Topology of Definite Fold Singularities

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§1. Special Generic Maps

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$$(x_1, x_2, \ldots, x_m) \mapsto \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_m^2 + c.$$

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They always appear if M^m is compact.

Reeb's theorem

Theorem 1.2 (Reeb, Smale, Cerf et al.)

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$\exists$Morse function $M^m \to \mathbf{R}$ with only critical points of index 0 or m

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(1) $M^m \approx S^m$ *(homeomorphic)* $(m \neq 4)$

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Remark 1.3
Generalized Poincaré conjecture is still open in dimension 4 in the C^∞ category.

Definition 1.4 A singularity of a C^∞ map $M^m \to N^n$, $m \geq n$, that has the normal form

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, \pm x_n^2 \pm x_{n+1}^2 \pm \cdots \pm x_m^2)$$

is called a **fold singularity**.

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Example 1.6 A function $f : M^m \to \mathbf{R}$ is a SGM iff it is a Morse function with only critical points of index 0 or m

Examples of SGMs

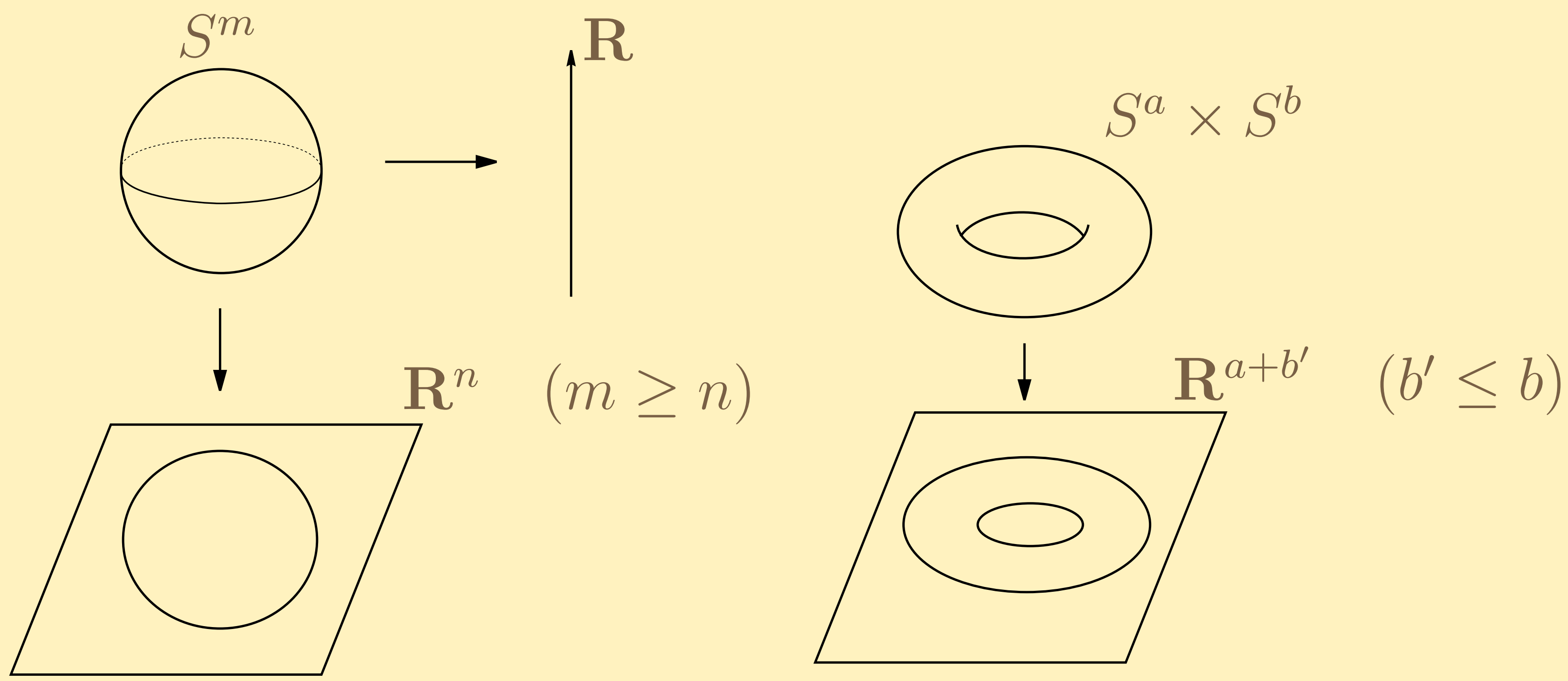

Figure 1: Examples of special generic maps

An invariant

Definition 1.7 M^m: compact

$$\mathcal{S}(M^m) = \{n \in \mathbf{Z} \,|\, 1 \leq n \leq m,\ \exists f : M^m \to \mathbf{R}^n \ \ \textbf{SGM}\}$$

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(2) $\mathcal{S}(S^a \times S^b) = \{a+1, a+2, \ldots, a+b\} \quad (a \leq b)$

Characterization of the sphere

Theorem 1.9 (S., 1993)

M^m: *compact* C^∞ *manifold of dimension* m

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Example 1.10
Σ^7 : Milnor's exotic 7-sphere
$\{1, 2, 7\} \subset \mathcal{S}(\Sigma^7) \subset \{1, 2, 3, 7\}$

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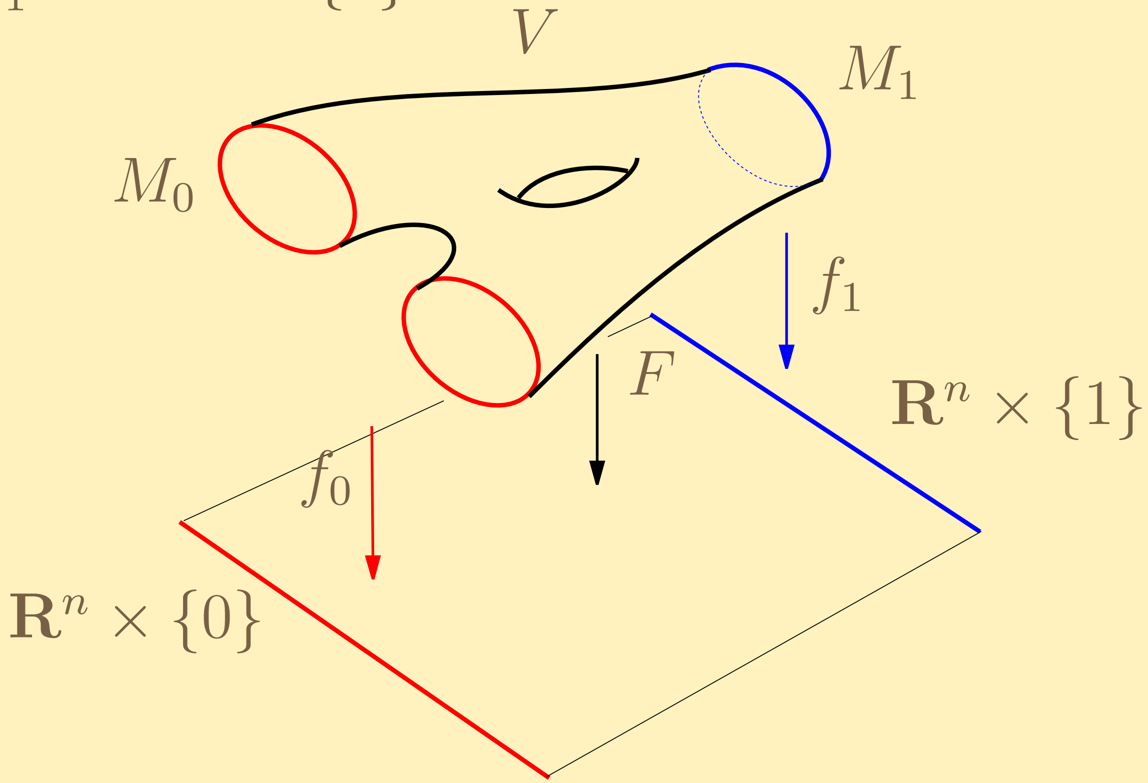

