# Ramifications arising from Drinfeld modules

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## Introduction

In this paper, we study various ramifications arising from division points of Drinfeld modules, abelian $T$-modules, formal modules, etc.. A motivation for this is to know how many isogeny classes and isomorphism classes of Drinfeld $A$-modules exist over a finite extension of the fraction field of $A$. We will see ( cf. Remark (3.4) ) that, modulo the isogeny conjecture, an isogeny class can contain infinitely many isomorphism classes and, without any restriction on ramification at the infinite places, there can be infinitely many isogeny classes.

To explane some of the results, let $F$ be a function field in one variable over a finite field, $\infty$ a fixed place of $F$, $A$ the ring of elements of $F$ which are regular outside $\infty$, and $K$ a finite extension of $F$. Given a Drinfeld $A$-module $\phi$ over $K$ and a prime $v$ of $A$, we denote by $K(\phi; v^n)$ the field of $v^n$-division points of $\phi$. Then it turns out ( Corollary 1.6 ) that the ramification at various primes in the tower $(K(\phi; v^n)/K)_{n\geq 1}$ is bounded at the places over $\infty$ by a divisor depending only on $\phi$, and at the finite places, it is controlled in a fairly precise way in terms of the "discriminant" $\Delta(\phi)$. Roughly speaking, $\Delta(\phi)$ is the coefficient of the leading term of the defining equation of $\phi$. For finite places, this result is analogous to the case of abelian varieties over number fields. ( At least one has the Hermite-Minkovski theorem for number fields, which assures the existence of an estimate of discriminants. ) But at infinite places, there occur new phenomena, which we describe by example in §2. We construct explicitely an infinite family of Drinfeld modules with everywhere good reduction and with ramification at infinity becoming arbitrarily large ( Example 2.1 ), as well as an infinite family of mutually non-isomorphic Drinfeld modules with everywhere good reduction and with bounded ramification at infinity ( Example 2.2 ). In §3, we give a proposition on $v$-adic Galois representations ( a positive characteristic version of a theorem of Faltings ), and discuss how many isomorphism and isogeny classes can exist. §4 and §5 are generalizations of §1 to the cases of finite submodules of higher dimensional formal modules. Theorem (4.6) is an $A$-module version of Théorème 1 of [5].

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**Notation.** Throughout this article, $A$ is a "basic" Dedekind ring and $F$ is its fraction field; in §§1 – 3 ( global context ), $F$ is a function field in one variable over a finite field, $\infty$ is a fixed place of $F$, and $A$ is the ring of elements of $F$ regular outside $\infty$, whereas in §§4 and 5 ( local context ), $A$ is a complete discrete valuation ring with finite residue field.

In either context, $K$ will mainly be used to denote a finite extension of $F$, and then $\mathfrak{O}_K$ will denote the integral closure of $A$ in $K$.

If we are in the global context, a *prime* of $\mathfrak{O}_K$ means a non-zero prime ideal of $\mathfrak{O}_K$, which is identified with a place of $K$, and called a *finite place* of $K$. A place of $K$ is called *infinite* if it extends the place $\infty$ of $F$. A place of $K$ will often be identified with a *normalized* valuation of $K$. A non-zero fractional ideal of $\mathfrak{O}_K$ is often regarded as a divisor of $K$ and denoted additively. So we use notations like, e.g., $\mathfrak{a} \leq \mathfrak{b}$ for such $\mathfrak{a}$ and $\mathfrak{b}$. If $w$ is a prime of $\mathfrak{O}_K$ or a place of $K$, $\mathfrak{O}_{K,w}$ and $K_w$ denote respectively the completions of $\mathfrak{O}_K$ and $K$ with respect to $w$.

For a field $K$, $K^{sep}$ denotes a fixed separable closure of $K$, and $G_K$ denotes the absolute Galois group $\mathrm{Gal}(K^{sep}/K)$. For a finite separable extension $L/K$, $\mathfrak{D}(L/K)$ ( resp. $\mathfrak{d}(L/K)$ ) denotes the different ( resp. discriminant ) of $L/K$ if it can be defined at all.

If $G$ is a group scheme and $v$ is a non-zero element or a non-zero ideal of $\mathrm{End}(G)$, then ${}_vG$ denotes the subgroup scheme $\mathrm{Ker}(v)$ of $G$.

## 1. Finite places

In this section, we estimate the differents at finite places of the field extensions arising from division points of Drinfeld modules.

Let $F$ be a function field in one variable over a finite field, $\infty$ a fixed place of $F$, and $A$ the ring of elements of $F$ regular outside $\infty$. We assume that the field of constants is $\mathbb{F}_q$, the finite field with $q = p^f$ elements. For $a \in A - 0$, we define $\deg(a)$ by $\mathrm{Card}(A/aA) = q^{\deg(a)}$.

Let $K$ be a field of characteristic $p > 0$, and $\mathbb{G}_a$ the additive group scheme over $K$. After choosing a coordinate $X$ of $\mathbb{G}_a$, we can identify $\mathrm{End}_K(\mathbb{G}_a)$ with the non-commutative ring of additive polynomials of $X$ with coefficients in $K$, where the product is the composition of maps. So in the following, if

$$\phi : A \longrightarrow \mathrm{End}_K(\mathbb{G}_a) \ ; \quad a \mapsto \phi_a$$

is a Drinfeld module over $K$, we think of $\phi_a$ as a polynomial $\phi_a(X) \in K[X]$ via this identification. If $\phi$ is of rank $r$, the degree of $\phi_a(X)$ as a polynomial of $X$ is $q^{r\deg(a)}$ for all $a \in A - 0$.

Let $R$ be, for example, a Dedekind ring over $A$, and $K$ its fraction field. For a Drinfeld module $\phi$ over $K$, we have a minimal model $(\tilde{\phi}, \mathfrak{m})$ of $\phi$ over $R$ ( [10], §2 ), where $\mathfrak{m}$ is a fractional ideal of $R$. If $R$ is a discrete valuation ring, we can take $\mathfrak{m} = R$, and $(\tilde{\phi}, R)$ is characterized as the unique ( up to isomorphisms ) Drinfeld module $\tilde{\phi}$ over $K$ such that $\tilde{\phi}_a(X) \in R[X]$ for all $a \in A$ and the valuations of the coefficients of $\tilde{\phi}_a(X)$ are minimal.

Lemma (1.1). *Let $R$ be a discrete valuation ring over $A$, and $K$ its fraction field. Let $\phi$ be a Drinfeld module over $K$ of rank $r$, and $(\tilde{\phi}, R)$ be the minimal* *model of $\phi$ over $R$. Then there exists an ideal $\mathfrak{n}$ of $R$ such that the leading coefficient of $\tilde{\phi}_a(X) \in R[X]$ divides $\mathfrak{n}^{\delta(r,a)}$ for any $a \in A - 0$, where $\delta(r,a) := (q^{r\deg(a)} - 1)/(q-1)$.*

Proof:— Clearly, we may assume $\tilde{\phi} = \phi$. Take a non-constant element $x \in A$, and let $y \in A - 0$. Since $y$ is algebraic over $\mathbb{F}_q[x]$, we have a non-trivial relation

$$\sum \alpha_{ij} x^i y^j \; = \; 0, \qquad \alpha_{ij} \in \mathbb{F}_q.$$

Let $z = \sum \beta_{ij} x^i y^j$ be the sum of the $\alpha_{ij} x^i y^j$ 's with degree $d := \max_{i,j}\{\deg(\alpha_{ij} x^i y^j)\}$ in the left side of this equality. We must have then

$$\deg(z) \; < \; d.$$

If $\phi_x(X) = xX + \cdots + x_m X^{q^m}$ and $\phi_y(X) = yX + \cdots + y_n X^{q^n}$ with $m = r\deg(x)$, $n = r\deg(y)$, and $x_m$, $y_n \in R$, then we have

$$\phi_{x^i y^j}(X) \; = \; x^i y^j X \; + \; \cdots \; + \; x_m^{1+q^m+\cdots+q^{(i-1)m}} y_n^{q^{im}(1+q^n+\cdots+q^{(j-1)n})} X^{q^{im+jn}}.$$

