On φ-modules

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Introduction. In this paper, we develop some basic formalisms of φ-modules and show how the result in [12] implies the familiar form of the Tate conjecture for Drinfeld modules. The proof is valid also for any φ-modules which presumably come from t-motives [1]. But the definition of t-motives over an imperfect base field is delicate and is not found in the published literature for the moment (cf. [3]). Thus in the Introduction, we restrict ourselves to the case of Drinfeld modules to explain the ideas of the proof, and we give a treatment for sufficiently general φ-modules in the body of the paper. For more general cases, see Tamagawa [13].[2]

(0.1) Let K be an algebraic function field in one variable over a finite field. Assume its subfield of constants is the finite field $\mathbb{F}_q$ of q elements. Fix a place ∞ of K, and let A be the ring of elements of K which are regular away from ∞. Let π be a non-zero prime ideal of A. We denote by A_π the π-adic completion of A. Let k be a finite extension of K. For a Drinfeld A-module ϕ over k, we denote by $T_\pi(\phi)$ the π-adic Tate module of ϕ. It is a free A_π-module of rank r if ϕ is of rank r. The absolute Galois group $G_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts continuously and A_π-linearly on $T_\pi(\phi)$. Thus we view $T_\pi(\phi)$ as an $A_\pi[G_k]$-module.

If $f : \phi \to \psi$ is a morphism of Drinfeld modules, then it gives rise to a homomorphism $T_\pi(\phi) \to T_\pi(\psi)$ of $A_\pi[G_k]$-modules. Hence there is a natural homomorphism $\mathrm{Hom}_k(\phi,\psi) \to \mathrm{Hom}_{A_\pi[G_k]}(T_\pi(\phi), T_\pi(\psi))$. Extending it A_π-linearly, we obtain a natural homomorphism

$$\mathrm{Hom}_k(\phi,\psi) \otimes_A A_\pi \;\to\; \mathrm{Hom}_{A_\pi[G_k]}(T_\pi(\phi), T_\pi(\psi))$$

of A_π-modules. It is the isomorphy of this map that the Tate conjecture for Drinfeld modules asserts. It is easy to see its injectivity (cf. (1.5.3)). To show the surjectivity, it is enough to show that $\mathrm{Hom}_k(\phi,\psi)$ is dense in $\mathrm{Hom}_{A_\pi[G_k]}(T_\pi(\phi), T_\pi(\psi))$, or that, for any $f \in \mathrm{Hom}_{A_\pi[G_k]}(T_\pi(\phi), T_\pi(\psi))$, there exists an $\tilde{f} \in \mathrm{Hom}_k(\phi,\psi)$ which is sufficiently close to f in the π-adic topology. Our method of proving this is totally different from Faltings' proof [7] for the case of Abelian varieties. As is explained below, we interpret the condition that f be in $\mathrm{Hom}_{A_\pi[G_k]}(T_\pi(\phi), T_\pi(\psi))$ (resp. in $\mathrm{Hom}_k(\phi,\psi)$) in terms of a certain "linear Frobenius equation"; call this equation (Φ). The equation (Φ) is the same for both f and $\tilde{f}$, but the solutions are sought in modules over different rings; for f, over a complete ring, whereas for $\tilde{f}$, over its dense subring of "rational" elements (compare (0.2.3) and (0.3.1)). For $f \in \mathrm{Hom}_{A_\pi[G_k]}(T_\pi(\phi), T_\pi(\psi))$, the equation (Φ) may be thought of as a "π-adic linear Frobenius equation", and is in fact a system of infinitely many algebraic equations. Since the Drinfeld modules ϕ and ψ are defined algebraically (rather than formally) over k, there are only finitely many coefficients, all in k, appearing in these equations. As we have done in [12], using a theorem of Anderson [2], we can bound the height (in the sense of Diophantine geometry) of the solutions to (Φ). By

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[2] Tamagawa seems to have reached a complete proof of the Tate conjecture earlier than the author.

the finiteness of heights, we conclude that, if a solution $F = (F_0, F_1, \cdots)$ is defined over k, then we can choose a *periodic* solution $\tilde{F} = (\tilde{F}_0, \tilde{F}_1, \cdots)$ which is arbitrarily close to the original solution F. The periodic solution $\tilde{F}$ defines a "rational" solution to (Φ), hence a morphism $\tilde{f}$ of Drinfeld modules (modulo isogeny).

(0.2) Now we assume that $K = \mathbb{F}_q(t)$, $\infty = (1/t)$, $A = \mathbb{F}_q[t]$, and $\pi = (t)$ (we may do so; see (3.3) and [12]), and explain how to obtain the linear Frobenius equation (Φ) from a homomorphism of Tate modules. An element $x \in T_t(\phi)$ is a projective system $x = (x_0, x_1, \cdots)$ with $x_i \in k^{\mathrm{sep}}$ such that $\phi_t(x_i) = x_{i-1}$ for all $i \geq 0$ (we put $x_{-1} := 0$). For such an x, form a power series $\sum_{i\geq 0} x_i t^i \in k^{\mathrm{sep}}[\![t]\!]$; we identify x with the power series to write $x = \sum_{i\geq 0} x_i t^i$. Then $T_t(\phi)$ is identified with a submodule of $k^{\mathrm{sep}}[\![t]\!]$ which is free of rank r over $\mathbb{F}_q[\![t]\!]$. For any power series $x = \sum_{i\geq 0} x_i t^i$ in $k^{\mathrm{sep}}[\![t]\!]$, we define $x^\sigma := \sum_{i\geq 0} x_i^q t^i$. If the Drinfeld module ϕ is defined by

$$(0.2.1) \qquad \phi_t \;=\; \theta + a_1\sigma + \cdots + a_r\sigma^r, \qquad a_i \in k,\; a_r \in k^\times,$$

where $\theta =$ (image of t in k) and $\sigma =$ (q-th power map), then, for any $x \in T_t(\phi)$, we have the relation

$$tx \;=\; \theta x + a_1 x^\sigma + \cdots + a_r x^{\sigma^r} \quad \text{in } k^{\mathrm{sep}}[\![t]\!].$$

Let M^{sep} be the space $k^{\mathrm{sep}}[\![t]\!]^r$ of column vectors over $k^{\mathrm{sep}}[\![t]\!]$ of rank r. We let σ act on M^{sep} entry-wise. For $x \in k^{\mathrm{sep}}[\![t]\!]$, put

$$\underline{x} \;:=\; \begin{pmatrix} x \\ x^\sigma \\ \vdots \\ x^{\sigma^{r-1}} \end{pmatrix} \quad \in M^{\mathrm{sep}}.$$

If $x \in T_t(\phi)$, we have

$$\underline{x}^\sigma \;=\; A\underline{x} \qquad \text{with } A \;:=\; a_r^{-1} \begin{pmatrix} & a_r & & \\ & & \ddots & \\ & & & a_r \\ t-\theta & -a_1 & \cdots & -a_{r-1} \end{pmatrix}.$$

Note that $A \in \mathrm{M}_r(k[t])$. Conversely, those elements $\underline{x} \in M^{\mathrm{sep}}$ which satisfy this equation come from $T_t(\phi)$ in this way. Thus we may identify $T_t(\phi)$ with a submodule of M^{sep} as follows:

$$(0.2.2) \qquad T_t(\phi) \;=\; \{\underline{x} \in M^{\mathrm{sep}};\; \underline{x}^\sigma = A\underline{x}\}.$$

Also, it can be shown, by the same arguments following (1.5.3), that $T_t(\phi)$ generates M^{sep} as a $k^{\mathrm{sep}}[\![t]\!]$-module.

