

Finiteness of an isogeny class of Drinfeld modules

— Correction to a previous paper —

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In this note, we prove the finiteness of an isogeny class of Drinfeld A -modules over a finite extension of the fraction field of A . This contradicts Remark (3.4(ii-1)) of my previous paper [4]. This is because Proposition (3.1) is wrong, on which §3 of that paper was based (the wrong point is in the “because” of the second last sentence of the proof). Besides, the property (ii) of Example (2.2) fails, which made use of a result in §3 (at present, its (in)validity is not known). The following replaces the whole §3 (except for Remarks (3.3) and (3.5)). To have a correct statement along the line of Proposition (3.1), one would need a more delicate analysis of Galois representations on division points of Drinfeld modules.

The strategy of our proof in this note (§1) is similar to that of Faltings’ [0], but in our case, we have to check that the Galois action on the determinants of certain subgroups of a Drinfeld module is just “as expected”; to do so, we shall show (§2) that a truncated v -divisible group is liftable to a v -divisible group (cf. [1]).

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1. Notation and convention are basically the same as in the “semi-simplicity” paper [3]. Let F be an algebraic function field in one variable over a finite field; we assume its field of constants is the field \mathbb{F}_q of q elements. Fix a place ∞ of F , and let A be the ring of elements of F which are regular outside ∞ . Let K be a finite extension of F , and \mathcal{O}_K the integral closure of A in K . Non-zero prime ideals of A (resp. \mathcal{O}_K) are identified with finite places of F (resp. K). The completion of K with respect to a place w is denoted by K_w . For any field k , we denote by G_k its absolute Galois group.

Theorem. *There are only finitely many K -isomorphism classes in each K -isogeny class of Drinfeld A -modules over K .*

Remark. The same statement is true in the “finite characteristic” case, since in this case the differential height is unchanged by separable isogenies ([3_f], Prop.(2.4)).

We begin with some reductions; in each reduction step, the point is that the number of isomorphism classes changes only “finitely” by doing so. First, we may assume that $A = \mathbb{F}_q[t]$ and $F = \mathbb{F}_q(t)$ (see [3_f], 2.1.2). Second, we may extend the base field K if necessary¹ and assume that it is so large that all Drinfeld modules

¹Since the isomorphism classes of Drinfeld modules over K which become isomorphic to a given one ϕ over a Galois extension K' of K is parametrized by $H^1(\text{Gal}(K'/K), \text{Aut}(\phi \otimes_K K'))$, which is finite if K'/K is finite (note that the endomorphism ring of a Drinfeld module is an order of a finite extension of F in which the infinite place ∞ does not split [Th.(4.9), (2) of Deligne-Husemüller: Survey of Drinfel’d modules, Contemp. Math. 67(1987), 25–91], so its unit group is finite.

ϕ under consideration (which are isogenous to each other (note that the lattices for isogenous Drinfeld modules differ only by finite index)) have everywhere stable reduction over \mathcal{O}_K and, further, have the following properties:

(A) For any infinite place (resp. bad finite place) w , the lattice ($\subset \overline{K}_w$) corresponding to $\phi \otimes_K K_w$ is K_w -rational (i.e. contained in K_w);

(B) K contains $(-t)^{1/(q-1)}$ (so the t -division points of the Carlitz module);

(C) The infinite places w are rational over the field of constants of K .

(A) implies, for those w , the triviality of the action of G_{K_w} on (certain quotients of) groups of division points of ϕ (cf. [3], §3). (B) means that the lattice for the Carlitz module at an infinite place w is rational and hence the action of G_{K_w} on its division points is trivial. (C) is equivalent to that the residue field of K_w is equal to the field of constants of K ; this together with (A) implies that there is no extension of the constant field in any extension of K by adjoining division points (because a constant field extension of K would result in that of K_w , and the latter should come from the lattice at w).

Fix a Drinfeld module ϕ over K of rank r . The Theorem follows from the following two Lemmas:

Lemma (1.1). *For any finite set S of finite places of K , there are only finitely many isomorphism classes (as $A_S[G_K]$ -modules) of G_K -invariant A_S -lattices in $\prod_{v \in S} T_v(\phi) \otimes_{A_v} F_v$.*

Here we put $A_S = \prod_{v \in S} A_v$.