So the coefficient of $X^{q^{rd}}$ in $0 = \sum \alpha_{ij} \phi_{x^i y^j}$ is

$$\sum \beta_{ij} x_m^{\frac{q^{im}-1}{q^m-1}} y_n^{q^{im} \frac{q^{jn}-1}{q^n-1}},$$

where the sum is over $i$ and $j$ with $im + jn = rd$. Since this sum must be zero, there exist two terms in the sum ( say, of indices $(i,j)$ and $(i',j')$, with $i \neq i'$ and $j \neq j'$ ) with the same valuation. Denoting by $v$ the valuation of $R$, we have

$$\frac{q^{im}-1}{q^m-1} v(x_m) \; + \; q^{im} \frac{q^{jn}-1}{q^n-1} v(y_n) \; = \; \frac{q^{i'm}-1}{q^m-1} v(x_m) \; + \; q^{i'm} \frac{q^{j'n}-1}{q^n-1} v(y_n).$$

Noticing the relation $rd = im + jn = i'm + j'n$, we see from this that

$$v(y_n) \; = \; \frac{q^n-1}{q^m-1} v(x_m).$$

Hence, if $x_m \mid \mathfrak{n}^{\frac{q^m-1}{q-1}}$, then $y_n \mid \mathfrak{n}^{\frac{q^n-1}{q-1}}$. Now the proof is complete.

Definition (1.2). Let $K$ be a finite extension of $F$. For a Drinfeld module $\phi$ over $K$ and a prime $w$ of $\mathfrak{O}_K$, consider its minimal model over $\mathfrak{O}_{K,(w)}$, and define $\Delta_w(\phi)$ to be the smallest ideal $\mathfrak{n}$ of $\mathfrak{O}_{K,(w)}$ with the property stated in Lemma (1.1). Define also $\Delta(\phi) := \sum_w \Delta_w(\phi)$, where the sum is over all primes of $\mathfrak{O}_K$ and $\Delta_w(\phi)$ is regarded as a divisor of $K$.

$\Delta(\phi)$ measures in a sense the "badness" of the reductions of $\phi$ at finite places. To estimate the differents, we begin with

Lemma (1.3). *Let $R$ be a complete discrete valuation ring, $K$ the fraction field, and $\phi(X) \in R[X]$ a separable polynomial with coefficients in $R$. Assume the coefficient of the leading term of $\phi$ is a unit. Let $\alpha$ be a root of $\phi$ in $K^{sep}$. Then the different $\mathfrak{D}(K(\alpha)/K)$ divides the principal ideal $(\phi'(\alpha))$.*

Note that $\phi'(\alpha) = a_0$ if $\phi$ is of the form $\phi(X) = \sum_i a_i X^{p^i}$ and $K$ is of positive characteristic $p$.

Proof:— Since the minimal polynomial of $\alpha$ divides $\phi(X)$ and all the roots of $\phi(X)$ are integral over $R$, this follows from Cor. 2 (p. 66) to Prop. 11 of §6, Chap. III of [8].

Lemma (1.4). *Let $R$ be a complete discrete valuation ring of characteristic $p > 0$ and $K$ the fraction field. Let $\phi(X) = \sum_{i=0}^{N} a_i X^{p^i}$ be a separable polynomial in $R[X]$ with $a_0 a_N \neq 0$, and $\alpha$ a root of $\phi$ in $K^{sep}$. Then the different $\mathfrak{D}(K(\alpha)/K)$ divides the principal ideal $(a_0 a_N^{p^N - 2})$.*

Proof:— $\tilde{\phi}(X) := a_N^{p^N-1}\phi(X/a_N) = \sum_{i=0}^{N} a_i a_N^{p^N-1-p^i} X^{p^i}$ is a separable monic polynomial in $R[X]$ and $a_N\alpha$ is a root of $\tilde{\phi}$. The assertion now follows from the previous lemma.

For a finite extension $K$ of $F$ and $a \in A - 0$, let $K(\phi; a) = K({}_a\phi(K^{sep}))$ denote the finite separable extension of $K$ obtained by adding the $a$-division points of $\phi$. Let $\mathfrak{D}_f(/)$ denote the finite part of $\mathfrak{D}(/)$, i.e., the sum of the components of $\mathfrak{D}(/)$ not lying over $\infty$. Since the extension $K(\phi; a)/K$ is obtained by adding $r$ roots of $\phi_a(X)$ which form an $(A/aA)$-base of ${}_a\phi(K^{sep}) \simeq (A/aA)^r$, we see from Lemmas (1.1) and (1.4) the following

Proposition (1.5). *Let $\phi$ be a Drinfeld module over a finite extension $K$ of $F$ of rank $r$. For $a \in A - 0$, we have*

$$\mathfrak{D}_f(K(\phi; a)/K) \leq r\big[(a) + \delta(r, a)(q^{r\deg(a)} - 2)\cdot\Delta(\phi)\big].$$

For an infinite place $w$ of $K$, let $\Lambda_w(\phi)$ denote the $A$-lattice in $K_w^{sep}$ correspondeing to $\phi \otimes_K K_w$ ( [3], §3 ). This is a $G_{K_w}$-stable projective $A$-module of rank $r$. In particular, it is finitely generated over $A$, and the fixed subfield $K_w(\Lambda_w(\phi))$ of $K_w^{sep}$ by the kernel of the natural representation $G_{K_w} \longrightarrow \mathrm{Aut}(\Lambda_w(\phi))$ is a finite extension of $K_w$. We have for $a \in A - 0$, ${}_a\phi(K_w^{sep}) \simeq \Lambda_w(\phi)/a\Lambda_w(\phi)$ as $G_{K_w}$-modules, which is rational over $K_w(\Lambda_w(\phi))$. Hence we have:

Corollary (1.6). *Let $r$ be a positive integer, $v$ a prime of $A$, and $\mathfrak{n}$ a non-zero ideal of $\mathfrak{O}_K$. Let $S$ be the set of finite places of $K$ consisting of the finite places lying above $v$ or dividing $\mathfrak{n}$. Then there exists a family $(N(w,n))_{w\in S, n\in\mathbb{N}}$ of non-negative integers which has the following property:*

*For any Drinfeld module $\phi$ over $K$ of rank $r$ with $\Delta(\phi) \leq \mathfrak{n}$ and for any $n \in \mathbb{N}$, we have*

$$\mathfrak{d}(K(\phi; v^n)/K) \leq \sum_{w\in S} N(w,n)\cdot(w) \;+\; M(\phi)\cdot\infty,$$

*where $M(\phi)$ is an integer $\geq 0$ depending on $\phi$ but not on $n$.*

## 2. Infinite places — Examples

In contrast to the classical case ( where we have only $\mathbb{C}/\mathbb{R}$ ), we have more complicated field extensions at infinity in the Drinfeld module case, if the rank $r$ is bigger than one. In this section, we give two typical examples which clarify this contrast.

Let $A$, $F$, and $\infty$ be as in §1. For a Drinfeld module $\phi$ over a finite extension $K$ of $F$ and an infinite place $w$ of $K$, let $\Lambda_w(\phi)$ denote, as in §1, the $A$-lattice corresponding to $\phi \otimes_K K_w$.

