CONNECTED COMPONENTS OF REGULAR FIBERS OF DIFFERENTIABLE MAPS

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Dedicated to Professors Satoshi Koike and Laurentiu Paunescu on the occasion of their sixtieth birthdays

Abstract. For a map between smooth manifolds, the space of the connected components of its fibers is called the Stein factorization. In our previous paper, we showed that for generic smooth maps, the Stein factorizations are triangulable. As an application, we show that every connected component of a regular fiber is null-cobordant if the top dimensional homology of the Stein factorization vanishes.

1. Introduction

Let $f\colon M \to N$ be a generic C^∞ map between smooth manifolds. The space of the connected components of fibers of f is denoted by W_f. Then, we have the canonical quotient map $q_f\colon M \to W_f$ and the natural map $\bar{f}\colon W_f \to N$ such that $f = \bar{f} \circ q_f$. Such a decomposition of f into the composition of q_f and $\bar{f}$ is called the *Stein factorization* of f. Sometimes the quotient space W_f is also called the Stein factorization of f.

It is known that when $\dim M > \dim N$, the Stein factorization of $f\colon M \to N$, or the quotient space W_f, is a very important tool in studying the topological properties of the map f. Refer to [1, 7, 8, 9, 10, 11, 15], for example. In our previous paper [5], the authors have shown that the Stein factorization, and in particular the quotient space W_f, is triangulable for a large class of generic smooth maps f.

In this paper, we use the triangulation of the Stein factorization in order to study the cobordism classes of the components of regular fibers of generic smooth maps. It is known that if the target manifold N of a smooth map $f\colon M \to N$ is connected, then the regular fibers of f are all cobordant. However, the components of regular fibers may not be cobordant to each other. We show that for a generic smooth map $f\colon M \to N$, we can associate a top dimensional homology class $\gamma_f \in H_n(W_f)$, $n = \dim N$, in such a way that if f has a regular fiber component that is not null-cobordant, then γ_f does not vanish, where the coefficient group is the cobordism group of manifolds of dimension $m - n$, $m = \dim M$.

The paper is organized as follows. In §2 we give a precise definition of the Stein factorization of a continuous map between topological spaces and its triangulation. We also recall the cobordism group of manifolds, and state our main theorem. In §3 we define the homology class γ_f and prove our main theorem. We also give some enlightening examples. In §4, we show that the above homology class γ_f gives a bordism invariant for maps whose fibers are connected. We also give some observations which show that the cobordism classes of regular fibers have little

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relation to the cobordism class of the source manifold in general. Finally we give some related problems.

Throughout the paper, we will often abuse the terminology "simplicial complex" (or "simplicial map") to indicate the corresponding polyhedron (resp. PL map). The symbol "$\approx$" denotes a homeomorphism between topological spaces.

2. Preliminaries

In this section, we define the notion of a triangulation of the Stein factorization of a map and state our main theorem.

Definition 2.1. Let $g\colon X \to Y$ be a continuous map between topological spaces X and Y. Two points $x, x' \in X$ are *g-equivalent* if $g(x) = g(x')$ and the points x and x' are in the same connected component of $g^{-1}(g(x)) = g^{-1}(g(x'))$. We denote by W_g the quotient space with respect to the g-equivalence, endowed with the quotient topology. The quotient map is denoted by $q_g\colon X \to W_g$. Then there exists a unique continuous map $\bar{g}\colon W_g \to Y$ such that $g = \bar{g} \circ q_g$. The quotient space W_g or the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "q_g"'] & \circlearrowleft & Y \\ & W_g \arrow[ur, "\bar{g}"'] & \end{tikzcd}$$

is called the *Stein factorization* of g.

There is a one-to-one correspondence between the quotient space and the space of the connected components of the fibers of g. Note that each fiber of the quotient map q_g is connected.

Remark 2.2. The space W_g is often called the *quotient space* or the *Reeb space* (or the *Reeb complex*) of g.