Lemma (1.2). *There exists a finite set S of finite places of K , depending only on the K -isogeny class of ϕ , such that any isogeny $\phi \rightarrow \phi'$ of degree² prime to S preserves the differential height h_d ((5.3) of [3]).*

Let us deduce the Theorem from these lemmas. Choose a finite set S of places as in Lemma (1.2). Let $\{T^{(1)}, \dots, T^{(m)}\}$ be a set of representatives, for this S , of the set of isomorphism classes of G_K -invariant A_S -lattices as in Lemma (1.1). We may assume that each $T^{(j)}$ contains $T_S := \prod_{v \in S} T_v(\phi)$, so that it comes from an isogeny $f^{(j)} : \phi \rightarrow \phi^{(j)}$. Let a K -isogeny $f : \phi \rightarrow \phi'$ be given. The Tate module $T_S(\phi')$ is isomorphic to $T^{(j)}$ for some j . Then by the Tate conjecture for Drinfeld modules ([9], [6], [7]), there exists an isogeny $f' : \phi^{(j)} \rightarrow \phi'$ of degree prime to S [This can be seen as follows: Since $T_v(\phi^{(j)}) \simeq T_v(\phi')$ for each $v \in S$, the Tate conjecture ensures that there is an $f_v \in \text{Hom}(\phi^{(j)}, \phi') \otimes_A A_v$ which yields an isomorphism $T_v(\phi^{(j)}) \xrightarrow{\sim} T_v(\phi')$, so $\prod_{v \in S} f_v \in \text{Hom}(\phi^{(j)}, \phi') \otimes_A A_S$ yields an isomorphism $T^{(j)} = T_S(\phi^{(j)}) \xrightarrow{\sim} T_S(\phi')$. Then any element f' of $\text{Hom}(\phi^{(j)}, \phi')$ which is sufficiently close to $\prod_{v \in S} f_v$ is an isogeny of degree prime to S]. By Lemma (1.2), we have $h_d(\phi^{(j)}/K) = h_d(\phi'/K)$. Hence the set of values of the differential height of Drinfeld modules over K which are K -isogenous to ϕ equals $\{h_d(\phi^{(1)}/K), \dots, h_d(\phi^{(m)}/K)\}$. This being finite, the Theorem follows now by (5.8) and (5.9) of [3].

Proof of Lemma (1.1). We may assume that S consists of only one prime v . Since $V_v(\phi) = T_v(\phi) \otimes_{A_v} F_v$ is semi-simple as an $F_v[G_K]$ -module ([3]), it is enough to consider the problem for each simple factor of $V_v(\phi)$. Let V be a simple factor of

²The degree of an isogeny of Drinfeld A -modules is well-defined as an ideal of A .

$V_v(\phi)$, and let $T := V \cap T_v(\phi)$. We claim that there exists a positive integer N such that, for any $x \in \partial T := T \setminus vT$, its orbit $A_v[G_K] \cdot x$ contains $v^N T$. To prove the claim, first note that, for each $x \neq 0$, the orbit $A_v[G_K] \cdot x$ is open since V is simple, so there is a positive integer such that $A_v[G_K] \cdot x \supset v^N T$. If we write $N(x)$ for the minimum of such N , then the mapping $x \mapsto N(x)$ is continuous (because $N(x) = N(x + \delta)$ if $\delta \in v^{N(x)+1}T$), hence has a maximum N on the compact space ∂T .

Suppose T' is a G_K -invariant A_v -lattice in V . Multiplying a power of v if necessary, we may assume that T' is “exactly” contained in T , i.e., $T' \subset T$ but $T' \not\subset vT$. Then we have $T \supset T' \supset A_v[G_K] \cdot x \supset v^N T$ with $x \in T' \cap \partial T$. Since there are only finitely many submodules in the finite module $T/v^{N+1}T$, and T' is generated by any subset which generate it modulo $v^{N+1}T$, there are only finitely many such lattices T' .

Proof of Lemma (1.2). Choose a finite place u of F such that ϕ has good reduction at all places of K lying above u . Let S be the union of $\{u\}$ and the places of F which either lie below the bad places of K for ϕ or ramify in K/F (so this S can be chosen depending only on the K -isogeny class of ϕ). We will later expand this S to get an S as in the Lemma.

Let v be any non-zero prime of A not contained in S . It is enough to prove the Lemma for an isogeny $f : \phi \rightarrow \phi'$ whose kernel is non-trivial and killed by v , since any isogeny as in the Lemma can be decomposed into a composition of such isogenies. Write W for $\text{Ker}(f)(K^{\text{sep}})$, the group of geometric points of $\text{Ker}(f)$. Let h denote the dimension of W over the residue field \mathbb{F}_v of v ; thus $1 \leq h \leq r$. Let $\Phi : G_K \rightarrow \mathbb{F}_v^\times$ be the character describing the action of G_K on $\det W := \wedge_{\mathbb{F}_v}^h W$.