Now let ϕ and ψ be two Drinfeld modules over k, of rank r and s respectively. By the above consideration, a homomorphism $f : T_t(\phi) \to T_t(\psi)$ of Tate modules is thought of as a $k^{\mathrm{sep}}[\![t]\!]$-linear map $F : k^{\mathrm{sep}}[\![t]\!]^r \to k^{\mathrm{sep}}[\![t]\!]^s$ which is G_k-equivariant and maps $T_t(\phi)$ into $T_t(\psi)$. If we think of F as a matrix, then the G_k-equivariance means that $F \in \mathrm{M}_{s\times r}(k[\![t]\!])$. The condition that $F(T_t(\phi)) \subset T_t(\psi)$ is interpreted as follows: Let $B \in \mathrm{M}_s(k[\![t]\!])$ be the matrix for ψ defined in the same way as A for ϕ. Then by (0.2.2), we must have $(F\underline{x})^\sigma = B(F\underline{x})$ for all $\underline{x} \in T_t(\phi)$. Since $\underline{x}^\sigma = A\underline{x}$ for $\underline{x} \in T_t(\phi)$ by (0.2.2), this implies the relation

$$(\Phi) \qquad F^\sigma A \;=\; BF,$$

where F^σ denotes the entry-wise application of σ. This is what we call a *linear Frobenius equation*. Conversely, the above arguments can be reversed and such an F yields an $\mathbb{F}_q[\![t]\!][G_k]$-module homomorphism $f : T_t(\phi) \to T_t(\psi)$. Thus we have

$$(0.2.3) \qquad \mathrm{Hom}_{\mathbb{F}_q[\![t]\!][G_k]}(T_t(\phi), T_t(\psi)) \;=\; \{F \in \mathrm{M}_{s\times r}(k[\![t]\!]);\ F^\sigma A \;=\; BF\}.$$

The equation (Φ) can be written down as a system of algebraic equations for the coefficients of the t-adic expansion of F. Indeed, write $F = \sum_{i\geq 0} F_i t^i$ with $F_i \in \mathrm{M}_{s\times r}(k)$, $A = A_0 + A_1 t$ with $A_i \in \mathrm{M}_r(k)$, and $B = B_0 + B_1 t$ with $B_i \in \mathrm{M}_s(k)$. Then (Φ) is equivalent to

$$(\Phi') \qquad F_i^\sigma A_0 + F_{i-1}^\sigma A_1 \;=\; B_0 F_i + B_1 F_{i-1} \qquad \text{for all} \;\; i \geq 0$$

(we put $F_{-1} := 0$). As the action of σ on each F_i is to raise the entries to the q-th power, each equation in (Φ') is an algebraic equation with coefficients in k, of which solutions F_i we look for also over k.

In fact, we can use the formalism of φ-modules (1.5.5) to reduce the Tate conjecture to the case where ϕ is a higher Carlitz module (for which the matrix A is a scalar matrix of the form $(t-\theta)^d$) and ψ is a rather general φ-module (so ϕ and ψ are no longer Drinfeld modules). Then the above equation is exactly the one we studied in [12].

(0.3) To obtain the equation (Φ) from a morphism of Drinfeld modules, we recall how to associate a *t-motive* M_ϕ ([1]) to a Drinfeld module ϕ (in [1], Anderson considers t-motives only over a perfect field; so the use of the word "t-motive" here may be an abuse; but cf. (3.2) and (3.4)). First, let $k[t,\sigma]$ be the non-commutative polynomial ring with commutation relation

$$\sigma(\sum a_i t^i) \;=\; (\sum a_i^q t^i)\sigma \qquad \text{for all} \;\; \sum a_i t^i \in k[t],$$

where the subring $k[t]$ is the usual commutative polynomial ring. Accordingly, we regard σ also as a ring endomorphism $\sum a_i t^i \;\mapsto\; \sum a_i^q t^i$ of $k[t]$. Next, we view the Drinfeld module ϕ as a group scheme over k which is isomorphic to to the additive group scheme $\mathbb{G}_\mathrm{a}$ equipped with an action of $\mathbb{F}_q[t]$. Then the t-motive M_ϕ associated to ϕ is defined to be the k-vector space $\mathrm{Hom}_{\mathbb{F}_q\text{-lin}/k}(\phi, \mathbb{G}_\mathrm{a})$ of $\mathbb{F}_q$-linear homomorphisms of group schemes defined over k, which we make into a left $k[t,\sigma]$-module by

$$tm \;:=\; m \circ \phi_t, \quad \sigma m \;:=\; m^q \qquad \text{for all } m \in M_\phi$$

(m^q is the function $x \mapsto m(x)^q$). As we see explicitly below, it is a free $k[t]$-module of rank r if the Drinfeld module ϕ is of rank r. We denote by $\varphi_\phi : M_\phi \to M_\phi$ the map induced by the action of σ on M_ϕ. Then φ_ϕ is a σ-semilinear map, and (M_ϕ, φ_ϕ) is a φ-module over $k[t]$ in the sense of Section 1.

If we identify ϕ with $\mathbb{G}_{\mathrm{a}}$ and choose a coordinate X of $\mathbb{G}_{\mathrm{a}}$ (thus $\phi = \mathbb{G}_{\mathrm{a}} = \operatorname{Spec} k[X]$), then we may write

$$M_\phi \;=\; \bigoplus_{i\ge 0} kX^{q^i}.$$

Morally speaking, it is the "essential part" of the structure sheaf

$$\mathcal{O}_\phi \;=\; \operatorname{Hom}_k(\phi, \mathbb{A}^1_k) \;=\; k[X] \;=\; \bigoplus_{j\ge 0} kX^j$$

of ϕ. We may choose a standard basis $(X, X^q, \cdots, X^{q^{r-1}})$ of M_ϕ as a $k[t]$-module. Suppose ϕ is defined by the equation (0.2.1). Then we have

$$\begin{cases} \varphi(X^{q^i}) \;=\; X^{q^{i+1}} & \text{if } \; 0 \le i \le r-2, \\ \varphi(X^{q^{r-1}}) \;=\; X^{q^r} \;=\; a_r^{-1}((t-\theta)X - a_1X^q - \cdots - a_{r-1}X^{q^{r-1}}). \end{cases}$$

In other words, the matrix of φ with respect to this basis is

$$ {}^tA \;=\; a_r^{-1} \begin{pmatrix} & & & t-\theta \\ a_r & & & -a_1 \\ & \ddots & & \vdots \\ & & a_r & -a_{r-1} \end{pmatrix}.$$

Let $f : \phi \longrightarrow \psi$ be a morphism of Drinfeld modules over k. It induces a k-algebra homomorphism ${}^tf : \mathcal{O}_\psi \longrightarrow \mathcal{O}_\phi$ of structure sheaves which is compatible with the action of $\mathbb{F}_q[t]$ and maps the submodule M_ψ into M_ϕ. Thus f induces a $k[t]$-linear homomorphism ${}^tF : M_\psi \longrightarrow M_\phi$. That ${}^tf$ is a k-algebra homomorphism corresponds to that ${}^tF$ is compatible with the action of Frobenius, i.e., $\varphi_\phi\, {}^tF = {}^tF \varphi_\psi$ (for example, $({}^tF(X))^q = {}^tF(X^q)$ etc.). If we view ${}^tF$ as a matrix in $\mathrm{M}_{r\times s}(k[t])$ (where r and s are respectively the ranks of ϕ and ψ) and if ${}^tB$ is the matrix for ψ defined in the same way as ${}^tA$ for ϕ, then we have ${}^tA\, {}^tF^\sigma = {}^tF\, {}^tB$ (note the σ-semilinearity of the φ's, whence ${}^tF^\sigma$ in the left-hand side), or

$$(\Phi) \qquad F^\sigma A \;=\; BF.$$

Conversely, the above arguments can be reversed and such an F yields a morphism $f : \phi \longrightarrow \psi$ of Drinfeld modules. Thus we have

$$(0.3.1) \qquad \operatorname{Hom}_k(\phi, \psi) \;=\; \{F \in \mathrm{M}_{s\times r}(k[t]);\; F^\sigma A \;=\; BF\}$$