Let $g\colon X \to Y$ be a continuous map between topological spaces. Then, g is said to be *triangulable* if there exist simplicial complexes K and L, a simplicial map $s\colon K \to L$, and homeomorphisms $\lambda\colon |K| \to X$ and $\mu\colon |L| \to Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \lambda\big\uparrow & & \big\uparrow\mu \\ |K| & \xrightarrow{|s|} & |L|, \end{array}$$

where $|K|$ and $|L|$ are polyhedrons associated with K and L, respectively, and $|s|$ is the PL map associated with s.

In [5], the authors have proved the following.

Theorem 2.3. *Let $g\colon X \to Y$ be a proper continuous map between locally compact topological spaces X and Y. If g is triangulable, then so is the Stein factorization* *of g. More precisely, we have the commutative diagram*

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "q_g"'] & & Y \\
& W_g \arrow[ur, "\bar{g}"'] & \\
& {|V|} \arrow[u, "\Theta"'] \arrow[dr, "|\psi|"] & \\
{|K'|} \arrow[uuu, "\lambda"] \arrow[ur, "|\varphi|"] \arrow[rr, "|s'|"] & & {|L'|} \arrow[uuu, "\mu"']
\end{tikzcd}$$

for some finite simplicial complexes K', L' and V, some simplicial maps $s' \colon K' \to L'$, $\varphi \colon K' \to V$ and $\psi \colon V \to L'$, and some homeomorphisms λ, μ and Θ.

Let us recall the notion of a cobordism of manifolds. Let M_0 and M_1 be closed oriented manifolds with $\dim M_0 = \dim M_1 (= m)$. We say that M_0 and M_1 are *oriented cobordant* if there exists a compact oriented $(m+1)$-dimensional manifold Q such that $\partial Q = (-M_0) \cup M_1$, where $-M_0$ denotes the manifold M_0 with the orientation reversed. Such a manifold Q is often called an *oriented cobordism* between M_0 and M_1. The above relation clearly defines an equivalence relation. The equivalence class of a manifold M will be denoted by $[M]$. We can define $[M] + [M'] = [M \cup M']$, in such a way that the set Ω_m of all oriented cobordism classes of closed oriented m-dimensional manifolds forms an additive group. This is called the *m-dimensional oriented cobordism group*.

In the above definition, if we ignore the orientations of the manifolds, then we get the *m-dimensional (unoriented) cobordism group*, denoted by $\mathfrak{N}_m$.

The groups Ω_m and $\mathfrak{N}_m$ have been extensively studied and their structures have been completely determined (see [16, 17]). For example, the following is known.

- Ω_m is a finitely generated abelian group.
- $\mathfrak{N}_m$ is a finitely generated $\mathbb{Z}_2$-module.
- Ω_m is a finite group unless m is a multiple of four.
- $\Omega_0 \cong \mathbb{Z}$, $\Omega_1 = \Omega_2 = \Omega_3 = 0$, $\Omega_4 \cong \mathbb{Z}$, $\Omega_5 \cong \mathbb{Z}_2$, ...
- $\mathfrak{N}_0 \cong \mathbb{Z}_2$, $\mathfrak{N}_1 = 0$, $\mathfrak{N}_2 \cong \mathbb{Z}_2$, $\mathfrak{N}_3 = 0$, $\mathfrak{N}_4 \cong \mathbb{Z}_2^2$, $\mathfrak{N}_5 \cong \mathbb{Z}_2$, ...

A closed (oriented) manifold M with $[M] = 0$ is said to be *(oriented) null-cobordant*.

Our main theorem of this paper is the following.

Theorem 2.4. *Let M be a closed manifold and $f \colon M \to N$ a smooth map into a manifold N with $m = \dim M \geq \dim N = n$. Assume that f is triangulable (e.g. a topologically stable map). Then, we have the following.*

(1) *If there exists a regular fiber component of f which is not null-cobordant, then $H_n(W_f; \mathbb{Z}_2) \neq 0$.*

(2) *Suppose that both M and N are oriented (note that then the regular fibers are naturally oriented). If there exists a regular fiber component of f which is not oriented null-cobordant, then $H_n(W_f; \Omega_{m-n}) \neq 0$.*

