Connected Components of Regular Fibers of Differentiable Maps

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§1. Introduction

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We denote by $q_f : M \to W_f$ the **quotient map**.

Stein factorization

There exists a unique continuous map $\bar{f} : W_f \to N$ that makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & {\scriptstyle q_f}\searrow \quad \nearrow{\scriptstyle \bar{f}} & \\ & W_f & \end{array}$$

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Note also that each fiber of q_f corresponds to a connected component of a fiber of f.

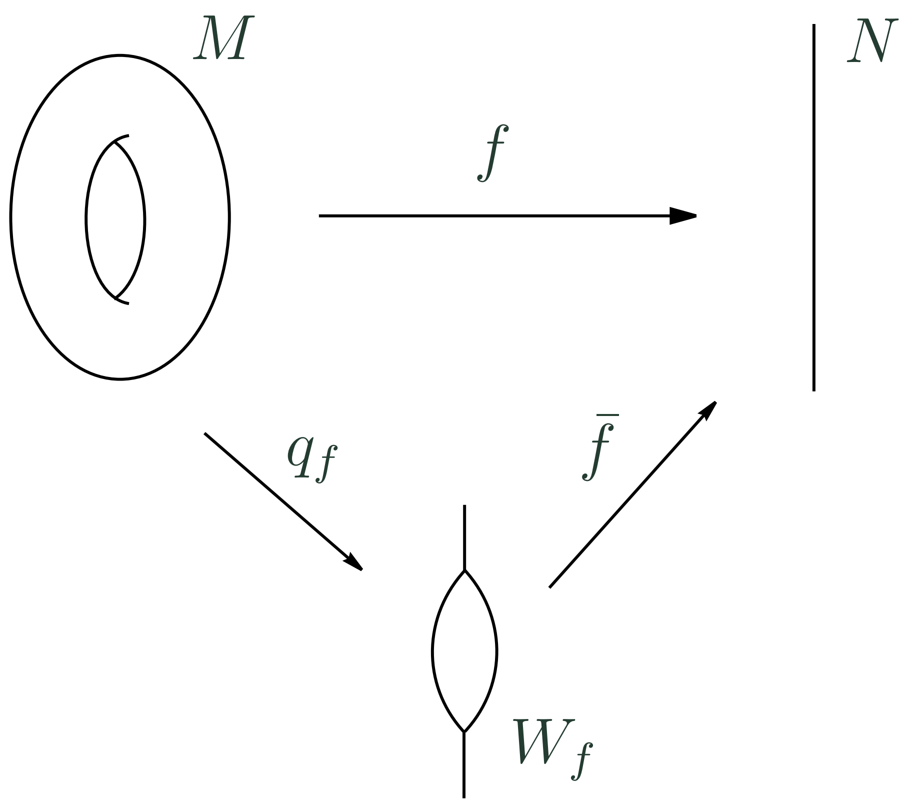

Figure 1: Stein factorization

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there exist **simplicial complexes** K and L,
a **simplicial map** $s : K \to L$, and
homeomorphisms $\lambda : |K| \to X$ and $\mu : |L| \to Y$
such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \lambda \uparrow & & \uparrow \mu \\ |K| & \xrightarrow{|s|} & |L|, \end{array}$$

where $|K|$ and $|L|$ are polyhedrons associated with K and L, respectively, and $|s|$ is the continuous map associated with s.

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We will show that the answer is "Yes" under certain mild conditions.

In the second part, we will apply the result for studying components of regular fibers of generic smooth maps.

§2. Triangulation of Stein Factorization

Lemma 2.1 *Let $s : K \to L$ be a simplicial map.*
We denote by L' the barycentric subdivision of L.
Then, there exists a subdivision K' of K and a simplicial map $s' : K' \to L'$ such that $|s| : |K| \to |L|$ coincides with $|s'| : |K'| \to |L'|$.

