しきい値モデルの漸近挙動の解析

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1 しきい値モデル

しきい値グラフ $G_n = (V_n, E_n)$ は，n 個のラベル付けされた頂点 $V_n = \{1, 2, \ldots, n\}$ からなる有限単純グラフである．辺集合 E_n を定義するために，n 個の実数（重み）$x_1, x_2, \ldots, x_n$ と，しきい値 $\theta \in \mathbb{R}$ を用意する．しきい値グラフでは，各頂点 $i \in V_n$ に重み x_i を対応させ，異なる 2 頂点 $i, j \in V_n$ はそれぞれの重みの和がしきい値を超えたときに限って辺で結ばれる．即ち，$E_n = \{(i,j) : x_i + x_j > \theta,\ 1 \le i < j \le n\}$ と定義する．しきい値グラフには何種類もの同値な定式化が存在することが知られている（例えば [8] を参照）．ここで，重みを独立同分布に従う確率変数列 $X_1, X_2, \ldots, X_n$ で与え，従って，辺集合を（確率的に）$E_n = \{(i,j) : X_i + X_j > \theta,\ 1 \le i < j \le n\}$ と定義した，ランダムなしきい値グラフをしきい値モデルと呼ぶことにする．このモデルは，Caldarelli *et al.* [1] と Söderberg [11] によって導入されたネットワークモデルの特別な場合であり，種々の特性量が調べられている（例えば [2–7, 9, 10] を参照）．特に，重みを与える確率変数の分布が指数分布やパレート分布の場合に，その次数分布がベキ則に従うこと（スケールフリー性）が示されており，ここ 10 年ほどの間で活発に研究されている，複雑ネットワークの分野で注目されているモデルの 1 つである．

本講演では，しきい値モデルの特性量のうち，特に，次数・クラスター係数・平均頂点間距離の 3 つについて，確率論的な立場から解析した結果 [3, 4, 6, 7] のうち，概収束定理を紹介する．講演の最後に，最近得られた次数分布に関する弱収束定理を紹介する．以下では，重みを与える確率変数の分布関数を F とする．即ち，任意の $x \in \mathbb{R}$ に対して，$F(x) = \mathbb{P}(X_1 \le x)$ と定義する．

2 極限定理

まず，頂点 i の次数 $D_n(i)$ と頂点 i を含む三角形の個数 $T_n(i)$ を以下で定義する．

$$D_n(i) \equiv \sum_{\substack{1 \le j \le n \\ j \ne i}} I_{\{X_i + X_j > \theta\}}, \quad T_n(i) \equiv \sum_{\substack{1 \le j < k \le n \\ j,k \ne i}} I_{\{X_i + X_j > \theta\}} \cdot I_{\{X_j + X_k > \theta\}} \cdot I_{\{X_k + X_i > \theta\}}.$$

さらに，不定元 w を用いて，頂点 i のクラスター係数 $C_n(i)$ を以下で定義する．

$$C_n(i) \equiv \frac{T_n(i)}{\binom{D_n(i)}{2}} \cdot I_{\{D_n(i) \ge 2\}} + w \cdot I_{\{D_n(i) = 0,1\}}.$$

このとき，次数・クラスター係数に関して以下が成り立つ．

定理 1 (次数に関する概収束定理).

$$\lim_{n \to \infty} \frac{D_n(1)}{n-1} = D(1) \equiv 1 - F(\theta - X_1), \quad a.s. \tag{1}$$

$$(2) \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} D_n(i) = \mathbb{E}\left[D(1)\right] = \mathbb{P}\left(X_1 + X_2 > \theta\right), \quad a.s.$$

定理 2 (クラスター係数に関する概収束定理).

$$\lim_{n\to\infty} C_n(1) = C(1) \equiv \frac{T(1)}{D(1)^2} \cdot I_{\{D(1)>0\}} + w \cdot I_{\{D(1)=0\}}, \quad a.s.$$

ここで，任意の $x \in \mathbb{R}$ に対して $T(1;x) \equiv \mathbb{P}(x + X_2 > \theta, X_2 + X_3 > \theta, X_3 + x > \theta)$ と定義するとき，$T(1) \equiv T(1;X_1)$ である．さらに，

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} C_n(i) = \mathbb{E}\left[C(1)\right], \quad a.s.$$

グラフの平均頂点間距離 L_n は，グラフの距離 $d(u,v)$ の可能な頂点対に関する相加平均で定義される量である．すなわち，平均頂点間距離 L_n を

$$L_n = \frac{1}{\binom{n}{2}} \sum_{1\le i<j\le n} d(i,j)$$

と定義する．但し，非連結なグラフでは，2 頂点間の距離が ∞ となる場合もあるが，その場合は不定元 w で置き換えることにする．この L_n に対して，以下が得られる．

定理 3 (平均頂点間距離に関する概収束定理).

$$\lim_{n\to\infty} L_n = 2 - \mathbb{P}\left(X_1 + X_2 > \theta\right) - (2-w)F\left(\theta - x^*\right)\left\{2 - F\left(\theta - x^*\right)\right\}, \quad a.s.$$

但し，$x^* = \min\{x \in \mathbb{R} : F\left(x\right) = 1\}$ である．

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Max-plus Stochastic Control and Risk-sensitivity

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(W.H. Fleming 氏 (Brown Univ., USA)，S.-J. Sheu 氏 (Academia Sinica, Taiwan) との共同研究）