Let ω be a prime of K dividing u , and let $P_{h,\omega}$ be the characteristic polynomial of the Frobenius Frob_ω at ω acting on $T := \wedge_{A_v}^h T_v(\phi)$. By [8], it has coefficients in A and all its roots have absolute value equal to $q_\omega^{h/r}$, where $q_\omega = \#(\mathcal{O}_K/\omega) = |\omega|_\infty$. Since $\det W$ is a one-dimensional subspace of T/vT , we have

$$P_{h,\omega}(\Phi(\text{Frob}_\omega)) \equiv 0 \pmod{v}.$$

Let T_o be the $A_v[G_{\mathbb{F}_u}]$ -module $\bigotimes_{\omega|u} T$ (tensor product of copies of T , one for each ω lying above u) on which Frob_u acts as $\bigotimes_{\omega|u} \text{Frob}_\omega^{e_\omega}$, where e_ω is the ramification index of ω/u . (Here, we could assume that ω/u is unramified.) The characteristic polynomial $P_{h,u}$ of Frob_u on T_o has coefficients in A and all its roots have absolute value equal to $\prod_{\omega|u} q_\omega^{e_\omega h/r} = q_u^{[K:F]h/r}$, where $q_u = \#(A/u) = |u|_\infty$.

Let $\text{Ver}_F^K : G_F^{\text{ab}} \rightarrow G_K^{\text{ab}}$ be the transfer map, and set $\Phi_o := \Phi \circ \text{Ver}_F^K$; this is the character which describes the action of G_F on the one-dimensional \mathbb{F}_v -subspace $\bigotimes_{\omega|u} \det W$ of T_o/vT_o . So we have

$$(*) \quad P_{h,u}(\Phi_o(\text{Frob}_u)) \equiv 0 \pmod{v}.$$

Lemma (1.3). *We have*

$$\Phi_o = \chi^d \quad \text{with} \quad d := \sum_{w|v} d_w [K_w : F_v],$$

where χ is the Carlitz character (mod v) (i.e., the character describing the action of G_F on the v -division points of the rank 1 Drinfeld module ρ over F defined by

$\rho_t(X) = tX + X^q$), and d_w is the dimension of $\text{Ker}(f) \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathcal{O}_{K_w}$ (see Remark (2.6)).

In the present case, $d_w = 0$ if $\text{Ker}(f)$ extends to an étale group scheme over \mathcal{O}_{K_w} and $d_w = 1$ otherwise; so we have in particular $d \leq [K : F]$.

Admitting this lemma, we see from (*) that

$$(**) \quad P_{h,u}(u^d) \equiv 0 \pmod{v},$$

where we identify u with a monic prime element of A . If we show that $d = [K : F]h/r$, then it will follow by the isogeny lemma ((5.5) of [3]; cf. also (5.6.3), loc. cit.) that the differential height $h_d(\phi/K)$ is left unchanged by the isogeny f , and the proof will be complete.

Now we extend the set S by adding to it the prime divisors of the following elements:

$$P_{h,u}(u^i) \quad \text{for } 1 \leq h \leq r \text{ and } 0 \leq i \leq [K : F], i \neq [K : F]h/r.$$

Note that these elements are not zero since all roots of $P_{h,u}$ have absolute value equal to $|u|_\infty^{[K:F]h/r}$. Thus if $v \notin S$, then it follows from (**) that $d = [K : F]h/r$. Finally, note that the above elements $P_{h,u}(u^i)$, and hence the set S , depend only on the isogeny class of ϕ and our choice of u .

Proof of Lemma (1.3). This follows from the following (1), (2), (3) via the formalism of 4.2.8 of [2] (cf. Proof of Théorème 4.2.9):

- (1) $\Phi|_{I_w} = (\chi|_{I_w})^{d_w}$ for all $w|v$;
- (2) $\Phi_\circ \cdot \chi^{-d}$ is unramified at all places of F ;
- (3) There is no constant field extension in $K(W)/K$.

(Note that there arise no constant field extensions from division points of the Carlitz module.)

(1) holds because, by (2.5) below, the finite flat group scheme $\text{Ker}(f)$ extends to a v -divisible group over \mathcal{O}_{K_w} of dimension at most 1, for which a similar statement is true (Prop. (1.9) of [3]). Then (2) follows because, at $w \nmid v$, both Φ and χ are unramified (recall the assumptions (A) and (B)). (3) follows from the assumptions (A) and (C).

2. Deformation of φ -sheaves. In this section, we use the language of φ - and ν -sheaves (cf. [5]; but we use the words φ - and ν -module rather than “sheaf”; see the definition below. Also, we use “ ν ” instead of “ v ”, which is used for a prime ideal or place in this paper). Our purpose is to prove Corollary (2.5), which was used in the proof of Lemma (1.3).

