Cobordism of algebraic knots defined by Brieskorn–Pham type polynomials

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§1. Introduction

Algebraic knot

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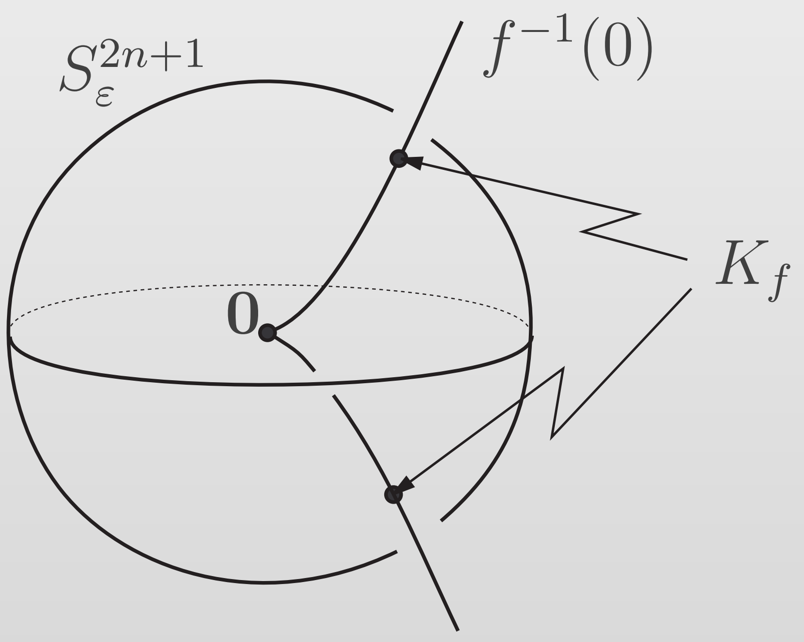

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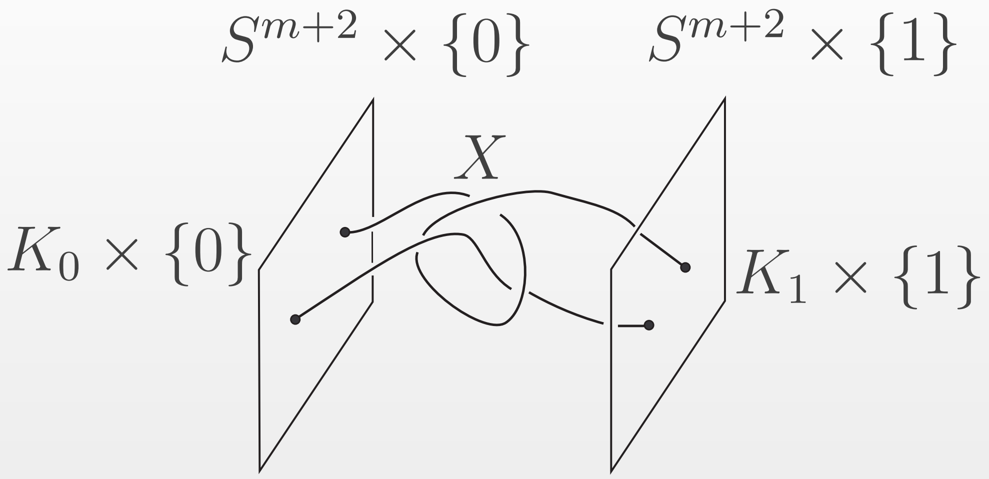
X is called a ***cobordism*** between K_0 and K_1.