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Special generic maps
$\updownarrow$ closely related !
Differentiable structures

§2. 4-Dimensional Case

Theorem 2.1 (Sakuma-S., 1990's)
$\exists(M_1^4, M_2^4)$: *pair of compact* C^∞ 4-*manifolds such that*
$M_1^4 \approx M_2^4$ *(homeomorphic),*
$\exists f_1 : M_1^4 \to \mathbf{R}^3$ *SGM,*
$\nexists f_2 : M_2^4 \to \mathbf{R}^3$ *SGM.*
In fact, there are infinitely many such pairs.

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SGMs can detect distinct differentiable structures on a given topological 4-manifold.

Compact 1-connected 4-manifolds

Theorem 2.2 (S. (1993) + 3-dim. Poincaré Conj.)

M^4: *compact simply connected* C^∞ 4-*manifold*

$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*

$\Longleftrightarrow M^4 \cong \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \mathbin{\widetilde{\times}} S^2)$ *(diffeomorphic)*

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Remark 2.5 M_1^4, M_2^4: compact orientable C^∞ 4-manifolds
If $M_1^4 \approx M_2^4$ (homeomorphic), then
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Fold maps cannot detect distinct differentiable structures.

Open 1-connected 4-manifolds

Theorem 2.6 (S., 2010)

M^4: *open simply connected* C^∞ 4*-manifold of "finite type"*

$\exists f : M^4 \to N^3$ *proper special generic map*

for some 3*-manifold* N^3 *with* $S(f) \neq \emptyset$

$\iff$ M^4 *is diffeomorphic to the connected sum of a finite number of copies of the following manifolds:*

$\mathbf{R}^4 (= S^4 \setminus \{\text{point}\})$,

$\mathrm{Int}\left(\natural^k (S^2 \times D^2)\right) = S^4 \setminus (\vee^k S^1)$,

$S^2 \times S^2$,

$S^2 \tilde{\times} S^2$,

$\mathbf{R}^2$*-bundle over* S^2

Standard $\mathbb{R}^4$

Corollary 2.7

M^4: C^∞ 4-manifold with $M^4 \approx \mathbf{R}^4$ (homeomorphic)

$\exists f : M^4 \to \mathbf{R}^p$ proper SGM with $S(f) \neq \emptyset$ for $1 \leq \exists p \leq 3$

$\Longleftrightarrow M^4 \cong \mathbf{R}^4$ (diffeomorphic)

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It is known that $\mathbf{R}^n$, $n \neq 4$, has a unique differentiable structure (Munkres, Stallings, $\sim$ 60's).
However, $\mathbf{R}^4$ admits **uncountably many** differentiable structures (Donaldson, Freedman, Taubes, $\sim$ 80's).

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Conjecture 2.10
M^4: topological 4-manifold
$\Longrightarrow$ *There exists at most one differentiable structure on M^4 that allows the existence of a proper SGM into $\mathbf{R}^3$.*

§3. Broken Lefschetz Fibrations

Broken Lefschetz fibration

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w.r.t. complex coordinates compatible with the orientations, is called a **Lefschetz singularity**.