本講演ではノイズが引き起こす最悪のシナリオに対応するための H^∞ 制御を max-plus 確率論（idempotent 確率論）の観点から捉えることにより，ある種の確率制御として展開することを目標とする．まず max-plus 確率論を説明するために，$T > 0$ は予め固定された有限な終末時刻，$t \in [0, T]$ を初期時刻として次の確率微分方程式を考える：

$$dX(s) = f(X(s))ds + \theta^{-1/2}\sigma(X(s))dW(s),\ t \le s \le T,\ X(t) = x \in \mathbb{R}^n.$$

$\{W(s)\}$ は d 次元 Brown 運動，$f : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to M(n, d)$ ($M(n, d)$ は $n \times d$ 行列全体) とする．$\theta > 0$ は大きいパラメータ，すなわち $\theta^{-1/2}$ は小さいとする．与えられた関数 $l : \mathbb{R}^n \to \mathbb{R}$ に対して，Freidlin-Wentzell の大偏差原理により $\theta \to \infty$ としたとき次のような漸近的性質が期待できる：

$$\frac{1}{\theta}\log E_{tx}\left[\int_t^T e^{\theta l(X(s))}ds\right] \to \sup_{v\in L^2[t,T]}\left\{\max_{t\le s\le T} l(x(s)) - \frac{1}{2}\int_t^T |v(s)|^2 ds\right\}\ (\theta \to \infty). \tag{1}$$

ここで $x(s)$ は $v \in L^2[t, T](= L^2([t, T]; \mathbb{R}^d))$ を与えるごとに決まる次の常微分方程式の解である：

$$\frac{dx}{ds}(s) = f(x(s)) + \sigma(x(s))v(s),\ t \le s \le T,\ x(t) = x. \tag{2}$$

H^∞ 制御では $v \in L^2[t, T]$ はノイズとして扱われ，(1) における極限は H^∞ 制御の立場から理解すると，システムの状態 $x(s)$ についての尺度 “$\max_{t\le s\le T} l(x(s))$” にノイズのエネルギー “$\int_t^T |v(s)|^2 ds$” を重みとして付けて，さらにノイズにより引き起こされる最悪のシナリオを sup を採って表現している．この意味において，$\Omega = L^2[t, T]$ をノイズに対応する「標本空間」，$Q(v) = -(1/2)\int_t^T |v(s)|^2 ds$ ($v \in \Omega$）を「確率」と思い，(1) における極限は汎関数 $\Phi(x(\cdot)) = \max_{t\le s\le T} l(x(s))$ の（標本に関する代表値という意味で）「期待値」と考えて次のように表す：

$$E^+_{tx}[\Phi(x(\cdot))] \equiv \sup_{v\in\Omega}\{\Phi(x(\cdot)) + Q(v)\} = \sup_{v\in L^2[t,T]}\left\{\Phi(x(\cdot)) - \frac{1}{2}\int_t^T |v(s)|^2 ds\right\}. \tag{3}$$

E^+_{tx} は max-plus 期待値と呼ばれている ([1] を参照)．本講演では (2), (3) に制御を加味した場合を考え，(3) を制御に関して最小化する問題を考察する．

ところで E^+_{tx} は Ω 上の一般の関数に対しても定義できるが，通常の演算に関して線形性を持たない．しかし，実数に新しい和・積

$$a \oplus b = \max\{a, b\},\ a \otimes b = a + b,\ a, b \in \mathbb{R}^- \equiv \mathbb{R} \cup \{-\infty\}$$

を導入すると，この演算に関して E^+_{tx} は線形性を持つ．$\mathbb{R}^- = \mathbb{R} \cup \{-\infty\}$ において新しい和・積として $\oplus, \otimes$ を考えたものは max-plus algebra と呼ばれ，和に関する逆元の存在以外は環の公理を満たし，(3) に対する制御問題を考えるときこの演算を用いると計算の見通しがよくなることが多い．また，max-plus 期待値に対して条件付き期待値が定義され，tower property など通常の確率論と同様の多くの性質が成り立ち，確率制御と類似のアイデアにより H^∞ 制御を議論できる．

(2) において $U \subset \mathbb{R}^m$ に値を取る制御 $u \in L^\infty([t, T]; U)$ に依存する常微分方程式を考えよう：

$$\frac{dx}{ds}(s) = f(x(s), u(s)) + \sigma(x(s), u(s))v(s),\ t \le s \le T,\ x(t) = x \in \mathbb{R}^n. \tag{4}$$

この方程式の解は制御 $u \in L^\infty([t,T];U)$ とノイズ $v \in L^2[t,T]$ を与えると決定される．制御はノイズの情報に基づいて決めるとし，$\alpha : L^2[t,T] \to L^\infty([t,T];U)$ を用いて $u(s) = \alpha[v](s)$ $(t \le s \le T,\ v \in L^2[t,T])$ と表す．さらに制御はノイズの情報に関して non-anticipating であるべきで，それを微分ゲームにおける Elliott-Kalton 戦略を使って定式化する．本講演における max-plus stochastic control は，次の評価関数を Elliott-Kalton 戦略のサブクラス $\Gamma(t,T)$ で最小化する問題である：

$$\begin{aligned} J(t,x;\alpha) &= E^+_{tx}\left[\int^{\oplus}_{[t,T]} l(x(s),\alpha[v](s))ds\right] \\ &= \sup_{v\in L^2[t,T]}\left\{\int^{\oplus}_{[t,T]} l(x(s),\alpha[v](s))ds - \frac{1}{2}\int_t^T |v(s)|^2 ds\right\}. \end{aligned} \tag{5}$$

