On the finiteness of various Galois representations

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Introduction. This is a written version of my talk under the same title at the Conference, except for the last section whose contents I did not mention in the talk. The first three sections reproduce rather closely the actual talk. This part is a survey on the finiteness of the number of isomorphism classes of various Galois representations. For mod p representations, we have already a survey article [**14**], but our emphasis is on the analogy with knot theory. Thus I tried to keep the article readable (at least in Section 1) to those with less background in number theory. The last section explains a certain relation between finiteness conjectures in the mod p and p-adic cases.

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Let us begin with the following vague

Problem. Classify Galois representations

$$\rho : G_{K,S} \to \mathrm{GL}_d(E).$$

Here and elsewhere, we denote by $G_{K,S}$ the Galois group of a field K (= our "base field") unramified outside a finite set S of places of K (resp. the fundamental group of a 3-manifold K minus a link S). E is a coefficient field. It is often a topological field, and if so, we always consider continuous representations ρ from the Krull topology (resp. discrete toporolgy) of $G_{K,S}$. In this broad sense, such a problem is prevalent in both sides of Knots and Primes (cf. other articles in this volume).

This "Problem" in general is too hard (or too "soft") for me, so let me consider the following modest problem:

Finiteness Problem. In particular, do there exist only finitely many isomorphism classes of $\rho : G_{K,S} \to \mathrm{GL}_d(E)$ such that $***$?

The condition $***$ depends on the context. The adjective "various" in my title means that we have various cases (K, E and the conditions $***$) to consider, as

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follows: As the base field K, we may consider a local field [1] or a global field [2]. It will also be interesting to consider the case where K is an infinite algebraic extension of one of these fields. In this paper, however, we shall mainly be concerned with the case of algebraic number fields of finite degree over $\mathbf{Q}$. As the coefficient field E, we may consider $\mathbf{C}$, $\overline{\mathbf{Q}}_p$, $\overline{\mathbf{F}}_p$, $\overline{\mathbf{F}_p((T))}$, etc. Here, the bar $^-$ means an algebraic closure. We remark that considering representations with coefficients in, say $\overline{\mathbf{Q}}_p$, means in effect considering representations with coefficients in various subfields of $\overline{\mathbf{Q}}_p$ of finite degree over $\mathbf{Q}_p$.

If the answer to the finiteness question is Yes, then the next problem will be to *enumerate* them. It will be interesting, for example, to consider the generating function (or the zeta function) for them. I understand that the work of Sink [**28**] has been done from such a point of view, in search of a version of the Casson invariant of knots. As Morishita points out, it would be interesting to define a Casson invariant for algebraic number fields.

Now let me mention some motivation for these problems form the side of Primes. First, we have:

Serre's Conjecture ([**26**]). Any odd and irreducible $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is modular of type $(N(\rho), k(\rho))$.

Here, a representaion ρ of $G_{\mathbf{Q}}$ is said to be *odd* if $\det \rho(\text{complex conjugation}) = -1$. The positive integers $N(\rho)$ and $k(\rho)$, called respectively the Artin conductor outside p and the Serre weight, will be defined in Section 2. Note that we have $k(\rho) \leq p^2-1$.

Some consequences of this conjecture are:

- Any such ρ lifts to characteristic 0. (In this direction, R. Ramakrishna proved the liftablity in many cases ([**19**], [**20**]).)
- Up to isomorphisms, there exist only finitely many such ρ's with $N(\rho)$ dividing a fixed integer N. (This is because the space of cusp forms of a given weight and level is of finite dimension.)
- For $p = 2, 3, 5, 7$, there exists no such ρ unramified outside p. (This is because, after twisting ρ by a power of the cyclotomic character, one can normalize the Serre weight $k(\rho)$ to be at most $p+1$, and there is no cusp form of level 1 and weight less than 12.)

Next, we have:

Fontaine-Mazur's Conjecture ([**6**]). Let K be an algebraic number field of finite degree and S a finite set of finite places of K. Then there exist only finitely many isomorphism classes of irreducible geometric p-adic representations with bounded inertial level and fixed Hodge-Tate type.

The precise meaning of this statement will be explained in Section 3. Morally, it asserts that there exist only finitely many irreducible p-adic representations which look like the ones coming from a certain class of smooth projective varieites over K with good reduction outside S. Behind this conjecture is a recognition that there should exist only finitely many varieties over K (up to K-isomorphisms) in a certain class. As a typical example, we have a theorem of Faltings [**5**] (conjectured by

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[1] A finite extension of either the p-adic field $\mathbf{Q}_p$ or the power series field $\mathbf{F}_p((t))$ over the finite field $\mathbf{F}_p$ of p elements.

[2] A finite extension of the rational number field $\mathbf{Q}$ or the rational function field $\mathbf{F}_p(t)$ in one variable over $\mathbf{F}_p$.

Shafarevich [**27**]) that there exist only finitely many isomorphism classes of principally polarized abelian varieties over K of a given dimension with good reduction outside S.

Throughout the paper, K denotes an algebraic number field of finite degree over $\mathbf{Q}$, S a finite set of places of K, and p a fixed prime number. A representation $\rho : G_{K,S} \to \mathrm{GL}_d(E)$ is often thought of as an E-vector space V of dimension d equipped with a continuous $G_{K,S}$-action, so that we may say "an E-representation V of $G_{K,S}$" in place of ρ.