Exactly the same, but with $k[\![t]\!]$ replacing $k[t]$, holds for t-divisible groups arising from Drinfeld modules. More precisely, if $\hat\phi$ and $\hat\psi$ are the t-divisible groups over k associated respectively to ϕ and ψ, then we have

$$(0.3.2) \qquad \operatorname{Hom}_k(\hat\phi, \hat\psi) \;=\; \{F \in \mathrm{M}_{s\times r}(k[\![t]\!]);\; F^\sigma A \;=\; BF\}.$$

Since étale t-divisible groups over k are equivalent to free $\mathbb{F}_q[\![t]\!]$-modules equipped with continuous G_k-action, it is only natural that we start from a morphism of t-divisible groups to arrive at the same equation (Φ) which we first arrived at in (0.2) starting from a homomorphism of Tate modules. Although it is apparently more transparent to work only with φ-modules forgetting about Galois representations by means of the functorial equivalence of the categories of certain π-adic φ-modules and π-adic Galois representations given in the Appendix to [9], we took the detour as in (0.2) going through both categories to stick to the "usual" form of the Tate conjecture.

In Section 1, the definition and basic formalisms of φ-modules are given. Section 2 contains some examples of φ-modules. In Section 3, we employ the formalisms to transform the result in [12] into the usual statement of the Tate conjecture for φ-modules of t-motive type.

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1. Formalism. In this section, we recall basic facts on φ-modules following [8] and develop some formalisms used in Section 3.

(1.1) Let (R, σ) be a pair consisting of a commutative ring R and an endomorphism $\sigma : R \to R$ (called the *Frobenius* of R). We write the action of σ as $a \mapsto a^\sigma$ for $a \in R$. We call such a pair (R, σ) a *Frobenius ring*. A *morphism* (or extension) $\lambda : (R, \sigma) \to (R', \sigma')$ of Frobenius rings is a ring homomorphism $\lambda : R \to R'$ such that $\lambda(a^\sigma) = \lambda(a)^{\sigma'}$ for all $a \in R$.

Definition. A *φ-module* (M, φ) over (R, σ) (or simply, M over R) is a pair consisting of an R-module M and a σ-semilinear map $\varphi : M \to M$ (i.e., φ is additive and satisfies $\varphi(ax) = a^\sigma \varphi(x)$ for $a \in R$ and $x \in M$). The map φ will be called the *Frobenius* of M. A *morphism* of φ-modules over R is an R-module homomorphism which is compatible with the φ's.

For an R-module M, we denote by $M^{(\sigma)}$ the scalar extension $R \otimes_R M$ of M by $\sigma : R \to R$. Then giving a σ-semilinear map $\varphi : M \to M$ is equivalent to giving an R-linear map $\varphi_\sigma : M^{(\sigma)} \to M$.

If R_0 denotes the subring of R consisting of elements which are fixed by σ, then the category of φ-modules over R is an R_0-linear category.

For a φ-module (M, φ), let M^φ denote the set of fixed points of M by φ. It is in fact an R_0-module.

For two φ-modules (M, φ_M) and (N, φ_N) over R, define their *tensor product* by setting $(M, \varphi_M) \otimes (N, \varphi_N) := (M \otimes_R N,\ \varphi_M \otimes_R \varphi_N)$. The *unit object* with respect to this tensor product in the category of φ-modules over R is $\mathbf{1}_R := (R, \sigma)$.

If $\lambda : (R, \sigma) \to (R', \sigma')$ is an extension of Frobenius rings, then the *scalar extension* $(R', \sigma') \otimes_{(R,\sigma)} (M, \varphi)$ (or simply, $R' \otimes M$) of a φ-module (M, φ) over (R, σ) is defined to be the φ-module over (R', σ') of which the underlying R'-module is $R' \otimes_R M$ and the Frobenius is $\sigma' \otimes \varphi$. By this scalar extension, we obtain a functor λ_* of the category of φ-modules over R to the category of φ-modules over R'.

A φ-module (M, φ) is said to be *étale* over R if M is of finite presentation over R and the map $\varphi_\sigma : M^{(\sigma)} \to M$ is an isomorphims. Assume in the rest of the paper that the map $\sigma : R \to R$ is flat. If (M, φ_M) and (N, φ_N) are two φ-modules over R and if M is étale, then we can define a φ-module $\mathrm{Hom}(M, N)$, whose underlying R-module is the space $H = \mathrm{Hom}_R(M, N)$ of R-module homomorphisms and $\varphi_\sigma : H^{(\sigma)} \to H$ is given by $f \mapsto \varphi_{N,\sigma} \circ f \circ (\varphi_{M,\sigma})^{-1}$ for $f \in H^{(\sigma)} \simeq \mathrm{Hom}_R(M^{(\sigma)}, N^{(\sigma)})$

(cf. [5], Chap. I, §2, Prop. 11). Thus the category of étale φ-modules over R has internal homs. If M is not étale, the internal hom may exist; this will be discussed in (1.4).

(1.2) A φ-module (M, φ) over R is said to be *free* if M is free of finite rank as an R-module. The *rank* of a free φ-module (M, φ) is by definition the rank of M as a free R-module. We denote by ΦM the category of free φ-modules over R. In the rest of the paper, *all φ-modules are assumed to be free.* (Here and in what follows, the condition "free" can be replaced by the weaker condition "projective", which is in fact more natural; our assumption is just for simplicity.)

For a φ-module (M, φ) over R, define $\det(M, \varphi) := (\det_R M, \det_R \varphi)$, where the $\det_R$ in the right-hand side means the maximum exterior product of free R-modules and σ-semilinear maps (or of R-linear maps if φ_σ is used).

For a φ-module (M, φ) over R of rank r, define

$$(M, \varphi)^* \;=\; (M^*, \varphi^*) \;:=\; (\wedge^{r-1} M, \wedge \varphi^{r-1}).$$

For an étale φ-module (M, φ) over R, set

$$(M, \varphi)^\vee \;=\; (M^\vee, \varphi^\vee) \;:=\; \mathrm{Hom}((M, \varphi), \mathbf{1}_R),$$

and call it the *dual* of (M, φ). We have a canonical isomorphism

$$M^* \;\simeq\; \det M \otimes M^\vee \tag{1.2.1}$$

in ΦM. If M and N are two φ-modules and M is étale, then we have a canonical isomorphism

$$\mathrm{Hom}(M, N) \;\simeq\; M^\vee \otimes N \tag{1.2.2}$$

in ΦM.