ここで $\int^{\oplus}_{[t,T]} l(x(s),\alpha[v](s))ds = \operatorname{ess.sup}_{t\le s\le T} l(x(s),\alpha[v](s))$, $x(s)$ は (4) の $u(s) = \alpha[v](s)$ に対応する解とする．最適制御理論においては (5) の α に関する最小値

$$V(t,x) = \inf_{\alpha\in\Gamma(t,T)} J(t,x;\alpha)$$

を特徴付けることが基本的問題である．本講演では制御理論で重要なフィードバック戦略を含むサブクラス $\Gamma(t,T)$ を新たに提案し，dynamic programming のアイデアを用いて $V(t,x)$ が次の準変分不等式 (quasi-variational inequality) の一意的な粘性解となることを報告する：

$$\min_{u\in U}\max\left\{\frac{\partial V}{\partial t} + H^u(x,\nabla V(t,x)), l(x,u) - V(t,x)\right\} = 0 \text{ in } (0,T)\times\mathbb{R}^n,$$
$$V(T,x) = \min_{u\in U} l(x,u),\ x\in\mathbb{R}^n.$$

ここで $H^u(x,p) = \max_{v\in\mathbb{R}^d}\{(f(x,u)+\sigma(x,u)v)\cdot p - (1/2)|v|^2\}$ とする．

さらに (1) に制御を組み込んだ問題を扱うために，フィルター付き確率空間 $(\Omega,\mathscr{F},P,\{\mathscr{F}_s\})$ において次の確率微分方程式を考える：

$$dX(s) = f(X(s),U(s))ds + \theta^{-1/2}\sigma(X(s),U(s))dW(s),\ t\le s\le T,\ X(t) = x\in\mathbb{R}^n.$$

$\{W(s)\}$ は $\{\mathscr{F}_s\}$-Brown 運動，$U(\cdot) = \{U(s)\}$ は制御過程を表し，U に値を取る $\{\mathscr{F}_s\}$-発展的可測過程とする．評価関数として

$$J_\theta(t,x;U(\cdot)) = E_{tx}\left[\int_t^T e^{\theta l(X(s),U(s))}ds\right]$$

を導入し，これを $U(\cdot) = \{U(s)\}$ に関して最小化する問題

$$\Psi_\theta(t,x) = \inf_{U(\cdot)} J_\theta(t,x;U(\cdot))$$

を考える．ある特別なモデルではこの問題は数理ファイナンスにおける巾効用関数に対する最適投資・消費問題と関連しており，$\theta\to\infty$ としたとき投資家は非常にリスク回避的 (risk-averse) な態度を取ることを意味する．本講演では $\Psi_\theta(t,x)$, およびその対数変換 $V_\theta(t,x) = (1/\theta)\log\Psi_\theta(t,x)$ が満たす偏微分方程式を通じて，粘性解理論における強力な安定性定理を用いることにより

$$V_\theta(t,x) = \frac{1}{\theta}\log\Psi_\theta(t,x) \to V(t,x)\ (\theta\to\infty)$$

が $(0,T)\times\mathbb{R}^n$ 上でコンパクト一様収束の意味で成立することを報告する．

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Perturbation of rough linear differential equations

Antoine Lejay (INRIA)

We show existence and uniqueness of a solution of a perturbation of a rough linear equation, which means a perturbation of a linear equation in the sense of rough paths. This have potential application for studying perturbation of rough differential equations.

Asymptotic expansion of the spectral partition function on Sierpinski carpets

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December 19 (Fri), 2008, 13:30-14:10

In this talk, I will talk about a detailed asymptotic behavior of the eigenvalues of the Laplacian associated with the Brownian motion on the Sierpinski carpet (Figure 1). All the results are valid for **any** *generalized Sierpinski carpets* modeled on $[0,1]^d$, $d \geq 2$, but we confine ourselves to the case of the usual Sierpinski carpet for simplicity.

Let $\{F_i\}_{i\in S}$, $S := \{1,\ldots,8\}$, be a family of similitudes on $\mathbb{R}^2$ as described in Figure 2 below, where the whole square denotes $[0,1]^2$. The Sierpinski carpet K is defined as the self-similar set associated with $\{F_i\}_{i\in S}$, that is, the unique non-empty compact subset of $\mathbb{R}^2$ such that $K = \bigcup_{i\in S} F_i(K)$. Let $V_0 := [0,1]^2 \setminus (0,1)^2$ (Figure 2), which should be regarded as the *boundary of* K: In fact, V_0 is the smallest subset of K that satisfies $F_i(K)\cap F_j(K) = F_i(V_0)\cap F_j(V_0)$ for any distinct $i, j \in S$. As $\#V_0 = \infty$, K is *infinitely ramified*. We also set $V_1 := \bigcup_{i\in S} F_i(V_0)$ (see Figure 3).