Example (2.1). Let $A = \mathbb{F}_q[T]$, $F = \mathbb{F}_q(T)$, $\infty = (\frac{1}{T})$, and $r$ an integer $\geq 2$. Then there exists an infinite family $(\phi^{(n)})_{n\geq 1}$ of Drinfeld modules over $F$ of rank $r$ which has the following properties:
(i) $\phi^{(n)}$ has everywhere good reduction over $A$;
(ii) the ramification of the corresponding lattice $\Lambda^{(n)} = \Lambda_\infty(\phi^{(n)})$ at $\infty$ becomes arbitrarily large, i.e., $\mathrm{ord}_\infty(\mathfrak{D}(F_\infty(\Lambda^{(n)})/F_\infty))$ tends to infinity as $n$ does.

Especially, for any finite place $v$ of $F$, there arise infinitely many isomorphism classes of $v$-adic $G_F$-representations $T_v(\phi) \otimes_{A_v} F_v$ from Drinfeld modules $\phi$ of rank $r$ over $F$ with everywhere good reduction.

Construction:— Consider a Drinfeld module $\phi^{(n)}$ over $A$ defined by

$$\phi_T^{(n)}(X) = TX + a_1 X^q + \cdots + a_r X^{q^r}, \qquad a_i \in A,$$

where we assume:
(1) $a_r \in A^\times = \mathbb{F}_q^\times$ ;
(2) $\mathrm{ord}_\infty(a_{r-1}) = -n(q^r - q^{r-1}) + 1$;
(3) $\mathrm{ord}_\infty(a_{r-1}) \leq \mathrm{ord}_\infty(a_i)$ for $1 \leq i \leq r-1$.

Let $v$ denote the normalized valuation $\mathrm{ord}_\infty(\cdot)$ extended uniquely to a fixed separable closure $F_\infty^{sep}$ of $F_\infty$. The Newton polygon of $\phi_T(X) \in F_\infty[X]$ shows that

$$(4) \qquad \phi_T(X) \text{ has } (q^r - q^{r-1}) \text{ roots } \lambda \text{ with } v(\lambda) = -n + \frac{1}{q^r - q^{r-1}}$$

( take and fix one such $\lambda$ ), and the other non-zero roots have non-negative valuations. Conseqently, $V := \{\lambda' \in {}_T\phi(F_\infty^{sep})\,;\, v(\lambda') \geq 0\}$ forms an $(r-1)$-dimensional $\mathbb{F}_q$-vector space.

By (4), the degree of the minimal polynomial of $\lambda$ over $F_\infty$ cannot exceed $(q^r - q^{r-1})$. On the other hand, the denominator of $v(\lambda)$, expressed as a reduced rational number, is $(q^r - q^{r-1})$. Hence the extension $F_\infty(\lambda)/F_\infty$ is totally ramified of degree $(q^r - q^{r-1})$, and in particular, it is wildly ramified.

Let $L$ be the Galois closure in $F_\infty^{sep}$ of $F_\infty(\lambda)/F_\infty$. Since $L/F_\infty$ is also wildly ramified, there exists an element $\sigma \in \mathrm{Gal}(L/F_\infty)$ of order $p$. Then $\sigma(\lambda)$ is of the form

$$\sigma(\lambda) = \alpha\lambda + \lambda', \qquad \text{for some } \alpha \in \mathbb{F}_q^\times \text{ and } \lambda' \in V.$$

Since $v(\sigma(\lambda')) = v(\lambda') > v(\lambda)$, we again have $\sigma(\lambda') \in V$. Hence $\sigma^p = 1$ implies $\alpha^p = 1$. Thus

$$\sigma(\lambda) = \lambda + \lambda', \qquad \lambda' \in V.$$

Set $\pi := \frac{\lambda}{T^n}$, so that $v(\pi) = \frac{1}{q^r - q^{r-1}}$ and $\pi$ is a uniformizer of $F_\infty(\lambda)$. Since

$$v(\sigma(\pi) - \pi) = v((\sigma(\lambda) - \lambda)/T^n) = n + v(\lambda') \geq n,$$

and

$$\mathfrak{D}(F_\infty(\lambda)/F_\infty) = \prod_{\tau \in \mathrm{Gal}(L/F_\infty)-1} (\tau(\pi) - \pi),$$

this different, and hence $\mathfrak{D}(F_\infty(\Lambda)/F_\infty)$, can become arbitrarily large, as asserted before.

Example (2.2). Let $A$, $F$, $\infty$, and $r$ be as in Example (2.1). Then there exists an infinite family $(\phi^{(n)})_{n\geq 0}$ of mutually non-isomorphic Drinfeld modules of rank $r$ over a finite extension $L$ of $F$ which has the following properties:
(i) $\phi^{(n)}$ has everywhere good reduction over the integral closure $\mathfrak{O}_L$ of $A$ in $L$.
(ii) Let $w$ be an infinite place of $L$, and set $\Lambda^{(n)} := \Lambda_w(\phi^{(n)})$. Then there are in fact only finitely many field extensions in the set $\{L_w(\Lambda^{(n)})/L_w \ ; \ n \in \mathbb{N}\}$.

Construction:— Let $K = \mathbb{F}_q(t)$, $\mathfrak{O}_K = \mathbb{F}_q[t]$, and $C$ the Carlitz $\mathfrak{O}_K$-module defined by

$$C_t(X) \ = \ tX \ + \ X^q.$$

Take an irreducible element $T = f(t) \in \mathfrak{O}_K$ of degree $r$, and let $A$ be the subring $\mathbb{F}_q[T]$ of $\mathfrak{O}_K$ ( $\infty := (\frac{1}{T}) = (\frac{1}{t^r})$ in $K$ ). Define a Drinfeld $A$-module $\phi$ over $\mathfrak{O}_K$ by

$$\phi_T \ := \ C_{f(t)}.$$

Then $\phi$ has rank $r$, and everywhere good reduction over $\mathfrak{O}_K$. By explicit class field theory ( [7] ), the field $L := K({}_{f(t)}C(K^{sep})) = K({}_T\phi(K^{sep}))$ is an abelian extension of $K$ with Galois group $(\mathfrak{O}_K/(f(t)))^\times \simeq \mathbb{F}_{q^r}^\times$, and the prime $(f(t))$ ramifies totally in $L$. In particular, the polynomial $C_{f(t)}(X)/X = \phi_T(X)/X$ over $\mathfrak{O}_K$ is Eisenstein at $(T)$; if we write

$$\phi_T(X) \ = \ TX \ + a_1 X^q \ + \cdots + \ a_{r-1} X^{q^{r-1}} \ + \ X^{q^r},$$

then we have

$$\mathrm{ord}_T(a_i) \geq 1, \quad 1 \leq i \leq r-1. \tag{1}$$

Write

$$\phi_T(X) = TX \prod_{\lambda \in \ {}_T\phi - 0} (1 - \frac{X}{\lambda}) \qquad \text{in } \ K^{sep}[X].$$

Looking at the Newton polygons of $\phi_T(X)$ at various finite places of $K$, we see that, for any $\lambda \in \ {}_T\phi(K^{sep}) - 0$, $(\lambda)$ is the unique prime ideal of $\mathfrak{O}_L$ lying over $(T)$. So we can take a prime element $\tau$ ( say $\tau :=$ one of the $\lambda$'s ) of $\mathfrak{O}_L$ and write

$$\lambda = \tau\lambda_1, \quad \lambda_1 \in \mathfrak{O}_L^\times$$

for each $\lambda \in \ {}_T\phi(K^{sep}) - 0$. Define Drinfeld modules $\phi^{(n)}$ ( $n = 0, 1, 2, \cdots$ ) over $L$ by

$$\phi_T^{(n)}(X) \ = \ TX \prod_{\lambda \in \ {}_T\phi - 0} (1 - \frac{X}{\tau\lambda_1^{p^n}}).$$

