Induction formula for the Artin conductors of mod ℓ Galois representations

Yuichiro Taguchi

Abstract. A formula is given to describe how the Artin conductor of a mod ℓ Galois representation behaves with respect to induction of the representation.

1. Introduction. In this paper, we give a formula which describes how the Artin conductor of a mod ℓ Galois representation behaves with respect to induction of the representation. Such a formula is well-known for representations over a field of characteristic zero, while there are some subtle problems in positive characteristics. We shall show that the classical formula still holds in the mod ℓ case.

Let K be a complete discrete valuation field with perfect residue field of characteristic $p > 0$. Let L/K be a finite Galois extension with Galois group G. Let H be a subgroup of G, and let $\rho : H \to \mathrm{GL}_k(V)$ be a finite-dimensional linear representation of H over a field k of characteristic $\ell \geq 0$. If ℓ does not divide the ramification index of L/K, then the exponent of the Artin conductor $f(\rho)$ of ρ is well-defined either as the inner product of ρ and the Artin character or as a weighted sum of the codimensions of the fixed subspaces of V by higher ramification groups, and it behaves nicely with respect to induction. If $\mathrm{Ind}_H^G\rho$ denotes the representation of G induced by ρ, then we have[1]

$$\text{(A)} \qquad f(\mathrm{Ind}_H^G\rho) \;=\; f_{K'/K}\cdot f(\rho) + v_K(d_{K'/K})\dim\rho,$$

where K' is the subfield of L corresponding to H, $f_{K'/K}$ (resp. $d_{K'/K}$) is the residual degree (resp. discriminant ideal) of K'/K, and v_K is the normalized valuation of K.

If k is of any characteristic, then the exponent of the Swan conductor $b(\rho)$ of ρ, defined as the inner product of ρ and the Swan character ([7], §19.3), still behaves nicely with respect to induction as long as ℓ is different from the residual characteristic p of K; we have ([5], Prop. 1; [2], Proof of Prop. 4.1)

$$\text{(B)} \qquad b(\mathrm{Ind}_H^G\rho) \;=\; f_{K'/K}\cdot b(\rho) + (v_K(d_{K'/K}) - [K':K] + f_{K'/K})\dim\rho.$$

Formulas (A) and (B) are deduced, by taking the inner products and using the Frobenius reciprocity, from the corresponding properties of the Artin and Swan characters with respect to restriction. Such a formula, however, does not seem to be

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[1]See Cor. to Prop. 4 of [6], Chap. VI, for the case $\ell = 0$. Most of the results there extend to the case $\ell > 0$ (*cf.* [9]) by employing the Brauer theory ([7], Part III) as long as ℓ does not divide the ramification index. For example, the inner product of two $k[H_0]$-modules U and V (or equivalently, their Brauer characters) is defined as $\dim_k \mathrm{Hom}_{k[H_0]}(U,V)$, and this inner product has enough properties to allow us the "usual" proof of (A) as in [6].

known for the exponent of the Artin conductor $n(\rho)$ of a mod ℓ Galois representation ρ as defined in [8] when $\ell \geq 0$ is arbitrary. It is defined as follows:

$$n(\rho) \ := \ \sum_{i=0}^{\infty} \frac{1}{(H_0 : H_i)} \dim_k(V/V^{H_i}),$$

where H_i is the ith ramification subgroup of H, and V^{H_i} is the fixed subspace of V by H_i. We also define

$$c(\rho) \ := \ \sum_{i=1}^{\infty} \frac{1}{(H_0 : H_i)} \dim_k(V/V^{H_i}),$$

so that

$$n(\rho) \ = \ \dim_k(V/V^{H_0}) + c(\rho).$$

This $n(\rho)$ coincides with the usual $f(\rho)$ if ℓ does not divide $|H_0|$, and $c(\rho)$ coincides with the usual $b(\rho)$ if $\ell \neq p$ (*cf.* the property (iii) of [7], §19.3). (If ℓ divides $|H_0|$, there are some technical difficulties in defining $f(\rho)$ as the inner product of ρ and the Artin character, and similarly for $c(\rho)$; *cf.* [9].) These non-negative rational numbers $n(\rho)$ and $c(\rho)$ are in fact integers if $\ell \neq p$ since so is $b(\rho)$, while they may not be so if $\ell = p$. We have $n(\rho) > 0$ if and only if ρ is ramified, and $c(\rho) > 0$ if and only if ρ is wildly ramified.

In this paper, we prove:

Theorem. *Let G, H and $\rho : H \to \mathrm{GL}_k(V)$ be as above. For any $\ell \geq 0$, we have*

$$\begin{aligned} n(\mathrm{Ind}_H^G \rho) \ &= \ f_{K'/K} \cdot n(\rho) + v_K(d_{K'/K}) \dim \rho, \\ c(\mathrm{Ind}_H^G \rho) \ &= \ f_{K'/K} \cdot c(\rho) + (v_K(d_{K'/K}) - [K' : K] + f_{K'/K}) \dim \rho. \end{aligned}$$

Thus $n(\rho)$ and $c(\rho)$ satisfy the same formulas as in (A) and (B) respectively.