1. Classical Case ($E = \mathbf{C}$). Let $G_K = \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group, and let $\rho : G_K \to \mathrm{GL}_{\mathbf{C}}(V)$ be a continuous representation, where V is a finite-dimensional $\mathbf{C}$-vector space. Here, we consider the coefficient field $\mathbf{C}$ with the discrete topology, so that the continuity means that ρ has finite image, since the Krull topology of G_K is totally disconnected. Note that ρ is unramified outside a finite set S of primes of K, so that ρ factors through the quotient $G_{K,S}$ of G_K. We define the *Artin conductor* $N(\rho)$ of ρ, which is an integral ideal of K, to be the product

$$N(\rho) \;:=\; \prod_{\mathfrak{q}} \mathfrak{q}^{n_{\mathfrak{q}}(\rho)}$$

over all prime ideals of K, with

$$n_{\mathfrak{q}}(\rho) \;:=\; \sum_{i=0}^{\infty} \frac{1}{(G_{\mathfrak{q},0} : G_{\mathfrak{q},i})} \dim_{\mathbf{C}}(V/V^{G_{\mathfrak{q},i}}),$$

where $V^{G_{\mathfrak{q},i}}$ denotes the fixed subspace of V by the action of the ith ramification subgroup $G_{\mathfrak{q},i}$ at $\mathfrak{q}$ of $G := \mathrm{Im}(\rho)$ (for the definition of ramification groups, see e.g. [**21**]). Although not obvious from the definition, $n_{\mathfrak{q}}(\rho)$ is an integer. The Artin conductor $N(\rho)$ measures how deeply ρ is ramified. In particular, we have $n_{\mathfrak{q}}(\rho) = 0$ if and only if ρ is unramified at $\mathfrak{q}$ (so $n_{\mathfrak{q}}(\rho) = 0$ if $\mathfrak{q} \notin S$). Thus the product in the definition of $N(\rho)$ is in fact a finite product, and the set S as above contains all the prime factors of $N(\rho)$.

One can easily prove the following proposition:

Proposition 1.1 (cf. [**1**]). *For any K, d, N, there exist only finitely many isomorphism classes of $\rho : G_K \to \mathrm{GL}_d(\mathbf{C})$ with $N(\rho)$ dividing N.*

Before giving a sketch of the proof, let us consider some possible link analogues of these things. Let $G = \pi_1(M \setminus L)$, where M is a 3-manifold and $L = K_1 \cup \cdots \cup K_r$ is a link in M. Let $m_i \in G$ be a meridian at K_i. Then the subgroup $\langle m_i \rangle$ generated by m_i will be the link analogue of the inertia subgroup $G_{\mathfrak{q},0}$ at $\mathfrak{q}$ (or perhaps its tame quotient $G_{\mathfrak{q},0}/G_{\mathfrak{q},1}$, since there is no "wild ramification" in characteristic 0). For a continuous representation $\rho : G \to \mathrm{GL}_{\mathbf{C}}(V)$, define $N(\rho)$ to be the formal sum $\sum_{i=1}^{r} n_i K_i$, where

$$n_i \;=\; \dim_{\mathbf{C}}(V/V^{\langle m_i \rangle}).$$

Or, one may simply think of $N(\rho)$ as an r-tuple $(n_1, ..., n_r)$. Then:

Question. Given an $N = (n_1, ..., n_r)$, how many $\rho : G \to \mathrm{GL}_{\mathbf{C}}(V)$ are there (up to isomorphisms) with $N(\rho) = N$?

Similarly, one may consider as follows: for $N = (n_1, ..., n_r)$, let $G(N)$ be the subgroup of G *normally* generated by $m_i^{n_i}$ for $r = 1, ..., r$. Then what is the structure of the quotient group $G/G(N)$? Is it finite or infinite? What (how many) representations $\rho : G/G(N) \to \mathrm{GL}_{\mathbf{C}}(V)$ are possible? Concerning these questions, the Referee communicated to me the following remark due to J. Hillman, for which I am grateful to both Hillman and the Referee.

Remark. The quotient group $G/G(N)$ is known as the orbifold fundamental group associated to a weighted link in M. In the most important special case where $r = 1$ and $M = S^3$, the group $G/G(N)$ maps onto $\mathbf{Z}/n\mathbf{Z}$ and the kernel is the orinary fundamental group of the n-fold branched cyclic cover of S^3, branched over the knot. In general such groups are usually infinite.

Now we turn to the proof of the above Proposition. By Jordan's Theorem, there exists an integer J depending only on d such that, for any finite subgroup G of $\mathrm{GL}_d(\mathbf{C})$, there exists an abelian normal subgroup H of G with index at most J. We apply this with $G = \mathrm{Im}(\rho)$. Let L be the extension of K corresponding to the kernel of ρ, and K_1 the fixed subfield by the subgroup H as in Jordan's theorem. Then the lower half K_1/K is bounded in degree, and the upper half L/K_1 is abelian. Both have bounded conductor. For the lower half, the Hermite-Minkowski Theorem, recalled below, implies that there exist only finitely many such extensions K_1/K. For the upper half, class field theory (or the finiteness of the ray class groups) implies that there exist only finitely many such extensions L/K_1.

Remark. We can prove similar statements for function fileds over finite fields or local fields, assuming suitable conditions such as the boundedness of the extension degree of the constant (or residue) fields.