Cobordism vs Isotopy



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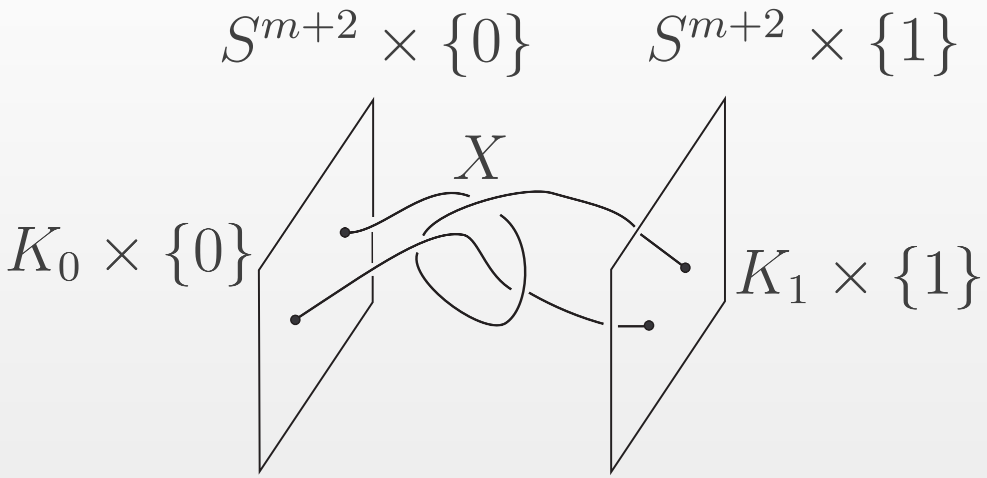

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$\Downarrow \quad \not\Uparrow$

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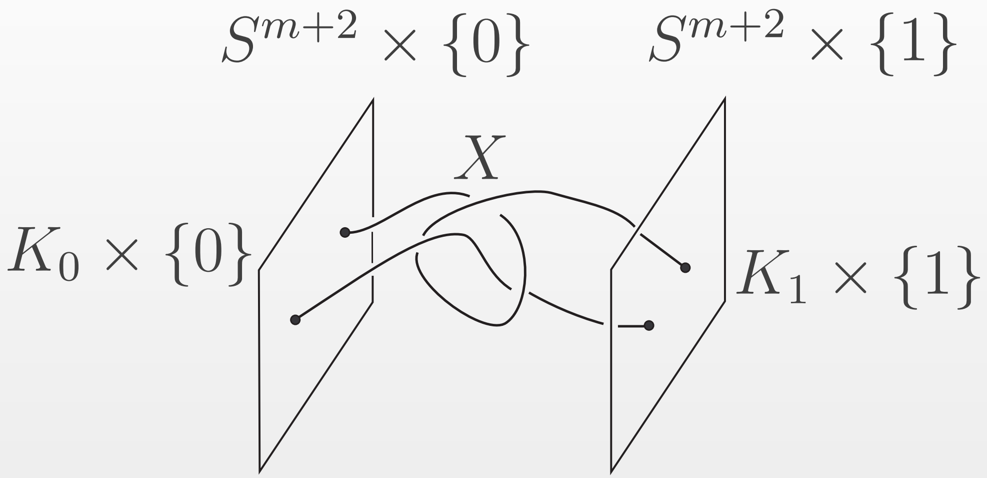

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Cobordant

If two algebraic knots K_f and K_g are **cobordant**, then the topological types of f and g are mildly related.

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Today's Topic: Problem 1.2 (2) for weighted homogeneous polynomials (in particular, Brieskorn–Pham type polynomials).

§2. Results

Two-variable case

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(1) K_f *and* K_g *are isotopic.*

(2) K_f *and* K_g *are cobordant.*

(3) *Alexander polynomials coincide:* $\Delta_f(t) = \Delta_g(t)$.

Higher dimensions

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du Bois–Michel, 1993

Examples of two algebraic (spherical) knots that are cobordant, but are not isotopic.

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A direct summand $M \subset G$ is called a ***metabolizer***
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Theorem 2.3 (Blanlœil–Michel, 1997) *For $n \geq 3$, two algebraic knots K_f and K_g are cobordant $\Longleftrightarrow$ Seifert forms L_f and L_g are algebraically cobordant.*

Witt equivalence

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Lemma 2.5 *If two algebraic knots K_f and K_g are cobordant, then their Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.*

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Furthermore, then the weights ≥ 2 are **analytic invariants** of the polynomial.
In the following, we will always assume $\forall$weights ≥ 2.