(1.3) Let (M, φ) be a φ-module of rank r over R, and let $(e_i)_{1 \le i \le r}$ be an R-basis of M. Suppose the map φ is represented with respect to the basis $(e_i)_{1 \le i \le r}$ by a matrix $A = (a_{ij}) \in \mathrm{M}_r(R)$, i.e., we have

$$\varphi(e_j) \;=\; \sum_i a_{ij} e_i.$$

If $(f_i)_{1 \le i \le r}$ is another R-basis of M and is related to $(e_i)_{1 \le i \le r}$ by the relation

$$f_j \;=\; \sum_i u_{ij} e_i \qquad \text{with } U = (u_{ij}) \;\in \mathrm{GL}_r(R),$$

then φ is represented with respect to $(f_i)_{1 \le i \le r}$ by the matrix

$$B \;=\; U^{-1} A U^\sigma,$$

where $U^\sigma := (u_{ij}^\sigma)$. The relation for two matrices A and B in $\mathrm{M}_r(R)$ defined by that $B = U^{-1} A U^\sigma$ with some $U \in \mathrm{GL}_r(R)$ is an equivalence relation. Thus the matrix representing the Frobenius φ is well-defined modulo this equivalence.

Admitting this ambiguity (i.e., with *some* choice of an R-basis of M), we say simply that φ is represented by a matrix $A \in \mathrm{GL}_r(R)$. If a φ-module (L, φ) is of rank one, then φ is represented by a scalar matrix $a \in R$, which is well-defined modulo $(R^\times)^{\sigma-1} = \{u^\sigma/u;\ u \in R^\times\}$. We express this situation by writing simply as $\varphi \equiv a$. This applies in particular to the determinant of a φ-module (M, φ); we shall write as $\det\varphi \equiv a$. If φ is represented by a matrix $A \in \mathrm{M}_r(R)$, then $\det\varphi$ is represented by $\det A$, and φ^* is represented by the transposed cofactor matrix A^*. If the φ-module (M, φ) is étale, then φ is represented by an invertible matrix $A \in \mathrm{GL}_r(R)$, and its dual $\varphi^\vee$ is represented by $A^\vee := {}^tA^{-1}$. We have then $A^* = \det A \cdot A^\vee$.

(1.4) Let $\widetilde{R}$ be the total fraction ring of R. Suppose the Frobenius σ on R extends to an endomorphism of $\widetilde{R}$ (then the extension is unique). A φ-module (L, φ) over R is said to be *cancellable* if it is of rank one and its scalar extension $\widetilde{R} \otimes_R L$ is étale over $\widetilde{R}$. If the Frobenius φ is represented by $a \in R$, this amounts to the same as to say that a is invertible in $\widetilde{R}$, or that a is a cancellable element in R ([4], Chap. I, §2, n° 2). For example, if R is integral, then any non-zero element in R is cancellable, and hence all non-trivial (i.e. $\varphi \neq 0$) φ-modules of rank one are cancellable.

Let (M, φ_M) and (N, φ_N) be two φ-modules over R. If the internal hom $\mathrm{Hom}(M, N)$ exists in $\Phi\mathrm{M}$ and if L is a cancellable φ-module over R, then $\mathrm{Hom}(L \otimes M, L \otimes N)$ also exists and we have a canonical isomorphism

$$\mathrm{Hom}(M, N) \ \xrightarrow{\sim}\ \mathrm{Hom}(L \otimes M, L \otimes N) \tag{1.4.1}$$

of φ-modules.

Assume, hereafter in (1.4), that $\det M$ is cancellable. Write $\widetilde{M}$ and $\widetilde{N}$ respectively for the scalar extensions of M and N to $\widetilde{R}$. Then the internal hom $\widetilde{H} := \mathrm{Hom}(\widetilde{M}, \widetilde{N}) = \widetilde{M}^\vee \otimes \widetilde{N}$ exists as a φ-module over $\widetilde{R}$. The R-module $H := \mathrm{Hom}_R(M, N)$ is naturally identified with a subspace of the underlying space $\mathrm{Hom}_{\widetilde{R}}(\widetilde{M}, \widetilde{N})$ of $\widetilde{H}$. If H is stable under the action of φ on $\widetilde{H}$, then H is a φ-module over R, and one may say that there exists the internal hom $H = \mathrm{Hom}(M, N)$ in $\Phi\mathrm{M}$. For example, this is the case if φ_N is divisible by $\det\varphi_M$ (i.e., if φ_N is represented by a matrix of which entries are all divisible in R by an element of R which represents $\det\varphi_M$). Especially, the φ-module $\mathrm{Hom}(M, \det M)$ exists in $\Phi\mathrm{M}$, and we have canonically

$$M^* \ \simeq\ \mathrm{Hom}(M, \det M).$$

If $\mathrm{Hom}(M, N)$ is defined, then we have a canonical isomorphism

$$\mathrm{Hom}(M, N) \ \xrightarrow{\sim}\ \mathrm{Hom}(\det M, M^* \otimes N). \tag{1.4.2}$$

Over $\widetilde{R}$, this isomorphism follows from (1.2.2) and (1.4.1), and restricts to the one over R.

(1.5) For two φ-modules (M, φ_M) and (N, φ_N) over R, let $\mathrm{Hom}_{\Phi\mathrm{M}}(M, N)$ denote the set of morphisms of (M, φ_M) to (N, φ_N). It is an R_0-module. If the internal hom $H = \mathrm{Hom}(M, N)$ exists, then the morphisms are the fixed points of H by the Frobenius;

$$H^\varphi \ =\ \mathrm{Hom}_{\Phi\mathrm{M}}(M, N).$$

Suppose that M and N are of rank m and n respectively, and that φ_M and φ_N are represented by matrices $A \in \mathrm{M}_m(R)$ and $B \in \mathrm{M}_n(R)$ respectively. Then we may identify

$$\text{(1.5.1)} \qquad \mathrm{Hom}_{\Phi\mathrm{M}}(M,N) \;=\; \{F \in \mathrm{M}_{n\times m}(R);\; FA = BF^{\sigma}\}.$$

If $\lambda : (R,\sigma) \to (R',\sigma')$ is an extension of Frobenius rings such that $\lambda : R \to R'$ is *injective*, then it follows from (1.5.1) that the natural map

$$\text{(1.5.2)} \qquad \mathrm{Hom}_{\Phi\mathrm{M}}(M,N) \;\to\; \mathrm{Hom}_{\Phi\mathrm{M}'}(\lambda_* M, \lambda_* N)$$

is injective. Here $\Phi\mathrm{M}'$ denotes the category of φ-modules over R'. Extending the scalars R'_0-linearly (where $R'_0 = (R')_0$), we obtain a map of R'_0-modules

$$\text{(1.5.3)} \qquad \lambda_\natural : \; R'_0 \otimes_{R_0} \mathrm{Hom}_{\Phi\mathrm{M}}(M,N) \;\to\; \mathrm{Hom}_{\Phi\mathrm{M}'}(\lambda_* M, \lambda_* N),$$

which we claim is also injective if R is integral with σ extending to its fraction field, and if $\det M$ is cancellable. To see this, we may localize R and R'. So we assume R is a field. Then the internal hom $H = \mathrm{Hom}(M,N)$ (resp. $H' = \mathrm{Hom}(\lambda_* M, \lambda_* N)$) exists in $\Phi\mathrm{M}$ (resp. $\Phi\mathrm{M}'$), and we have $H^{\varphi} = \mathrm{Hom}_{\Phi\mathrm{M}}(M,N)$ and $(H')^{\varphi} = \mathrm{Hom}_{\Phi\mathrm{M}'}(\lambda_* M, \lambda_* N)$. Note that $H' \simeq R' \otimes_R H$. The injectivity amounts to that any elements $h_1, \cdots, h_s$ in H^{φ} which are linearly independent over R_0 remain linearly independent over R in H. Suppose the contrary and take a shortest non-trivial relation (*) $a_1 h_1 + \cdots + a_s h_s = 0$ with $a_i \in R$. We may assume $a_1 = 1$. Then computing $\varphi(*) - (*)$, we have $(a_2^{\sigma} - a_2)h_2 + \cdots + (a_s^{\sigma} - a_s)h_s = 0$. By the minimality of the length s, we have $a_i^{\sigma} - a_i = 0$, i.e., $a_i \in R_0$ for all i. This contradicts the linear independence of $h_1, \cdots, h_s$ over R_0.