If $\phi_T^{(n)}(X) = TX + a_1^{(n)}X^q + \cdots + a_r^{(n)}X^{q^r}$, then

$$a_i^{(n)} \;=\; T\sum\prod_{(q^i-1)} \frac{1}{-\tau\lambda_1^{p^n}} \;=\; T\tau^{1-q^i}(\sum\prod_{(q^i-1)} \frac{1}{\lambda_1})^{p^n},$$

where $\sum\prod_{(q^i-1)}$ denotes the sum of the products of $(q^i-1)$ elements, the sum taken over all possible choices of $(q^i-1)$ $\lambda$'s from ${}_T\phi(K^{sep}) - 0$. Set $b_i := \sum\prod_{(q^i-1)} \frac{1}{\lambda_1}$. Since $(T) = (\tau^{q^r-1})$ and, for $1 \le i \le r-1$,

$$a_i = a_i^{(0)} = T\tau^{1-q^i}b_i$$

is by (1) an element of $\mathfrak{O}_K$ divisible by $T$, $b_i$ is integral; $b_i \in \mathfrak{O}_L$. Note that

$$\text{(2)} \qquad \mathrm{ord}_\tau(b_i) = \mathrm{ord}_\tau(a_i) - (q^r - q^i) > 0.$$

Since

$$a_i^{(n)} = T\tau^{1-q^i}b_i^{p^n} = a_i b_i^{p^n-1},$$

we have for a prime $w$ of $\mathfrak{O}_L$,

$$\text{(3)} \qquad \mathrm{ord}_w(a_i^{(n)}) \;=\; \begin{cases} \mathrm{ord}_\tau(a_i) + (p^n-1)\mathrm{ord}_\tau(b_i) & \text{if} \quad w \mid T \\ p^n \mathrm{ord}_w(a_i) & \text{if} \quad w \nmid T. \end{cases}$$

Moreover we have for $i = r$,

$$a_r^{(n)} \;=\; T\tau^{1-q^r}(\prod_{(q^r-1)} \frac{1}{\lambda_1})^{p^n} \;\in\; \mathfrak{O}_L^\times.$$

We have thus obtained an infinite family $(\phi^{(n)})_{n\in\mathbb{N}}$ of Drinfeld modules over $\mathfrak{O}_L$ with everywhere good reduction. (2) and (3) imply that these are mutually non-isomorphic, and yet the $T$-division points ${}_T\phi^{(n)}(L^{sep})$ are rational over $L$. Further, it will be shown in §3 ( Cor. (3.2), (ii) ) that there arise in fact only finitely many extensions $L_w(\Lambda_w(\phi^{(n)}))/L_w$ for all $w \mid \infty$.

## 3. Some finiteness and infiniteness

In this section, $A$, $F$ and $\infty$ are as in §1. Let $v$ be a prime of $A$, $K$ a finite extension of $F$, and $\mathfrak{n}$ a positive divisor of $K$. Let $V$ be a finite dimensional $F_v$-vector space, and $\rho : G_K \longrightarrow \mathrm{GL}(V)$ an $F_v$-linear continuous representation of $G_K$. Consider the following condition for $\rho$:

(*) $\rho$ is unramified outside $\mathrm{Supp}(\mathfrak{n})$, and there exists in $V$ a $G_K$-stable $A_v$-lattice $T$ such that $\mathfrak{d}(K'/K) \le \mathfrak{n}$, where $K'$ is the fixed subfield of $K^{sep}$ by the kernel of the map $G_K \longrightarrow \mathrm{Aut}(T/vT)$ induced by $\rho$.

Proposition (3.1). *Let $r$ be a positive integer and $\mathfrak{n}$ a positive divisor of $K$. Then there exists a finite set $S$ of finite places of $K$ disjoint from Supp($\mathfrak{n}$) which has the following property:*

*Let $\rho_i : G_K \longrightarrow GL(V_i)$, $i$ = 1, 2, be two $r$-dimensional $v$-adic representations which are semi-simple and satisfy the above condition (*). If the characteristic polynonial of $\rho_1(Frob_w)$ and $\rho_2(Frob_w)$ coincide for all $w \in S$, then we have $\rho_1 \simeq \rho_2$.*

Proof:— ( Cf. Proof of Theorem 5 of [4]. ) Since there are only finitely many separable extensions of $K$ with given degree and discriminant, there exists a finite Galois extension $K_{\mathfrak{n}}/K$ which contains all separable extensions $K'/K$ such that $[K':K] \leq \mathrm{Card}(\mathrm{GL}_r(\mathbb{F}_{q_v}))$ and $\mathfrak{d}(K'/K) \leq \mathfrak{n}$, where $q_v := \mathrm{Card}(A/vA)$. By Čebotarev, there exists a finite set $S$ of finite places of $K$ disjoint from $\mathrm{Supp}(\mathfrak{n})$ such that $\mathrm{Gal}(K_{\mathfrak{n}}/K)$ is filled with the images of the conjugacy classes of $Frob_w$ for $w \in S$. We will show that this $S$ has the required property. Choose $G_K$-stable $A_v$-lattices $T_i$ of $V_i$ for $i = 1, 2$, with the property as in the assumption (*), and let $M_j$, $1 \leq j \leq r$, be the $A_v$-subalgebra of $\mathrm{End}_{A_v}(\wedge^j T_1 \times \wedge^j T_2)$ generated by the image of $\wedge^j \rho_1 \times \wedge^j \rho_2$. By the assumption of semi-simplicity and by a version of the Brauer-Nesbitt theorem ( cf. [9] ), it suffices to show $\mathrm{Tr}(m_1; \wedge^j V_1) = \mathrm{Tr}(m_2; \wedge^j V_2)$ for all $m = (m_1, m_2) \in M_j$, $1 \leq j \leq r$, which is already true by assumption if $m$ is conjugate to the image of $Frob_w$ for some $w \in S$. It remains to show that these images together with their conjugates generate $M_j$ over $A_v$. They generate the $A_v/vA_v$-module $M_j/vM_j$, because $G_{K_{\mathfrak{n}}}$ acts trivially on $T_1/vT_1 \times T_2/vT_2$ according to our choice of $K_{\mathfrak{n}}$. By Nakayama's lemma, they generate the $A_v$-module $M_j$.

Hereafter in §3, $w$ denotes an infinite place of $K$.