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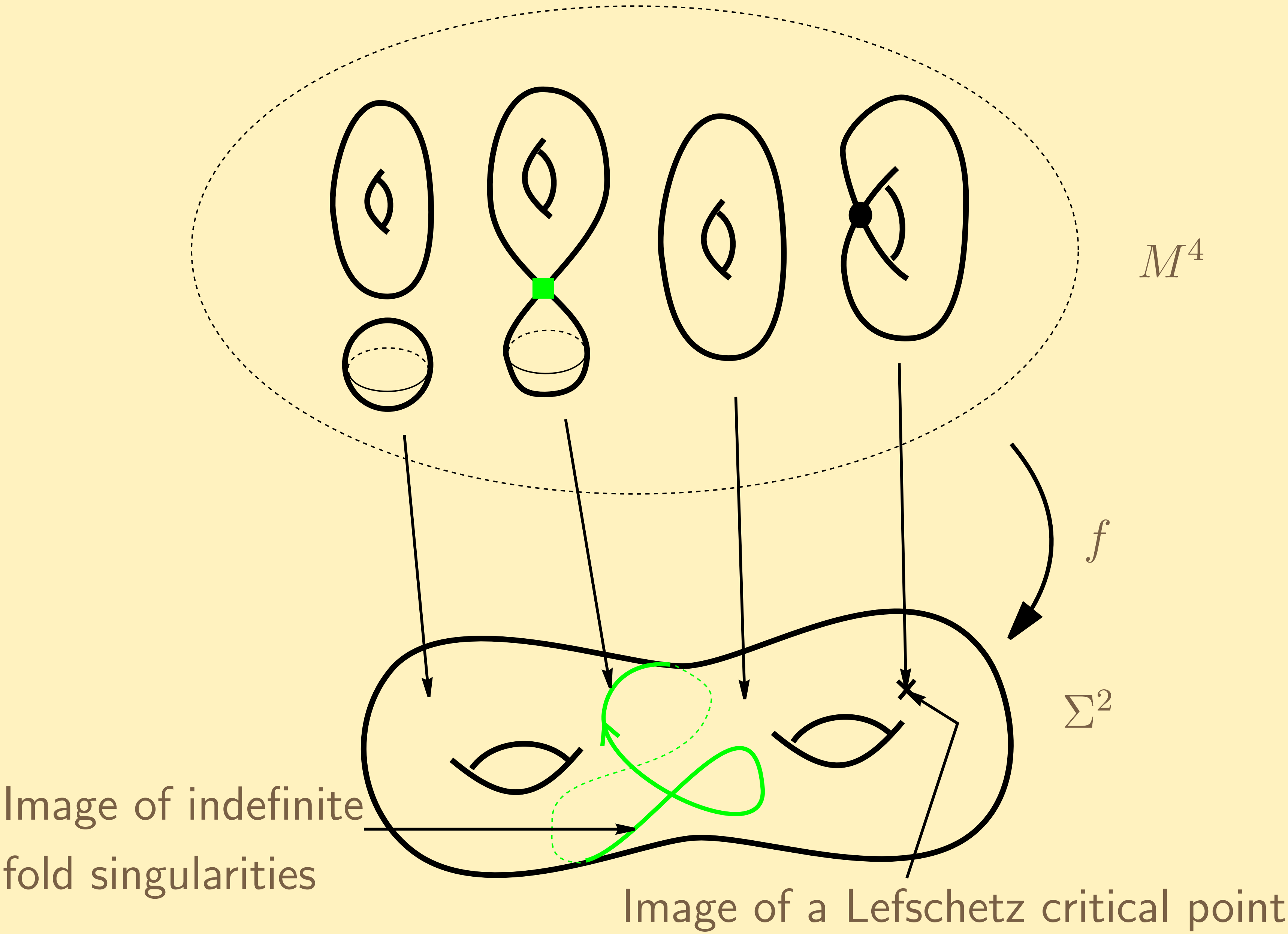
Definition 3.2 (Auroux–Donaldson–Katzarkov 2005, etc.)
Let $f : M \to \Sigma$ be a C^∞ map.
f is a **broken Lefschetz fibration** (**BLF**, for short) if
it has at most Lefschetz and **indefinite** fold singularities.

Fibers of a BLF



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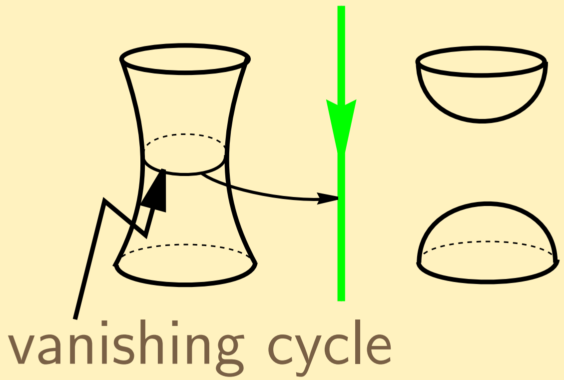