One may then be interested in the surjectivity of the map $\lambda_\natural$. It is this surjectivity, in various special situations, that the Tate conjecture asserts. We will deal with the case of t-motives in Section 3.

Let (L, φ_L) be a cancellable φ-module over R and let φ_L be represented by $a \in R$. Then we have

$$\begin{aligned} \mathrm{Hom}_{\Phi\mathrm{M}}(L\otimes M, L\otimes N) \;&=\; \{F \in \mathrm{M}_{n\times m}(R);\; FaA = aBF^{\sigma}\} \\ &=\; \{F \in \mathrm{M}_{n\times m}(R);\; FA = BF^{\sigma}\}. \end{aligned}$$

Thus we have a canonical isomorphism

$$\text{(1.5.4)} \qquad \mathrm{Hom}_{\Phi\mathrm{M}}(M,N) \;\simeq\; \mathrm{Hom}_{\Phi\mathrm{M}}(L\otimes M, L\otimes N)$$

of R_0-modules. If the internal hom $\mathrm{Hom}(M,N)$ exists in $\Phi\mathrm{M}$, this is obtained also from (1.4.1) by taking the φ-fixed part.

If $\det M$ is cancellable, then we have a canonical isomorphism

$$\text{(1.5.5)} \qquad \mathrm{Hom}_{\Phi\mathrm{M}}(M,N) \;\simeq\; \mathrm{Hom}_{\Phi\mathrm{M}}(\det M, M^*\otimes N)$$

of R_0-modules (this is a trick used in the proof of Theorem 1 of [2]). Indeed, in explicit terms, if $A^* = (a^*_{ij})$, $B = (b_{ij})$, and $d = \det A$, then both modules are the set of solutions (f_{ij}) of the system of equations

$$df_{ij} \;=\; \sum_{k,l} b_{ik} a^*_{jl} f^{\sigma}_{kl},$$

with a suitable interpretation of (f_{ij}) (i.e., as elements of $\mathrm{M}_{n\times m}(R)$ or $\mathrm{M}_{mn\times 1}(R)$). If the internal hom $\mathrm{Hom}(M,N)$ exists in $\Phi\mathrm{M}$, this is obtained also from (1.4.2) by taking the φ-fixed part.

2. Examples. We give some examples of φ-modules. The first one (2.1) is the motivating example for Section 3.

(2.1) Let k be a perfect field containing $\mathbb{F}_q$ (in Section 3, we will be interested in the case where k is not perfect). Fix an element $\theta \in k$. Let $A := \mathbb{F}_q[t]$, and $K := \mathbb{F}_q(t)$. Set $A_k := A \otimes_{\mathbb{F}_q} k = k[t]$ and $K_k := K \otimes_{\mathbb{F}_q} k = k[t][a^{-1}; a \in A - \{0\}]$. Let $A_k[\sigma] = k[t,\sigma]$ be the non-commutative polynomial ring with commutation relation $\sigma a = a^\sigma \sigma$, where $a^\sigma = \sum a_i^q t^i$ if $a = \sum a_i t^i$ with $a_i \in k$. A *t-motive* over k is defined ([1]) as a left $A_k[\sigma]$-module which is free of finite rank over A_k and satisfies certain conditions. If M is a t-motive over k, then the action of σ on M can be thought of as a σ-semilinear map $\varphi : M \to M$ of the free A_k-module M. Thus in our terminology, a t-motive over k is a φ-module over A_k. A φ-module coming in this way from a t-motive has the property, among others, that $\det\varphi \equiv u(t-\theta)^d$ for a $u \in k^\times$ and a positive integer d.

An *isogeny* of t-motives is an injective map of $k[t,\sigma]$-modules whose cokernel is killed by a non-zero element of A. If M is a t-motive over k, then $K_k \otimes_{A_k} M$ is a φ-module over K_k. The φ-moduels of this form form the category of t-motives *modulo isogeny*, which is a full subcategory of the category of φ-modules over K_k.

(2.2) Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$, and let S be an $\mathbb{F}_q$-scheme. A *right F-sheaf* (or *shtuka*) on S is defined ([6]) as a diagram

$$\mathrm{Fr}_S^*\mathcal{E} \xrightarrow{\varphi} \mathcal{F} \xleftarrow{\iota} \mathcal{E}$$

in which $\mathcal{E}$ and $\mathcal{F}$ are locally free $\mathcal{O}_{X\times S}$-modules of finite rank, Fr_S is the Frobenius endomorphism of S relative to $\mathbb{F}_q$, and φ and ι are injective morphisms of $\mathcal{O}_{X\times S}$-modules having certain properties. In fact, $\mathcal{E}$ and $\mathcal{F}$ are not so different. Suppose there exists an open affine subset $U = \mathrm{Spec}R$ of $X \times S$ such that (1) U is stable under $\mathrm{id}_X \times \mathrm{Fr}_S$, (2) ι induces an isomorphism $\mathcal{E}|_U \xrightarrow{\sim} \mathcal{F}|_U$, and (3) $\mathcal{E}|_U$ and $\mathcal{F}|_U$ are free over $\mathcal{O}_U$. Then the map $\varphi : \mathrm{Fr}_S^*\mathcal{E}|_U \to \mathcal{F}|_U$ together with the identification $\iota : \mathcal{E}|_U \xrightarrow{\sim} \mathcal{F}|_U$ allows us to regard $\mathcal{E}|_U$ as a φ-module over R, where R is a Frobenius ring with σ induced by $\mathrm{id}_X \times \mathrm{Fr}_S$.

In the case where the F-sheaf comes from a Drinfeld A-module over S, where A is such that $X = \mathrm{Spec}A \cup \{\infty\}$, the two sheaves $\mathcal{E}$ and $\mathcal{F}$ are identified over $\mathrm{Spec}A \times S$.

(2.3) Let k be a perfect field of characteristic $p > 0$, and $W(k)$ the ring of Witt vectors with coefficients in k. Let σ be the Frobenius automorphism of $W(k)$. Let $W(k)[F,V]$ be the Dieudonné ring. It is a non-commutative ring with commutation relation $FV = VF = p$, $Fa = a^\sigma F$, $Va^\sigma = aV$, for all $a \in W(k)$. Usually, a *Dieudonné module* is defined (e.g. [11]) as a left $W(k)[F,V]$-module of certain type. For example, the Dieudonné module M corresponding to a p-divisible group over k is a free $W(k)$-module of rank equal to the height of the p-divisible group. Such an M is a φ-module over $W(k)$ in our sense, with the Frobenius $\varphi = F : M \to M$. More generally, an *F-crystal* over k (e.g. [10]) is by definition a φ-module over $W(k)$.

3. Function field case. In this section, we deduce the familiar form of the Tate conjecture for certain φ-modules from the main result of [12].