Since the Galois representations $T_v(\phi) \otimes_{A_v} F_v$ are semi-simple ( [10] ), Cor. (1.6) and Prop. (3.1) imply

Corollary (3.2). *Let $v$ be a prime of $A$, $r$ a positive integer, and $\mathfrak{n}$ a positive divisor of $K$. Then:*
*(i) There arise only finitely many isomorphism classes of Galois representations $T_v(\phi) \otimes_{A_v} F_v$ from Drinfeld modules $\phi$ over $F$ of rank $r$ such that $\Delta(\phi) \leq \mathfrak{n}$ and $\mathfrak{d}(K_w({}_v\phi(K_w^{sep})/K_w) \leq \mathfrak{n}$ for all infinite places $w$ of $K$.*
*(ii) For a Drinfeld module $\phi$ over $K$ and an infinite place $w$ of $K$, let $\Lambda_w(\phi)$ be the $A$-lattice ( $\subset K_w^{sep}$ ) corresponding to $\phi \otimes_K K_w$. Then there arise only finitely many field extensions $K_w(\Lambda_w(\phi))/K_w$ from Drinfeld modules $\phi$ as in (i). Especially, $\mathfrak{d}(K_w(\Lambda_w(\phi))/K_w)$ is bounded.*

Remark (3.3). Over $K_w$, there may exist an infinite family $(\phi^{(n)})_{n\in\mathbb{N}}$ of Drinfeld modules of rank $r \geq 2$ such that ${}_v\phi^{(n)}(K_w^{sep})$ is rational over $K_w$ but the ramification of the corresponding lattices $\Lambda^{(n)}$ is not bounded. For example, let $A := \mathbb{F}_q[T]$ and $\lambda^{(n)}$ a root of the Artin-Schreier equation $X^q - X = T^n$. Consider the rank two Drinfeld module $\phi^{(n)}$ over $F_\infty$ corresponding to the $A$-lattice $\Lambda^{(n)} := A \cdot \lambda^{(n)} + A \cdot \frac{1}{T}$ ( $\subset F_\infty^{sep}$ ). Then $G_{F_\infty}$ acts trivially on ${}_T\phi^{(n)}(F_\infty^{sep}) \simeq \Lambda^{(n)}/T\Lambda^{(n)}$ ( since $\sigma(\lambda^{(n)}) - \lambda^{(n)} \in \mathbb{F}_q \subset A \subset T\Lambda^{(n)}$ for $\sigma \in G_{F_\infty}$ ), but $\mathfrak{d}(F_\infty(\lambda^{(n)})/F_\infty)$ is not bounded.

Remark (3.4). It is conjectured that the isogeny classes of Drinfeld modules $\phi$ over $K$ are in one to one correspondence with the Galois representations $T_v(\phi) \otimes_{A_v} F_v$. If this is true, then Ex. (2.1), Ex. (2.2) and Cor. (3.2) imply the following:

For a Drinfeld module $\phi$ over $K$, consider the positive divisor of $K$

$$\overline{\Delta}(\phi) := \Delta(\phi) \; + \; \sum_{w|\infty} \mathfrak{d}(K_w(\Lambda_w(\phi))/K_w).$$

Suppose we are given a positive divisor $\mathfrak{n}$ of $K$. Then:
(i) There exist only finitely many isogeny classes of Drinfeld modules $\phi$ of rank

$r \geq 1$ with $\overline{\Delta}(\phi) \leq \mathfrak{n}$ ( Cor. (3.2) ).
(ii) Let $A = \mathbb{F}_q[T]$ and $F = \mathbb{F}_q(T)$. Then

(ii-1) there exist infinitely many isogeny classes of Drinfeld modules $\phi$ of rank $r \geq 2$ over $F$ with $\Delta(\phi) \leq \mathfrak{n}$ ( Ex. (2.1) );

(ii-2) there exists an isogeny class of Drinfeld modules of rank $r \geq 2$ over some finite extension of $F$ which contains infinitely many isomorphism classes ( Ex. (2.2) + Cor. (3.2) ).

Remark (3.5). The above definition of $\overline{\Delta}(\phi)$ is not good. We hope to find a definition of the infinite component of $\overline{\Delta}(\phi)$ which is calculated directly from the defining eqation of $\phi$ and with which we can bound $\sum \mathfrak{d}(K_w(\Lambda_w(\phi))/K_w)$.

## 4. Higher dimensional cases

The content of this section is an $A$-module version of a theorem of Fontaine ( Théorème 1 of [5] ), which can be regarded as a higher dimensional generalization of Lemma (1.3).

First we give a preliminary on Taylor expansions.

Let $R$ be a commutative ring and $R[[X]] = R[[X_1, \cdots, X_h]]$ the ring of formal power series over $R$ in $h$ variables. For a multi-index $n = (n_1, \cdots, n_h) \in \mathbb{N}^h$ ( $\mathbb{N}$ is the set of natural numbers including 0 ), we define a "differential operator" $\frac{\delta^n}{\delta X^n}$ as follows:

If $f(X) = \sum a_m X^m = \sum a_{m_1,\cdots,m_h} X_1^{m_1} \cdots X_h^{m_h} \in R[[X]]$, then

$$\frac{\delta^n}{\delta X^n} f(X) := \sum a_m \binom{m}{n} X^{m-n}$$

$$= \sum a_{m_1,\cdots,m_h} \binom{m_1}{n_1} \cdots \binom{m_h}{n_h} X_1^{m_1-n_1} \cdots X_h^{m_h-n_h},$$

where $\binom{m}{n} = \binom{m_1}{n_1} \cdots \binom{m_h}{n_h}$ is the "multi-binomial coefficient" with $\binom{m_i}{n_i} := 0$ if $n_i > m_i$.

Remarks (4.1). (1) $\frac{\delta^n}{\delta X^n}$ is $R$-linear.
(2) $\frac{\partial^n}{\partial X^n} = n! \frac{\delta^n}{\delta X^n}$ ( where $n! := n_1! \cdots n_h!$ ) is the usual differential operator, and $\frac{\delta^n}{\delta X^n} = \frac{1}{n!}(\frac{\delta}{\delta X})^n$ if $n!$ is invertible in $R$. In particular, we have $\frac{\partial}{\partial X} = \frac{\delta}{\delta X}$.
(3) For $f(X) \in R[[X]]$, put $f_Y(X) := f(X+Y) \in R[[X,Y]] = R[[X]][[Y]]$. We have

$$\frac{\delta^n}{\delta X^n} f_Y(X) = (\frac{\delta^n}{\delta X^n} f)(X+Y) \quad \text{in } R[[X,Y]].$$

(4)
$$\frac{\delta^n}{\delta X^n}(fg) = \sum_{k+l=n} (\frac{\delta^k}{\delta X^k} f)(\frac{\delta^l}{\delta X^l} g) \quad \text{for } f,\ g \in R[[X]].$$

(5) Let $S$ be an $R$-algebra and $I$ an ideal of $S$. Assume $S$ is complete with respect to the $I$-adic topology. If $f(X) \in R[[X]]$ has the value $f(x) \in S$ at a point $x = (x_1, \cdots, x_h) \in S^h$, then $\frac{\delta^n}{\delta X^n} f(X)$ also has the value $\frac{\delta^n}{\delta X^n} f(x)$ at $x$ for any $n \in \mathbb{N}^h$

Proposition (4.2). *For $f(X)\in R[[X]]$, we have the formal Taylor expansion ( or rather, the binomial expansion )*

$$f(X+Y)=\sum_{|n|\geq 0}\frac{\delta^n}{\delta X^n}f(X)\cdot Y^n \quad \text{in } R[[X,Y]]. \tag{4.2.1}$$

*If $f(X)$ has the value $f(x)\in S$ at $x\in S^h$ and $y$ is an element of $I^h$, then $f(x+y)\in S$ also exists and we have*

$$f(x+y)=\sum_{|n|\geq 0}\frac{\delta^n}{\delta X^n}f(x)\cdot y^n \quad \text{in } S. \tag{4.2.2}$$

Proof:— Write $f(X+Y)=\sum a_n(X)Y^n$ with $a_n(X)\in R[[X]]$. Applying $\frac{\delta^n}{\delta Y^n}$ to both sides and reducing modulo $Y$, we obtain ( cf. Remark (4.1), (3) )

$$\frac{\delta^n}{\delta X^n}f(X)=a_n(X)$$

and hence (4.2.1).

The latter half of the Proposition is obvious.