Here we recall:

Theorem 1.2 (Hermite-Minkowski). *For any K, S and a natural number n, there exist only finitely many extensions of K of degree n which are unramified outside S.*

Motivated by the above discussion, let us agree to say[3] that a topologcal group G is *of type (HM)* if it has, for each $n \geq 1$, only finitely many closed subgroups of index n. A finitely generated group is of type (HM) (here and elsewhere, "finitely generated" means "topologically finitely generated"). The Galois group $G_{K,S}$ and the fundamental group $\pi_1(M \setminus L)$ are of type (HM), but it is not known whether $G_{K,S}$ is finitely generated or not. There are infinitely generated profinite groups of type (HM), such as the direct product of ${\mathbf{Z}_p}^{\oplus p}$ for infinitely many different primes p (cf. [**23**], Chap. III, §4.1, Exercice 2; this example was communicated to me by M. Yamagishi, to whom I am grateful). What are other infinitely generated profinite groups of type (HM)? Can one find a nice condition for a profinite group of type (HM) to be finitely generated?

Also, to have the analogue for links of the finiteness of ideal class groups, we want $H_1(M', \mathbf{Z})$ to be finite (i.e. M' be a $\mathbf{Q}$-homology sphere) for any finite covering M' of M involved in the above discussion. Suppose M is a $\mathbf{Q}$-homology sphere and $M' \to M$ a finite covering of M ramified along a link in M. When is M' again a $\mathbf{Q}$-homology sphere?

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[3] I thank A. Tamagawa who informed me that such a profinite group was said to be of type (F) in [**23**], Chap. III, §4.1, and that, recently, the terminology "small" seemed standard for such a group. Thus I use the terminology "type (HM)" here only to preserve the atmosphere of the Conference.

2. Mod p Case ($E = \overline{\mathbf{F}}_p$). In this section, we consider similar problems as above for representations with coefficients in $\overline{\mathbf{F}}_p$. This case is interesting not only in its own right but also as the "residual case" of the p-adic case, because problems in the p-adic case (as in Sect. 3) are often solved by first considering their reduction modulo p and then lifting (or deforming) to characteristic 0. This kind of relations between the mod p and p-adic cases will be discussed in Section 4.

Let G_K denote the absolute Galois groups of K. For a continuous representation $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$, its *Artin conductor outside* p is defined as follows:

$$N(\rho) \ := \ \prod_{\mathfrak{q} \nmid p} \mathfrak{q}^{n_{\mathfrak{q}}(\rho)}.$$

Here, the exponent $n_{\mathfrak{q}}(\rho)$ is defined by the same formula as in the classical case (Sect. 1), but the product is only over prime ideals $\mathfrak{q}$ which do not divide p. As before, the ρ in fact factors through the quotient $G_{K,S}$ of G_K if S contains the primes of K which divide $N(\rho)$.

We are concerned with:

Conjecture ($\mathbb{F}$). For any K, d, p, and any integral ideal N of K, there exist only finitely many isomorphism classes of semisimple representations $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ with $N(\rho)$ dividing N.

Such a statement may have been considered vaguely since before, but as far as I know, Khare [**7**] and Moon [**12**] (independently) were the first to jot down this statement (as a "Conjecture" in the former, and as a "Problem" in the latter). First results in the case $d > 2$ were given in [**12**]. Some quantitative and effective results on the number of monomial mod p representations are given in [**13**].

Remarks. (1) We need to assume the semisimplicity. (Otherwise, for a fixed finite Galois extension L/K, there may exist infinitely many mutually unisomorphic embeddings $\mathrm{Gal}(L/K) \hookrightarrow \mathrm{GL}_d(\overline{\mathbf{F}}_p)$.)
(2) We need to bound the conductor $N(\rho)$.
(3) We do *not* need to bound the ramification of ρ at p. (This is because the ramification at p is automatically bounded because of the group structure of $\mathrm{GL}_d(\overline{\mathbf{F}}_p)$.)
(4) If we put a finite field $\mathbf{F}_q$ instead of $\overline{\mathbf{F}}_p$ in the Conjecture, then the finiteness follows from the Hermite-Minkowski Theorem.
(5) The case of $d = 1$ is OK by class field theory (or rather the finiteness of the ray class groups).
(6) So one may say that this statement generalizes both the Hermite-Minkowski Theorem and the finiteness of ideal class groups.
(7) The case of $d = 2$ is related to Serre's conjecture.
(8) The case of $d \geq 3$ is also related to a conjecture of Ash et al. ([**3**], [**2**]), which is a higher dimensional generalization of Serre's conjecture.
(9) If K is a local field, the finiteness holds true if we impose suitable conditions on the representations (such as the boundedness of the degree of the residual extensions).

Our first result in this section is:

Theorem 2.1 ([**15**]). *For any K, d, p, N, there exist only finitely many isomorphism classes of semisimple representations $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ with $N(\rho)$ dividing N and $\mathrm{Im}(\rho)$ solvable.*

A similar statement is also true for function fields over finite fields under suitable conditions.

For the proof, we use a theorem of Larsen and Pink [**8**] instead of Jordan's. It states that there exist two constants J_1, J_2 depending only on d such that, for any finite subgroup G in $\mathrm{GL}_d(\overline{\mathbf{F}}_p)$ (in fact, the field $\overline{\mathbf{F}}_p$ here may be any field of characteristic $p \geq 0$) there exist normal subgoups G_i of G with $G \supset G_1 \supset G_2 \supset G_3$ such that
(1) G_1 has index at most J_1;
(2) G_1/G_2 is a product of a finite number of finite simple groups of Lie type in characteristic p, and the number of components is at most J_2;
(3) G_2/G_3 is an abelian group of order prime to p; and
(4) G_3 is a p-group.