(3.1) Let K be an algebraic function field in one variable over a finite field. We assume its field of constants is the field $\mathbb{F}_q$ of q elements. For any field k containing $\mathbb{F}_q$, set $K_k := K \otimes_{\mathbb{F}_q} k$. For a place π of K, let K_π (resp. $K_{k,\pi}$) be the π-adic completion of K (resp. K_k). The ring $K_{k,\pi}$ is a product of a finite number of complete discrete valuation fields whose residue fields are finite extensions of k. As we have assumed that $\mathbb{F}_q$ is the field of constants in K, the ring K_k is integral, and its field of fractions $\widetilde{K_k}$ is the function field of the curve $C \otimes_{\mathbb{F}_q} k$ over k if C is a model of K over $\mathbb{F}_q$ (i.e., a curve over $\mathbb{F}_q$ with function field K). The π-adic completion of $\widetilde{K_k}$ coincides with $K_{k,\pi}$. In the simplest case $K = \mathbb{F}_q(t)$, we have $\widetilde{K_k} = k(t)$, and K_k is its subring which consists of elements whose denominators are in $\mathbb{F}_q[t]$; $K_k = k[t][\mathbb{F}_q(t)] = k[t][a^{-1}; a \in \mathbb{F}_q[t] - \{0\}]$.

Let σ denote the q-th power Frobenius endomorphism of k. It induces the endomorphism $\mathrm{id}_K \otimes \sigma$ on K_k, $\widetilde{K_k}$ and $K_{k,\pi}$. We think of these rings as Frobenius rings (1.1). We write simply σ for these endomorphisms. The action of σ will be denoted $a \mapsto a^\sigma$ as before. The σ-fixed subrings of these rings are respectively K, K, and K_π.

Let $\Phi\mathrm{M}_k$ (resp. $\widetilde{\Phi\mathrm{M}}_k$, $\Phi\mathrm{M}_{k,\pi}$) be the category of φ-modules (M, φ) over K_k (resp. $\widetilde{K_k}$, $K_{k,\pi}$) such that the map φ is *injective* (the assumption of injectivity is not essential at all, but we assume so in order to avoid inessential discussions. Also, recall that we have assumed all φ-modules are free (1.2)). Then all non-zero objects in $\Phi\mathrm{M}_k$ (resp. in $\widetilde{\Phi\mathrm{M}}_k$) have cancellable determinants (resp. are étale), since $\widetilde{K_k}$ is a field. The categories $\Phi\mathrm{M}_k$ and $\widetilde{\Phi\mathrm{M}}_k$ are K-linear categories, and $\Phi\mathrm{M}_{k,\pi}$ is a K_π-linear category. We have a commutative diagram of morphisms of Frobenius rings

$$\begin{array}{ccc} K_k & \xrightarrow{\rho} & K_{k,\pi} \\ {\scriptstyle\lambda}\searrow & & \nearrow{\scriptstyle\widetilde{\rho}} \\ & \widetilde{K_k} & \end{array}$$

in which all arrows are injective, and accordingly a commutative diagram of functors

$$\begin{array}{ccc} \Phi\mathrm{M}_k & \xrightarrow{\rho_*} & \Phi\mathrm{M}_{k,\pi} \\ {\scriptstyle\lambda_*}\searrow & & \nearrow{\scriptstyle\widetilde{\rho}_*} \\ & \widetilde{\Phi\mathrm{M}}_k & . \end{array} \tag{3.1.1}$$

Hence, for any M, N in $\Phi\mathrm{M}_k$ and $\widetilde{M}$, $\widetilde{N}$ in $\widetilde{\Phi\mathrm{M}}_k$, we have canonical homomorphisms

$$\begin{aligned} \mathrm{Hom}_{\Phi\mathrm{M}_k}(M, N) &\to \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\lambda_* M, \lambda_* N), \\ \mathrm{Hom}_{\Phi\mathrm{M}_k}(M, N) &\to \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N), \\ \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\widetilde{M}, \widetilde{N}) &\to \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\widetilde{\rho}_* \widetilde{M}, \widetilde{\rho}_* \widetilde{N}), \end{aligned}$$

which are injective by (1.5.2) and such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\Phi\mathrm{M}_k}(M, N) & \xrightarrow{\qquad\qquad} & \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N) \\ \searrow & & \nearrow \\ & \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\lambda_* M, \lambda_* N) & \end{array}$$

is commutative. Extending the scalars K_π-linearly, we obtain canonical homomorphisms

$$\begin{aligned}
\lambda_\natural &: K_\pi \otimes_K \mathrm{Hom}_{\Phi\mathrm{M}_k}(M,N) \to K_\pi \otimes_K \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\lambda_* M, \lambda_* N),\\
\rho_\natural &: K_\pi \otimes_K \mathrm{Hom}_{\Phi\mathrm{M}_k}(M,N) \to \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N),\\
\widetilde{\rho}_\natural &: K_\pi \otimes_K \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\widetilde{\rho}_* \widetilde{M}, \widetilde{\rho}_* \widetilde{N}),
\end{aligned}$$

such that the diagram

(3.1.2)

$$\begin{array}{ccc}
K_\pi \otimes_K \mathrm{Hom}_{\Phi\mathrm{M}_k}(M,N) & \xrightarrow{\quad\rho_\natural\quad} & \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N)\\
\lambda_\natural \searrow & & \nearrow \widetilde{\rho}_\natural\\
& K_\pi \otimes_K \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\lambda_* M, \lambda_* N) &
\end{array}$$

is commutative. The map $\lambda_\natural$ is of course injective. The other maps $\rho_\natural$ and $\widetilde{\rho}_\natural$ are also injective by (1.5.3).

(3.2) We shall see the variance of φ-fixed points of a φ-module under finite extensions k'/k of the base field. Let R be one of the rings K_k, $\widetilde{K_k}$ and $K_{k,\pi}$, and let R' be the ring similarly defined as R with k' replacing k. Let M be a φ-module over R. It is identified canonically with a submodule of $R' \otimes_R M = k' \otimes_k M$, which is a φ-module over R'.

First we consider the case of a purely inseparable extension.

Lemma (3.2.1). *Let k'/k be a purely inseparable extension of finite degree. With the notations as above, we have*

$$(k' \otimes_k M)^\varphi \;=\; M^\varphi.$$

Proof. It is clear that $(k' \otimes_k M)^\varphi \supset M^\varphi$. Conversely, let x be any element of $(k' \otimes_k M)^\varphi$. Write $x = \sum a_i \otimes m_i$ with $a_i \in k'$ and $m_i \in M$. Take a positive integer n such that $a_i^{q^n} \in k$ for all i. Then we have $x = \varphi^n(x) = \sum a_i^{q^n} \otimes \varphi^n(m_i) \in M$, hence $x \in M^\varphi$.