3. Proof of Theorem 2.4

Proof. Let $s \colon K \to L$ be a triangulation of $f \colon M \to N$. Then, for the barycentric subdivision L' of L, there exist a subdivision K' of K and a simplicial map $s' \colon K' \to L'$ such that $|s'| = |s|$ (for example, see [6]). By Theorem 2.3 (see also [5]), we have a triangulation of the Stein factorization as in the commutative diagram

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\
& W_f \arrow[ur, "\bar{f}"'] & \\
& {|V|} \arrow[u, "\Theta"'] \arrow[dr, "|\psi|"] & \\
{|K'|} \arrow[uuu, "\lambda"] \arrow[ur, "|\varphi|"] \arrow[rr, "|s'|"] & & {|L'|,} \arrow[uuu, "\mu"']
\end{tikzcd}$$

where V is a finite simplicial complex of dimension n, $\varphi\colon K' \to V$ and $\psi\colon V \to L'$ are simplicial maps with ψ being non-degenerate, and λ, Θ and μ are homeomorphisms. Here, a simplicial map is *non-degenerate* if it preserves the dimension of each simplex.

For each n-simplex $\sigma \in V$, λ maps $|\varphi|^{-1}(b_\sigma)$ homeomorphically onto $q_f^{-1}(\Theta(b_\sigma))$, which is a component of a fiber of f, where b_σ is a point in the interior of σ. By the Sard theorem, we may assume that $\bar{f}(\Theta(b_\sigma))$ is a regular value of f. Then, define

$$\omega_\sigma = [q_f^{-1}(\Theta(b_\sigma))] \in \mathfrak{N}_{m-n},$$

which is the cobordism class of the regular fiber component corresponding to $\sigma \subset |V| \approx W_f$.

Lemma 3.1. *The cobordism class $[q_f^{-1}(\Theta(b_\sigma))]$ does not depend on a choice of $b_\sigma \in \operatorname{Int}\sigma$.*

Proof. Let b'_σ be another point in $\operatorname{Int}\sigma$ such that $\bar{f}(\Theta(b'_\sigma))$ is a regular value of f. We can choose an embedded arc γ in $\operatorname{Int}\sigma$ connecting b_σ and b'_σ in such a way that $\bar{f}(\Theta(\gamma))$ is transverse to f (for example, see [3, §4.3]). Then, $\lambda(|\varphi|^{-1}(\gamma))$ gives a smooth cobordism between $q_f^{-1}(\Theta(b_\sigma))$ and $q_f^{-1}(\Theta(b'_\sigma))$. $\square$

Set

$$c_f = \sum_\sigma \omega_\sigma \sigma \in C_n(V; \mathfrak{N}_{m-n}),$$

where σ runs over all n-simplices of V, and $C_n(V; \mathfrak{N}_{m-n})$ denotes the n-th chain group of V with coefficients in $\mathfrak{N}_{m-n}$.

Lemma 3.2. *We have $\partial c_f = 0$ in $C_{n-1}(V; \mathfrak{N}_{m-n})$, i.e. c_f is an n-cycle.*

Proof. Let τ be an arbitrary $(n-1)$-simplex of V, and let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the n-simplices of V containing τ as a face (see Fig. 1). We have only to show

$$\sum_{j=1}^{r} \omega_{\sigma_j} = 0$$

in $\mathfrak{N}_{m-n}$, i.e. the vanishing of the coefficient of τ in ∂c_f.

Let $\bar{\alpha}$ be a small arc in $|L'|$ which intersects $\bar{\tau} = \psi(\tau)$ transversely in one point (see Fig. 1), and let $\bar{\sigma}_1$ and $\bar{\sigma}_2$ be the n-simplices of L' adjacent to $\bar{\tau}$. Note that $\psi(\sigma_i)$ coincides with either $\bar{\sigma}_1$ or $\bar{\sigma}_2$. We take $\bar{\alpha}$ so that $\mu(\bar{\alpha})$ is a smooth arc in N transverse to f. Let α be the component of $|\psi|^{-1}(\bar{\alpha})$ that intersects τ. Then, $Q = q_f^{-1}(\Theta(\alpha))$ is an $(m-n+1)$-dimensional compact manifold and