We apply this to $G = \mathrm{Im}(\rho)$. If G is solvable, then the factor G_1/G_2 is trivial. So the extension L/K corresponding to $\mathrm{Ker}(\rho)$ is a successive extension $K \subset K_1 \subset K_3 \subset L$, with K_i corresponding to G_i. The bottom part K/K_1 is taken care of by Hermite-Minkowski, and the middle part K_3/K_1 by class field theory. The top part L/K_3 is a successive extension of elementary p-extensions, and the number of steps is bounded in terms of d, so this is also done by class field theory.

Next let us look at the non-existence or finiteness in the case where $K = \mathbf{Q}$, d and p are small, and $N = 1$.

In 1973, Tate [**30**] showed that there exists no irreducible representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$ unramified outside 2. Soon after this, Serre pointed out that the same method applies to $p = 3$.

In 1999, Brueggeman [**4**] proved the same for $p = 5$, assuming the GRH (= Generalized Riemann Hypothesis).

In 2000, Moon proved the following:

Theorem 2.2 ([**12**]). *Besides the results cited above, the finiteness* ($\mathbb{F}$) *for* $K = \mathbf{Q}$ *and* $N = 1$ *holds true in the following cases:*
(1) ($d \leq 2$; $p = 5$; totally real),
    ($d \leq 2$; $p = 7, 11, 13$; totally real and GRH).
(2) ($d \leq 4$; $p = 2$; totally real),
    ($d \leq 4$; $p = 2$; GRH),
    ($d \leq 4$; $p = 3$; totally real and GRH).
(3) ($d \leq 8$; $p = 2$; totally real and GRH).

Here, the condition "totally real" (resp. "GRH") in the last component of each of the triples describing the cases means that the finiteness holds true if we restrict to those ρ's such that the field L corresponding to $\mathrm{Ker}(\rho)$ is totally real (resp. if we assume the GRH).

These results are proved by comparing two different inequalities, called the Tate and Odlyzko bounds, for the discriminant $d_{L/\mathbf{Q}}$ (or the different $\mathcal{D}_{L/\mathbf{Q}}$) of $L/\mathbf{Q}$. The Tate bound is an upper bound and is proved algebraically. The Odlyzko bound is a lower bound and is proved analytically. One shows that these two inequalities contradict when the degree of L goes to infinity. Then one knows that the degree is bounded, and hence the set of extensions $L/\mathbf{Q}$ is finite by Hermite-Minkowski. So the set of isomorphism classes of semisimple representations is also finite.

The Tate bound is the following: If $d = 2$, then ([**30**]):

$$v_p(\mathcal{D}_{L/\mathbf{Q}}) \ \le\ 2 + \frac{1}{p} - \frac{1}{(p-1)p^{m-1}}.$$

For a general d, if d is at most 2^r, then ([**12**]):

$$v_p(\mathcal{D}_{L/\mathbf{Q}}) \ \le\ r + 1 + \frac{r}{p-1}.$$

There are several versions of the Odlyzko bound, one of which is the following asymptotic inequality ([**24**]):

$$\log|d_{L/\mathbf{Q}}| \ >\ r_1 \log(8\pi e^{\gamma+\pi/2}) + 2r_2 \log(8\pi e^{\gamma}) + o(n)$$

as $n = [L : \mathbf{Q}] \to \infty$. Effective versions of the Odlyzko bound are found, e.g., in [**17**].

To go further (by the same method) beyond the above theorem, we need to improve these inequalities, but it seems almost impossible to improve the Odlyzko bound. So we try to refine the Tate bound in terms of the Serre weight. For a mod p representation $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ of the Galois group of the p-adic field $\mathbf{Q}_p$, Serre defined the Serre weight $k(\rho)$, which is an integer taking values between 1 and $p^2 - 1$. We define the *reduced Serre weight* $\tilde{k}(\rho)$ to be the minimum of the Serre weights of the twists of ρ by powers of the mod p cyclotomic character χ;

$$\tilde{k}(\rho) \ :=\ \min_{\alpha \in \mathbf{Z}} k(\chi^{-\alpha} \otimes \rho).$$

Note that χ, taking values in $\mathbf{F}_p^{\times}$, has order $p - 1$.

If ρ is *wildly ramified*, then ρ has the following form when restricted to the inertia group I_p at p:

$$\rho|_{I_p} \ \sim\ \begin{pmatrix} \chi^{\beta} & * \\ & \chi^{\alpha} \end{pmatrix} \ =\ \chi^{\alpha} \begin{pmatrix} \chi^{k-1} & * \\ & 1 \end{pmatrix},$$

with $* \neq 0$ and some integers α, β, k; we normalize k by $2 \le k \le p$. Then we have

$$\tilde{k}(\rho) \ =\ \begin{cases} p+1, & \text{if } k = 2 \text{ and } \chi^{-\alpha} \otimes \rho \text{ is not finite,} \\ k, & \text{otherwise.} \end{cases}$$

Here, a representation of $G_{\mathbf{Q}_p}$ over $\overline{\mathbf{F}}_p$ is said to be *finite* if it comes from a finite flat groups scheme over $\mathbf{Z}_p$.