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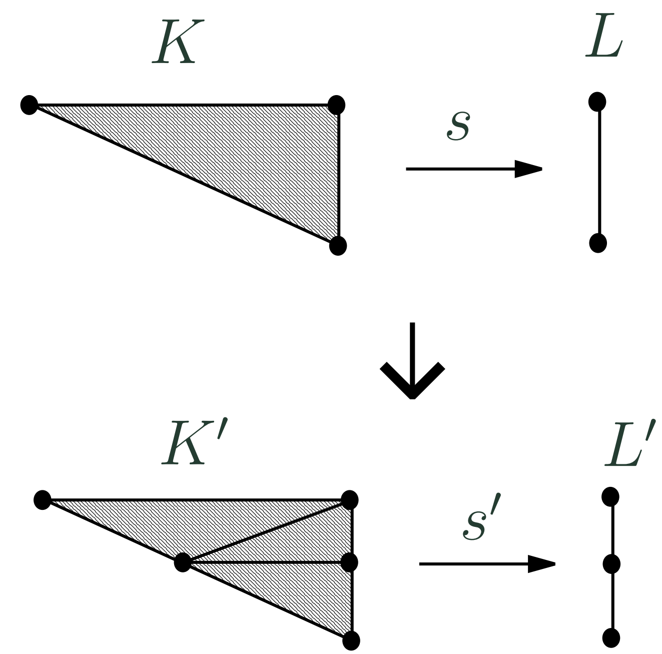

Triangulation of a Stein factorization

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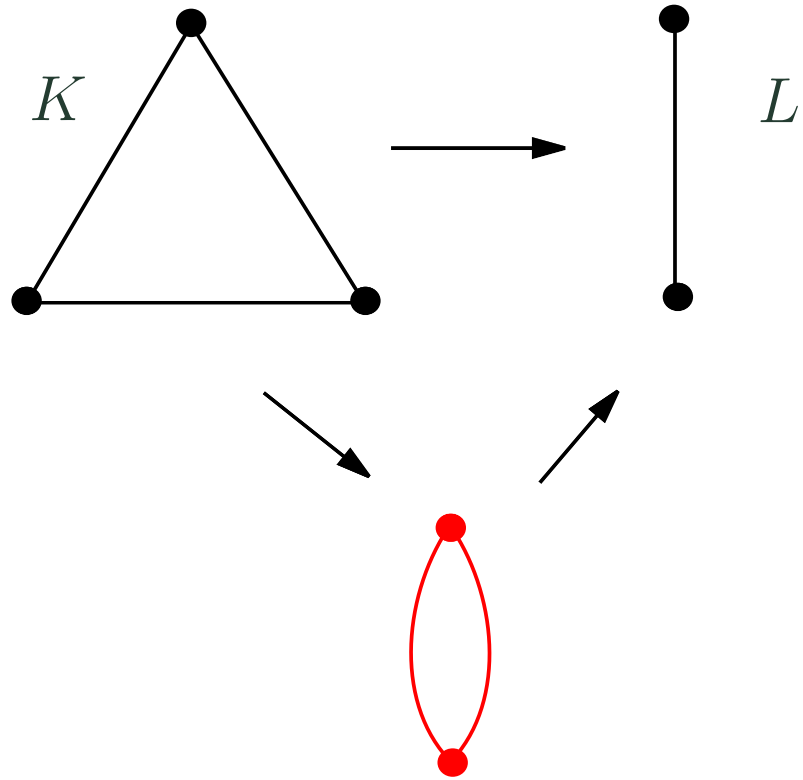
That is, we have the commutative diagram