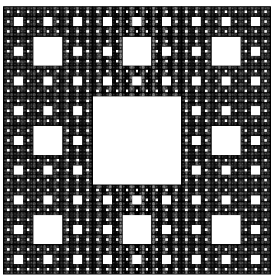

Figure 1 the Sierpinski carpet

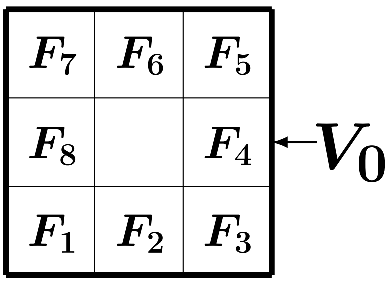

Figure 2

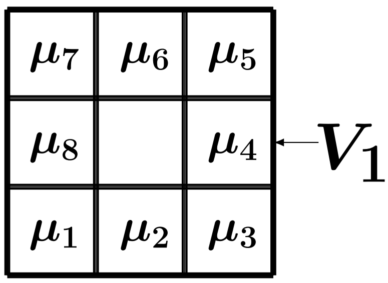

Figure 3

Framework 1 (Barlow-Bass, Kusuoka-Zhou) (1) Let $\boldsymbol{\nu}$ be the self-similar measure on K with weight $(1/8,\ldots,1/8)$, that is, the unique Borel probability measure on K satisfying $\nu(K_w) = (1/8)^{|w|}$ for any $w \in \bigcup_{m\in\mathbb{N}\cup\{0\}} S^m$, where $F_w := F_{w_1}\circ\cdots\circ F_{w_m}$, $K_w := F_w(K)$ and $|w| := m$ for $w = w_1\ldots w_m \in S^m$, $m \in \mathbb{N}\cup\{0\}$.
(2) There exists a (unique good) strong local regular Dirichlet form $\boldsymbol{(\mathcal{E},\mathcal{F})}$ on $L^2(K,\nu)$ such that

$$\boldsymbol{\mathcal{E}(u,v) = \sum_{i\in S}\frac{1}{r}\mathcal{E}(u\circ F_i, v\circ F_i), \quad u,v\in\mathcal{F}} \qquad \textbf{(SSDF)}$$

for some $\boldsymbol{r} \in (0,1)$. By $r < 1$, it can be shown that $\boldsymbol{\mathcal{F} \subset C(K)}$. We also set $\boldsymbol{\tau := 8/r}$.

This Dirichlet space $(\mathcal{L} := (K, S, \{F_i\}_{i\in S}), \nu, \mathcal{E}, \mathcal{F}, r)$ is the framework of our study. Note that $\boldsymbol{\tau}$ **is the *time scaling factor* for the corresponding diffusion process**.

Definition 1 (The eigenvalue counting function and the (spectral) partition function) Let $\{\lambda_n^N\}_{n\in\mathbb{N}}$ be the eigenvalues of the non-negative self-adjoint operator $-A$ (*'Laplacian'*) on $L^2(K,\nu)$ associated with $(\mathcal{E},\mathcal{F})$. We define the *eigenvalue counting function* $\boldsymbol{N_N}$ by $\boldsymbol{N_N(x) := \#\{n\in\mathbb{N} \mid \lambda_n \leq x\}, x \in [0,\infty)}$, and the *(spectral) partition function* $\boldsymbol{Z_N}$ by

$$\boldsymbol{Z_N(t) := \sum_{n\in\mathbb{N}} e^{-\lambda_n t} = \int_{[0,\infty)} e^{-tu}dN_N(u) = \mathrm{Tr}(e^{tA}) = \int_K p_t^N(x,x)d\nu(x), \quad t > 0,} \quad (1)$$

*JSPS Research Fellows DC (20·6088): Supported by Japan Society for the Promotion of Science where $\{p_t^N\}_{t>0}$ denotes the (unique) jointly continuous *heat kernel*, i.e. the integral kernel of e^{tA}.

Let $\mathcal{F}_D := \{u \in \mathcal{F} \mid u|_{V_0} = 0\}$ and $\mathcal{F}_{V_1} := \{u \in \mathcal{F} \mid u|_{V_1} = 0\}$. For $b \in \{D, V_1\}$, let N_b and Z_b be the eigenvalue counting function and the partition function, respectively, associated with $(\mathcal{E}|_{\mathcal{F}_b \times \mathcal{F}_b}, \mathcal{F}_b)$. The following result has been obtained by B. M. Hambly [1]. (Below $\dim_{\mathrm{B}}$ denotes Box-counting dimension with respect to the Euclidean metric, which is also called Minkowski dimension.)

Theorem 2 *There exists a* $\boldsymbol{(\log\tau)}$***-periodic continuous function*** $\boldsymbol{G_0 : \mathbb{R} \to (0,\infty)}$ *such that, for* $b \in \{N, D\}$*, as* $t \downarrow 0$*,*

$$\boldsymbol{Z_b(t) = t^{-d_f/d_w} G_0(\log t^{-1}) + O(t^{-1/d_w}),} \tag{2}$$

where $d_f := \log_3 8 = \dim_{\mathrm{B}} K$ *and* $d_w := \log_3 \tau$*. Note that* $d_f/d_w = \log_\tau 8$ *and* $1 = \dim_{\mathrm{B}} V_0$*.*

Theorem 2 is proved by showing $\boldsymbol{Z_b(t) \asymp t^{-d_f/d_w}}$ (easily proved from the self-similarity) and

$$(0 \le) \boldsymbol{Z_D(t) - Z_{V_1}(t) \le c_1 t^{-1/d_w}}, \quad t \in (0,1], \tag{3}$$

via the sub-Gaussian heat kernel upper bound. Since $\boldsymbol{Z_{V_1} = 8Z_D(\tau t)}$ by the self-similarity of the diffusion (8: the number of cells, τ: time scaling factor), (3) immediately yields

$$\boldsymbol{0 \le t^{d_f/d_w} Z_D(t) - (\tau t)^{d_f/d_w} Z_D(\tau t) \le c_1 t^{(d_f-1)/d_w}, \quad t \in (0,1],} \tag{4}$$

which is the essense of the statement of Theorem 2. Moreover, by having a closer look at the asymptotic behavior of $Z_D(t) - Z_{V_1}(t)$, we have the following main theorem of this talk.