The proof will be given in §2. Our proof is different from the standard one as in [6] even in the classical case $k = \mathbb{C}$. As an application of the Theorem, we give in §3 a modular version of the Führerdiskriminantenproduktformel.

Such a consideratаion has been motivated by Proposition 4.1 of [2] and Lemma 2.1 of [3]. I thank Hyunsuk Moon for discussions on this subject, and Hiro Yamada for discussions on the Brauer theory. I also thank the Inamori Foundation for its financial support. Finally, I thank Professor Jean-Pierre Serre for communicating to me that the use of Hom (rather than $\otimes$) in the definition of the induced module simplifies the exposition of the proof.

2. Proof of the Theorem. Let $W = \mathrm{Ind}_H^G V := \mathrm{Hom}_{k[H]}(k[G], V)$ be the representation space[2] for $\mathrm{Ind}_H^G \rho$. For any $k[G]$-module U, we denote by $\mathrm{Res}_H^G U$ its restriction to H. By Mackey's theorem ([7], §7.3, Prop. 22; [4], Chap. 3, Th. 1.9), the $k[G_i]$-module $\mathrm{Res}_{G_i}^G \mathrm{Ind}_H^G V$ decomposes as follows:

$$\mathrm{Res}_{G_i}^G \mathrm{Ind}_H^G V \ \simeq \bigoplus_{s \in G_i \backslash G/H} \mathrm{Ind}_{H_{i,s}}^{G_i} V_{i,s},$$

[2] One may use $W = k[G] \otimes_{k[H]} V$ here. Note that $k[G] \otimes_{k[H]} V$ is isomorphic (non-canonically) to $\mathrm{Hom}_{k[H]}(k[G], V)$ since $(G : H)$ is finite.

where s moves through a complete set of representatives for $G_i\backslash G/H$, $H_{i,s} := sHs^{-1} \cap G_i = sH_is^{-1}$, and the representation $\rho_{i,s} : H_{i,s} \to \mathrm{GL}_k(V_{i,s})$, with $V_{i,s} := V$, is defined by $g \mapsto \rho(s^{-1}gs)$. By applying the Frobenius reciprocity law

$$\mathrm{Hom}_{k[G]}(U, \mathrm{Ind}_H^G V) \;\simeq\; \mathrm{Hom}_{k[H]}(\mathrm{Res}_H^G U, V)$$

(*cf.* [1], Chap. II, Prop. 5.2′) to our G_i, $H_{i,s}$ and $V_{i,s}$ with $U = k$ (with trivial G_i-action), we have $\dim_k(\mathrm{Ind}_{H_{i,s}}^{G_i} V_{i,s})^{G_i} = \dim_k V_{i,s}^{H_{i,s}}$. This equals $\dim_k V^{H_i}$, since $V_{i,s} = V$ and the fixed subspace $V_{i,s}^{H_{i,s}}$ is identified with V^{H_i}. Hence $\dim_k W^{G_i} = |G_i\backslash G/H| \dim_k V^{H_i}$. Then we have

$$\begin{aligned} & n(\mathrm{Ind}_H^G \rho) - f_{K'/K} \cdot n(\rho) \\ = & \sum_{i=0}^{\infty} \left(\frac{1}{(G_0 : G_i)} \dim_k(W/W^{G_i}) - \frac{f_{K'/K}}{(H_0 : H_i)} \dim_k(V/V^{H_i}) \right) \\ = & \sum_{i=0}^{\infty} \left(\frac{(G : H)}{(G_0 : G_i)} - \frac{f_{K'/K}}{(H_0 : H_i)} \right) \dim_k V \\ & - \sum_{i=0}^{\infty} \left(\frac{|G_i\backslash G/H|}{(G_0 : G_i)} - \frac{f_{K'/K}}{(H_0 : H_i)} \right) \dim_k V^{H_i}. \end{aligned}$$

Here, the last sum vanishes because we have

$$\frac{|G_i\backslash G/H|}{(G_0 : G_i)} \;=\; \frac{f_{K'/K}}{(H_0 : H_i)}.$$

This follows from the equality $f_{K'/K} = |G_0\backslash G/H|$ and the "exact sequence"

$$1 \;\to\; H_i\backslash H_0 \;\to\; G_i\backslash G_0 \;\to\; G_i\backslash G/H \;\to\; G_0\backslash G/H \;\to\; 1$$

of homogeneous spaces.[3] (Or, one may identify $G_i\backslash G/H = G/G_iH$ and $G_0\backslash G/H = G/G_0H$ since G_i are normal in G, and use $(G_0H : G_iH) = (G_0 : G_i)/(H_0 : H_i)$ to show the desired equality.) Thus we have

$$n(\mathrm{Ind}_H^G \rho) - f_{K'/K} \cdot n(\rho) \;=\; \sum_{i=0}^{\infty} \left(\frac{[K' : K]}{(G_0 : G_i)} - \frac{f_{K'/K}}{(H_0 : H_i)} \right) \dim_k V.$$