$$\partial Q = \bigcup_{j=1}^{r} \lambda(|\varphi|^{-1}(b_{\sigma_j})).$$

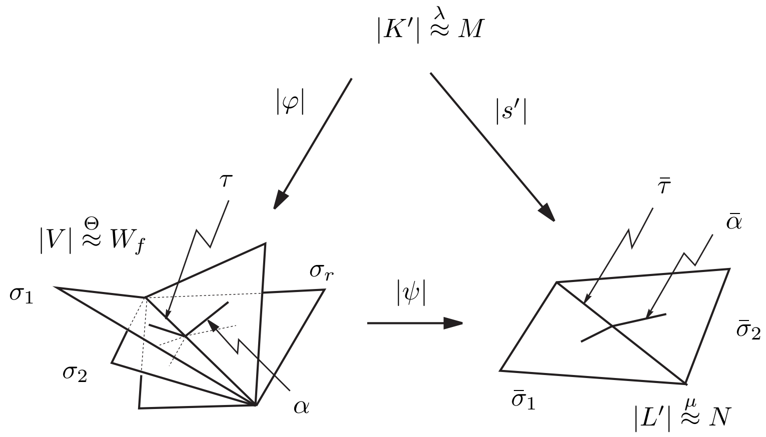

FIGURE 1. The small arc $\bar{\alpha}$

Therefore, we have

$$\sum_{j=1}^{r} \omega_{\sigma_j} = \sum_{j=1}^{r} \left[\lambda(|\varphi|^{-1}(b_{\sigma_j}))\right] = 0$$

in $\mathfrak{N}_{m-n}$. □

Thus, c_f defines a homology class

$$\gamma_f \in H_n(W_f; \mathfrak{N}_{m-n}). \tag{3.1}$$

Furthermore, since $\dim W_f = n$, we have $\gamma_f \neq 0$ if and only if $c_f \neq 0$. Moreover, $c_f \neq 0$ if and only if there exists a component of a regular fiber of f which is not null-cobordant. Therefore, if such a regular fiber component exists, we have $H_n(W_f; \mathbb{Z}_2) \neq 0$, since $\mathfrak{N}_{m-n}$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, the direct sum of a finite number of copies of $\mathbb{Z}_2$.

The case where both M and N are oriented can be treated similarly. (In this case, the orientation of N induces an orientation of each n-simplex of V. Therefore, the argument also works with coefficients in the abelian group Ω_{m-n}.) This completes the proof of Theorem 2.4. □

Example 3.3. Let us consider a tree T. Then, since $H_1(T; \mathbb{Z}_2) = 0$, there exists no Morse function $f_1 \colon M_1^5 \to \mathbb{R}$ on a closed 5-dimensional manifold M_1^5 such that the quotient space W_{f_1} is homeomorphic to T and that f_1 has $\mathbb{C}P^2$ as a component of a regular fiber. (Recall that $\mathbb{C}P^2$ is not null-cobordant.)

Example 3.4. There exists a Morse function $f_2 \colon M_2^5 \to \mathbb{R}$ on a closed 5-dimensional manifold M_2^5 whose quotient space is as depicted in Fig. 2. The integer at each vertex denotes the index of the corresponding critical point, and the 4-manifold attached to each edge denotes the corresponding regular fiber component.

Note that $H_1(W_{f_2}; \mathbb{Z}) \cong H_1(W_{f_2}; \Omega_4) \cong \mathbb{Z}$ is generated by the homology class γ_{f_2} of (3.1).

We note that the 5-dimensional manifold M_2^5 can be chosen to be diffeomorphic to $S^1 \times \mathbb{C}P^2$, which is null-cobordant.