Criterion for Witt equivalence over $\mathbb{R}$

Set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1}.$$

$P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbf{Z}$ for some integer $m > 0$.

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Two non-degenerate weighted homogeneous polynomials f and g have the **same weights** if and only if $P_f(t) = P_g(t)$.

Theorem 2.6 *Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are* ***Witt equivalent over $\mathbf{R}$*** *iff*

$$P_f(t) \equiv P_g(t) \mod t+1.$$

Criterion for isomorphism over $\mathbf{R}$

The above theorem should be compared with the following.

Remark 2.7 *The Seifert forms L_f and L_g associated with non-degenerate weighted homogeneous polynomials f and g are* ***isomorphic over $\mathbf{R}$*** *iff*

$$P_f(t) \equiv P_g(t) \mod t^2 - 1.$$

Brieskorn–Pham type polynomials

Proposition 2.8 *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

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be Brieskorn–Pham type polynomials.
Then, their Seifert forms are ***Witt equivalent over*** $\mathbf{R}$ *iff*

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integers ℓ.

Cobordism invariance of exponents

Theorem 2.9 *Suppose that for each of the Brieskorn–Pham type polynomials*

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no exponent is a multiple of another one.
Then, the knots K_f and K_g are ***cobordant*** *iff*

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

up to order.

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If K_f and K_g are ***cobordant****, then the* ***multiplicities*** *of f and g coincide.*

Case of two or three variables

Proposition 2.11 *Let f and g be weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w'_1, w'_2), respectively, with $w_j, w'_j \geq 2$.*
If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w'_j$, $j = 1, 2$, up to order.

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If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w_j'$, $j = 1, 2$, up to order.

Proposition 2.12 *Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn–Pham type polynomials of three variables.*
If the Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$, then $a_j = b_j$, $j = 1, 2, 3$, up to order.

§3. Proofs

Proof of Theorem 2.6

Theorem 2.6 Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are **Witt equivalent over $\mathbf{R}$** iff

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Let $\Delta_f(t)$ be the **characteristic polynomial of the monodromy**

$$h_* : H_n(\mathrm{Int}\, F_f; \mathbf{C}) \to H_f(\mathrm{Int}\, F_f; \mathbf{C}),$$

where $h : \mathrm{Int}\, F_f \to \mathrm{Int}\, F_f$ is the characteristic diffeomorphism of the Milnor fibration $\varphi_f : S_\varepsilon^{2n+1} \setminus K_f \to S^1$.

Proof of Theorem 2.6 (Continued)

We have

$$H^n(F_f;\mathbf{C}) = \oplus_\lambda H^n(F_f;\mathbf{C})_\lambda,$$

where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f;\mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

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The **intersection form** $S_f = L_f + L_f^T$ of F_f on $H^n(F_f; \mathbf{C})$ decomposes as the orthogonal direct sum of $(S_f)|_{H^n(F_f;\mathbf{C})_\lambda}$.

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Let $\mu(f)^+_\lambda$ (resp. $\mu(f)^-_\lambda$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F;\mathbf{C})_\lambda}$.

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Let $\mu(f)_\lambda^+$ (resp. $\mu(f)_\lambda^-$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F;\mathbf{C})_\lambda}$.
The integer

$$\sigma_\lambda(f) = \mu(f)_\lambda^+ - \mu(f)_\lambda^-$$

is called the ***equivariant signature*** of f with respect to λ.

Proof of Theorem 2.6 (Continued)

Lemma 3.1 (Steenbrink, 1977)

Set $P_f(t) = \sum c_\alpha t^\alpha$. Then we have

$$\sigma_\lambda(f) = \sum_{\substack{\lambda=\exp(-2\pi i\alpha) \\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi i\alpha), \\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, where $i = \sqrt{-1}$, and $\lfloor\alpha\rfloor$ is the largest integer not exceeding α.