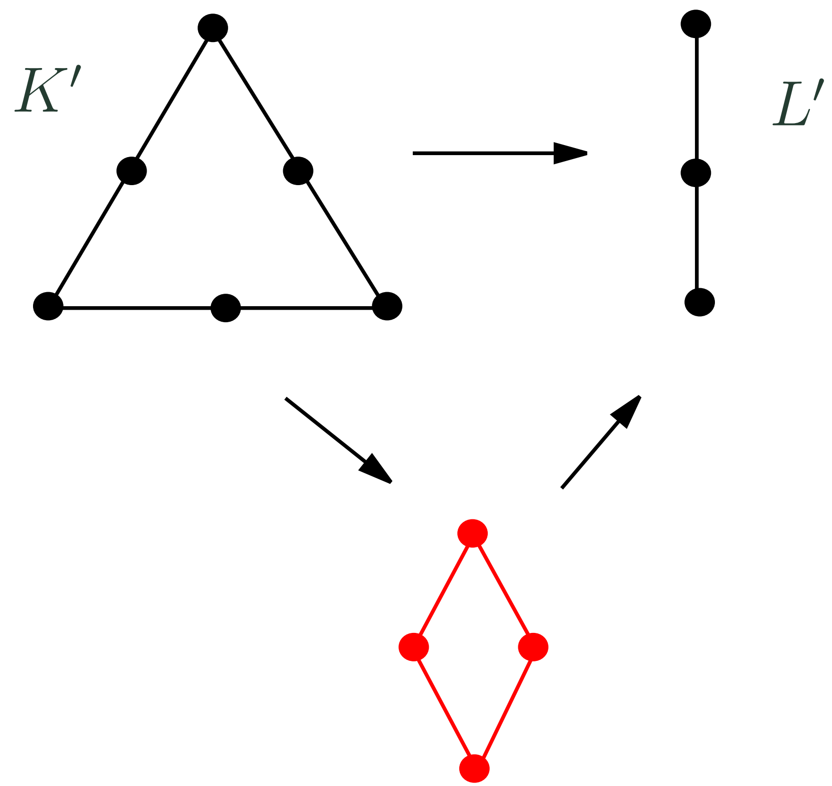
$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "q_g"'] & & Y \\
& W_g \arrow[ur, "\bar{g}"'] & \\
& {|V|} \arrow[u, "\Theta"] \arrow[dr, "|\psi|"] & \\
{|K'|} \arrow[uuu, "\lambda"] \arrow[ur, "|\varphi|"] \arrow[rr, "|s'|"] & & {|L'|} \arrow[uuu, "\mu"']
\end{tikzcd}$$

for some simplicial complex V, simplicial maps $\varphi : K' \to V$, $\psi : V \to L'$, and a homeomorphism Θ, where K', L', s', etc. are as before.

Why barycentric subdivision?



No Good!



OK!

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Corollary 2.4
For smooth manifolds M and N, the set of smooth maps $M \to N$ whose Stein factorization is triangulable contains an open and dense subset of the set of all proper smooth maps $C^\infty(M, N)_{\mathrm{prop}}$ endowed with the Whitney C^∞-topology.

§3. Application

M_0, M_1 : closed oriented manifolds with $\dim M_0 = \dim M_1 = m$.
We say that M_0 and M_1 are **oriented cobordant**
if $\exists$ compact oriented $(m+1)$-dimensional manifold W
such that $\partial W = (-M_0) \cup M_1$,
where $-M_0$ denotes the manifold M_0 with the orientation reversed.

Cobordism of manifolds

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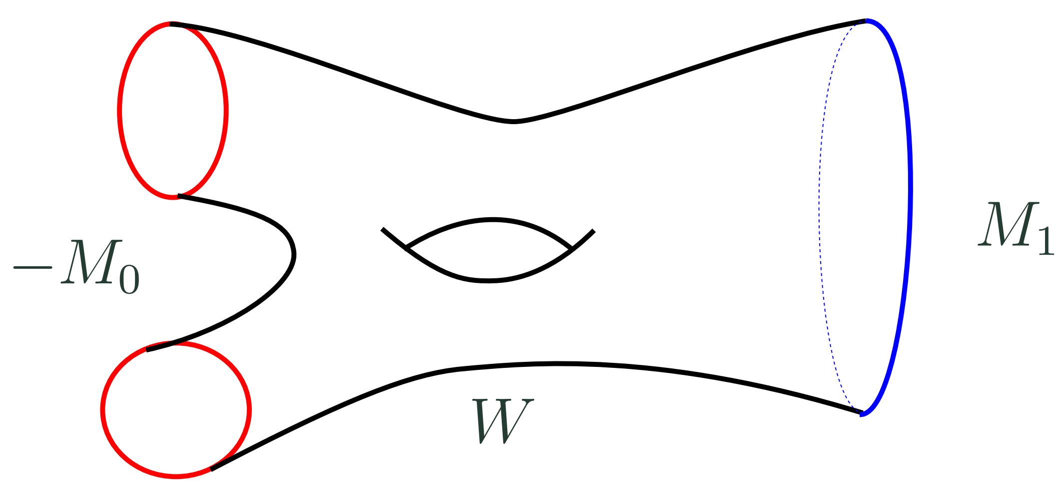

