On potentially abelian geometric representations

Yuichiro Taguchi

Intorduction. In this paper, we prove the potentially abelian case of the finiteness conjecture of Fontaine and Mazur (Conjectures 2a, 2b and 2c of [5]; N.B. 2a implies 2b and 2c). Let K be an algebraic number field of finite degree over the rational number field $\mathbb{Q}$ and $G_K = \mathrm{Gal}(\overline{K}/K)$ its absolute Galois group. Let d be a positive integer, p a prime number, and $\overline{\mathbb{Q}}_p$ an algebraic closure of the p-adic number field $\mathbb{Q}_p$. Then we prove:

Theorem. *There exist only finitely many isomorphism classes of potentially abelian geometric representations $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ with a fixed Hodge-Tate type and bounded inertial level.*

Note that we do not need the assumption of semisimplicity in the potentially abelian case, since potentially abelian geometric representations are automatically semisimple.

Fontaine-Mazur's "Main Conjecture" (Conjecture 1 of [5]) is known for potentially abelian representations (Proposition in §6 of [5]). However, this conjecture combined with Faltings' arguments in the proof of Satz 5 of [4] (*cf.* Theorem 6.1 of [7]) does not seem to imply Conjectures 2b and 2c (at least directly), because if we consider geometric representations with a fixed coefficient field E_π of finite degree over $\mathbb{Q}_p$, they may arise from motives over K with global coefficient fields E which are contained in E_π but of, *a priori*, larger and larger degree over $\mathbb{Q}$, and thus the possibilities of the local L-factors may not be bounded.

A similar result to the above Theorem is proved for *admissible systems* of ℓ-adic representations in [1], 4.5; they prove it as a corollary to their finiteness theorem on representations of the Weil group of $\mathbb{Q}$ into $\mathrm{GL}_d(\mathbb{C})$ with bounded conductor. Here we give a direct p-adic proof.

We are also motivated by a similar question in the mod p case ([8], [10], [11], [12], [13], [14]).

After giving some preliminaries in §1, we prove the Theorem in §2.

All algebraic number fields and algebraic extensions of $\mathbb{Q}_p$ in this paper are considered as subfields of the completion $\mathbb{C}_p$ of a fixed algebraic closure of $\mathbb{Q}_p$. For any field k, its algebraic closure is denoted by $\overline{k}$.

1. Preliminaries. First we recall the terminologies used in the Theorem. A continuous represenation $\rho : G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$ is said to be *geometric* ([5], §1) if it is unramified outside a finite set of places of K and, for each place v of K lying above p, its restriction to the decomposition group of (an extension to $\overline{K}$ of) v is potentially semistable. Let E_π be an algebraic extention of $\mathbb{Q}_p$. A continuous representation $\rho : G_K \to \mathrm{GL}_d(E_\pi)$ is said to be *geometric* if it is defined over a finite extension of $\mathbb{Q}_p$ and is geometric as a (finite-dimensional) representation over $\mathbb{Q}_p$.

An *inertial level* $\mathcal{L}$ is a collection $(\mathcal{L}_v)_v$ of open normal subgroups $\mathcal{L}_v$ of an inertia group I_v for each finite place v of K such that $\mathcal{L}_v = I_v$ for almost all v. We say that a representation ρ of G_K has *inertial level bounded by* $(\mathcal{L}_v)_v$ if, for each v, the restriction of ρ to the decomposition group G_{K_v} becomes semistable when the local field K_v is replaced by a finite extension whose inertia group is contained in $\mathcal{L}_v$. Here and elsewhere, K_v denotes the completion of K at the place v, and we identify its absolute Galois group G_{K_v} with the decomposition group of an extension of v to $\overline{K}$, $\overline{K}$ being identified with a subfield of the algebraic closure $\overline{K}_v$ of K_v.