Proposition 2.3 ([**16**]). *If $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is wildly ramified, then the valuation of the different $\mathcal{D}_{L/\mathbf{Q}_p}$ of the extension $L/\mathbf{Q}_p$ corresponding to the kernel of ρ is:*

$$v_p(\mathcal{D}_{L/\mathbf{Q}_p}) \ =\ \begin{cases} 1 + \frac{k-1}{p-1} - \frac{k-1+\delta}{(p-1)p^m}, & \text{if } 2 \le k \le p, \\ 2 + \frac{2}{(p-1)p} - \frac{2}{(p-1)p^m}, & \text{if } k = p+1, \end{cases}$$

where the valuation v_p of L is normalized by $v_p(p) = 1$, $k = \tilde{k}(\rho)$ and $\delta = \gcd(\alpha, \beta, p-1) = \gcd(\alpha, k-1, p-1)$.

To prove this, we consider the subextensions $L/K_1/K_0/\mathbf{Q}_p$ such that $K_0/\mathbf{Q}_p$ is unramified, K_1/K_0 is tamely ramified, and L/K_1 is wildly ramified. The tame part is well-known. The point is the calculation of the different of the wild part, which is done by the Führerdiskriminantenproduktformel:

$$d_{L/K_1} \ =\ \prod_{\psi} \mathfrak{f}_{\psi}.$$

Since $\mathrm{Gal}(K_1/K_0)$ acts on $\left(\begin{smallmatrix}1 & * \\ & 1\end{smallmatrix}\right)$ by $\chi^{\beta-\alpha} = \chi^{k-1}$, one knows which quotient of the unit group of K_1 the characters ψ factor through, and can calculate the conductor $\mathfrak{f}_\psi$.

Using this, we can prove the following theorem:

Theorem 2.4 ([**16**]). *When* $K = \mathbf{Q}$, $d = 2$, $N = 1$, $2 \leq p \leq 31$, *and* $k \leq p + 1$, *the non-existence/finiteness of irreducible* $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ *is summarized as follows:*

$k\backslash p$	2	3	5	7	11	13	17	19	23	29	31
2	×	×	×	×	×	×	×	×	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$
3	×	×	×	×	×	×	×	×	f	f	f
4		×	×	×	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	f_{R}	f_{R}
5			×	×	×	×	×	×	f	f	f
6			$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	f_{R}	f_{R}	?	?	?
7				×	×	×	×	×	f	f	f_{R}
8				?	?	?	?	?	?	?	?
9					$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	f_{R}	f_{R}	f_{R}
10					?	?	?	?	?	?	?
11					$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	f_{R}	f_{R}	f_{R}
12					∃	∃	∃	∃	?	∃	∃
13						f_{R}	$\times_{\mathrm{R}}$	$\times_{\mathrm{R}}$	f_{R}	f_{R}	f_{R}
14						?	?	?	?	?	?
15							f_{R}	f_{R}	f_{R}	f_{R}	f_{R}
16							∃	∃	∃	∃	?
17							?	?	?	f_{R}	f_{R}
18							∃	∃	∃	∃	∃
19								?	?	?	?
20								∃	∃	∃	∃

Here, the meaning of the notations are as follows: There exists no such ρ which is unramified outside p and of reduced Serre weight k in the cases marked with ×, and the same is true if we assume the GRH in the cases marked with $\times_{\mathrm{R}}$; An f (resp. f_{R}) means that, unconditionally (resp. under GRH), there exist only finitely many such ρ in that case, and an ∃ means that there does exist an irreducible representation in that case. A ? means that the non-existence/finiteness is unknown (at present) in that case. For $k > 20$, the results are either ∃ or ?.

3. *p*-adic case ($E = \overline{\mathbf{Q}}_p$). Let us first recall some terminologies used in the conjecture of Fontaine and Mazur. Let E be a subfield of $\overline{\mathbf{Q}}_p$. A representation $\rho : G_K \to \mathrm{GL}_d(E)$ is said to be *geometric* if it is
(1) unramified outside a finite number of places, and
(2) potentially semistable at all places v dividing p.
(The condition "potentially semistable" is equivalent to being de Rham according to a recent work of L. Berger.)

Let $\rho : G_{K,S} \to \mathrm{GL}_E(V)$ be a geometric representation. The *inertial level* $\mathcal{L}(\rho)$ of ρ is a family $(\mathcal{L}_v(\rho))_v$, where, for each finite place v of K, $\mathcal{L}_v(\rho)$ is the maximal open normal subgroup of an inertia group I_v at v such that ρ becomes semistable when restricted there. One has $\mathcal{L}_v(\rho) = I_v$ is $v \notin S$. An *inertial level* $\mathcal{L} = (\mathcal{L}_v)_v$ for (K, S) is a family of open normal subgroups $\mathcal{L}_v$ of I_v for all finite places v of K such that $\mathcal{L}_v = I_v$ if $v \notin S$. The *E-Hodge-Tate type* $h(\rho)$ of ρ is a family $(h_v(\rho))_{v|p}$

of $h_v(\rho)$'s for all primes v dividing p, where $h_v(\rho)$ is the isomorphism class of the graded $(E \otimes_{\mathbf{Q}_p} K_v)$-module $\mathrm{Hom}_{\mathbf{Q}_p[G_{K_v}]}(V, \oplus_{r \in \mathbf{Z}} \mathbf{C}_p(r))$. An *$E$-Hodge-Tate type* for K at p is a family $(h_v)_{v|p}$, for all primes v dividing p, of isomorphism classes of graded free $(E \otimes_{\mathbf{Q}_p} K_v)$-modules h_v, all having a same rank (called the *degree* of h).