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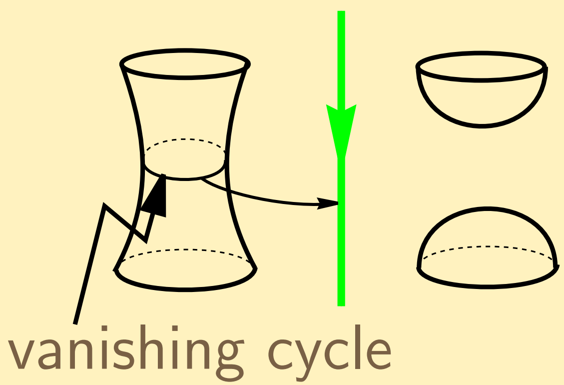

Figure 2: Regular fibers near **indefinite fold**

For a BLF $f : M^4 \to \Sigma^2$, we denote by $S_I(f)$ $(\subset M^4)$ the **oriented** 1-dimensional submanifold of M^4 consisting of the indefinite fold singularities.

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Near-symplectic structures

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$\Downarrow$

Gauge theoretic invariants can be defined.

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Not every 4-manifold admits a symplectic structure.
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Symplectic vs Near-symplectic

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In fact, there are a variety of such structures on a given 4-manifold M^4.

Cusps

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$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.

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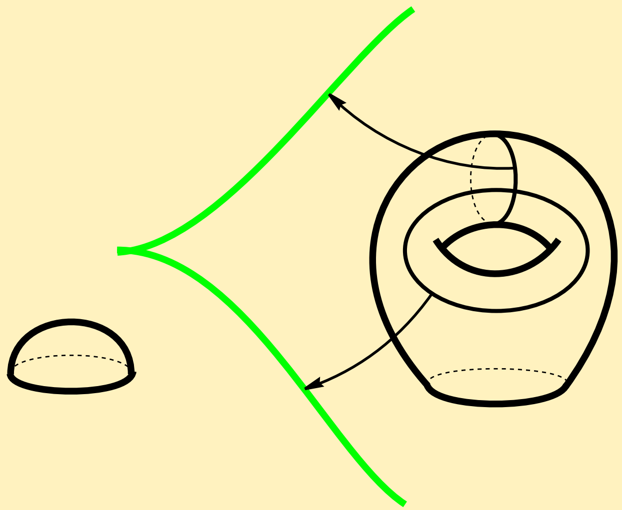

Figure 3: Indefinite cusp

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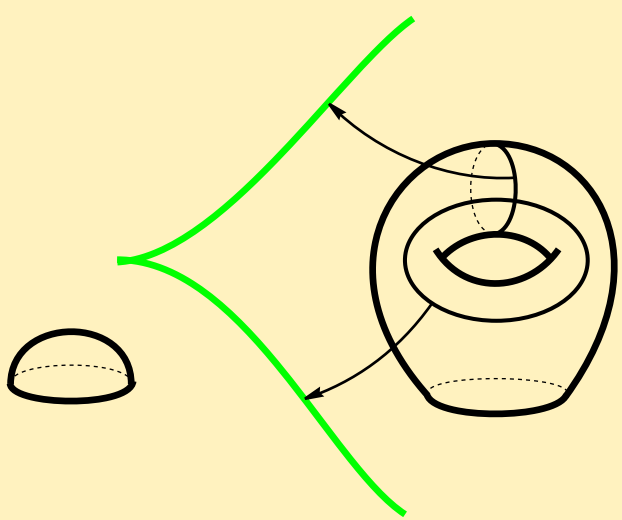