The E_π*-Hodge-Tate type* of the representation $\rho : G_K \to \mathrm{GL}_d(E_\pi)$ at a place v of K lying above p is the isomorphism class of the graded $(E_\pi \otimes_{\mathbb{Q}_p} K_v)$-module $\mathrm{Hom}_{\mathbb{Q}_p[G_{K_v}]}(V, B_{\mathrm{HT}})$, where V is the representation space for $\rho|_{G_{K_v}}$ and B_{HT} is the graded ring $\oplus_{r\in\mathbb{Z}}\mathbb{C}_p(r)$. If E_π is of finite degree over $\mathbb{Q}_p$, then the E_π-Hodge-Tate type determines in particular the *Hodge-Tate weights*, *i.e.,* the integers $h_{v,i}$ such that

$$V \otimes_{\mathbb{Q}_p} \mathbb{C}_p \;\simeq\; \mathbb{C}_p(h_{v,1}) \oplus \cdots \oplus \mathbb{C}_p(h_{v,d[E_\pi:\mathbb{Q}_p]}).$$

Now suppose that E is an algebraic number field (of either finite or infinite degree over $\mathbb{Q}$) contained in E_π. A representation $\rho : G_K \to \mathrm{GL}_d(E_\pi)$ is said to be E-*rational* if, for each place v of K where ρ is unramified, the image by ρ of a Frobenius element at v has characteristic polynomial with coefficients in E. Geometric $\overline{\mathbb{Q}}_p$-representations are conjectured (as part of the Fontaine-Mazur Conjectures) to be $\overline{\mathbb{Q}}$-rational, and this is known for potentially abelian representations ([5], §6, Prop.)

When ρ is potentially abelian, geometricity is equivalent (*loc. cit.*) to local algebraicity in the sense of Serre ([16], III, §2.1; note that the notion of local algebraicity makes sense (not only for abelian but) also for potentially abelian p-adic representations). As we shall work with this notion, let us recall its definition in some detail, as well as the algebraic groups $T_{\mathfrak{m}}$ and $S_{\mathfrak{m}}$.

Local algebraicity should be invariant by finite extensions of the base field K, so it is enough to define the notion for abelian representations $\rho : G_K^{\mathrm{ab}} \to \mathrm{GL}_d(E_\pi)$, where E_π is an algebraic extension of $\mathbb{Q}_p$. Roughly speaking, such a ρ is said to be locally algebraic if, locally at p, it arises from an algebraic representation of the torus $T = \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}/K})$. To be precise, let T be the torus over $\mathbb{Q}$ obtained from the multiplicative group $\mathbb{G}_{\mathrm{m}}$ over K by restriction of scalars to $\mathbb{Q}$. We write $T_{/E_\pi}$ for the base change $T \otimes_{\mathbb{Q}} E_\pi$. Let $K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{v|p} K_v$, and regard its multiplicative group as a subgroup of the idèle group I_K of K. Note that $K_p^\times = T_{/\mathbb{Q}_p}(\mathbb{Q}_p) \subset T_{/E_\pi}(E_\pi)$. An abelian representation $\rho : G_K^{\mathrm{ab}} \to \mathrm{GL}_d(E_\pi)$ may be thought of as a representation of I_K by means of class field theory. Now ρ is said to be *locally algebraic* (*cf.* [16], III, §2.1, Prop.; [15], I, §5) if there exists a morphism $r : T_{/E_\pi} \to \mathrm{GL}_{d/E_\pi}$ of algebraic groups defined over E_π such that

$$\rho(x) \;=\; r(x^{-1})$$

for all $x \in K_p^\times$ close enough to 1.

Let $\mathfrak{m} = (m_v)_{v\in S}$ be a modulus of K, *i.e.*, a family of positive integers m_v for v in a finite set S of finite places of K (we set $m_v = 0$ if $v \notin S$). We say that $\mathfrak{m}$ is a *modulus of definition* for an abelian locally algebraic representation ρ if:

(1) ρ is trivial on $U_{v,\mathfrak{m}}$ for each $v \in S$ with $v \nmid p$; and
(2) $\rho(x) = r(x^{-1})$ for $x \in U_{p,\mathfrak{m}}$.

Here, $U_{v,\mathfrak{m}}$ is the group of units u of $K_v^\times$ such that $u \equiv 1$ modulo the m_vth power of a uniformizer of K_v if v is a finite place (resp. it is the connected component of $K_v^\times$ if v is an infinite place), $U_{p,\mathfrak{m}} := \prod_{v|p} U_{v,\mathfrak{m}}$, and $r : T_{/E_\pi} \to \mathrm{GL}_{d/E_\pi}$ is as above.