Example 3.5. There exists a Morse function $f_3 \colon M_3^5 \to \mathbb{R}$ on a closed 5-dimensional manifold M_3^5 whose quotient space is as depicted in Fig. 3. Note that the quotient space W_{f_3} is homeomorphic to W_{f_2}; however, $\gamma_{f_3} = 0$ in $H_1(W_{f_3}; \mathbb{Z}) \cong \mathbb{Z}$, while $\gamma_{f_2} \neq 0$ in $H_1(W_{f_2}; \mathbb{Z})$. This means that even if the top dimensional homology

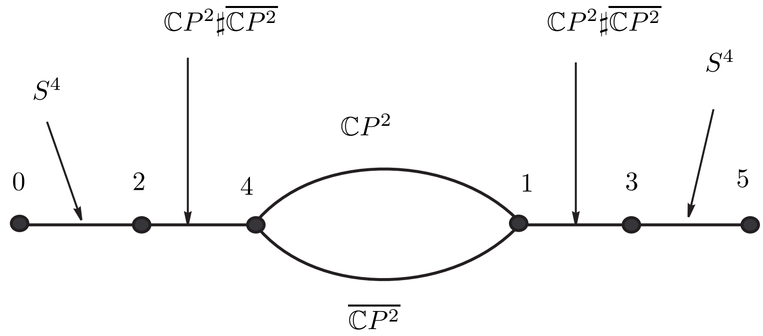

FIGURE 2. An example with non-vanishing γ_{f_2}

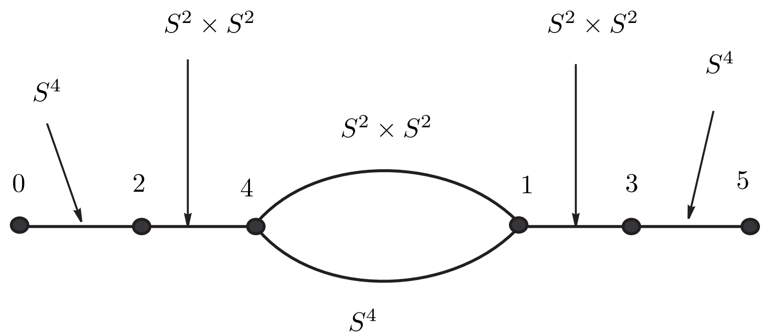

FIGURE 3. An example with vanishing γ_{f_3}

group of the quotient space does not vanish, the map may not have a regular fiber component that is not null-cobordant.

We note that the 5-dimensional manifold M_3^5 can be chosen to be diffeomorphic to $S^1 \times S^4$, which is null-cobordant.

By considering the product maps $\widetilde{f_i} = f_i \times \mathrm{id}_{S^k} : M_i^5 \times S^k \to \mathbb{R} \times S^k$, $k \geq 1$, $i = 1, 2, 3$, where id_{S^k} denotes the identity map of S^k, we can construct examples of higher dimensional quotient spaces as well.

Remark 3.6. For a smooth map $f : M \to N$ as in Theorem 2.4 with N being a closed manifold, we see that $\bar{f}_* \gamma_f \in H_n(N; \mathfrak{N}_{m-n}) \cong H_n(N; \mathbb{Z}_2) \otimes \mathfrak{N}_{m-n}$ coincides with $[N] \otimes F_f$, where $\bar{f} : W_f \to N$ is the continuous map that appears in the Stein factorization of f, $[N] \in H_n(N; \mathbb{Z}_2)$ is the fundamental class of N, and $F_f \in \mathfrak{N}_{m-n}$ is the unoriented cobordism class of a regular fiber of f. Note that when N is a closed manifold, the unoriented cobordism class F_f can be determined by the Stiefel-Whitney classes of M together with $f^*(1^*) \in H^n(M; \mathbb{Z}_2)$, where $1^* \in H^n(N; \mathbb{Z}_2)$ is the Poincaré dual of the canonical generator $1 \in H_0(N; \mathbb{Z}_2)$. A similar remark is also valid in the oriented case. For details, see [12, §3].

Remark 3.7. Even if every component of every regular fiber is null-cobordant, the source manifold may not be null-cobordant.

For example, consider a C^∞ stable map $f : \mathbb{C}P^2 \to \mathbb{R}^3$. Every component of every regular fiber is diffeomorphic to S^1, which is null-cobordant. However, $\mathbb{C}P^2$ is not null-cobordant.