Suppose next that k'/k is a finite Galois extension, with Galois group G. Then M is recovered from $k' \otimes_k M$ as the G-fixed points of $k' \otimes_k M$;

$$(k' \otimes_k M)^G \;=\; M.$$

Since the action of G on $k' \otimes_k M$ commutes with that of φ, we have

(3.2.2) $$((k' \otimes_k M)^\varphi)^G \;=\; M^\varphi.$$

(3.3) We shall see the variance of the φ-fixed points of a φ-module when the coefficient field K is replaced by a subfield. Let K' be a subfield of K over which K is of finite degree. Let π' be the restriction of the place π to K'. A φ-module M over $\widetilde{K_k}$ can be regarded naturally as a φ-module over $\widetilde{K'_k}$. Consider the natural maps

$$\begin{aligned}
\widetilde{\rho}_\natural &: \; K_\pi \otimes_K M^\varphi \;\to\; (K_{k,\pi} \otimes_{\widetilde{K_k}} M)^\varphi,\\
\widetilde{\rho}_\natural{}' &: \; K'_{\pi'} \otimes_{K'} M^\varphi \;\to\; (K'_{k,\pi'} \otimes_{\widetilde{K'_k}} M)^\varphi.
\end{aligned}$$

Lemma (3.3.1). *If $\widetilde{\rho}_\natural'$ is an isomorphism, then $\widetilde{\rho}_\natural$ is also an isomorphism.*

Proof. We have

$$K'_{\pi'} \otimes_{K'} M^\varphi \;=\; K'_{\pi'} \otimes_{K'} K \otimes_K M^\varphi \;=\; (\prod_{\pi|\pi'} K_\pi) \otimes_K M^\varphi,$$

$$(K'_{k,\pi'} \otimes_{K'_k} M)^\varphi \;=\; (K'_{k,\pi'} \otimes_{K'_k} K_k \otimes_{K_k} M)^\varphi \;=\; ((\prod_{\pi|\pi'} K_{k,\pi}) \otimes_{K_k} M)^\varphi,$$

where the products are over all extensions π of π' to K. Hence the Lemma follows.

(3.4) Let d be a non-zero element of K_k. We denote by $\Phi\mathrm{M}_k(d)$ the category of étale φ-modules over $K_k[d^{-1}]$. The inclusion $\lambda' : K_k[d^{-1}] \hookrightarrow \widetilde{K_k}$ yields a functor $\lambda'_* : \Phi\mathrm{M}_k(d) \to \widetilde{\Phi\mathrm{M}}_k$. The natural map $\mathrm{Hom}_{\Phi\mathrm{M}_k(d)}(M, N) \to \mathrm{Hom}_{\widetilde{\Phi\mathrm{M}}_k}(\lambda'_* M, \lambda'_* N)$ of K-vector spaces is injective (1.5.2).

Now we specialize to the case in which K is the rational function field $\mathbb{F}_q(t)$. Fix an elemenent θ of k. Let $t\text{-}\mathrm{Mot}_k^+$ be the full subcategory of $\Phi\mathrm{M}_k$ consisting of objects (M, φ) such that the Frobenius φ is represented by a matrix $A \in \mathrm{M}_r(k[t])$ whose determinant is of the form $u(t-\theta)^d$ with $u \in k^\times$ and d a non-negative integer. It contains essentially the category of t-motives over k modulo isogeny (since we have inverted non-zero elements of $\mathbb{F}_q[t]$). In the case where the base field k is *not* perfect (which is the case we will be interested in), there are delicate problems in giving a good definition of t-motives. Here, we only remark that any t-motive over k should become a free K_k-module after a purely inseparable extension k'/k of finite degree (cf. (3.2.1)).

There is a natural functor $\lambda_* : t\text{-}\mathrm{Mot}_k^+ \to \Phi\mathrm{M}_k(t-\theta)$ induced by the localization $\lambda : K_k \to K_k[(t-\theta)^{-1}]$. The dual of $\lambda_* M$ exists in $\Phi\mathrm{M}_k(t-\theta)$ if $M \neq 0$. In fact, $\Phi\mathrm{M}_k(t-\theta)$ is generated as a $\otimes$-category by $\lambda_* M$ and $(\lambda_* M)^\vee$ for all non-zero objects M in $t\text{-}\mathrm{Mot}_k^+$; we write $t\text{-}\mathrm{Mot}_k$ instead of $\Phi\mathrm{M}_k(t-\theta)$. Let $t\text{-}\widetilde{\mathrm{Mot}}_k$ be the full subcategory of $\widetilde{\Phi\mathrm{M}}_k$ which has the same objects as $t\text{-}\mathrm{Mot}_k$ (it will turn out (3.6.1) that the morphisms, as well as the objects, are the same in $t\text{-}\mathrm{Mot}_k$ and $t\text{-}\widetilde{\mathrm{Mot}}_k$). An object of $\widetilde{\Phi\mathrm{M}}_k$ is in $t\text{-}\widetilde{\mathrm{Mot}}_k$ if and only if its Frobenius is represented by a matrix A in $\mathrm{GL}_r(k[t, (t-\theta)^{-1}])$. Note that, if this is the case, A and A^{-1} are both of the form $(t-\theta)^d B^{-1}$ with some $d \geq 0$ and

$$B \in \mathrm{M}_r(k[t]) \quad \text{such that} \quad \det B = u(t-\theta)^e,\ u \in k^\times,\ e \geq 0.$$

In the following, we work only with $t\text{-}\widetilde{\mathrm{Mot}}_k$ (rather than $t\text{-}\mathrm{Mot}_k$), but all results hold true also for $t\text{-}\mathrm{Mot}_k$.

Let π be a place of K. As in (3.1.1), we have a commutative diagram

(3.4.1)
$$\begin{array}{ccc} t\text{-}\mathrm{Mot}_k^+ & \xrightarrow{\rho_*} & \Phi\mathrm{M}_{k,\pi} \\ {\scriptstyle\lambda_*} \searrow & & \nearrow {\scriptstyle\widetilde{\rho}_*} \\ & t\text{-}\widetilde{\mathrm{Mot}}_k & . \end{array}$$

We have also natural maps

$$\lambda_\natural : K_\pi \otimes_K \mathrm{Hom}_{t\text{-}\mathrm{Mot}_k^+}(M, N) \to K_\pi \otimes_K \mathrm{Hom}_{t\text{-}\widetilde{\mathrm{Mot}}_k}(\lambda_* M, \lambda_* N),$$

$$\rho_\natural : K_\pi \otimes_K \mathrm{Hom}_{t\text{-}\mathrm{Mot}_k^+}(M, N) \to \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N),$$

$$\widetilde{\rho}_\natural : K_\pi \otimes_K \mathrm{Hom}_{t\text{-}\widetilde{\mathrm{Mot}}_k}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\widetilde{\rho}_* \widetilde{M}, \widetilde{\rho}_* \widetilde{N})$$

for any M and N in $t\text{-Mot}_k^+$ (resp. $\widetilde{M}$ and $\widetilde{N}$ in $t\text{-}\widetilde{\text{Mot}}_k$). These maps are all injective as shown in (1.5.3), and the diagram

$$(3.4.2)\qquad \begin{array}{ccc} K_\pi \otimes_K \operatorname{Hom}_{t\text{-Mot}_k^+}(M,N) & \xrightarrow{\rho_\natural} & \operatorname{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N) \\ \lambda_\natural \downarrow & \nearrow \widetilde{\rho}_\natural & \\ K_\pi \otimes_K \operatorname{Hom}_{t\text{-}\widetilde{\text{Mot}}_k}(\lambda_* M, \lambda_* N) & & \end{array}$$

is commutative.

In what follows, we assume that k is an algebraic function field in one variable over a finite field, and that θ is a non-zero element of k. In [12], we proved

Theorem (3.5). *Assume the place π is different from $\infty = (1/t)$. Let A be a matrix in $\mathrm{GL}_r(k[t,(t-\theta)^{-1}])$. Let V (resp. V_π) be the K-vector space (K_π-vector space) consisting of the solutions X in $K_k^{\oplus r}$ (resp. $K_{k,\pi}^{\oplus r}$) of the linear Frobenius equation*

$$AX^\sigma \;=\; X.$$

Then the natural map

$$K_\pi \otimes_K V \;\to\; V_\pi$$

is an isomorphism.