We denote by $\mathrm{Geom}(K, S, h; E)$ (resp. $\mathrm{Geom}(K, S, \mathcal{L}, h; E)$) the set of geometric irreducible E-representations of $G_{K,S}$ with E-Hodge-Tate type h (resp. with inertial level $\mathcal{L}$ and E-Hodge-Tate type h). Then the Finiteness Conjecture of Fontaine and Mazur ([**6**], Conj. 2) is:

Conjecture. For any $K, S, p, \mathcal{L}, h$, we have:
FM(a). The set $\mathrm{Geom}(K, S, \mathcal{L}, h; \overline{\mathbf{Q}}_p)$ is finite.
Further, if E is a *finite* extension of $\mathbf{Q}_p$, we have:
FM(b). The set $\mathrm{Geom}(K, S, \mathcal{L}, h; E)$ is finite.
FM(c). The set $\mathrm{Geom}(K, S, h; E)$ is finite.

We have the relations: FM(a) $\Rightarrow$ FM(b) $\Leftrightarrow$ FM(c). Here, the only nontrivial one is (b) $\Rightarrow$ (c), and this is proved by showing that, if E is of finite degree over $\mathbf{Q}_p$, then the inertial level of a geometric E-representation is automatically bounded. Our result is that FM(a) is true if we restrict ourselves to *potentially abelian* ρ's, i.e. those which become abelian after a finite extension of the base field K:

Theorem 3.1 ([**29**]). *There exist only finitely many isomorphism classes of irreducible potentially abelian geometric representations* $\rho : G_{K,S} \to \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$ *with bounded inertial level and given Hodge-Tate type.*

This is proved by combining Serre's theory of locally algebraic representations ([**22**]) and the following lemma:

Lemma 3.2. *All such ρ become abelian after a* fixed *finite extension of K.*

This lemma, in turn, follows from a Jordan type theorem on "potentially toric" algebraic subgroups of the general linear group GL_d, together with the Hermite-Minkowski Theorem.

4. Relation between the mod p and p-adic cases.

There seems to be a substantial difference between Conjectures FM(a) and FM(b) of the last section. We expect that, in some sense, Conjecture ($\mathbb{F}$) of Section 2 fills this gap, i.e., that FM(b) + ($\mathbb{F}$) $\Rightarrow$ FM(a). At the moment of writing this paper, however, some technicality concerning the universal deformation of Galois representations has not been fully checked, so we give here a proof of a variant (Prop. 4.3 below) of the expected relation in a restricted situation, leaving the details to a future publication. Let E be an algebraic extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_E$, valuation ideal $\mathfrak{m}_E$, and residue field $k_E = \mathcal{O}_E/\mathfrak{m}_E$. We shall restrict the objective of the finiteness conjecture of Fontaine and Mazur to residually absolutely irreducible representations. Here we say that a continuous E-representation V of $G_{K,S}$ is *residually absolutely irreducible* if there is a $G_{K,S}$-stable $\mathcal{O}_E$-lattice T of V of which the reduction $\overline{V} = T/\mathfrak{m}_E T$ modulo $\mathfrak{m}_E$ is absolutely irreducible as a k_E-representation of $G_{K,S}$. Then the isomorphism class of $\overline{V}$ does not depend on the choice of the lattice T.

For our purpose, it is convenient to employ the language of deformation theory of Galois representations ([**10**], [**11**]). So let k be a finite field of characteristic p, and let $\overline{\rho} : G_{K,S} \to \mathrm{GL}_d(k)$ be an absolutely irreducible representation. (The condition "absolutely irreducible" can be weakned to that "End$(\overline{\rho}) \simeq k$", but for simplicity, we content ourselves with this stronger condition.) Let $W = W(k)$ be the ring of Witt vectors over k. Let $\mathcal{D}$ be a "deformation data", i.e., a full subcategory of the category $\underline{\mathrm{Rep}}^{\mathrm{f}}_W(G_{K,S})$ of continuous $W[G_{K,S}]$-modules of finite length which is closed under taking subobjects, quotients and direct sums. Suppose that our $\overline{\rho}$ is in $\mathcal{D}$. Then by Ramakrishna ([**18**] or [**11**], Sect. 25), there exist a universal deformation ring $R_{\mathcal{D}}(\overline{\rho})$ of $\overline{\rho}$ of type $\mathcal{D}$ in the category $\mathcal{C}_W$ of complete noetherian local W-algebras with residue field equal to k, and a universal deformation $\rho_{\mathcal{D}} : G_{K,S} \to \mathrm{GL}_d(R_{\mathcal{D}}(\overline{\rho}))$ of $\overline{\rho}$ of type $\mathcal{D}$. The universality refers to the property that there exists a canonical bijection

$$\{\text{type-}\mathcal{D} \text{ deformations of } \overline{\rho} \text{ to } A\} \ \simeq \ \mathrm{Hom}_{W\text{-alg}}(R_{\mathcal{D}}(\overline{\rho}), A),$$

functorially in $A \in \mathcal{C}_W$.

Fontaine and Mazur formulated a deformation-theoretic version of their finiteness conjecture. To state this, for an inertial level $\mathcal{L}$ for (K, S) and two integers $a \leq b$, let us take, as our deformation data $\mathcal{D}$, the full subcategory $\underline{\mathrm{Rep}}^{\mathrm{f}}_W(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$ of $\underline{\mathrm{Rep}}^{\mathrm{f}}_W(G_{K,S})$ consisting of objects T such that, for each $v \in S$,

(1) if $v \nmid p$, then $\mathcal{L}_v$ acts trivially on T; and

(2) if $v \mid p$, there exists a semistable p-adic representation V of $\mathcal{L}_v$ such that $(\mathbf{C}_v(r) \otimes_{\mathbf{Q}_p} V)^{\mathcal{L}_v} = 0$ if $r \notin [a, b]$ and that T is isomorphic to a subquotient of V as a $\mathbf{Z}_p[\mathcal{L}_v]$-module.