Let $A = \mathbb{F}_q[t]$ as in the proof of the Theorem in §1. To state the definition of ν -modules, let \mathcal{O} be an A -algebra, and $\alpha : A \rightarrow \mathcal{O}$ its structure morphism. Denote by θ the image of t in \mathcal{O} . In the following, all \otimes with no reference to the base ring are over \mathbb{F}_q . Let R be either $(A/\mathfrak{a}) \otimes \mathcal{O}$, where \mathfrak{a} is an ideal of A (possibly 0), or $A_v \widehat{\otimes} \mathcal{O} := \text{projlim}_n ((A/v^n) \otimes \mathcal{O})$, where v is a non-zero prime ideal of A . We use the same letter t for the images of t in A/\mathfrak{a} and A_v . Also, we simply write t for the element $t \otimes 1 \in R$, and θ for the element $1 \otimes \theta \in R$ (thus, e.g., $t - \theta$ is actually $t \otimes 1 - 1 \otimes \theta$). For any R -module M , we write $M^{(\sigma)}$ for the base extension $R \otimes_R M$ by $\sigma := \text{id} \otimes (q\text{-th power map of } \mathcal{O}) : R \rightarrow R$. The R -module structure of $M^{(\sigma)}$ is of course given by the R -action on the first factor.

Definition. ³ A ν -module over R is an R -module M which is equipped with two R -module homomorphisms

$$\varphi : M^{(\sigma)} \rightarrow M, \quad \nu : M \rightarrow M^{(\sigma)}$$

such that $\varphi \circ \nu$ and $\nu \circ \varphi$ are both equal to multiplication by $t - \theta$ on the R -modules M and $M^{(\sigma)}$ respectively. To specify the underlying module M , we may sometimes write $\varphi = \varphi_M$ and $\nu = \nu_M$.

A *morphism* $f : M \rightarrow N$ of ν -modules is an R -module homomorphism f which makes the diagrams

$$\begin{array}{ccc} M^{(\sigma)} & \xrightarrow{f^{(\sigma)}} & N^{(\sigma)} \\ \varphi_M \downarrow & & \downarrow \varphi_N \\ M & \xrightarrow{f} & N \end{array} \quad \begin{array}{ccc} M & \xrightarrow{f} & N \\ \nu_M \downarrow & & \downarrow \nu_N \\ M^{(\sigma)} & \xrightarrow{f^{(\sigma)}} & N^{(\sigma)} \end{array}$$

commutative. Here $f^{(\sigma)} := \text{id}_R \otimes f$.

For a ν -module M over R (not necessarily flat; in particular, torsion is allowed), consider the following property (cf. Gabber's proof (b) of Prop. (1.7) of [1]):

(D) There exist R -submodules M_ν of M and $M_\varphi^{(\sigma)} = (M^{(\sigma)})_\varphi$ of $M^{(\sigma)}$ such that

$$\begin{cases} M = M_\nu \oplus \varphi(M_\varphi^{(\sigma)}) & \text{and} & M^{(\sigma)} = \nu(M_\nu) \oplus M_\varphi^{(\sigma)}; \\ \nu \text{ is injective on } M_\nu & \text{and} & \varphi \text{ is injective on } M_\varphi^{(\sigma)}. \end{cases}$$

Here we do *not* mean that $M_\varphi^{(\sigma)}$ is a base extension of some " M_φ ".

Remark (2.1). If M is free of rank r over R , then M_ν and $M_\varphi^{(\sigma)}$ are projective over R . Assume for simplicity that they are free of constant rank (later in (2.3) and (2.4), R will be a semi-local ring, and a situation similar to this Remark will arise locally, i.e., component-wise). If we choose R -bases (e_1, \dots, e_d) of M_ν and (e_{d+1}, \dots, e_r) of $M_\varphi^{(\sigma)}$, then $(e_1, \dots, e_d, \varphi(e_{d+1}), \dots, \varphi(e_r))$ is a basis of M and $(\nu(e_1), \dots, \nu(e_d), e_{d+1}, \dots, e_r)$ is a basis of $M^{(\sigma)}$. Using the fact that $\varphi \circ \nu$ and $\nu \circ \varphi$ are equal to multiplication by $t - \theta$ on M and $M^{(\sigma)}$ respectively, we see that, with respect to these bases, the maps $\varphi : M^{(\sigma)} \rightarrow M$ and $\nu : M \rightarrow M^{(\sigma)}$ are represented by matrices as

$$(\star) \quad \varphi = \begin{pmatrix} (t - \theta)I_d & \\ & I_{r-d} \end{pmatrix}, \quad \nu = \begin{pmatrix} I_d & \\ & (t - \theta)I_{r-d} \end{pmatrix},$$

where I_d is the identity matrix of size d . The number d is somehow related to the "dimension" of M (if it is defined). For example, if \mathcal{O} is finite over $\mathbb{F}_q[[\theta]]$ and $R = A/(t^n) \otimes \mathcal{O}$, then d is indeed the dimension of (the finite t -module scheme corresponding to) M (see (2.6)).