Theorem 3 (K., in preparation) *There exist* $\boldsymbol{(\log\tau)}$***-periodic continuous functions*** $\boldsymbol{G_k :}$ $\mathbb{R} \to \mathbb{R}$*,* $k = 1, 2$ *and* $c \in (0,\infty)$ *such that* $\boldsymbol{G_1 < 0}$ *and, as* $t \downarrow 0$*,*

$$\boldsymbol{Z_D(t) = t^{-d_f/d_w} G_0(\log t^{-1}) + t^{-1/d_w} G_1(\log t^{-1}) + G_2(\log t^{-1}) + O\big(\exp(ct^{-\frac{1}{d_w-1}})\big).} \tag{5}$$

Here the existence of G_1, for example, follows by showing the existence of a periodic function $\boldsymbol{G_1'}$ such that

$$\boldsymbol{Z_D(t) - Z_{V_1}(t) = t^{-1/d_w} G_1'(\log t^{-1}) + O(1)} \quad \text{as } t \downarrow 0, \tag{6}$$

whereas $\boldsymbol{G_1 < 0}$ follows from the fact that $\boldsymbol{Z_D(t) - Z_{V_1}(t) \ge c_2 t^{-1/d_w}}$ $^\exists c_2 > 0$, hence

$$\boldsymbol{t^{-d_f/d_w} G_0(\log t^{-1}) - Z_D(t) \asymp t^{-1/d_w}} \quad \textbf{as } \boldsymbol{t \downarrow 0} \tag{7}$$

(see Kajino [2]). Finally, we remark that the similar (or more detailed) result is valid for any generalized Sierpinski carpet $\boldsymbol{K_{\mathrm{GSC}}} \subset \mathbb{R}^d$, as follows.

Theorem 4 ($^\exists \boldsymbol{G_0 > 0}$***: Hambly [1, Theorem 4.1], the others: K., in preparation)*** *There exist* ***continuous*** $\boldsymbol{(\log\tau)}$***-periodic functions*** $\boldsymbol{G_k :}$ $\mathbb{R} \to \mathbb{R}$*,* $0 \le k \le d$ *and* $c \in (0,\infty)$ *such that* $\boldsymbol{G_0 > 0}$*,* $\boldsymbol{G_1 < 0}$ *and, as* $t \downarrow 0$*,*

$$\boldsymbol{Z_D(t) = \sum_{k=0}^{d} t^{-d_k/d_w} G_k(\log t^{-1}) + O\big(\exp(ct^{-\frac{1}{d_w-1}})\big),} \tag{8}$$

where $d_k := \dim_{\mathrm{B}}\big(K_{\mathrm{GSC}} \cap \{x_1 = \cdots = x_k = 0\}\big)$ *for* $0 \le k \le d$*.*

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ON DOUBLY FELLER PROPERTY

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1. Main Theorem

Let $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ be a strong Markov process on a locally compact separable metric space E. Let ∂ be a point added to E so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of E. The point ∂ also serves as the cemetery point for X. Recall that X is said to have *Feller property* if $P_t(C_\infty(E)) \subset C_\infty(E)$ for every $t > 0$ and $\lim_{t\to\infty} \|P_t f - f\|_\infty = 0$ for every $f \in C_\infty(E)$, where $\{P_t; t \geq 0\}$ defined by $P_t f(x) := \mathbf{E}_x[f(X_t)]$ is the semigroup of X. Here $C_\infty(E)$ is the space of continuous functions on E that vanishes at infinity and $\|f\|_\infty := \sup_{x\in E} |f(x)|$. The space of bounded continuous functions on E will be denoted as $C_b(E)$. The process X is said to have *strong Feller property* if $P_t(\mathcal{B}_b(E)) \subset C_b(E)$ for every $t > 0$. We say X (or its transition semigroup) has *doubly Feller property* if it has both Feller and strong Feller property. Clearly the above terminology can be formulated for any semigroup $\{T_t, t \geq 0\}$ acting on $\mathcal{B}_b(E)$.

Let $\{Z_t, t \geq 0\}$ be a positive multiplicative functional of X. It defines a semigroup

$$T_t f(x) := \mathbf{E}_x\left[Z_t f(X_t)\right] \qquad \text{for } t > 0 \text{ and } f \geq 0. \tag{1}$$

For an open subset B of E, we also define a semigroup T_t^B by

$$T_t^B f(x) := \mathbf{E}_x\left[Z_t f(X_t) : t < \tau_B\right] \qquad \text{for } t > 0 \text{ and } f \geq 0 \text{ on } B, \tag{2}$$

where $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ is the first exit time from B. Let $\mathcal{B}^*(E)$ be the σ-field of universally measurable subsets of E and denote by $\mathcal{B}_b^*(E)$ the family of bounded universally measurable functions on E. Note that $T_t f \in \mathcal{B}^*(E)$ when f is Borel measurable ($T_t f$ is Borel for Borel function f if Z_t is $\mathcal{F}_t^0$-measurable). An open set $B(\subset E)$ is said to be *regular* if $\mathbf{P}_x(\tau_B = 0) = 1$ for any $x \in B^c = E \setminus B$. Fix an open set B.

$$\lim_{t\to 0} \sup_{x\in D} \mathbf{E}_x\left[|Z_t - 1| : t < \tau_D\right] = 0 \quad \text{for any relatively compact open set } D \subset B, \tag{3}$$

$$\sup_{s\in[0,t]} \sup_{x\in B} \mathbf{E}_x[Z_s : s < \tau_B] < \infty \qquad \text{for some (and hence for every) } t > 0, \tag{4}$$

and for $\forall t > 0$, $\exists p > 1$ (which may depend on t) such that

$$\sup_{x\in B} \mathbf{E}_x\left[Z_t^p : t < \tau_B\right] < \infty. \tag{5}$$

The following theorem extends [1].