Figure 3: Indefinite cusp

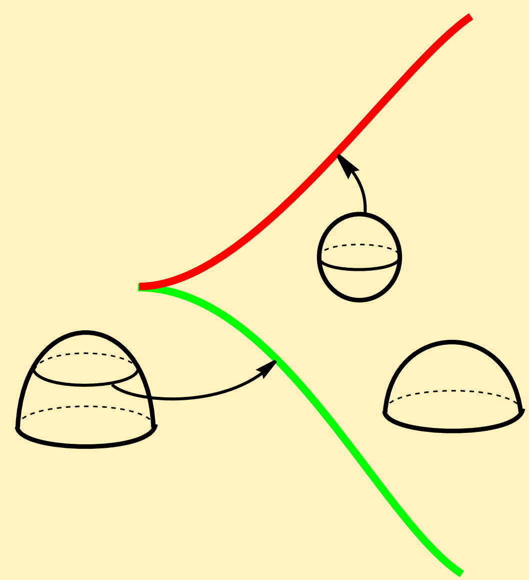

Figure 4: Definite cusp

Excellent maps

Facts.

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Every smooth map $M \to \Sigma$ is homotopic to a map with at most definite fold, indefinite fold, and cusp singularities.

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Levine (1965)

Every smooth map $M \to \Sigma$ is homotopic to an excellent map without a cusp if $\chi(M)$ is even, and with exactly one cusp if $\chi(M)$ is odd.

Elimination of definite fold

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In other words, we can eliminate **definite fold singularities** by homotopy.

Existence of BLF

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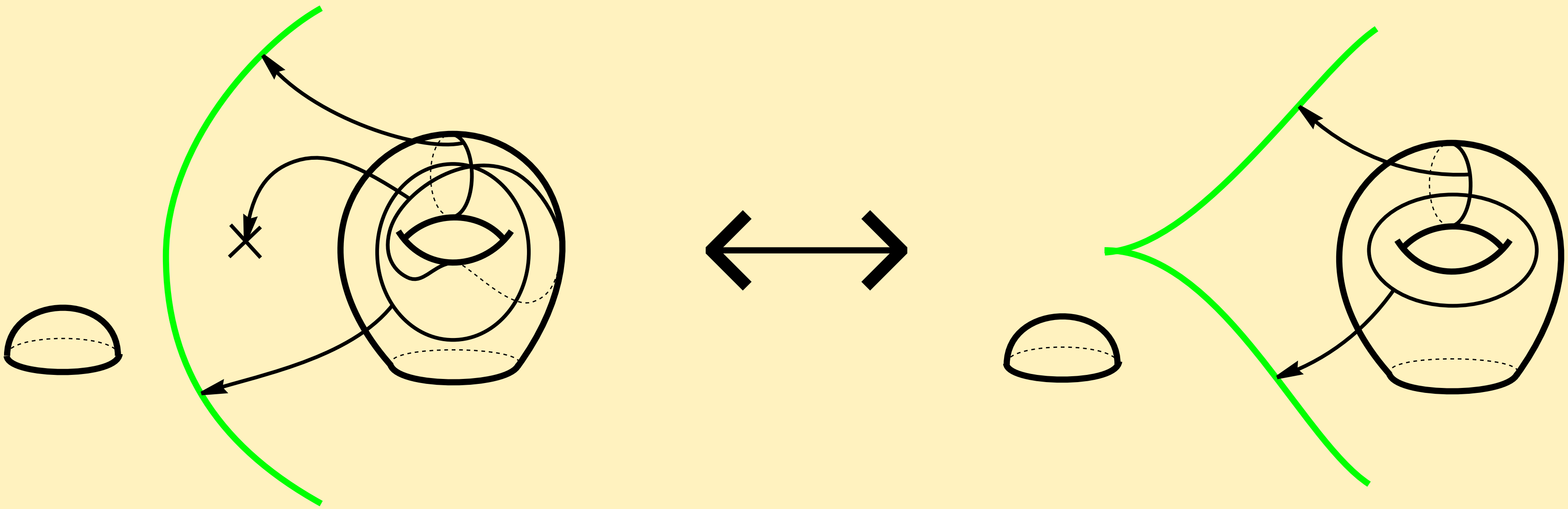

Figure 5: Sinking and Unsinking (Lekili 2009)

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Using similar techniques in the context of near-symplectic structures (Perutz, 2006; Lekili, 2009), we can prove the following.