Corollary (3.5.1). *With the same notation as above, let $\widetilde{V}$ be the K-vector space consisting of the solutions X in $\widetilde{K_k}^{\oplus r}$ of the equation $AX^\sigma = X$.*
(1) The natural map

$$K_\pi \otimes_K \widetilde{V} \;\to\; V_\pi$$

is an isomorphism.
(2) We have $\widetilde{V} = V$.

Proof. Since $V \subset \widetilde{V}$, the surjectivity of the map in (1) follows from the Theorem. The injectivity is proved in the same way as in (1.5.3). By (1) and the Theorem, it follows that $\widetilde{V}$ and V have the same dimension over K, hence $\widetilde{V} = V$.

From these results, we deduce the following

Theorem (3.6). *The natural maps of K_π-vector spaces*

$$\rho_\natural : K_\pi \otimes_K \operatorname{Hom}_{t\text{-Mot}_k^+}(M,N) \;\to\; \operatorname{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N) \qquad \textit{and}$$
$$\widetilde{\rho}_\natural : K_\pi \otimes_K \operatorname{Hom}_{t\text{-}\widetilde{\text{Mot}}_k}(\widetilde{M},\widetilde{N}) \;\to\; \operatorname{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\widetilde{\rho}_* \widetilde{M}, \widetilde{\rho}_* \widetilde{N})$$

are isomorphisms for any M and N in $t\text{-Mot}_k^+$ (resp. $\widetilde{M}$ and $\widetilde{N}$ in $t\text{-}\widetilde{\text{Mot}}_k$).

Proof. For $\widetilde{\rho}_\natural$, this is reduced immediately to (1) of Corollary (3.5.1) by using the internal hom $\widetilde{H} = \operatorname{Hom}(\widetilde{M},\widetilde{N})$ in $t\text{-}\widetilde{\text{Mot}}_k$ and interpreting $\widetilde{V}$ as $\widetilde{H}^\varphi$. Almost the same can be done in $t\text{-Mot}_k^+$, but strictly speaking, we proceed as follows: by (1.5.5), we have canonical isomorphisms

$$\operatorname{Hom}_{t\text{-Mot}_k^+}(M,N) \;\simeq\; \operatorname{Hom}_{t\text{-Mot}_k^+}(\det M, M^* \otimes N),$$
$$\operatorname{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N) \;\simeq\; \operatorname{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\det(\rho_* M), (\rho_* M)^* \otimes \rho_* N).$$

Since the functor ρ_* commutes with the operations det, $\otimes$ and $(\ )^*$, we only need to prove the isomorphy of $\rho_\natural$ assuming that the first object M is of rank one. Let φ_M be represented by $a \in k[t]$ (resp. φ_N by $B \in \mathrm{M}_r(k[t])$). Note that a and $\det B$ are both of the form $u(t-\theta)^d$ with some $u \in k^\times$ and $d \geq 0$. By (1.5.1), we have

$$\begin{aligned}\mathrm{Hom}_{t\text{-Mot}_k^+}(M,N) &= \{X \in \mathrm{M}_{r\times 1}(K_k);\ Xa = BX^\sigma\},\\ \mathrm{Hom}_{\Phi\mathrm{M}_{k,\pi}}(\rho_* M, \rho_* N) &= \{X \in \mathrm{M}_{r\times 1}(K_{k,\pi});\ Xa = BX^\sigma\}.\end{aligned}$$

If we put $A := a^{-1}B$ ($\in \mathrm{GL}_r(k[t,(t-\theta)^{-1}])$), then these spaces are respectively the spaces V and V_π in Theorem (3.5). Thus Theorem (3.5) is equivalent to Theorem (3.6).

Corollary (3.6.1). *The functor*

$$\lambda_* : t\text{-Mot}_k^+ \ \to\ t\text{-}\widetilde{\mathrm{Mot}}_k$$

is fully faithful, and the functor

$$\lambda'_* : t\text{-Mot}_k \ \to\ t\text{-}\widetilde{\mathrm{Mot}}_k$$

is an equivalence.

Proof. This is in fact equivalent to (2) of Corollary (3.5.1). But here is another way of showing the isomorphy of the natural map

$$\mathrm{Hom}_{t\text{-Mot}_k^+}(M,N) \ \to\ \mathrm{Hom}_{t\text{-}\widetilde{\mathrm{Mot}}_k}(\lambda_* M, \lambda_* N)$$

of K-vector spaces. It is enough to show this after tensoring K_π. This follows from the commutative diagram (3.4.2), in which both $\rho_\natural$ and $\widetilde{\rho}_\natural$ are isomorphisms by the above Theorem. The latter assertion is now clear, as t-Mot_k is "in between" t-Mot_k^+ and t-$\widetilde{\mathrm{Mot}}_k$.

Let G_k be the absolute Galois group of k. In the Appendix to [9], we gave an equivalence of a certain subcategory of $\Phi\mathrm{M}_{k,\pi}$ containing the essential image of $\widetilde{\rho}_* : t\text{-}\widetilde{\mathrm{Mot}}_k \to \Phi\mathrm{M}_{k,\pi}$ and the category $\mathrm{Rep}_{K_\pi}(G_k)$ of finite dimensional K_π-linear continuous representations of G_k. Composing this equivalence and our functor $\widetilde{\rho}_* : t\text{-}\widetilde{\mathrm{Mot}}_k \to \Phi\mathrm{M}_{k,\pi}$, we obtain a covariant functor $V_\pi : t\text{-}\widetilde{\mathrm{Mot}}_k \to \mathrm{Rep}_{K_\pi}(G_k)$. The above theorem implies

Theorem (3.7). *For $\widetilde{M}$ and $\widetilde{N}$ in t-$\widetilde{\mathrm{Mot}}_k$, the natural map*

$$K_\pi \otimes_K \mathrm{Hom}_{t\text{-}\widetilde{\mathrm{Mot}}_k}(\widetilde{M}, \widetilde{N}) \ \to\ \mathrm{Hom}_{\mathrm{Rep}_{K_\pi}(G_k)}(V_\pi(\widetilde{M}), V_\pi(\widetilde{N}))$$

is an isomorphism. The same holds true with t-Mot_k^+ replacing t-$\widetilde{\mathrm{Mot}}_k$.

References

[1] G. W. Anderson, *t-Motives*, Duke Math. J. **53** (1986), 457–502.
[2] G. W. Anderson, *On Tate modules of formal t-modules*, Intl. Math. Res. Notices **2** (1993), 41–52.
[3] G. W. Anderson, unpublished lecture notes (1987).
[4] N. Bourbaki, *Algèbre*, Hermann, Paris, 1970.

[5] N. Bourbaki, *Algèbre Commutative*, Masson, Paris, 1985.

[6] V. G. Drinfeld, *Moduli varieties of F-sheaves*, Funct. Anal. Appl. **21** (1987), 107–122.

[7] G. Faltings, *Endlichkeitssätze für Abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366.

[8] J.-M. Fontaine, *Représentation p-adiques des corps locaux*, in: The Grothendieck Festschrift, Vol. II, Birkhäuser, 1990, pp. 249–309.

[9] D. Goss, *The adjoint of the Carlitz module and Fermat's Last Theorem*, J. of Finite Fields.

[10] N. Katz, *Slope filtration of F-crystals*, Astérisque **63** (1979), 113–163.

[11] Y. I. Manin, *The theory of commutative formal groups over fields of finite characteristic*, Russian Math. Surveys **18** (1963), 1–83.

[12] Y. Taguchi, *The Tate conjecture for t-motives*, to appear in Proc. AMS.

[13] A. Tamagawa, *The Tate conjecture for A-premotives*, preprint.

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