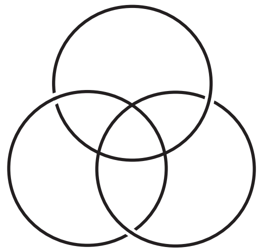

FIGURE 4. The singular fiber that determines the cobordism class

In fact, for a C^∞ stable map $f\colon M^4 \to \mathbb{R}^3$ of a closed oriented 4-dimensional manifold M^4, the cobordism class of M^4 is determined by singular fibers as depicted in Fig. 4 (see [13, 14]).

Remark 3.8. Let M be a closed connected m-dimensional manifold. For a given Morse function $f'\colon M \to \mathbb{R}$, we can modify it by homotopy so that we get an ordered Morse function $f\colon M \to \mathbb{R}$. Here, a Morse function f is *ordered* if for every pair of critical points p and q of f with $\mathrm{index}(p) < \mathrm{index}(q)$, we have $f(p) < f(q)$. As is observed in [4], if f is an ordered Morse function with $m \geq 3$, then every fiber of f is connected, and hence W_f is homeomorphic to a line segment. In this case, we have $\gamma_f = 0$, although the source manifold M may not be null-cobordant.

4. Further results

Let $f'\colon M \to S^1$ be a continuous map of a smooth closed connected manifold M of dimension $m \geq 3$ into the circle. We assume that the induced homomorphism $f'_*\colon \pi_1(M) \to \pi_1(S^1)$ is surjective. Then, it is known that f' is homotopic to a Morse map $f\colon M \to S^1$ whose fibers are all non-empty and connected, where a *Morse map* is a smooth map whose critical points are all non-degenerate (for example, see [4, Theorem 1.3]). In this case, the quotient space W_f is canonically homeomorphic to S^1 through $\bar{f}$. In the following, a map is said to be *fiber-connected* if all of its fibers are non-empty and connected. For such a map, the quotient space is canonically homeomorphic to the target manifold.

Definition 4.1. Let $f_i\colon M_i \to N$ be smooth maps of closed m-dimensional manifolds M_i into a manifold N, $i = 0, 1$. We say that f_0 and f_1 are *bordant* if there exists a cobordism Q between M_0 and M_1 (i.e. Q is a compact $(m+1)$-dimensional manifold with ∂Q being identified with $M_0 \cup M_1$), and a smooth map $F\colon Q \to N \times [0, 1]$ such that $f_i = F|_{M_i}\colon M_i \to N \times \{i\}$, $i = 0, 1$. Such a map F is often called a *bordism* between f_0 and f_1. If everything is oriented, then we say that f_0 and f_1 are *oriented bordant* (for details, see [2], for example).

For example, if two smooth maps are homotopic, then they are bordant.

Proposition 4.2. *Let $f_i\colon M_i \to N$ be smooth maps of m-dimensional closed connected manifolds M_i into a connected n-dimensional manifold N with $m \geq n \geq 1$, $i = 0, 1$. We suppose that f_i are topologically stable and are fiber-connected, $i = 0, 1$. If f_0 and f_1 are bordant, then we have $\gamma_{f_0} = \gamma_{f_1} \in H_n(N; \mathfrak{N}_{m-n})$. If M_0, M_1 and N are oriented, and f_0 and f_1 are oriented bordant, then we have $\gamma_{f_0} = \gamma_{f_1} \in H_n(N; \Omega_{m-n})$.*

Proof. Let $F\colon Q \to N \times [0, 1]$ be the map as in Definition 4.1. By the Sard theorem, we can choose a point $y \in N$ which is a common regular value of f_0 and f_1. Then, by slightly perturbing F on the interior of Q, we may assume that F is transverse to the line segment $\{y\} \times [0,1]$. Then, we see that $Q' = F^{-1}(\{y\} \times [0,1])$ gives a cobordism between regular fibers of f_0 and f_1. Since, for each $i = 0, 1$, the regular fibers of f_i are all cobordant, we get the required result. $\square$

The above proposition implies that if two topologically stable fiber-connected maps $f_i \colon M_i \to N$, $i = 0, 1$, satisfy $\gamma_{f_0} \neq \gamma_{f_1}$, then they are not bordant. In particular, when $M_0 = M_1$, f_0 and f_1 cannot be homotopic.