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We can define $[M] + [M'] = [M \cup M']$, so that

$$\Omega_m = \{[M] \mid M \text{ is a closed oriented } m\text{-dim. manifold}\}$$

forms an additive group. This is called the **m-dim. oriented cobordism group**.

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The groups Ω_m and $\mathfrak{N}_m$ have been extensively studied and their structures have been completely determined.

- Ω_m is a finitely generated abelian group.
- $\mathfrak{N}_m$ is a finitely generated $\mathbf{Z}_2$-module.
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dim.	0	1	2	3	4	5	$\cdots$
Ω_*	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\cdots$
$\mathfrak{N}_*$	$\mathbf{Z}_2$	0	$\mathbf{Z}_2$	0	$\mathbf{Z}_2^2$	$\mathbf{Z}_2$	$\cdots$

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A closed manifold M with $[M] = 0$ is said to be **(oriented) null-cobordant**.

Cobordism classes of regular fiber components

M: closed manifold (compact and $\partial M = \emptyset$)
$f : M \to N$ a smooth map with $m = \dim M \geq \dim N = n$.
Assume that f is **triangulable** (e.g. a topologically stable proper map).

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(1) *If a component of a regular fiber of f is not null-cobordant, then $H_n(W_f; \mathbf{Z}_2) \neq 0$.*

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Theorem 3.1
(1) *If a component of a regular fiber of f is not null-cobordant, then $H_n(W_f; \mathbf{Z}_2) \neq 0$.*
(2) *Suppose f is an oriented map (i.e. the regular fibers are consistently oriented). If a component of a regular fiber of f is not oriented null-cobordant, then $H_n(W_f; \Omega_{m-n}) \neq 0$.*

Corollary 3.2

(1) *If $H_n(W_f; \mathbf{Z}_2) = 0$, then every component of every regular fiber of f is null-cobordant.*

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(2) *If* f *is an oriented map and* $H_n(W_f; \Omega_{m-n}) = 0$*, then every component of every regular fiber of* f *is oriented null-cobordant.*

Let $s : K \to L$ be a triangulation of $f : M \to N$.

By Theorem 2.2, we have a triangulation of the Stein factorization:

$$
\begin{tikzcd}
{|K'|} \arrow[rr, "|s'|"] \arrow[dr, "|\varphi|"'] & & {|L'|} \\
& {|V|} \arrow[ur, "|\psi|"'] &
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For each n-simplex $\sigma \in V$, define

$$\omega_\sigma := [|\varphi|^{-1}(b_\sigma)] \in \mathfrak{N}_{m-n},$$

where $b_\sigma \in \sigma$ is the barycenter of σ.

Proof of Theorem 3.1

Let $s : K \to L$ be a triangulation of $f : M \to N$.
By Theorem 2.2, we have a triangulation of the Stein factorization:

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ω_σ : cobordism class of the regular fiber component corresponding to $\sigma \subset |V| = W_f$.