³This definition differs slightly from that in [5], where the compatibility of ν ($= v$ in the notation of [5]) with t and φ is lacking. The present definition should replace that of [5]. Also in [5], we restricted ourselves to those M which are finite over \mathcal{O} , whereas here we do not. I thank the referee for his criticism on these points.

Example (2.2). Let ϕ be a Drinfeld A -module of rank r over \mathcal{O} . Then the ν -module M over $R = A \otimes \mathcal{O}$ corresponding to ϕ (cf. [5]) has property (D). To see this, suppose that ϕ is defined by the equation

$$\phi_t(X) = \theta X + a_1 X^q + \cdots + a_r X^{q^r}, \quad a_i \in \mathcal{O}, a_r \in \mathcal{O}^\times.$$

(In general, ϕ is of this form locally on $\text{Spec}\mathcal{O}$.) For simplicity, assume $a_r = 1$. Then we have $M = \bigoplus_{i=0}^{r-1} R \cdot X^{q^i}$ and the effect of φ and ν is as follows:

$$\varphi: \begin{cases} 1 \otimes X^{q^i} \mapsto X^{q^{i+1}} & \text{for } i = 0, \dots, r-2, \\ 1 \otimes X^{q^{r-1}} \mapsto (t - \theta)X - a_1 X^q - \cdots - a_{r-1} X^{q^{r-1}}, \end{cases}$$

$$\nu: \begin{cases} X \mapsto a_1 \otimes X + \cdots + a_r \otimes X^{q^{r-1}}, \\ X^{q^i} \mapsto (t - \theta) \otimes X^{q^{i-1}} & \text{for } i = 1, \dots, r-1. \end{cases}$$

Put

$$M_\nu = R \cdot X \quad M_\varphi^{(\sigma)} = \bigoplus_{i=0}^{r-2} R \cdot X^{q^i}.$$

Then M has property (D) with these M_ν and $M_\varphi^{(\sigma)}$. As in Remark (2.1), with respect to the bases $(X, \varphi(1 \otimes X), \dots, \varphi(1 \otimes X^{q^{r-2}}))$ of M and $(\nu(X), 1 \otimes X, \dots, 1 \otimes X^{q^{r-2}})$ of $M^{(\sigma)}$, the R -linear maps φ and ν are represented respectively by the matrices

$$\begin{pmatrix} t - \theta & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & & \\ & t - \theta & & \\ & & \ddots & \\ & & & t - \theta \end{pmatrix}.$$

In the following, relevant is the case where \mathcal{O} is a complete discrete valuation ring, and we will be concerned with ν -modules over $R = (A/v^n) \otimes \mathcal{O}$ and $A_v \widehat{\otimes} \mathcal{O}$. Such a ring R is semi-local, hence a product of local rings. It will be convenient to know explicitly the presentation of R as such a product. So in the rest of the paper, let \mathcal{O} be a complete discrete valuation ring equipped with an injective ring homomorphism $\alpha: A \rightarrow \mathcal{O}$. Let v be a non-zero prime ideal of A , and assume its image $\alpha(v)$ by α is contained in the maximal ideal \mathfrak{m} of \mathcal{O} . Let $k = \mathcal{O}/\mathfrak{m}$ be the residue field of \mathcal{O} , which we identify canonically with the field of constants of \mathcal{O} . Denote by $\theta_o \in k$ the class modulo \mathfrak{m} of $\theta \in \mathcal{O}$. Let $v(t)$ be the monic irreducible polynomial in $A = \mathbb{F}_q[t]$ which generates the prime ideal v . If we set $\delta = \deg v(t)$, then $v(t)$, considered as a polynomial in $k[t]$ or $\mathcal{O}[t]$ by means of $\mathbb{F}_q \hookrightarrow k \hookrightarrow \mathcal{O}$, has roots $\theta_o, \theta_o^q, \dots, \theta_o^{q^{\delta-1}}$ in k (hence in \mathcal{O}). We consider the case $R = (A/v^n) \otimes \mathcal{O}$ (the case of $A_v \widehat{\otimes} \mathcal{O}$ is similar). Thinking of R as $\mathcal{O}[t]/(v(t)^n)$, we have