Theorem 1.1 ([2]). *Let X be a doubly Feller process and B an open set in E. Suppose that* (3) *holds and that*

$$\text{for } \forall t > 0 \text{ and compact set } K \subset B,\ \exists p > 1 \text{ such that } \sup_{x\in K} \mathbf{E}_x\left[Z_t^p : t < \tau_B\right] < \infty. \tag{6}$$

Then $\{T_t^B, t \geq 0\}$ defined by (1) *has strong Feller property. Assume further that* (4) *and* (5) *hold for every $t > 0$, B is regular and that $\lim_{t\to 0} \mathbf{E}_x[|Z_t - 1| : t < \tau_B] = 0$ for every $x \in B$. Then $\{T_t^B, t \geq 0\}$ has Feller property.*

Corollary 1.1 ([2]). *Let X be a doubly Feller process and assume* (3) *holds. Let B an open regular set. Suppose that*

$$(7) \qquad \exists p>1 \text{ such that } \sup_{s\in[0,t]}\sup_{x\in B}\mathbf{E}_x[Z_s^p : s<\tau_B]<\infty \text{ for some (hence every) } t>0.$$

Then $\{T_t^B; t\geq 0\}$ has doubly Feller property. If in addition, B is relatively compact, then $T_t^B g\in C_\infty(B)$ for every $t>0$ and $g\in\mathcal{B}_b(B)$.

2. Feynman-Kac transform

A positive additive functional (PAF in short) A is said to be *of Dynkin class* if $\sup_{x\in E}\mathbf{E}_x[A_t]<\infty$ for some $t>0$, or equivalently, $\sup_{x\in E}\mathbf{E}_x[\int_0^\infty e^{-\alpha t}dA_t]<\infty$ for some $\alpha>0$. A PAF A is said to be *of Kato class* (resp. *of extended Kato class* or *generalized Kato class*) if $\lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x[A_t]=0$ (resp. $\lim_{t\to 0}\sup_{x\in E}\mathbf{E}_x[A_t]<1$); or equivalently, $\lim_{\alpha\to\infty}\sup_{x\in E}\mathbf{E}_x[\int_0^\infty e^{-\alpha t}dA_t]=0$ (resp. $\lim_{\alpha\to\infty}\sup_{x\in E}\mathbf{E}_x[\int_0^\infty e^{-\alpha t}dA_t]<1$). A PAF A is said to be *of local Kato class* if for each compact set K, a PAF $\mathbf{1}_K A$ defined by $(\mathbf{1}_K A)_t=\int_0^t \mathbf{1}_K(X_{s-})dA_s$ is of Kato class. For a PAF A, let $\mathrm{Exp}(A)_t$ be the Stieltjes exponential of A, that is, $\mathrm{Exp}(A)_t$ is the unique solution of Z_t of $Z_t=1+\int_{]0,t]}Z_{s-}dA_s$.

Theorem 2.1. *Let A be a PAF of local and extended Kato class. Suppose that a PAF B defined by $B_t:=\sum_{0<s\leq t}(\Delta A_s)^2$ is of Dynkin class. Put $Z_t:=\mathrm{Exp}(A)_t$. Let B an open regular set. Then* (3) *and* (7) *hold for Z and B. Consequently, $\{T_t^B, t\geq 0\}$ defined by* (2) *has doubly Feller property provided X is doubly Feller.*

3. Girsanov transform

Assume that X is an m-symmetric doubly Feller process, where m is a positive Radon measure on E with full support and that the Dirichlet form $(\mathcal{E},\mathcal{F})$ of X is regular on $L^2(E;m)$. We fix a continuous locally square integrable MAF M^c and a Borel function $\phi: E_\partial\times E_\partial\to\mathbb{R}$ with $\phi(x,y)>-1$ for all $x,y\in E_\partial$ and $\phi(x,x)=0$ for $x\in E_\partial$. We write $N(\psi)(x):=\int_{E_\partial}\psi(x,y)N(x,dy)$, where $N(x,dy)$ is the kernel of Lévy system (N,H). We use $\mu_{\langle M^c\rangle}$ to denote the Revuz measure of $\langle M^c\rangle$. Let S_1 be the class of smooth measures in the strict sense (see [3]).