An n-cycle of the quotient space

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$$c_f = \sum_{\sigma} \omega_\sigma \sigma \in C_n(V; \mathfrak{N}_{m-n}),$$

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Lemma 3.3 *$\partial c_f = 0$, i.e. c_f is an n-cycle.*

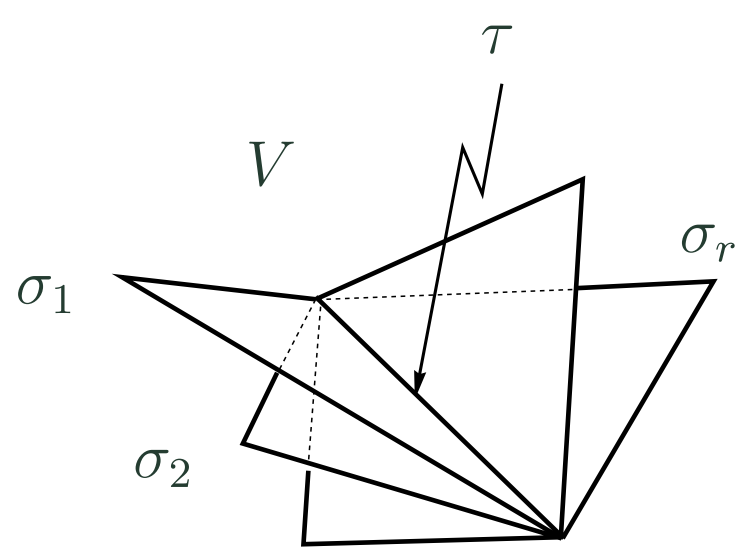
Proof of Lemma 3.3.

Let τ be an arbitrary $(n-1)$-simplex of V, and let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the n-simplices of V containing τ as a face.

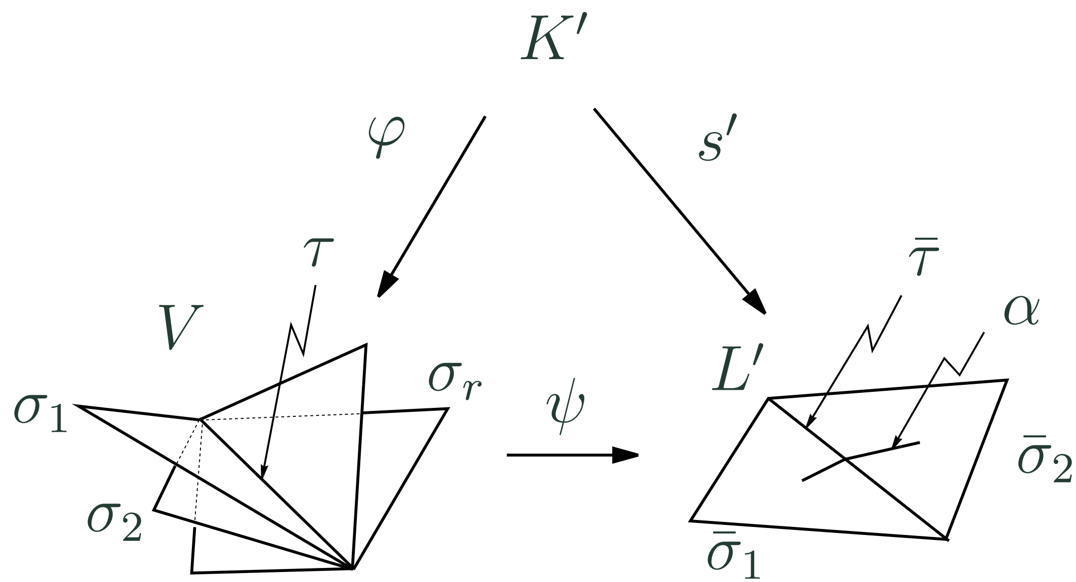
We have only to show

$$\sum_{j=1}^{r} \omega_{\sigma_j} = 0.$$

(The coefficient of τ in ∂c_f.)



Proof of Lemma 3.3



Then, $|s'|^{-1}(\alpha)$ is an $(m-n+1)$-dim. compact manifold and

$$\partial(|s'|^{-1}(\alpha)) = |s'|^{-1}(b_{\bar{\sigma}_1}) \cup |s'|^{-1}(b_{\bar{\sigma}_2}) = \bigcup_{j=1}^{r} |\varphi|^{-1}(b_{\sigma_j}).$$

Therefore, we have $\sum_{j=1}^{r} \omega_{\sigma_j} = \sum_{j=1}^{r} \left[|\varphi|^{-1}(b_{\sigma_j})\right] = 0$ in $\mathfrak{N}_{m-n}$. $\square$

Thus, c_f defines a homology class $\gamma_f \in H_n(W_f; \mathfrak{N}_{m-n})$.