Let $\overline{\rho} \in \mathcal{D}$ be an absolutely irreducible representation. Then the deformation-theoretic version of the finiteness conjecture of Fontaine and Mazur ([**6**], Conj. 5) is:

Conjecture FM(D). The universal deformation ring $R_{\mathcal{D}}(\overline{\rho})$ is finite as a W-algebra.

A priori, this $R_{\mathcal{D}}(\overline{\rho})$ is not known to be flat over W (although it seems quite possible that such a deformation ring is always flat). Since we are interested in E-representations (rather than torsion $\mathcal{O}_E$-representations), let us consider the image $R_{\mathcal{D}}(\overline{\rho})_0$ of the natural map $R_{\mathcal{D}}(\overline{\rho}) \to R_{\mathcal{D}}(\overline{\rho}) \otimes_W F$, where F is the fraction field of W. Then FM(D) implies:

Conjecture FM($\mathbf{D}_0$). The W-algebra $R_{\mathcal{D}}(\overline{\rho})_0$ is finite.

In order to see the relation of FM(D_0) with FM(b), it is convenient to introduce a slightly wider (a priori) class of representations than geometric ones. Let $E/\mathbf{Q}_p$ be a *finite* extension, $\mathcal{L}$ an inertial level for (K, S), and $a \leq b$ two integers. We say that an E-representation V of $G_{K,S}$ is *piecewise geometric* with inertial level bounded by $\mathcal{L}$ and Hodge-Tate weights in $[a, b]$ if it admits a $G_{K,S}$-stable $\mathcal{O}_E$-lattice of which all quotients of finite length lie in $\mathcal{D} = \underline{\mathrm{Rep}}^{\mathrm{f}}_W(G_{K,S})_{\mathrm{st},\mathcal{L},[a,b]}$. A typical example of such a representation is one which arises from the type-$\mathcal{D}$ universal deformation of some residual representation $\overline{\rho}$ by way of a continuous W-algebra homomorphism $\phi : R_{\mathcal{D}}(\overline{\rho}) \to E$.

Let $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a, b]; E)$ denote the set of isomorphism classes of d-dimensional irreducible piecewise geometric representations with inertial level bounded by $\mathcal{L}$ and Hodge-Tate weights in $[a, b]$. We also denote by $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a, b]; \overline{\mathbf{Q}}_p)$ the inductive limit of these sets when E moves through the set of finite extensions of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$. A geometric representation is trivially piecewise geometric, so that $\mathrm{Geom}(K, S, \mathcal{L}, h; E) \subset \mathrm{Geom}'_d(K, S, \mathcal{L}, [a, b]; E)$ if d is the degree of the

Hodge-Tate type h and $[a,b]$ contains all the Hodge-Tate weights of h. Conversely, Fontaine and Mazur point out ([**6**], Rem. (a) after Conj. 5) that it should not be very hard to prove that:

Conjecture FM(P). A piecewise geometric representation with bounded inertial level and Hodge-Tate weights is geometric.

At this moment, I do not know to what extent this statement is proved.

On the contrary, a piecewise geometric representation with *unbounded* inertial level or Hodge-Tate weights may not be geometric (i.e. an E-representation may not be geometric no matter how its subquotients of finite length come from geometric representations, if their inertial levels are unbounded or their Hodge-Tate weights do not stay in a fixed interval $[a,b]$).

Now following Fontaine and Mazur, we may state:

Conjecture. For any $K, S, \mathcal{L}, a \leq b$, we have:
FM′(a). The set $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a,b]; \overline{\mathbf{Q}}_p)$ is finite.
FM′(b). The set $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a,b]; E)$ is finite for any finite extension E of $\mathbf{Q}_p$.

We have:

$$\begin{tikzcd}
\mathrm{FM}'(\mathrm{a}) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{FM}(\mathrm{a}) \arrow[d, Rightarrow] \\
\mathrm{FM}'(\mathrm{b}) \arrow[r, Rightarrow] & \mathrm{FM}(\mathrm{b}) \arrow[d, Leftrightarrow] \\
 & \mathrm{FM}(\mathrm{c}).
\end{tikzcd}$$

In the rest of this paper, we concentrate on residually absolutely irreduible representations. For a set (or a conjecture) X, we denote by X_{rair} the corresponding object which concerns only with such representations; thus $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a,b]; \overline{\mathbf{Q}}_p)_{\mathrm{rair}}$ is the subset of isomorphism classes of residulally absolutely irreducible representations in $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a,b]; \overline{\mathbf{Q}}_p)$, and the conjecture $\mathrm{FM}'(\mathrm{a})_{\mathrm{rair}}$ states that this set is finite.

After Conj. 5 in [**6**], it is explained how FM(D) [4] implies FM(b), but what is actually implied is a little bit weaker (the assertion "the fact that V is irreducible implies that one can find an $\mathcal{O}_E$-lattice T of V stable under $G_{K,S}$ and such that, if $\overline{V} = k \otimes_{\mathcal{O}_E} T$, the map $k \to \mathrm{End}_{k[G_{K,S}]}(\overline{V})$ is an isomorphism" is not true in general, so FM(D) implies nothing about those V which do not have the above mentioned property). However, the proof there does show that FM(D) (or even $\mathrm{FM}(\mathrm{D}_0)$) implies $\mathrm{FM}'(\mathrm{b})_{\mathrm{rair}}$. In fact, the converse is also true:

Proposition 4.1. $\mathrm{FM}'(\mathrm{b})_{\mathrm{rair}} \iff \mathrm{FM}(\mathrm{D}_0)$.

