) &= \sum_{r\equiv\mu}q^{\frac{r^2}{4m}}\xi^r q^{\frac{r}{2m}}=\sum_{r\equiv\mu}q^{\frac{(r+1)^2}{4m}}q^{-\frac{1}{4m}}\xi^{(r+1)-1}\\ &= q^{-\frac{1}{4m}}\xi^{-1}\sum_{r\equiv\mu+1}q^{\frac{r^2}{4m}}\xi^r=q^{-\frac{1}{4m}}\xi^{-1}\theta_{m,\mu+1}(\tau,z).\end{aligned}$$

(2) can be check even more easily and the proof is omitted. □

Finally we can derive the following well-known modular transformation formula:

Theorem 3.3. *We have*

$$\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau})\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}$$

$$=\theta_{m,0}(\tau,z)+\zeta_{2m}^{-\mu}\theta_{m,1}(\tau,z)+...+\zeta_{2m}^{-(2m-1)\mu}\theta_{m,2m-1}(\tau,z)$$

Proof We prove this result by three steps.

(Step 1) First note that, from Lemma 3.1, we can let

(3.1)

$$\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau})\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}=c_{0,\mu}\theta_{m,0}(\tau,z)+c_{1,\mu}\theta_{m,1}(\tau,z)+...+c_{2m-1,\mu}\theta_{m,2m-1}(\tau,z).$$

Replacing z by $z+\tau/2m$ in this equation, we have

$$\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau}+\frac{1}{2m})\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{(z+\frac{\tau}{2m})^2}{\tau}}=c_{0,\mu}\theta_{m,0}(\tau,z+\frac{\tau}{2m})+...+c_{2m-1,\mu}\theta_{m,2m-1}(\tau,z+\frac{\tau}{2m}).$$

Using Lemma 3.2 and multiplying $q^{\frac{1}{4m}}\xi$ with the both sides, we have

(3.2)

$$\zeta_{2m}^{\mu}\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau})\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}=(c_{2m-1,\mu}\theta_{m,0}(\tau,z)+c_{0,\mu}\theta_{m,1}(\tau,z)+...+c_{2m-2,\mu}\theta_{m,2m-1}(\tau,z)).$$

By comparing the two equations (3.1) and (3.2), we conclude that

$$\sqrt{\frac{2mi}{\tau}}e^{-2\pi im\frac{z^2}{\tau}}\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau})=c_{0,\mu}\sum_{\lambda=0}^{2m-1}\zeta_{2m}^{-\lambda\mu}\theta_{m,\lambda}(\tau,z). \tag{3.3}$$

(Step 2) Next, we need to determine the constant $c_{0,\mu}$ in Equation (3.3) for each $\mu, 0\le\mu\le(2m-1)$. We shall use (3.3) twice to deduce that $c_{0,\mu}^2=1$. By replacing z by τz and then τ by $-\frac{1}{\tau}$ in (3.3), we have

$$\theta_{m,\mu}(\tau,z)=c_{0,\mu}\sqrt{\frac{-1/\tau}{2mi}}e^{-2\pi im\frac{z^2}{\tau}}\sum_{\lambda=0}^{2m-1}\zeta_{2m}^{-\lambda\mu}\theta_{m,\lambda}(-\frac{1}{\tau},-\frac{z}{\tau}).$$

Since

$$\theta_{m,\lambda}(\tau,-z)=\theta_{m,2m-\lambda}(\tau,z)$$

for each λ, we have

$$\theta_{m,\mu}(\tau,z) = c_{0,\mu}\sqrt{\frac{-1/\tau}{2mi}}e^{-2\pi i m\frac{z^2}{\tau}}\sum_{\lambda=0}^{2m-1}\zeta_{2m}^{\lambda\mu}\theta_{m,\lambda}(-\frac{1}{\tau},\frac{z}{\tau}).$$

Applying (3.3) again to the $\theta_{m,\lambda}(-\frac{1}{\tau},\frac{z}{\tau})$ on the right-hand side, we derive

$$\theta_{m,\mu}(\tau,z) = \frac{c_{0,\mu}}{2m}\sum_{\nu=0}^{2m-1}(\sum_{\lambda=0}^{2m-1} c_{0,\lambda}\zeta_{2m}^{\lambda(\mu-\nu)})\theta_{m,\nu}(\tau,z).$$

Since the $\theta_{m,\nu}(\tau,z)$'s are linearly independent over $\mathbb{C}$ (as can be seen from the Fourier series expansion), this implies that, for each μ and ν,

$$\frac{c_{0,\mu}}{2m}\sum_{\lambda=0}^{2m-1} c_{0,\lambda}\zeta_{2m}^{\lambda(\mu-\nu)} = \delta_{\mu,\nu}.$$

Here $\delta_{\mu,\nu}$ denotes the Kronecker delta function. Taking $\mu=\nu$ and setting $\alpha=\sum_{\lambda=0}^{2m-1}c_{0,\lambda}$, we have $\frac{c_{0,\mu}}{2m}\alpha=1$. In particular, the $c_{0,\mu}$ is independent of μ. It follows that $\alpha=2mc_{0,0}$ and $c_{0,0}^2=1$.

(Step 3) The positivity of $c_{0,0}$ can be seen as follows: looking at the Fourier series expansion we note that $\theta_{m,\mu}(it,0)>0$ for any real number $t>0$ and for every μ. So in the functional equation

$$\sqrt{\frac{2m}{t}}\theta_{m,0}(\frac{i}{t},0) = c_{0,0}(\theta_{m,0}(it,0)+\theta_{m,1}(it,0)+\ldots+\theta_{m,2m-1}(it,0)),$$

everything except $c_{0,0}$ is positive. This implies that $c_{0,0}>0$. By (Step 2) we conclude that $c_{0,0}=1$. This ends our proof. □

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