Lemma 3.1. *Suppose $N(\phi-\log(1+\phi))\mu_H\in S_1$ and assume that $\nu:=N(\phi^2)\mu_H+\frac{1}{2}\mu_{\langle M^c\rangle}$ is a Radon measure of extended Kato class. Then there exists a locally square integrable MAF M^d of purely discontinuous type such that $\Delta M_t^d=\phi(X_{t-},X_t)$ $t\in]0,\infty[$ $\mathbf{P}_x$-a.s.*

Theorem 3.1. *Assume that $\log(1+\phi)$ is bounded on $K\times E$ for each compact set K and $\nu:=N(\phi^2)\mu_H+\frac{1}{2}\mu_{\langle M^c\rangle}$ is a positive Radon measure of local and extended Kato class. Put $Z_t:=\mathrm{Exp}(M)_t$, the solution of Doléan-Dade SDE $Z_t=1+\int_{]0,t]}Z_{s-}dM_s$ for $M_t:=M_t^c+M_t^d$. Then* (3) *and* (7) *hold for Z and for every regular open set B. In particular, Z is a martingale under the conditions. Consequently, $\{T_t^B, t\geq 0\}$ defined by* (1) *has doubly Feller property.*

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ON GENERAL PERTURBATIONS OF SYMMETRIC MARKOV PROCESSES

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1. Main Results

Let $X = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \zeta, \mathbf{P}_x, x \in E)$ be an m-symmetric Hunt process on a locally compact metric space E, whose associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular in $L^2(E; m)$, and that m is a positive Radon measure on E with full topological support. Let $E_\partial := E \cup \{\partial\}$ be the one point compactification of E and Ω the totality of right-continuous, left-limited sample paths from $[0, \infty[$ to E_∂ that hold the value ∂ once attaining it. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \partial\}$ be the life time of X. For a Borel subset B of E, $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ (the *exit time* of B) is an $(\mathcal{F}_t)$-stopping time. The transition semigroup $\{P_t : t \geq 0\}$ of X is defined by $P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t) : t < \zeta]$, $t \geq 0$ Let $(\mathcal{E}, \mathcal{F})$ be the associated Dirichlet form with X. A positive continuous additive functional (PCAF in abbreviation) of X (call it A) determines a measure $\nu = \nu_A$ on the Borel subsets of E. $\int_E f(x)\nu(dx) =\uparrow \lim_{t\to 0} t^{-1}\mathbf{E}_m\left[\int_0^t f(X_s)\, dA_s\right]$, in which $f : E \to [0, \infty]$ is Borel measurable. The measure ν is necessarily *smooth*, in the sense that ν charges no exceptional set and there is an $\mathcal{E}$-nest $\{F_k\}$ of closed subsets of E such that $\nu(F_k) < \infty$ for each $n \in \mathbb{N}$. In the sequel we refer to this bijection between smooth measures and PCAFs as the *Revuz correspondence*, and to ν as the Revuz measure of A^ν. We write $\mathbf{K}(X)$ for the Kato class smooth measures and define $\mathbf{K}_0(X) := \{\nu \in \mathbf{K}(X) \mid \nu(E) < \infty\}$. If M is a locally square-integrable (local) martingale additive functional on $[\![0, \zeta[\![$ (with respect to X), then the process $\langle M \rangle$ (the predictable quadratic variation of M) is a PCAF, and the associated Revuz measure is denoted by $\mu_{\langle M \rangle}$. More generally, if M^u is the martingale part in the Fukushima decomposition of $u \in \mathcal{F}$, then $\langle M^u, M \rangle$ is a CAF locally of bounded variation, and we have the associated Revuz measure $\mu_{\langle M^u, M\rangle}$, which is locally the difference of smooth (positive) measures.

Now let M and $\widehat{M}$ be two locally square-integrable local martingale additive functionals (MAFs) on $[\![0, \zeta[\![$, and let A be a CAF locally of bounded variation with (signed) Revuz measure μ. By the discussion of the preceding paragraph, there is a nest $\{F_k\}$ such that $\mathbf{1}_{F_k}(\mu_{\langle M\rangle} + \mu_{\langle \widehat{M}\rangle} + |\mu|) \in \mathbf{K}_0(X)$ for all n. Our main results concern the form perturbation $\mathcal{Q}$ of $(\mathcal{E}, \mathcal{F})$ defined on $\cup_k \mathcal{F}_{F_k}$ by

$$\begin{aligned}\mathcal{Q}(f, g) \quad &= \quad \mathcal{E}(f, g) - \int_E f(x)\, \mu_{\langle M^g, \widehat{M}\rangle}(dx) - \int_E g(x)\, \mu_{\langle M^f, M\rangle}(dx) - \int_E f(x)g(x)\, \mu(dx) \\ (1) \qquad &\qquad - \int_{E\times E} f(y)g(x)\varphi(x, y)\psi(y, x)N(x, dy)\mu_H(dx).\end{aligned}$$

Here φ and ψ are Borel functions defined on $E \times E$, vanishing on the diagonal and bounded below away from -1 such that $M_t - M_{t-} = \varphi(X_{t-}, X_t)$, $\widehat{M}_t - \widehat{M}_{t-} = \psi(X_{t-}, X_t)$ for every $t \in]0, \zeta[$, $\mathbf{P}_m$-a.e. Let r_t denote the time-reversal operator defined on the path space Ω of X as follows: For $\omega \in \{t < \zeta\}$, $r_t(\omega) := \omega((t - s)-)$ if $0 \leq s < t$ and $r_t(\omega) := \omega(0)$ if $s \geq t$. Now define, for $0 \leq t < \zeta$,

$$(2) \qquad Z_t = \mathrm{Exp}(M_t + A^\mu_t + \langle M^c, \widehat{M}^c\rangle_t) \cdot \mathrm{Exp}(\widehat{M}_t) \circ r_t \cdot (1 + \psi(X_t, X_{t-})),$$

wherein Exp denotes the familiar Doléans-Dade stochastic exponential. Finally, define

$$T_t f(x) := \mathbf{E}_x \left[Z_t f(X_t) \right], \tag{3}$$

One of the purpose of this talk is to establish the following extension of the results in [1].