Next we recall Fontaine's numbering of the ramification groups of a local field and some of his results ( [5], §1 ). In the rest of this section, if $L$ is a discrete valuation field, $\mathfrak{O}_L$ ( resp. $\mathfrak{m}_L$, resp. $k_L$ ) denotes the integer ring of $L$ ( resp. the maximal ideal of $\mathfrak{O}_L$, resp. the residue field $\mathfrak{O}_L/\mathfrak{m}_L$ ).

In the following, $K$ is a complete discrete valuation field with perfect residue field $k$ of characteristic $p\neq 0$. Let $v_K$ denote the valuation on $K$ normalized by $v_K(K^\times)=\mathbb{Z}$, and also its unique extension to any algebraic extension of $K$. If $\mathfrak{a}$ is a subset of an algebraic extension of $K$, we put $v_K(\mathfrak{a}):=\inf\{v_K(x);x\in\mathfrak{a}\}$.

For a finite Galois extension $L/K$, Fontaine defines a filtration with lower ( resp. upper ) numbering $G_{(i)}$ ( resp. $G^{(u)}$ ) ( $i,u\in\mathbb{R}$ ) on the Galois group $G=\mathrm{Gal}(L/K)$, which is connected with the usual filtration $G_i$ ( resp. $G^u$ ) defined in Chapitre IV of [8] by

$$G_i=G_{((i+1)/e)}, \qquad \text{resp.} \quad G^u=G^{(u+1)},$$

where $e=e_{L/K}$ is the ramification index of $L/K$.

He also defines a real number $i_{L/K}$ ( resp. $u_{L/K}$ ), which is characterized as the largest real number $i$ (resp. $u$ ) such that $G_{(i)}\neq 1$ ( resp. $G^{(u)}\neq 1$ ). $i_{L/K}$ and $u_{L/K}$ are connected by

$$u_{L/K}=\int_0^{i_{L/K}}(G_{(x)}:1)dx.$$

Then he proves the following

Proposition (4.3). *Let $L$ be a finite Galois extension of $K$.*
*(1) ( [5], 1.3 )* $v_K(\mathfrak{D}(L/K))=u_{L/K}-i_{L/K}$.

*(2) ( [5], 1.5 ) For a real number* $m \geq 0$, *consider the following property* $(P_m)$ *on the extension* $L/K$:

$$(P_m)\begin{cases} \text{For any algebraic extension } E \text{ of } K, \text{ if there exists} \\ \text{an } \mathfrak{O}_K\text{-algebra homomorphism : } \mathfrak{O}_L \to \mathfrak{O}_E/\mathfrak{a}^m_{E/K} \\ (\text{ where } \mathfrak{a}^m_{E/K} := \{x \in \mathfrak{O}_E; v_K(x) \geq m\} \text{ }), \\ \text{then there exists a } K\text{-embedding : } L \hookrightarrow E. \end{cases}$$

*Then*

*(i) if* $m > u_{L/K}$, $L/K$ *has the property* $(P_m)$;

*(ii) if* $L/K$ *has the property* $(P_m)$, *we have* $m > u_{L/K} - e^{-1}_{L/K}$.

Now we shall refine Fontaine's Proposition 1.7 of [5] as follows. The main point is that it works, *mutatis mutandis*, even in positive characteristics.

Proposition (4.4). *Let* $B$ *be a finite flat* $\mathfrak{O}_K$*-algebra which is locally of complete intersection over* $\mathfrak{O}_K$. *Suppose that there exists an element* $a \in \mathfrak{O}_K$ *such that* $\Omega^1_{B/\mathfrak{O}_K}$ *is a flat* $(B/aB)$*-module.*

*(i) Let* $S$ *be a finite flat* $\mathfrak{O}_K$*-algebra and* $I$ *an ideal of* $S$. *Suppose either the* $S$*-submodule* $a^{-1}I^{p-1}$ *of* $K \otimes_{\mathfrak{O}_K} S$ *is topologically nilpotent ( i.e.,* $\cap_{n\geq 1}(a^{-1}I^{p-1})^n = 0$ *), or* $I$ *has a PD-structure such that* $\cap_{n \geq 1} I^{[n]} = 0$.

*(a) For any* $\mathfrak{O}_K$*-algebra homomorphism* $u : B \longrightarrow S/aI$, *there exists an* $\mathfrak{O}_K$*-algebra homomorphism* $\hat{u} : B \longrightarrow S$ *which is uniquely determined by* $u$ *(mod.* $I$*) and makes the following diagram commutative:*

$$B@ > u >> S/aI$$

$$@V\hat{u}VV@VVV$$

$$S@ >>> S/I \text{ .}$$

*(b) The canonical map of sets*

$$Hom_{\mathfrak{O}_K\text{-alg}}(B, S) \longrightarrow Hom_{\mathfrak{O}_K\text{-alg}}(B, S/I)$$

*is injective.*

*(ii) The* $K$*-algebra* $B_K := K \otimes_{\mathfrak{O}_K} B$ *is étale. Let* $L$ *be the smallest subfield of a separable closure* $K^{sep}$ *of* $K$ *which contains the images* $u(B)$ *for all* $u \in Hom_{K\text{-alg}}(B_K, K^{sep})$. *Then* $L/K$ *is a finite Galois extension and* $u_{L/K} \leq v_K(a) + \frac{1}{p-1} \cdot min\{v_K(a), v_K(p)\}$.

The proof is essentially the same as the original one due to Fontaine, but here we reproduce his proof of (i) to make clear the meaning of the condition in (i).

Proof:— We may and do suppose $B$ is a local ring, because $B$ is the product of a finite number of local rings. Let $\mathfrak{m}_B$ be the maximal ideal of $B$. Replacing $K$ by an unramified extension if necessary, we may also suppose $B/\mathfrak{m}_B = k$, the residue field of $\mathfrak{O}_K$.

Then $\Omega^1_{B/\mathfrak{O}_K}$ is a free $(B/aB)$-module. Let $x_1, \cdots, x_h$ be elements of $\mathfrak{m}_B$ the images of which form a $k$-base of $\mathfrak{m}_B/(\mathfrak{m}_B^2 + \mathfrak{m}_K B)$. We see from the definition of differential modules that $dx_1, \cdots, dx_h$ generate $\Omega^1_{B/\mathfrak{O}_K}$, and further, they form a $(B/aB)$-base of $\Omega^1_{B/\mathfrak{O}_K}$ because of the canonical isomorphisms

$$\Omega^1_{B/\mathfrak{O}_K} \otimes_B B_o @>\sim>> \Omega^1_{B_o/k} \qquad (B_o := B/\mathfrak{m}_K B),$$

$$\mathfrak{m}_B/(\mathfrak{m}_B^2 + \mathfrak{m}_K B) @>\sim>> \mathfrak{m}_{B_o}/\mathfrak{m}_{B_o}^2 @>\sim>> \Omega^1_{B_o/k} \otimes_{B_o} k,$$

where $\mathfrak{m}_{B_o} = \mathfrak{m}_B/\mathfrak{m}_K B$ is the maximal ideal of $B_o$.

Now let

$$\alpha : \mathfrak{O}_K[[X_1, \cdots, X_h]] \longrightarrow B$$

be the unique continuous $\mathfrak{O}_K$-algebra homomorphism such that $\alpha(X_j) = x_j$, and let $J := \mathrm{Ker}(\alpha)$. Since $B$ is finite of complete intersection over $\mathfrak{O}_K$, $J$ is generated by $h$ elements, say $P_1, \cdots, P_h \in \mathfrak{O}_K[[X_1, \cdots, X_h]]$.