An abelian representation ρ is locally algebraic with a modulus of definition $\mathfrak{m}$ if and only if it becomes semi-stable when restricted to $U_{p,\mathfrak{m}}$ (in fact, crystalline in this case (*cf.* Proposition in §6 of [5]); it even looks like a product of representations coming from Lubin-Tate groups (*cf.* the last part of §2 below)). Thus, for a set of abelian geometric (or locally algebraic) representations, the following are equivalent:

(i) they have bounded inertial levels;
(ii) they have bounded moduli of definition.

We shall later reduce the proof of the Theorem to the case of abelian geometric representations. Then, by the above equivalence, we may work with locally algebraic representations with bounded moduli of definition (rather than geometric representations with bounded inertial level).

Put $U_{\mathfrak{m}} = \prod_v U_{v,\mathfrak{m}}$ (v running through all the places of K), $I_{\mathfrak{m}} = I_K/U_{\mathfrak{m}}$, and $T_{\mathfrak{m}}$ the quotient of T by the Zariski closure of the group of units of K which belong to $U_{\mathfrak{m}}$. Let $C_{\mathfrak{m}} = I_K/(K^\times U_{\mathfrak{m}})$ be the ray class group of K of modulus $\mathfrak{m}$. Then there exist ([16], II, §2.2) a commutative algebraic group $S_{\mathfrak{m}}$ over $\mathbb{Q}$ sitting in an exact sequence

$$1 \to T_{\mathfrak{m}} \to S_{\mathfrak{m}} \to C_{\mathfrak{m}} \to 1$$

and a group homomorphism $\varepsilon : I_{\mathfrak{m}} \to S_{\mathfrak{m}}(\mathbb{Q})$ which make the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times & \longrightarrow & I_{\mathfrak{m}} & \longrightarrow & C_{\mathfrak{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \varepsilon & & \| & & \\ 1 & \longrightarrow & T_{\mathfrak{m}}(\mathbb{Q}) & \longrightarrow & S_{\mathfrak{m}}(\mathbb{Q}) & \longrightarrow & C_{\mathfrak{m}} & \longrightarrow & 1 \end{array}$$

A representation of $S_{\mathfrak{m}}$ gives rise to a representation of G_K as follows: Let α_π be the composite map

$$\alpha_\pi : I_K \to T(\mathbb{Q}_p) \to S_{\mathfrak{m}}(\mathbb{Q}_p) \hookrightarrow S_{\mathfrak{m}}(E_\pi),$$

where the first arrow is the projection of I_K to its pth factor $T(\mathbb{Q}_p)$, and the latter two arrows are the canonical maps. On the other hand, ε may be regarded as a map $\varepsilon : I_K \to S_{\mathfrak{m}}(E_\pi)$. Then the product $\varepsilon_\pi := \varepsilon\alpha_\pi^{-1}$ is trivial on $K^\times$, and defines a continuous homomorphism from the idèle class group to $S_{\mathfrak{m}}(E_\pi)$. By topological reasons, it descends to a map $\varepsilon_\pi : G_K^{\mathrm{ab}} \to S_{\mathfrak{m}}(E_\pi)$. Now each representation $\phi : S_{\mathfrak{m}/E_\pi} \to \mathrm{GL}_{d/E_\pi}$ gives rise to a representation $\rho := \phi \circ \varepsilon_\pi : G_K^{\mathrm{ab}} \to \mathrm{GL}_d(E_\pi)$, which is locally algebraic by construction.

The following theorem ([16], III, §2.3, Th. 2 and [15], I, §6, Th. (MT 1)) will play an essential role in the proof of our Theorem.

Theorem. *Any abelian locally algebraic E-rational representation $\rho : G_K^{\mathrm{ab}} \to \mathrm{GL}_d(E_\pi)$ with a modulus of definition $\mathfrak{m}$ arises from a unique semisimple representation $\phi : S_{\mathfrak{m}} \to \mathrm{GL}_d$, defined over E, of the algebraic group $S_{\mathfrak{m}}$, in such a way that $\rho = \phi_{/E_\pi} \circ \varepsilon_\pi$.*

2. Proof of the Theorem. In this section, we prove our main Theorem. The following proposition is a key to the proof; it is a version of Jordan's theorem ([6], [9]; *cf.* also [1], Prop. 2.3).