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Furthermore, $c_f \neq 0$ iff there exists a component of a regular fiber which is not null-cobordant.

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Furthermore, $c_f \neq 0$ iff there exists a component of a regular fiber which is not null-cobordant.

Therefore, if such a regular fiber component exists, we have $H_n(W_f; \mathbf{Z}_2) \neq 0$, since $\mathfrak{N}_{m-n} \cong \mathbf{Z}_2 \oplus \cdots \oplus \mathbf{Z}_2$.

The case of an oriented map can be treated similarly. $\square$

Example 1

(1) Let us consider a tree T.

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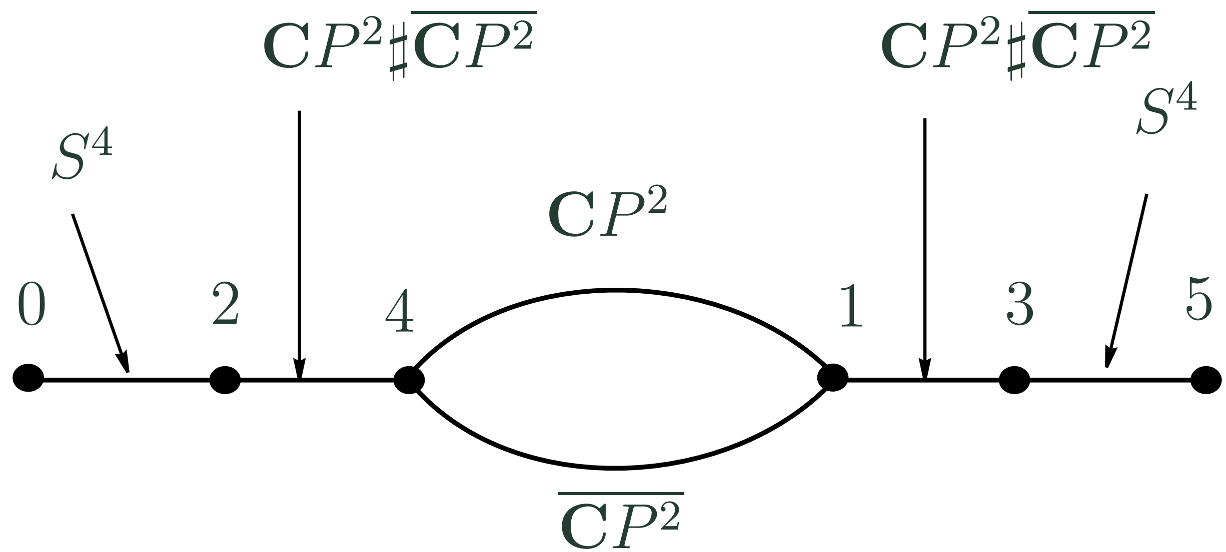
Then, since $H_1(T) = 0$, there exists no Morse function $f_1 : M_1^5 \to \mathbf{R}$ whose quotient space is homeomorphic to T and which has $\mathbf{C}P^2$ as a component of a regular fiber.

(2) $\exists$Morse function $f_2 : M_2^5 \to \mathbf{R}$ whose quotient space is:



The integer at each vertex denotes the index of the corresponding critical point, and the 4-manifold attached to each edge denotes the corresponding regular fiber component.