For each $i$, we have $\sum_j \frac{\delta P_i}{\delta X_j}(x_1, \cdots, x_h)dx_j = 0$ ( note $\frac{\delta}{\delta X_j} = \frac{\partial}{\partial X_j}$ ), which implies $\frac{\delta P_i}{\delta X_j}(x_1, \cdots, x_h) \in aB$. Hence there are $p_{ij} \in B$ such that $\frac{\delta P_i}{\delta X_j}(x_1, \cdots, x_h) = ap_{ij}$. The fact that $\Omega^1_{B/\mathfrak{O}_K}$ is a free $(B/aB)$-module means that the free $B$-submodule of $\oplus_{j=1}^h B dX_j$ generated by $\sum_j \frac{\delta P_i}{\delta X_j}(x_1, \cdots, x_h)dX_j$, $1 \le i \le h$, coincides with the one generated by $adX_j$, $1 \le j \le h$. We can therefore find $q_{li} \in B$ such that

$$adX_l = \sum_i q_{li}(\sum_j \frac{\delta P_i}{\delta X_j}(x_1, \cdots, x_h)dX_j), \quad 1 \le l \le h,$$

*i.e.*, $a1_h = (q_{li})(ap_{ij})$. ( $1_h$ is the unit matrix of degee $h$. ) Since $B$ is a free $\mathfrak{O}_K$-module, we can divide both sides by $a$. Thus the matrix $(p_{ij})$ is invertible in $\mathrm{M}_h(B)$ and $(q_{li}) = (p_{ij})^{-1}$.

The case of PD-ideals is proved in [5], so we suppose $a^{-1}I^{p-1}$ is topologically nilpotent. Then the ideal $a^{-1}I^{p-1} + I$ is also topologically nilpotent. Set $I_n := (a^{-1}I^{p-1} + I)^{n-1}I$, $n \ge 1$ ( so that $a^{-1}I_n^{p-1}$ is again topologically nilpotent, and $S$ is canonically isomorphic to the projective limit of the system $(S/I_n)_{n\ge 1}$ ). It is easily seen that $I_n^p \subset aI_{2n}$ and $I_n^2 \subset I_{2n}$. To show the assertion, it is enough to verify:

For any integer $n \ge 1$ and an $\mathfrak{O}_K$-algebra homomorphism $u : B \longrightarrow S/aI_n$, there exists an $\mathfrak{O}_K$-algebra homomorphism $u' : B \longrightarrow S/aI_{2n}$ such that $u'(\mathrm{mod.}\, I_{2n})$ is uniquely determined by $u(\mathrm{mod.}\, I_n)$ and $u'$ makes the following diagram commutative:

$$B @>u>> S/aI_n$$

$$@Vu'VV @VVV$$

$$S/aI_{2n} @>>> S/I_n.$$

In other words, writing $I$ for $I_n$ and $I_2$ for $I_{2n}$ :

For any elements $u_1, \cdots, u_h$ of $S$ such that

$$P_i(u_1, \cdots, u_h) = a\lambda_i \quad \text{with some } \lambda_i \in I \quad (1 \le i \le h),$$

there exist $\mu_1, \cdots, \mu_h \in I$ such that $\mu_j\,(\mathrm{mod.}\, I_2)$ are uniquely determined by $u_j\,(\mathrm{mod.}\, I)$ and

$$(4.4.1) \qquad P_i(u_1 + \mu_1, \cdots, u_h + \mu_h) \in aI_2 \quad (1 \le i \le h).$$

If $\mu_j \in I$, we have the Taylor expansion (4.2.2)

$$P_i(u_1+\mu_1,\cdots,u_h+\mu_h) = a\lambda_i + \sum_j \frac{\delta P_i}{\delta X_j}(u_1,\cdots,u_h)\mu_j + R_i \tag{4.4.2}$$

with $R_i := \sum_{|r|\geq 2} \frac{\delta^r P_i}{\delta X^r}(u_1,\cdots,u_h)\mu^r$.

For any element $P \in J$, we have $\frac{\delta P}{\delta X_j}(x_1,\cdots,x_h) \in aB$, i.e.,

$$\frac{\delta P}{\delta X_j}(X_1,\cdots,X_h) \in a\mathfrak{O}_K[[X_1,\cdots,X_h]] + J.$$

If $|r| \geq 1$ and $r!$ is invertible in $\mathfrak{O}_K$, we see inductively ( cf. Remark (4.1), (2) )

$$\frac{\delta^r P}{\delta X^r}(X_1,\cdots,X_h) \in a\mathfrak{O}_K[[X_1,\cdots,X_h]] + J,$$

so

$$\frac{\delta^r P}{\delta X^r}(u_1,\cdots,u_h) \in aS + aI = aS.$$

Since $I^2 \subset I_2$, we have

$$\frac{\delta^r P}{\delta X^r}(u_1,\cdots,u_h)\cdot\mu^r \in aI_2,$$

if $|r| \geq 2$ and $r!$ is invertible in $\mathfrak{O}_K$.

On the other hand, we have $\mu^r \in I^{|r|} \subset I^p \subset aI_2$ if $p$ divides $r!$, and $\frac{\delta^r P}{\delta X^r}(u_1,\cdots,u_h)$ are always in $S$ ( Remark (4.1), (5) ). Thus we have

$$R_i \in aI_2. \tag{4.4.3}$$

Take an element $P_{ij} \in \mathfrak{O}_K[[X_1,\cdots,X_h]]$ such that $\alpha(P_{ij}) = p_{ij} \in B$ for each $(i,j)$. We have

$$\frac{\delta P_i}{\delta X_j}(x_1,\cdots,x_h) = ap_{ij},$$

i.e., $\frac{\delta P_i}{\delta X_j} = aP_{ij} + R_{ij}$ with some $R_{ij} \in J$, from which follows the congruence

$$\frac{\delta P_i}{\delta X_j}(u_1,\cdots,u_h) \equiv aP_{ij}(u_1,\cdots,u_h) \pmod{aI},$$

and

$$\frac{\delta P_i}{\delta X_j}(u_1,\cdots u_h)\cdot\mu_j \equiv aP_{ij}(u_1,\cdots,u_h)\cdot\mu_j \pmod{aI_2}. \tag{4.4.4}$$

Putting (4.4.3) and (4.4.4) into (4.4.2), we have

$$P_i(u_1+\mu_1,\cdots,u_h+\mu_h) \equiv a(\lambda_i + \sum_j P_{ij}(u_1,\cdots,u_h)\cdot\mu_j) \pmod{aI_2}.$$

Since $S$ is flat over $\mathfrak{O}_K$, the condition (4.4.1) for $\mu_j$ is now equivalent to

$$\lambda_i + \sum_j P_{ij}(u_1,\cdots,u_h)\cdot\mu_j \equiv 0 \pmod{I_2}, \quad 1 \leq i \leq h.$$

Since the matrix $(p_{ij}) = (P_{ij}(x_1,\cdots,x_h))$ is invertible, the matrix $(P_{ij}(u_1,\cdots,u_h))$ is invertible modulo $aI$. Now the existence of $\mu_j \in I$ satisfying (4.4.1) is clear. Moreover $u_j \pmod{I}$, $1 \leq j \leq h$, determine $\mu_j \pmod{I_2}$, $1 \leq j \leq h$, uniquely, because they determine $\lambda_i \equiv 0 \pmod{I}$ and $P_{ij}(u_1,\cdots,u_h) \pmod{I}$ uniquely and $I^2 \subset I_2$.

Part (b) of (i) follows immediately from Part (a).

The proof of (ii) is exactly the same as in [5].

Corollary (4.5). *Let the notation and hypothesis be as in Proposition (4.4). Then we have* $v_K(\mathfrak{D}(L/K)) < v_K(a) + \frac{1}{p-1} \min\{v_K(a), v_K(p)\}$ *unless* $v_K(\mathfrak{D}(L/K)) = 0$.