Proposition. *Let k be a field of characteristic zero. Then there exists a constant J, depending only on d, such that any potentially toric subgroup G of $\mathrm{GL}_d(k)$ has a normal abelian subgroup A with $(G : A) \leq J$.*

Here we mean by a *potentially toric* subgroup of $\mathrm{GL}_d(k)$ a subgroup G which has a subgroup H of finite index whose Zariski closure in GL_d is a torus. This is equivalent to saying that the connected component $\mathbb{G}^{\mathrm{o}}$ of the Zariski closure $\mathbb{G}$ of G is a torus. For example, the image of a potentially abelian representation $\rho : G_K \to \mathrm{GL}_d(E_\pi)$ is potentially toric if it is locally algebraic.

Proof. We may and do assume that k is algebraically closed. Given a potentially toric subgroup G, let $\mathbb{G}$ be its Zariski closure in the k-algebraic group GL_d, and $\mathbb{G}^{\mathrm{o}}$ its connected component of the identity element. By assumption, $\mathbb{G}^{\mathrm{o}}$ is a torus. The quotient $\mathbb{G}/\mathbb{G}^{\mathrm{o}}$ is a finite algebraic group. We have the the following sequence of maps:

$$(*) \qquad G \;\to\; \mathbb{G}(k)/\mathbb{G}^{\mathrm{o}}(k) \;\hookrightarrow\; (\mathbb{G}/\mathbb{G}^{\mathrm{o}})(k).$$

The latter map is injective (in fact, it is bijective since we assumed k to be algebraically closed (*cf.* [2], II, §6.8, Caution)).

Lemma. *There exists an integer $N \geq 1$, depending only on d, such that, for any subtorus $\mathcal{Q}$ of GL_d, if $\mathcal{N}$ denotes its normalizer in GL_d, there exists an immersive morphism $\mathcal{N}/\mathcal{Q} \hookrightarrow \mathrm{GL}_N$ of algebraic groups.*

Proof. Let $\mathcal{C}$ be the centralizer of $\mathcal{Q}$ in GL_d. It is a reducive group ([3], Exp. XIX, §1.6) of rank $\leq d$. The order m of the Weyl group $\mathcal{N}/\mathcal{C}$ of GL_d with respect to $\mathcal{Q}$ is bounded in terms of d, because it is a finite group acting faithfully on the character group $X(\mathcal{Q})$ of $\mathcal{Q}$, which is a free $\mathbb{Z}$-module of rank $\leq d$. Put $\overline{\mathcal{N}} = \mathcal{N}/\mathcal{Q}$ and $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{Q}$. The latter is again reductive of rank $\leq d$. Let $\overline{\mathcal{T}}$ be a maximal torus of $\overline{\mathcal{C}}$, and W the Weyl group of $\overline{\mathcal{C}}$ with respect to $\overline{\mathcal{T}}$. The representation theory of the split reductive group $\overline{\mathcal{C}}$ may be identified with the "W-symmetric part" of that of $\overline{\mathcal{T}}$ (*e.g.* [17], §3.6, Th. 4). Thus any finite subset S of the character group $X(\overline{\mathcal{T}})$ of $\overline{\mathcal{T}}$ which contains a set of basis of $X(\overline{\mathcal{T}})$ and stable under the action of W gives rise to a faithful representation of $\overline{\mathcal{C}}$ of dimension $|S|$ (i.e. immersive morphism $\overline{\mathcal{C}} \to \mathrm{GL}_{|S|}$ of algebraic groups). One can take such an S of cardinality at most $\mathrm{rank}_{\mathbb{Z}} X(\overline{\mathcal{T}}) \cdot |W|$, which is bounded by an integer n depending only on d. Now we have a faithful representation $\rho : \overline{\mathcal{C}} \hookrightarrow \mathrm{GL}_n$, whence $\mathrm{Ind}_{\overline{\mathcal{C}}}^{\overline{\mathcal{N}}} \rho : \overline{\mathcal{N}} \hookrightarrow \mathrm{GL}_{mn}$. Q.E.D.