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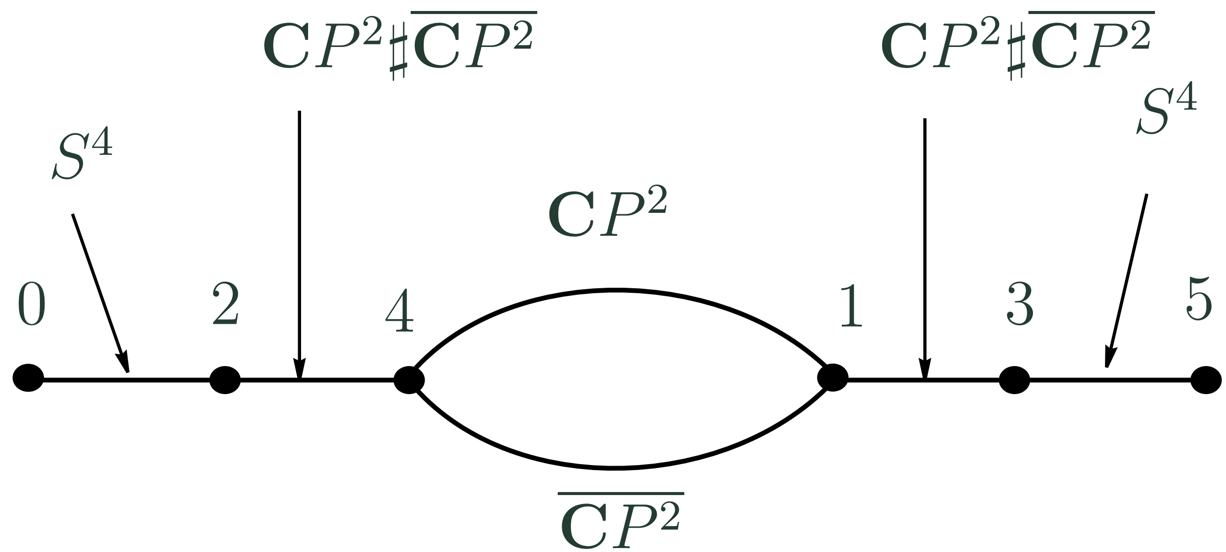
(2) $\exists$Morse function $f_2 : M_2^5 \to \mathbf{R}$ whose quotient space is:



The integer at each vertex denotes the index of the corresponding critical point, and the 4-manifold attached to each edge denotes the corresponding regular fiber component.
Note that $H_1(W_{f_2}; \mathbf{Z}) \cong H_1(W_{f_2}; \Omega_4) \cong \mathbf{Z}$ is generated by γ_{f_2}.

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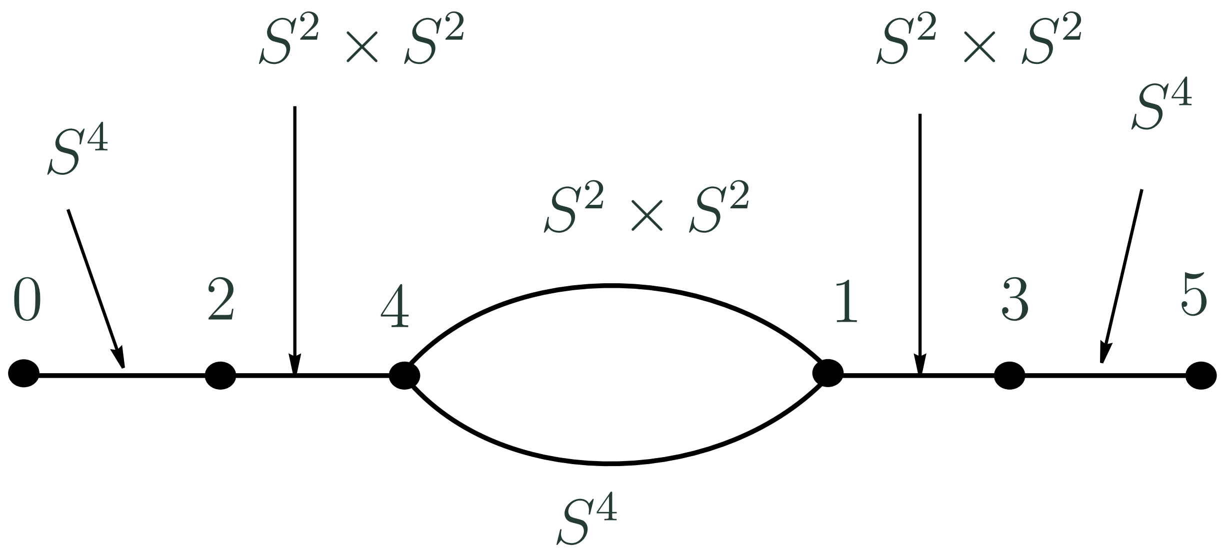
(3) $\exists$Morse function $f_3 : M_3^5 \to \mathbf{R}$ whose quotient space is:



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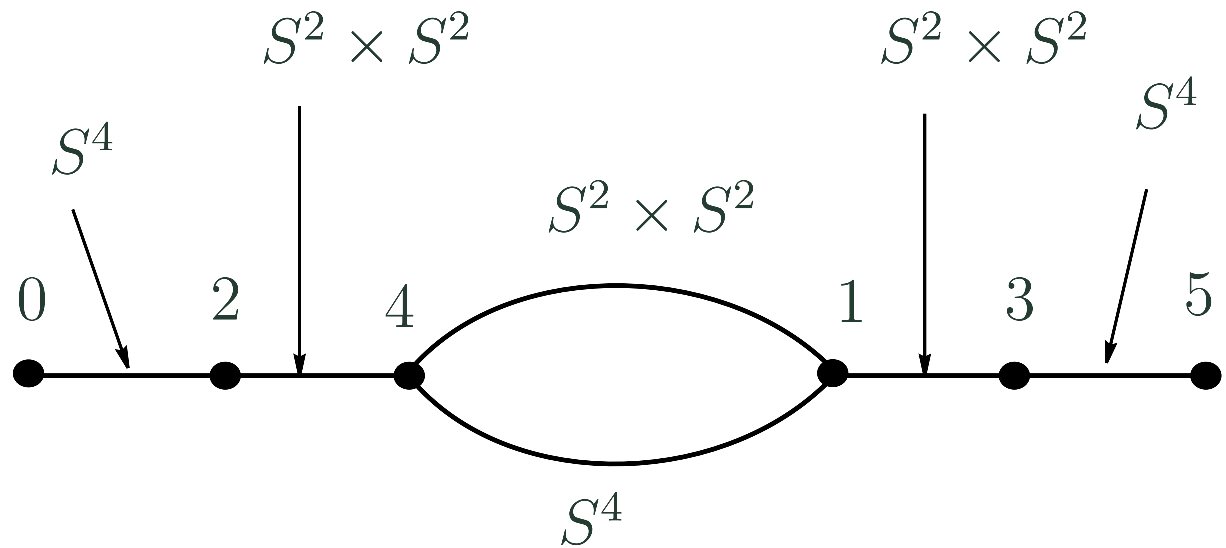

Note that $W_{f_3} \cong W_{f_2}$, but $\gamma_{f_3} = 0$ in $H_1(W_{f_3}; \mathbf{Z}) \cong \mathbf{Z}$, while $\gamma_{f_2} \neq 0$ in $H_1(W_{f_2}; \mathbf{Z})$.

Remark

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For example, consider a stable map $f : \mathbf{C}P^2 \to \mathbf{R}^3$.
Every component of every regular fiber is diffeomorphic to S^1, which is null-cobordant.
However, $\mathbf{C}P^2$ is not null-cobordant.

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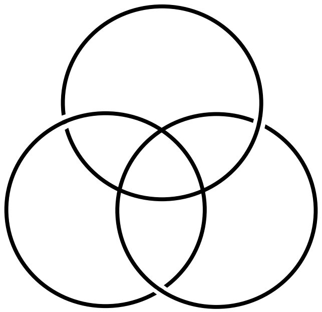
In fact, for a stable map $f : M^4 \to \mathbf{R}^3$, the cobordism class of M^4 is determined by **singular fibers**.



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Problem 3.4
Study such kind of homology classes and their relations to the geometry and topology of the manifolds and the map.

Thank you!