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Remark 3.2 The equivariant signature for $\lambda = 1$ is always equal to zero.

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Set

$$P_f(t) = P_f^0(t) + P_f^1(t), \quad \text{where}$$

$$P_f^0(t) = \sum_{\lfloor \alpha \rfloor \equiv 0 \pmod 2} c_\alpha t^\alpha,$$

$$P_f^1(t) = \sum_{\lfloor \alpha \rfloor \equiv 1 \pmod 2} c_\alpha t^\alpha.$$

We define $P_g^0(t)$ and $P_g^1(t)$ similarly.

Proof of Theorem 2.6 (Continued)

Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.
$\Longrightarrow \quad \sigma_\lambda(f) = \sigma_\lambda(g)$ for all λ.

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We define $P_g^0(t)$ and $P_g^1(t)$ similarly.
Since the equivariant signatures of f and g coincide, we have

$$\begin{aligned} tP_f^0(t) - P_f^1(t) &\equiv tP_g^0(t) - P_g^1(t) \mod t^2 - 1, \\ tP_f^1(t) - P_f^0(t) &\equiv tP_g^1(t) - P_g^0(t) \mod t^2 - 1. \end{aligned}$$

Proof of Theorem 2.6 (Continued)

Adding up these two congruences we have

$$(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2-1, \tag{1}$$

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Conversely, suppose that (2) holds.

$\Longrightarrow$ (1) holds.

$\Longrightarrow$ f and g have the same equivariant signatures.

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Conversely, suppose that (2) holds.

$\Longrightarrow$ (1) holds.

$\Longrightarrow$ f and g have the same equivariant signatures.

Then, we can prove that they are Witt equivalent over $\mathbf{R}$.

This completes the proof.

Proof of Proposition 2.8

Proposition 2.8 Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham type polynomials. Then, their Seifert forms are Witt equivalent over $\mathbf{R}$ iff

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integers ℓ.

Proof of Proposition 2.8 (Continued)

Proof.

$P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some m.

Put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$.

Then, $P_f(t) \equiv P_g(t) \mod t+1$ holds

$\iff \quad Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$.

Proof of Proposition 2.8 (Continued)

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Then, $P_f(t) \equiv P_g(t) \mod t+1$ holds
$\Longleftrightarrow \quad Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$.

Note that ξ is of the form

$$\exp(\pi\sqrt{-1}\ell/m)$$

with ℓ odd and that

$$\frac{-1-\exp(\pi\sqrt{-1}\ell/a_j)}{\exp(\pi\sqrt{-1}\ell/a_j)-1} = \sqrt{-1}\cot\frac{\pi\ell}{2a_j}.$$

Then, we immediately get Proposition 2.8.

Proof of Theorem 2.9

Theorem 2.9 Suppose that for each of the Brieskorn–Pham type polynomials

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.
Then, the knots K_f and K_g are **cobordant** iff

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

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Then, the knots K_f and K_g are **cobordant** iff

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up to order.

This is a consequence of the "**Fox–Milnor type relation**" for the Alexander polynomials of cobordant algebraic knots.

Open problem

Problem 3.3 *Are the exponents cobordism invariants for Brieskorn–Pham type polynomials?*

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$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \quad \forall \text{odd integers } \ell$$

$$\Longrightarrow \quad a_j = b_j \quad \text{up to order ?}$$

Cobordism and isotopy for Brieskorn–Pham type polynomials

Remark 3.4 Theorem 2.9 implies that two algebraic knots K_f and K_g associated with certain Brieskorn–Pham type polynomials are **isotopic** if and only of they are **cobordant**.

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According to **Yoshinaga–Suzuki**, two algebraic knots associated with Brieskorn–Pham type polynomials in general are isotopic if and only if they have the same set of exponents.

In fact, they showed that the **characteristic polynomials coincide if and only if the Brieskorn–Pham type polynomials have the same set of exponents**.

Thank you!