$$R = \mathcal{O}[t]/\prod_{\iota \in \mathbb{Z}/\delta\mathbb{Z}} (t - \theta_o^{q^\iota})^n \simeq \prod_{\iota \in \mathbb{Z}/\delta\mathbb{Z}} \mathcal{O}[t - \theta_o^{q^\iota}]/(t - \theta_o^{q^\iota})^n,$$

in which the components are cyclically permuted by σ (σ maps the ι -th component into the $(\iota+1)$ -st). Set $x_\iota := t - \theta_o^{q^\iota}$ and $R_\iota := \mathcal{O}[x_\iota]/x_\iota^n$. Then R_ι is a local ring with

maximal ideal (\mathfrak{m}, x_ι) (here, we write x_ι also for its image in R_ι). Suppose that an element of R represented by $f(t) \in \mathcal{O}[t]$ corresponds by the above isomorphism to $(f_0, \dots, f_{\delta-1}) \in \prod R_\iota$; thus f_ι is the image of $f(t)$ in R_ι . For example, (1) the image of $v(t)$ in R_ι is x_ι times a unit; (2) the image of $t - \theta$ in R_ι is $(\theta_0^{q^\iota} - \theta) + x_\iota$. For $n = 1$, we have $R_\iota = \mathcal{O}[t]/(t - \theta_0^{q^\iota}) \simeq \mathcal{O}$, and the image here of $f(t)$ is $f(\theta_0^{q^\iota})$; thus the map $\sigma : R \rightarrow R$, in terms of this expression, is $(\dots, f(\theta_0^{q^\iota}), \dots) \mapsto (\dots, f^\sigma(\theta_0^{q^\iota}), \dots) = (\dots, f(\theta_0^{q^{\iota-1}})^q, \dots)$, i.e., in short, $(r_\iota) \mapsto (r_{\iota-1}^q)$. In particular, the image of $t - \theta$ in the ι -th component is $\theta_0^{q^\iota} - \theta$. For $\iota = 0$, the element $\theta_0 - \theta$ is in \mathfrak{m} , whereas for $\iota > 0$, the element $\theta_0^{q^\iota} - \theta$ is a unit in \mathcal{O} . Note that $(\theta_0^{q^\iota} - \theta) + x_\iota$ is not a zero-divisor of R_ι in all cases.

For a ν -module M over $R = \prod R_\iota$, set $M_\iota := R_\iota \otimes_R M$ by using the ι -th projection $R \rightarrow R_\iota$. Then the R_ι -module $(M^{(\sigma)})_\iota = R_\iota \otimes_R M^{(\sigma)}$ is canonically isomorphic to $(M_{\iota-1})^{(\sigma)}$, which we simply denote $M_{\iota-1}^{(\sigma)}$. The R -module homomorphisms φ and ν induce R_ι -module homomorphisms

$$\varphi_\iota : M_{\iota-1}^{(\sigma)} \rightarrow M_\iota, \quad \nu_\iota : M_\iota \rightarrow M_{\iota-1}^{(\sigma)}$$

such that $\varphi_\iota \circ \nu_\iota$ and $\nu_\iota \circ \varphi_\iota$ are multiplication by $(\theta_0^{q^\iota} - \theta) + x_\iota$ on M_ι and $M_{\iota-1}^{(\sigma)}$ respectively. Conversely, a ν -module M can be thought of as a collection of such $(M_\iota, \varphi_\iota, \nu_\iota)$ for $\iota \in \mathbb{Z}/\delta\mathbb{Z}$. Thus the property (D) for M can be “decomposed” into the product of property (D $_\iota$), where

(D $_\iota$) There exist R_ι -submodules $M_{\iota,\nu}$ of M_ι and $M_{\iota-1,\varphi}^{(\sigma)} = (M_{\iota-1}^{(\sigma)})_\varphi$ of $M_{\iota-1}^{(\sigma)}$ such that

$$\begin{cases} M_\iota = M_{\iota,\nu} \oplus \varphi(M_{\iota-1,\varphi}^{(\sigma)}) & \text{and} & M_{\iota-1}^{(\sigma)} = \nu(M_{\iota,\nu}) \oplus M_{\iota-1,\varphi}^{(\sigma)}; \\ \nu_\iota \text{ is injective on } M_{\iota,\nu} & \text{and} & \varphi_\iota \text{ is injective on } M_{\iota-1,\varphi}^{(\sigma)}. \end{cases}$$

Here again we do *not* mean that $M_{\iota-1,\varphi}^{(\sigma)}$ is a base extension of some “ $M_{\iota-1,\varphi}$ ”.