By the Lemma applied with $\mathcal{Q} = \mathbb{G}^{\mathrm{o}}$, we can find an integer N, depending only on d, and an embedding $\mathbb{G}/\mathbb{G}^{\mathrm{o}} \hookrightarrow \mathrm{GL}_N(k)$. By Jordan's theorem ([6], [9]), there exists an integer J, depending only on N, such that any finite subgroup of $\mathrm{GL}_N(k)$ has a normal abelian subgroup C of index $\leq J$. Apply this to $\mathbb{G}/\mathbb{G}^{\mathrm{o}} \subset \mathrm{GL}_N(k)$. Then the inverse image of C in $\mathbb{G}$ is commutative, being an extension of a finite commutative group by a torus in characteristic zero (*cf.* [16], II, §1.3, Rem. 2). Hence the inverse image A of C in G by the map (*) is an abelian normal subgroup of index $\leq J$. Q.E.D.

Now we turn to the proof of the Theorem. It is known that the validity of the Finiteness Conjecture of Fontaine-Mazur is unchanged by replacing the base field K by a finite extension ([5], §4 (b)). By the above Proposition, all representations ρ under consideration become abelian after replacing K by a finite extension of degree $\leq J$ which is unramified outside a fixed finite set of places of K (= the set where ρ's are allowed to ramify). By the Hermite-Minkowski theorem, there exist only finitely many such extensions. Thus we may assume all the ρ's are abelian.

By assumption of the Theorem, our ρ's are locally algebraic and have moduli of definition bounded by some fixed modulus $\mathfrak{m}$. Such a ρ arises form a unique representation $\phi : S_{\mathfrak{m}} \to \mathrm{GL}_d$, defined over $\overline{\mathbb{Q}}$, of the algebraic group $S_{\mathfrak{m}}$ (§1, Theorem), which is in fact defined over a *fixed* finite extension E of $\mathbb{Q}$ (*e.g.* the splitting field of the algebraic group $S_{\mathfrak{m}}$ of mutiplicative type; *cf.* [16], II, §1.3, Rem. 2); note in passing that our representations $\rho : G_K \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ are all defined over the topological closure E_π of E in $\mathbb{C}_p$. It is now enough to show that the possibilities of such ϕ are finite.

Since the ray class group $C_{\mathfrak{m}} = S_{\mathfrak{m}}/T_{\mathfrak{m}}$ is finite, the possibilities of extensions of a representation $\varphi : T_{\mathfrak{m}} \to \mathrm{GL}_d$ to the semisimple group $S_{\mathfrak{m}}$ is finite. Hence it is enough to show that the possibilities of $\phi|_{T_{\mathfrak{m}}} : T_{\mathfrak{m}} \to \mathrm{GL}_d$ are finite. This follows from the fact that a linear representation of a torus is determined by its weights ([18], Th. 7.2), and that the weights of $\phi|_{T_{\mathfrak{m}}}$ are determined by the Hodge-Tate weights of $\rho : G_K \to \mathrm{GL}_d(E_\pi)$ viewed as a $\mathbb{Q}_p$-linear representation. The last fact should be well-known to experts, but we include a proof here for the sake of completeness.

Since the representations ρ and ϕ are semisimple (*cf.* [16], II, §2.5), we may assume that they are in fact simple and are given by homomorphisms $\rho : G_K \to {E_\pi}^\times$ and $\phi : S_{\mathfrak{m}} \to \mathrm{GL}_{1/E} \subset \mathrm{GL}_{e/\mathbb{Q}}$, with $e := [E : \mathbb{Q}]$. For our purpose, we may replace K again by a finite extension to assume that K contains all the conjugates of E over $\mathbb{Q}$ (but we do not need to change E and E_π, since the objects (such as $T_{\mathfrak{m}}$ and its representations) for the new K are the base change lifts (= pull-backs by the norm map) of the old ones (*cf.* [16], III, §1.1)).

Suppose $\phi|_{T_{\mathfrak{m}}}$ is equivalent over $\overline{\mathbb{Q}}$ to the diagonal representation with diagonals $\psi_1, ..., \psi_e$, where ψ_i are characters of $T_{\mathfrak{m}}$. Thus $\phi|_{T_{\mathfrak{m}}}$ is identified with one of them via the tacitly fixed embedding $E \hookrightarrow \overline{\mathbb{Q}}$. Each $\psi = \psi_i$ is of the form

$$\psi \;=\; \prod_{\sigma\in\Gamma_K} [\sigma]^{n_\sigma},$$

where $\Gamma_K = \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}_p)$, $n_\sigma \in \mathbb{Z}$, and $[\sigma]$ is the character of $T_{\mathfrak{m}}$ induced by $\sigma : K^\times \to \mathbb{C}_p^\times$. Note that there is a canonical bijection $\Gamma_K \simeq \coprod_{v|p} \Gamma_{K_v}$ if we set $\Gamma_{K_v} = \mathrm{Hom}_{\mathbb{Q}_p}(K_v, \mathbb{C}_p)$.