But it is well-known that

$$\sum_{i=0}^{\infty} \left(\frac{[K' : K]}{(G_0 : G_i)} - \frac{f_{K'/K}}{(H_0 : H_i)} \right) \;=\; v_K(d_{K'/K}).$$

For example, this can be deduced from the classical formula

$$v_K(\mathcal{D}_{L/K}) \;=\; \sum_{i=0}^{\infty} (|G_i| - 1)/|G_0|$$

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[3]Meaning that it is a sequence of maps compatible with the group actions of $H_i\backslash H_0 \hookrightarrow G_i\backslash G_0 \hookrightarrow G_i\backslash G \twoheadrightarrow G_0\backslash G$ and, at each step $X \to Y \to Z$, the inverse image of any element of $\mathrm{Im}(Y \to Z)$ is a translation of $\mathrm{Im}(X \to Y)$.

([6], Chap. IV, §1, Prop. 4), where $\mathcal{D}_{L/K}$ is the different of L/K, and that $v_K(d_{K'/K}) = [K' : K]v_K(\mathcal{D}_{L/K}\mathcal{D}_{L/K'}^{-1})$. (Note also that this is a special case of the formula (A) of the Introduction, because its left-hand side is equal to

$$\sum_{i=0}^{\infty} \frac{(G:H) - |G_i\backslash G/H|}{(G_0 : G_i)} \;=\; \sum_{i=0}^{\infty} \frac{\dim_k(k[G/H]/k[G/H]^{G_i})}{(G_0 : G_i)}$$

for any field k, which is the exponent of the Artin conductor of the permutation representation $k[G/H]$ of G.) Thus the formula for $n(\mathrm{Ind}_H^G \rho)$ is proved.

The formula for $c(\mathrm{Ind}_H^G \rho)$ is proved similarly by omitting the $i = 0$ term $((G : H) - f_{K'/K}) \dim_k V$.

3. Globalization. In this section, K may be a global (as well as local) field. Let A be a Dedekind ring and K its field of fractions. Assume that the residue fields of all non-zero prime ideals of A are perfect (*e.g.* A is a localization of the ring of integers of an algebraic number field of finite degree). Let L/K be a finite Galois extension with Galois group G, and let B be the integral closure of A in L. For a representation $\rho : G \to \mathrm{GL}_k(W)$, we define its Artin conductor $\mathfrak{n}(\rho)$, as an ideal of A, by

$$\mathfrak{n}(\rho) \;=\; \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}(\rho)} \qquad \text{with} \;\; n_{\mathfrak{p}}(\rho) := n(\rho|_{D_{\mathfrak{P}}}),$$

where $\mathfrak{p}$ runs through the non-zero prime ideals of A and $D_{\mathfrak{P}}$ is the decomposition subgroup of G for a prime ideal $\mathfrak{P}$ of B lying above $\mathfrak{p}$.

Suppose that H is a subgroup of G and $\rho : H \to \mathrm{GL}_k(V)$ is now a representation of H. Let K' be the subfield of L corresponding to H, and $d_{K'/K}$ the discriminant ideal of K'/K over A (= discriminant ideal of B/A). Then our Theorem globalizes (*cf.* [6], Chap. VI, §3) to:

Corollary 1. $\mathfrak{n}(\mathrm{Ind}_H^G \rho) \;=\; N_{K'/K}(\mathfrak{n}(\rho)) \cdot d_{K'/K}^{\dim \rho}.$

Here $N_{K'/K}$ is the norm map from K' to K.

In the special case where ρ is the unit representation of H, we have

$$d_{K'/K} \;=\; \mathfrak{n}(s_{G/H}),$$

where $s_{G/H}$ is the permutation representation $k[G/H]$ of G on G/H. If further $H = \{1\}$, then

$$\text{(C)} \qquad d_{L/K} \;=\; \mathfrak{n}(r_G),$$

where r_G is the regular representation $k[G]$ of G.

Next write r_G (uniquely) as a direct sum of indecomposable $k[G]$-modules:

$$r_G \;=\; \bigoplus_{\chi \in \mathrm{S}_k(G)} m_\chi \cdot P_\chi \qquad \text{with} \;\; m_\chi \in \mathbb{Z}_{\geq 1}.$$

Here the direct sum is over the set $\mathrm{S}_k(G)$ of isomorphism classes of simple $k[G]$-modules, and P_χ is the projective envelope of $\chi \in \mathrm{S}_k(G)$ ([7], Part III). Noticing that $n(\rho)$ is additive[4] in ρ, we deduce from (C) a modular version of the Führerdiskriminantenproduktformel:

Corollary 2. $d_{L/K} \;=\; \prod_{\chi \in \mathrm{S}_k(G)} \mathfrak{n}(P_\chi)^{m_\chi}.$

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[4] *i.e.* $n(\rho_1 \oplus \rho_2) = n(\rho_1) + n(\rho_2)$. Note, however, that $n(\rho)$ is *not* additive with respect to short exact sequences.

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Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-8581, Japan

taguchi@math.kyushu-u.ac.jp