Theorem 1.1. *Let $\mu_{\langle M\rangle}$, $\mu_{\langle \widehat{M}\rangle}$, and $|\mu|$ be of Hardy class smooth measures, and let $\{F_k\}$ be an $\mathcal{E}$-nest such that $\mathbf{1}_{F_k}\left(\mu_{\langle M\rangle} + \mu_{\langle \widehat{M}\rangle} + |\mu|\right)$ is in the Kato class, for each $k \geq 1$. Suppose that the quadratic form defined on $\cup_k \mathcal{F}_{F_k}$ by (1) is bounded below in the sense that there is a constant $\alpha_0 \geq 0$ such that for every $f \in \cup_k \mathcal{F}_{F_k}$*

$$\mathcal{Q}_{\alpha_0}(f,f) \geq 0. \tag{4}$$

Suppose also that there is a constant K such that for every $f, g \in \cup_k \mathcal{F}_{F_k}$

$$|\mathcal{Q}_{\alpha_0}(f,g)| \leq K \mathcal{Q}_{\alpha_0}(f,f)^{1/2} \mathcal{Q}_{\alpha_0}(g,g)^{1/2}. \tag{5}$$

The formula (3) then defines a strongly continuous semigroup $\{T_t : t \geq 0\}$ of bounded operators on $L^2(E;m)$. Conversely, if $\{T_t, t \geq 0\}$ defined by (3) is a strongly continuous semigroup on $L^2(E;m)$ with $\|T_t\| \leq e^{\alpha_0 t}$ for some $\alpha_0 \geq 0$, then (4) holds, provided $\{F_k, k \geq 1\}$ is an $\mathcal{E}$-nest such that $\mathbf{1}_{F_k}\left(\mu_{\langle M\rangle} + \mu_{\langle \widehat{M}\rangle} + |\mu|\right) \in \mathbf{K}_0(X)$ for every $k \geq 1$.

The following $\mathcal{Q}$ on $\mathcal{F}_b$ is a well-defined symmetric form provided $\mu_{\langle M\rangle}$ is of Hardy:

$$\mathcal{Q}(f,g) = \mathcal{E}(f,g) + \frac{1}{2}\nu_{\langle M^{fg}, M\rangle}(E) \text{ for } f, g \in \mathcal{F}_b, \tag{6}$$

where $\nu_{\langle M^u\rangle} := \mu_{\langle M^u + M^{u,\kappa}\rangle}$. Define $\check{\varphi}(x,y) := \varphi(y,x)$ and $N(\mathbf{1}_{E\times E}\check{\varphi}^2)(x) := \int_E \check{\varphi}(x,y)^2 N(x,dy)$, and write $\mathcal{F}_{F_k,b}$ for the set of bounded elements of $\mathcal{F}_{F_k}$. Suppose $N(\mathbf{1}_{E\times E}\check{\varphi}^2)\mu_H$ is smooth and let K be the local MAF on $[\![0,\zeta[\![$ such that $K_t - K_{t-} = -\varphi(X_{t-}, X_t) - \varphi(X_t, X_{t-})$ $t \in]0,\zeta[$ $\mathbf{P}_m$-a.e. Define $\Lambda(M)_0 = 0$ and $\Lambda(M)_t := -\frac{1}{2}(M_t + M_t \circ r_t + \varphi(X_t, X_{t-}) + K_t)$, $t \in]0,\zeta[$ under $\mathbf{P}_m$.

Theorem 1.2. *Let M be a locally square-integrable MAF on $[\![0,\zeta[\![$ with jump function φ satisfying $M_t - M_{t-} = \varphi(X_{t-}, X_t)$ for $t \in]0,\zeta[$ $\mathbf{P}_m$-a.e. Let $\{F_k\}$ be an $\mathcal{E}$-nest such that*

$$\mathbf{1}_{F_k}\left(\mu_{\langle M\rangle} + N(\mathbf{1}_{E\times E}\check{\varphi}^2)\mu_H\right) \in \mathbf{K}_0(X) \qquad \text{for every } k \geq 1.$$

Then $\mathcal{Q}$ defined by (6) for $f, g \in \bigcup_{k\geq 1} \mathcal{F}_{F_k,b}$ is well-defined. Suppose that $(\mathcal{Q}, \bigcup_{k\geq 1} \mathcal{F}_{F_k,b})$ is bounded below in the sense that there is a constant $\alpha_0 \geq 0$ such that for every $f \in \bigcup_k \mathcal{F}_{F_k,b}$

$$\mathcal{Q}(f,f) + \alpha_0 (f,f)_{L^2} \geq 0. \tag{7}$$

Then the symmetric semigroup $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{\Lambda(M)_t} f(X_t)\right]$ is strongly continuous on $L^2(E;m)$, and the associated quadratic form $(\mathcal{C}, \mathcal{D}(\mathcal{C}))$ is the largest closed symmetric form bounded below by $-\alpha_0$ that is less than $(\mathcal{Q}, \bigcup_{k\geq 1} \mathcal{F}_{F_k,b})$. Conversely, if $\overline{P}_t f(x) := \mathbf{E}_x\left[e^{\Lambda(M)_t} f(X_t)\right]$ is a strongly continuous semigroup on $L^2(E;m)$, then there is some $\alpha_0 \geq 0$ so that (7) holds.

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