Theorem 3.9 M^4: *closed oriented* 4*-manifold with* $b_2^+(M^4) > 0$
$L \subset M^4$: *a non-empty closed oriented* 1*-dim. submanifold*
$\exists$**near-symplectic structure** ω *whose zero locus coincides with* L
$\Longleftrightarrow [L] = 0$ *in* $H_1(M^4; \mathbf{Z})$

Remark 3.10 For the existence of a BLF, several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

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Elimination of definite fold for generic homotopy is possible.

Lekili's moves

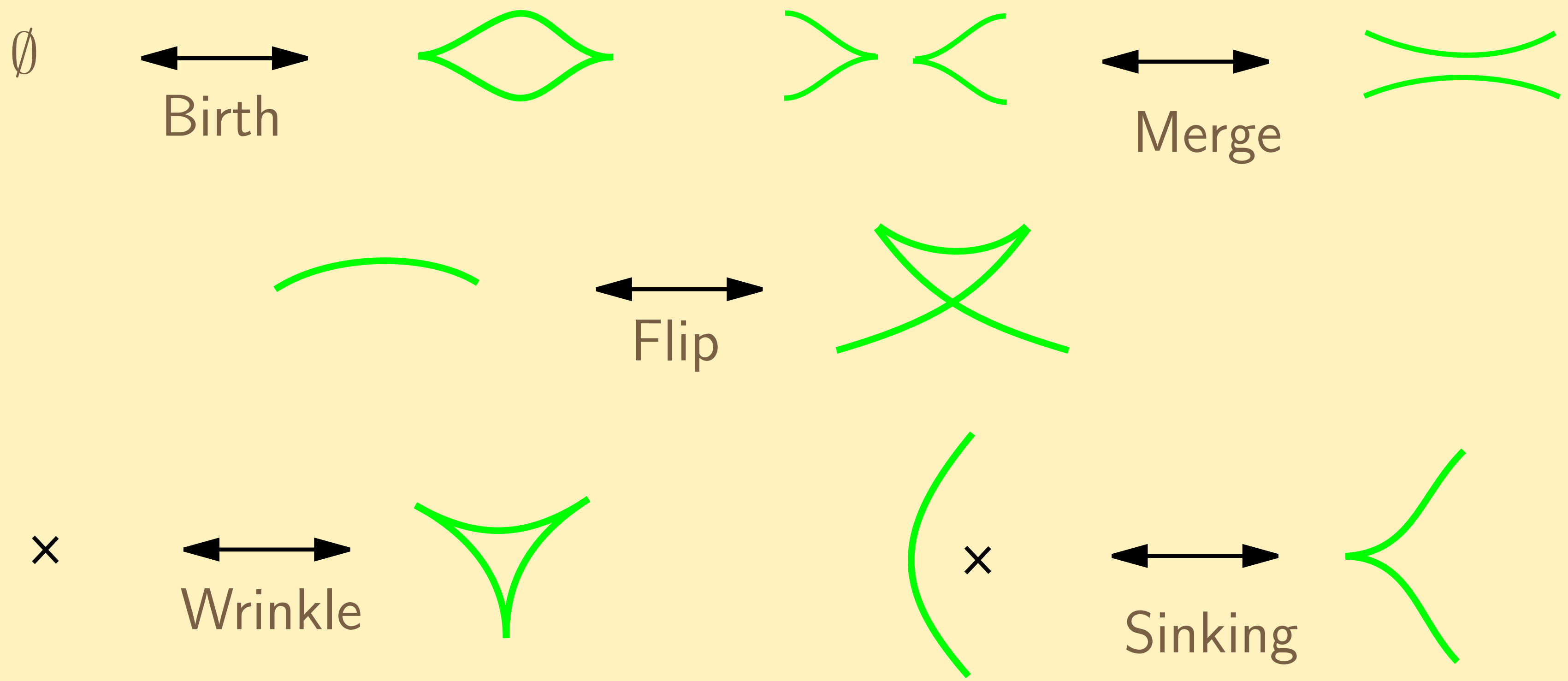

Figure 6: Lekili's moves

Concluding remarks

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Singularities of C^∞ maps are closely related to **differentiable structures** of manifolds!

Thank you!