Proof:— If $L/K$ is unramified, then $v_K(\mathfrak{D}(L/K)) = 0$. If not, we have $i_{L/K} > 0$ and ( Prop. (4.3), (1) )

$$v_K(\mathfrak{D}(L/K)) = u_{L/K} - i_{L/K} < u_{L/K} \leq v_K(a) + \frac{1}{p-1}\min\{v_K(a), v_K(p)\}.$$

Theorem (4.6). *Let $A$ be a complete discrete valuation ring with finite residue field, and fix a prime element $\pi$ of $A$. Let $K$ be a local field of "mixed characteristic" over $A$, i.e., a complete discrete valuation field $K$ with perfect residue field which is endowed with an injective ring homomorphism $A \longrightarrow K$ inducing a local homomorphism $A \longrightarrow \mathfrak{O}_K$. Let $n \geq 1$ be an integer and $J$ a finite flat $\pi$-module scheme over $\mathfrak{O}_K$ ( [10], §1 ) such that the invariant differential module $\omega_J$ of $J$ is a free $(\mathfrak{O}_K/\pi^n\mathfrak{O}_K)$-module. ( A typical example of such a $\pi$-module is the kernel of $\pi^n$ on a $\pi$-divisible group ( loc. cit. ) ). Let $u_o := nv_K(\pi) + \frac{1}{p-1} \min\{nv_K(\pi), v_K(p)\}$, $H$ the kernel of the action of $G_K = Gal(K^{sep}/K)$ on $J(K^{sep})$, $L := (K^{sep})^H$. Then we have $G_K^{(u)} \subset H$ for all $u > u_o$, and $v_K(\mathfrak{D}(L/K)) < u_o$.*

Proof:— As in the proof of Théorème 1 of [5], the affine ring $B$ of $J$ is locally of complete intersection. Since $\Omega^1_{B/\mathfrak{O}_K} = B \otimes_{\mathfrak{O}_K} \omega_J$ is a free $(B/\pi^n B)$-module, we can apply Prop. (4.4) and Cor. (4.5) with $a = \pi^n$ and obtain the theorem.

Remark (4.7). In some simple cases, direct calculations yield sharper results. For example, let $A$ and $\pi$ be as above, $F$ the fraction field of $A$, and $F_n$, $n \geq 0$, the field of $\pi^n$-division points of a Lubin-Tate group over $A$ associated with $\pi$. If $L/K = F_m/F_n$ with $m > n$, we have

$$u_{L/K} = \begin{cases} m, & \text{if } n = 0, \\ q^n + (m-n-1)(q^n - q^{n-1}), & \text{if } n \geq 1, \end{cases}$$

$$i_{L/K} = \begin{cases} \frac{1}{q-1}, & \text{if } n = 0, \\ q^{n-1}, & \text{if } n \geq 1, \end{cases}$$

$$v_K(\mathfrak{D}(L/K)) = \begin{cases} m - \frac{1}{q-1}, & \text{if } n = 0, \\ (m-n)(q^n - q^{n-1}), & \text{if } n \geq 1. \end{cases}$$

## 5. The case of non-scalar $A$-actions on tangent spaces

Important classes of abelian $T$-modules ( [1] ), such as higher Carlitz modules $C^{\otimes n}$ ( [2] ) and tensor products of Drinfeld modules ( [1], [6] ), are such that the actions of $T$ on the tangent spaces are not just multiplication by $T$, but $T$ plus nilpotent linear maps. In this section, we study the ramification arising from division points of such objects.

Let $A$, $\pi$ and $K$ be as in Th. (4.6); $A$ a complete discrete valuation ring with finite residue field, $\pi$ a uniformizer of $A$, and $K$ a local field of "mixed characteristic" over $A$. Consider a smooth connected commutative formal group $J$ over $\mathfrak{O}_K$ with an $A$-action

$$\phi : A \longrightarrow \mathrm{End}_{\mathfrak{O}_K}(J) \ ; \qquad a \mapsto \phi_a$$

such that, for all $a \in A$, $\phi_a$ induces a linear map Lie($\phi_a$) on Lie($J$) of the form ( multiplication by $a$ ) + ( nilpotent map ). If $J$ is, for example, the tensor product of abelian $T$-modules with scalar $T$-actions on their tangent spaces, Th. (4.6) for $J_n = \mathrm{Ker}(\phi_{T^n})$ would be valid because $\otimes$ and $T_v(\cdot)$ should be compatible ( this is shown in [6] at least for the tensor products of two Drinfeld modules). What can be said on the ramification of the geometric points of $J_n := \mathrm{Ker}(\phi_{\pi^n})$ in other cases ?

**(5.1).** First assume that the nilpotent map $\mathrm{Lie}(\phi_{\pi^n}) - \pi$ is divisible by $\pi$ in $\mathrm{End}_{\mathfrak{O}_K}(\mathrm{Lie}(J))$. Then, since the image of Lie( $\phi_{\pi^n}$ ) is $\pi^n\,\mathrm{Lie}(J)$, we have

$$\Omega^1_{J_n} \ = \ \Omega^1_J/\pi^n\Omega^1_J,$$

which is a flat $B/\pi^n B$-module ( $B$ is the affine ring of $J_n$ ). So we can apply Prop. (4.4), and Th. (4.6) remains valid for such $J_n$.

**(5.2).** We now return to a general $J$. Let $d := \dim(J) = \mathrm{rank}_{\mathfrak{O}_K}(\mathrm{Lie}(J))$, and let $p^k$ be the smallest power of $p$, the residue characteristic of $A$, such that $p^k \geq d$. Then for any multiple $m$ of $p^k$, the nilpotent map $\mathrm{Lie}(\phi_{\pi^m}) - \pi^m$ is divisible by $\pi^m$ in $\mathrm{End}_{\mathfrak{O}_K}(\mathrm{Lie}(J))$, as is easily seen by looking at the binomial expansion of $[\pi + (\mathrm{Lie}(\phi_\pi) - \pi)]^{p^k}$. In view of (5.1), we have

Theorem (5.3). *Let $J$ and $p^k$ be as above. For a positive integer $n$, let $n'$ be the smallest multiple of $p^k$ not less than $n$, $u_o := n'v_K(\pi) + \frac{1}{p-1} min\{n'v_K(\pi),\ v_K(p)\}$, $H$ the kernel of the action of $G_K$ on $J_n(K^{sep})$, and $L := (K^{sep})^H$. Then we have $G_K^{(u)} \subset H$ for all $u > u_o$, and $v_K(\mathfrak{D}(L/K)) < u_o$.*

This theorem reduces to Th. (4.6) when $n$ is divisible by $p^k$. But if $n$ is not divisible by $p^k$, the ramification can be bigger than expected from Th. (4.6), as the following example shows:

Example (5.4). Let $A = \mathbb{F}_q[[T]]$ and $K = \mathbb{F}_q((T))$, and consider the $d$-dimensional formal group $\widehat{\mathbb{G}}_a^{\oplus d}$ over $A$ with an $A$-action defined by

$$\phi_T \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix} = \begin{pmatrix} T & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & T \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix} + \begin{pmatrix} X_1^q \\ \vdots \\ X_d^q \end{pmatrix}.$$

Then we have

$$K(J_1(K^{sep})) \ = \ K({}_{T^d}C(K^{sep})),$$

where $C$ is the Carlitz module defined by $C_T(X) = TX + X^q$. This means that $\mathfrak{D}(K(J_1(K^{sep}))/K)$ can become arbitrarily large according to $d$.

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