Fix a place v of K lying above p. Think of ρ as a representation of the idèle group I_K by means of class field theory. Since ρ comes from ϕ (as described in §1) with $\phi|_{T_{\mathfrak{m}}}$ of the above form, its restriction to the subgroup $U_{v,\mathfrak{m}}$ has the following form:

$$(**) \qquad \rho|_{U_{v,\mathfrak{m}}} \;=\; \prod_{\sigma\in\Gamma_{K_v}} \sigma^{-n_\sigma}.$$

Here, σ is thought of as a homomorphism $\sigma : U_{v,\mathfrak{m}} \to \overline{\mathbb{Q}}_p^\times$.

On the other hand, the Hodge-Tate weights of $\rho : G_K \to {E_\pi}^\times$ can be "seen" as follows: By Theorem 2 of the Appendix to Chapter III of [16], the restriction of ρ to G_{K_v} coincides, modulo a character of finite order, with

$$\prod_{\tau\in\Gamma_{E_\pi}} \tau^{-1} \circ \chi_{\tau E_\pi}^{h_\tau}$$

for some $h_\tau \in \mathbb{Z}$, where $\chi_{\tau E_\pi} : G_{\tau E_\pi} \to \tau E_\pi^\times$ is the character of $G_{\tau E_\pi}$ (and its restriction to G_{K_v}) which describes the $G_{\tau E_\pi}$-action on the Tate module of the Lubin-Tate group associated to a uniformizer of τE_π, and $\tau^{-1} \circ \chi_{\tau E_\pi}$ is the composite map $G_{K_v} \xrightarrow{\text{restr.}} G_{\tau E_\pi} \xrightarrow{\chi_{\tau E_\pi}} \tau E_\pi^\times \xrightarrow{\tau^{-1}} {E_\pi}^\times$. (Though $\chi_{\tau E_\pi}$ may depend on the choice of the uniformizer of τE_π, its restriction to the inertia group does not.) The integers $(h_\tau)_{\tau \in \Gamma_{E_\pi}}$ appearing here are the Hodge-Tate weights of ρ (*cf.* Th. 2, *loc. cit.*).

By Lubin-Tate theory and local class field theory, the map $\chi_{\tau E_\pi}$ restricted to the inertia subgroup of G_{K_v} is identified with the inverse of the norm map $N_{K_v/\tau E_\pi}$. Now our ρ coincides on an open subgroup of U_v with

$$\prod_{\tau \in \Gamma_{E_\pi}} \tau^{-1} \circ N_{K_v/\tau E_\pi}^{-h_\tau} \;=\; \prod_{\tau \in \Gamma_{E_\pi}} \tau^{-1} \circ \prod_{\sigma} \sigma^{-h_\tau}$$

(where the last product is over $\sigma \in \mathrm{Hom}_{\tau E_\pi}(K_v, \mathbb{C}_p)$)

$$= \prod_{\sigma \in \Gamma_{K_v}} \sigma^{-h_\tau} \qquad \text{with} \;\; \tau = \sigma|_{E_\pi}.$$

Comparing this with (**) and noticing the multiplicative linear independence of the characters σ on any open subgroup of U_v, we have

$$n_\sigma \;=\; h_{\sigma|_{E_\pi}}$$

(in particular, n_σ depends only on $\sigma|_{E_\pi}$). Thus the Hodge-Tate weights $(h_\tau)_{\tau \in \Gamma_{E_\pi}}$ of ρ at v are determined by the weights $(n_\sigma)_{\sigma \in \Gamma_{K_v}}$ of $\phi|_{T_\mathfrak{m}}$, and vice versa. This completes the proof of the Theorem.

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Graduate School of Mathematics
Kyushu University 33
Fukuoka 812-8581, Japan

taguchi@math.kyushu-u.ac.jp