Proof. Suppose $R_{\mathcal{D}}(\overline{\rho})_0$ is not finite over the Witt ring W. Then by factoring by a minimal prime, we would have an integral quotient R of $R_{\mathcal{D}}(\overline{\rho})_0$ which contains W and is not finite over W. By the structure theorem on complete noetherian local integral ring (e.g. [**9**], Sect. 29), such an R is finite over a subring R_0 which is isomorphic to a power series ring $W[\![X_1, ..., X_m]\!]$ over W in $m \geq 1$ variables. Recall that F denotes the fraction field of W. We claim that:

Lemma 4.2. *For any finite extension E_0/F, there exists a finite extension E/E_0 such that any W-algebra homomorphism $\phi_0 : R_0 \to \mathcal{O}_{E_0}$ extends to a W-algebra homomorphism $\phi : R \to \mathcal{O}_E$.*

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[4] Exactly speaking, a more general version of FM(D) for $\overline{\rho}$ with $\mathrm{End}(\overline{\rho}) \simeq k$.

It will follow from this lemma that the set $\mathrm{Hom}_{W\text{-alg}}(R, \mathcal{O}_E)$ is infinite since, R_0 being a power series ring over W in $m \geq 1$ variables, there are infinitely many $\phi_0 : R_0 \to \mathcal{O}_{E_0}$. This proves that $\mathrm{FM}'(\mathrm{b})_{\mathrm{rair}} \Rightarrow \mathrm{FM}(\mathrm{D}_0)$.

Now we prove the lemma. Since R is finite over R_0, it can be obtained from R_0 by a successive extension

$$R_0 \subset R_1 \subset \cdots \subset R_n = R$$

with

$$R_j = R_{j-1}[Y_j]/I_j,$$

where I_j is an ideal of $R_{j-1}[Y_j]$. For simplicity, assume that I_j is principal and generated by a monic polynomial $f_j(Y_j) \in R_{j-1}[Y_j]$ (see a future publication for the general case). Let $d_j := \deg(f_j)$. Since there exist only finitely many extensions of E_0 (contained in $\overline{\mathbf{Q}}_p$) of a given degree ([**25**]), we can find inductively a series of finite extensions $E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that E_j contains all finite extensions of E_{j-1} of degree $\leq d_j$. Let $\phi_0 : R_0 \to \mathcal{O}_{E_0}$ be any W-algebra homomorphism. By definition of E_1, the image $\phi_0 f_1 \in \mathcal{O}_{E_0}[Y_1]$ of the monic polynomial $f_1 \in R_0[Y_1]$ has a root in $\mathcal{O}_{E_1}$. Thus ϕ_0 extends to a W-algebra homomorphism $\phi_1 : R_1 = R_0[Y_1]/(f_1) \to \mathcal{O}_{E_1}$. Proceeding inductively, we obtain a W-algebra homomorphism $\phi : R \to \mathcal{O}_E$ which extends ϕ_0.

Finally, we prove:

Proposition 4.3. $\mathrm{FM}'(\mathrm{b})_{\mathrm{rair}} + (\mathbb{F}) \Longrightarrow \mathrm{FM}'(\mathrm{a})_{\mathrm{rair}}$.

Proof. By $(\mathbb{F})$, there appear only finitely many isomorphism classes of $\overline{\rho} : G_{K,S} \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ as the reduction of elements of $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a,b]; \overline{\mathbf{Q}}_p)_{\mathrm{rair}}$. So it is enough to show that, for each absolutely irreducible representation $\overline{\rho} : G_{K,S} \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$, there are only finitely many elements ρ of $\mathrm{Geom}'_d(K, S, \mathcal{L}, [a,b]; \overline{\mathbf{Q}}_p)_{\mathrm{rair}}$ reducing to the class of $\overline{\rho}$. The $\overline{\mathbf{F}}_p$-representation $\overline{\rho}$ is in fact defined over some finite field k. Then the $\overline{\mathbf{Q}}_p$-representation ρ, being residually absolutely irreducible, is in fact defined over a ring in the category $\mathcal{C}_W$, with $W = W(k)$. Indeed, if ρ is defined over a finite extension E of $\mathbf{Q}_p$, then by a lemma of Carayol and Serre (cf. e.g. Prop. in Sect. 6 of [**11**]), it is actually defined over the subring generated over W by the traces $\mathrm{Tr}\ \rho(g)$ for all $g \in G_{K,S}$. But since $\mathrm{Tr}\ \rho(g) \pmod{\mathfrak{m}_E}$ are all in k, this ring is contained in the inverse image $\mathcal{O}_{E,k}$ of k by the reduction map $\mathcal{O}_E \to k_E$. The ring $\mathcal{O}_{E,k}$ is in $\mathcal{C}_W$. Thus the ρ comes from some W-algebra homomorphism $\phi : R_{\mathcal{D}}(\overline{\rho})_0 \to \mathcal{O}_{E,k} \hookrightarrow \overline{\mathbf{Q}}_p$. By $\mathrm{FM}(\mathrm{D}_0)$, which is equivalent to $\mathrm{FM}'(\mathrm{b})_{\mathrm{rair}}$, there are only finitely many such ϕ's.

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