Remark 4.3. Suppose that $f_i \colon M_i \to N$, $i = 0, 1$, are bordant as in Definition 4.1. We further assume that f_i are topologically stable, $i = 0, 1$, and that γ_{f_0} does not vanish. Under these assumptions, we have a sufficient condition for the non-vanishing of γ_{f_1} as follows.

Let $F \colon Q \to N \times [0,1]$ be the smooth map that gives a bordism between f_0 and f_1. Let y be a regular value of f_0 such that $f_0^{-1}(y)$ contains a component which is not null-cobordant. Let $\alpha \colon [0,1] \to N \times [0,1]$ be a smooth embedding with $\alpha(0) = y \times \{0\}$ such that $\alpha(1) \in N \times \{1\}$ corresponds to a regular value of f_1 and that α is transverse to F. Let R be the component of $F^{-1}(\alpha([0,1]))$ which contains the component of $f_0^{-1}(y)$ which is not null-cobordant. By choosing α and F generic enough, we may further assume that the map $h = \alpha^{-1} \circ F \colon R \to [0,1]$ is a Morse function. If this Morse function has no critical points of index 1, then we can show that γ_{f_1} does not vanish.

Proposition 4.4. *Let m be an integer with $m \geq 3$. For an arbitrary $c \in \mathfrak{N}_m$ and for an arbitrary finite number of elements $c_j \in \mathfrak{N}_{m-1}$, $j = 1, 2, \ldots, k$, there exist a smooth closed connected m-dimensional manifold M and fiber-connected Morse maps $f_j \colon M \to S^1$ such that $[M] = c$ and $\gamma_{f_j} \in H_1(S^1; \mathfrak{N}_{m-1}) \cong \mathfrak{N}_{m-1}$ corresponds to c_j, $j = 1, 2, \ldots, k$.*

We also have a corresponding proposition for maps between oriented manifolds.

Proof of Proposition 4.4. Take a closed connected $(m-1)$-dimensional manifold F_j in the cobordism class c_j for each j, and a closed connected m-dimensional manifold M' with $[M'] = c$. Let us consider the closed connected m-dimensional manifold M given by

$$M = M' \sharp \left(\sharp_{j=1}^{k} (S^1 \times F_j) \right).$$

Since $S^1 \times F_j$ bounds $D^2 \times F_j$, it is null-cobordant, and hence we have $[M] = [M'] = c$.

Note that for each j, M naturally decomposes as $((S^1 \times F_j) \smallsetminus \operatorname{Int} D^m) \cup M'_j$, where $(S^1 \times F_j) \smallsetminus \operatorname{Int} D^m$ and M'_j are attached along their sphere boundaries. Let us construct a continuous map $f'_j \colon M \to S^1$ as follows. On $(S^1 \times F_j) \smallsetminus \operatorname{Int} D^m$, it is homotopic to the restriction of the projection $S^1 \times F_j \to S^1$ to the first factor and is a constant map on the boundary of $(S^1 \times F_j) \smallsetminus \operatorname{Int} D^m$. We define f'_j on M'_j to be the constant map to the same value in S^1. We may assume that f'_j is smooth on $(S^1 \times F_j) \smallsetminus \operatorname{Int} D^m$ and it has $\{*\} \times F_j$ as a regular fiber. Then by [4, Theorem 1.3], f'_j is homotopic to a fiber-connected Morse map $f_j \colon M \to S^1$. By construction, the regular fibers of f_j are all cobordant to F_j. Hence, f_j satisfy the desired properties. $\square$

Proposition 4.4 shows that at least for fiber-connected maps f, γ_f carries no information on the cobordism class of the source manifold.

We end this paper by posing some problems.

Problem 4.5. (1) How about the case of maps of manifolds with non-empty boundaries?

(2) By associating an "invariant" of a (regular or singular) fiber component corresponding to certain dimensional simplices of W_f, can we define a homology class of W_f ?

(3) Study such kind of homology classes and their relations to the geometry and topology of the manifolds and the map. For example, can we define Vassiliev type invariants for maps using such homology classes?

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