Proposition (2.3). *Let v , \mathcal{O} and $n \geq 1$ be as above. Assume further that \mathcal{O} is unramified over A_v , i.e., its maximal ideal \mathfrak{m} is generated by the image $\alpha(v)$ of v . Let M be a ν -module which is free of finite rank over $R = (A/v^n) \otimes \mathcal{O}$ or $A_v \widehat{\otimes} \mathcal{O}$. Then M has property (D).*

This proposition is applicable in particular to a ν -module which corresponds (by Dieudonné theory [5]) to a finite⁴ A -submodule of a Drinfeld module over \mathcal{O} of which the A -module of geometric points is of the form $(A/v^n)^{\oplus h}$.

Proof. Let the notations R_ι, M_ι, \dots be the same as before. We shall construct R_ι -submodules $M_{\iota,\nu}$ and $M_{\iota-1,\varphi}^{(\sigma)}$ with property (D $_\iota$) for each ι . Observe that, by Nakayama’s lemma, it is enough to check the property (D $_\iota$) modulo the maximal ideal of R_ι . In particular, we may and do assume $R = (A/v) \otimes \mathcal{O}$; then the maximal ideal of R_ι is (the image of) \mathfrak{m} . We distinguish the two cases $\iota = 0$ and $\iota \neq 0$.

*Case $\iota = 0$:*⁵ Set

$$\overline{M}_0 := M_0/\mathfrak{m}M_0, \quad \overline{M}_{-1}^{(\sigma)} := M_{-1}^{(\sigma)}/\mathfrak{m}M_{-1}^{(\sigma)}.$$

⁴Finite as a scheme over \mathcal{O} .

⁵As the referee points out, one could simply use the elementary divisor theorem here.

By the assumption of unramifiedness, the element $\theta_o - \theta$ is a uniformizer of $R_0 = \mathcal{O}$, so $\mathfrak{m} = (\theta_o - \theta)$. Then since $\varphi_0 \circ \nu_0$ and $\nu_0 \circ \varphi_0$ are equal to multiplication by $\theta_o - \theta$, which is injective on M_0 and $M_{-1}^{(\sigma)}$, the sequences of k -vector spaces

$$\overline{M}_0 \xrightarrow{\overline{\nu}_0} \overline{M}_{-1}^{(\sigma)} \xrightarrow{\overline{\varphi}_0} \overline{M}_0, \quad \overline{M}_{-1}^{(\sigma)} \xrightarrow{\overline{\varphi}_0} \overline{M}_0 \xrightarrow{\overline{\nu}_0} \overline{M}_{-1}^{(\sigma)}$$

are exact. Here, $\overline{\nu}_0$ and $\overline{\varphi}_0$ are respectively the natural maps induced by ν_0 and φ_0 . So we have

$$\text{Im}(\overline{\nu}_0) = \text{Ker}(\overline{\varphi}_0), \quad \text{Im}(\overline{\varphi}_0) = \text{Ker}(\overline{\nu}_0).$$

Choose k -subspaces $\overline{M}_{0,\nu}$ of \overline{M}_0 and $\overline{M}_{-1,\varphi}^{(\sigma)}$ of $\overline{M}_{-1}^{(\sigma)}$ such that

$$\overline{M}_{0,\nu} \oplus \overline{\varphi}_0(\overline{M}_{-1}^{(\sigma)}) = \overline{M}_0, \quad \overline{\nu}_0(\overline{M}_0) \oplus \overline{M}_{-1,\varphi}^{(\sigma)} = \overline{M}_{-1}^{(\sigma)}.$$

Note that $\overline{\nu}_0$ is injective on $\overline{M}_{0,\nu}$ and $\overline{\varphi}_0$ is injective on $\overline{M}_{-1,\varphi}^{(\sigma)}$. Applying $\overline{\nu}_0$ and $\overline{\varphi}_0$ respectively to the above equalities, we see

$$\overline{\nu}_0(\overline{M}_{0,\nu}) = \overline{\nu}_0(\overline{M}_0), \quad \overline{\varphi}_0(\overline{M}_{-1,\varphi}^{(\sigma)}) = \overline{\varphi}_0(\overline{M}_{-1}^{(\sigma)}).$$

Getting back to those equalities, we have

$$\overline{M}_{0,\nu} \oplus \overline{\varphi}_0(\overline{M}_{-1,\varphi}^{(\sigma)}) = \overline{M}_0, \quad \overline{\nu}_0(\overline{M}_{0,\nu}) \oplus \overline{M}_{-1,\varphi}^{(\sigma)} = \overline{M}_{-1}^{(\sigma)}.$$

Now lift these k -subspaces arbitrarily to R_0 -direct summands $M_{0,\nu}$ of M_0 and $M_{-1,\varphi}^{(\sigma)}$ of $M_{-1}^{(\sigma)}$. Then using Nakayama's lemma, one checks that the R_0 -module M_0 with these submodules has property (D0).

Case $\iota \neq 0$: In this case, multiplication by $\theta_o^{q^\iota} - \theta$ is injective. Then one may choose $M_{l,\nu}$ and $M_{l-1,\varphi}^{(\sigma)}$ rather freely; for example, $M_{l,\nu} = M_l$ and $M_{l-1,\varphi}^{(\sigma)} = 0$ will do.

Proposition (2.4). *Let v , \mathcal{O} and $n \geq 1$ be as above (\mathcal{O} may be ramified over A_v). Let M be a ν -module which is free of rank r over $(A/v^n) \otimes \mathcal{O}$. If M has property (D), then there exists a ν -module \widehat{M} free of rank r over $A_v \widehat{\otimes} \mathcal{O}$ such that $M \simeq \widehat{M} \otimes_{A_v} (A/v^n)$.*

Proof. Set $R = \prod R_\iota = (A/v^n) \otimes \mathcal{O}$ and $\widehat{R} = \prod \widehat{R}_\iota = A_v \widehat{\otimes} \mathcal{O}$. We shall lift the R_ι -module $M_\iota = R_\iota \otimes_R M$ to an \widehat{R}_ι -module \widehat{M}_ι for each ι to obtain a ν -module $\widehat{M} := \prod \widehat{M}_\iota$ over \widehat{R} . To simplify notation, we omit the suffix ι from R_ι , M_ι etc. (so $M = M_\iota$ and $M^{(\sigma)} = M_{\iota-1}^{(\sigma)}$), and write $t - \theta$ for its image (which is actually $\theta_o^{q^\iota} - \theta + x_\iota$) in R_ι (resp. \widehat{R}_ι). Then by Remark (2.1), there exist elements e_1, \dots, e_d of M and e_{d+1}, \dots, e_r of $M^{(\sigma)}$ such that $(g_i) := (e_1, \dots, e_d, \varphi(e_{d+1}), \dots, \varphi(e_r))$ is an R -basis of M and $(h_i) := (\nu(e_1), \dots, \nu(e_d), e_{d+1}, \dots, e_r)$ is an R -basis of $M^{(\sigma)}$, and with respect to these bases, the maps φ and ν are represented by the matrices (\star) in Remark (2.1).

As $(1 \otimes g_i)$ is also an R -basis of $M^{(\sigma)}$, we have

$$h_j = \sum_{i=1}^r u_{ij} \otimes g_i$$

for some $U = (u_{ij}) \in \mathrm{GL}_r(R)$. Choose a matrix $\widehat{U} = (\widehat{u}_{ij}) \in \mathrm{GL}_r(\widehat{R})$ such that $\widehat{U} \pmod{v^n} = U$. Let \widehat{M} be a free \widehat{R} -module of rank r with basis (\widehat{g}_i) , which we think of as a lifting of M by mapping \widehat{g}_i to g_i . Define a basis (\widehat{h}_i) of $\widehat{M}^{(\sigma)}$ by

$$\widehat{h}_j = \sum_{i=1}^r \widehat{u}_{ij} \otimes \widehat{g}_i.$$

Let $\widehat{\varphi} : \widehat{M}^{(\sigma)} \rightarrow \widehat{M}$ and $\widehat{\nu} : \widehat{M} \rightarrow \widehat{M}^{(\sigma)}$ be the \widehat{R} -linear maps represented respectively by the matrices (\star) with respect to the bases (\widehat{g}_i) and (\widehat{h}_i) . Then these are liftings of φ and ν such that $\widehat{\varphi} \circ \widehat{\nu}$ and $\widehat{\nu} \circ \widehat{\varphi}$ are equal to multiplication by $t - \theta \in \widehat{R}$; thus we obtain (the ι -th component of) a ν -module $(\widehat{M}, \widehat{\varphi}, \widehat{\nu})$ over \widehat{R} which lifts (M, φ, ν) .

Corollary (2.5). *Let v and \mathcal{O} be as above. Assume further that \mathcal{O} is unramified over A_v . Let ϕ be a Drinfeld module over \mathcal{O} . Let G be a finite A -submodule scheme of ϕ of which the A -module of geometric points is of the form $(A/v^n)^{\oplus h}$ with some n and $h \geq 1$. Then G extends to a v -divisible group \widehat{G} over \mathcal{O} .*

This follows from Propositions (2.3) and (2.4).

Remark (2.6). For a v -divisible group over \mathcal{O} , its *dimension* can be defined ([3], §1). Although the extension \widehat{G} of G is not unique, its dimension is determined by G .

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