Desingularizing Special Generic Maps

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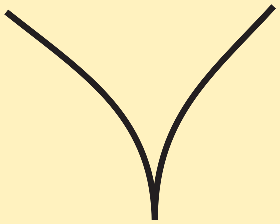
November 7, 2012

§1. Desingularizing Singular Maps

Desingularizing a singular curve

This is a **singular** plane curve.



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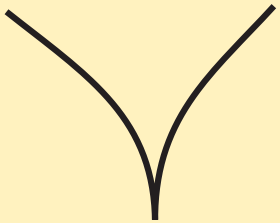
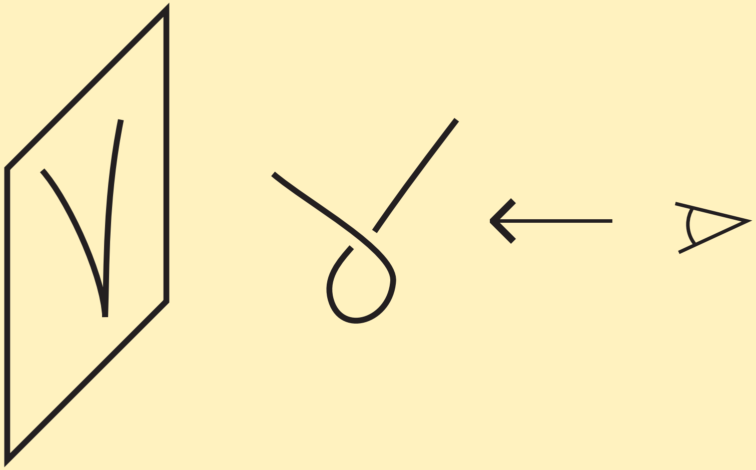


But, this might be the projected image of a **non-singular** space curve.



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M^n: **closed** n-dim. C^∞ manifold,
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In this talk, we consider the case $m = n + 1$.

Surface case

Theorem 1.2 (Haefliger, 1960) $f : M^2 \to \mathbf{R}^2$ *generic*
$\exists$**immersion** $\eta : M^2 \to \mathbf{R}^3$ *s.t.* $f = \pi \circ \eta$
$\Longleftrightarrow$ *For every singular set component* S $(\cong S^1)$ *of* f*:*
if S *has an annulus nbhd,* S *contains an even number of cusps,*
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Equi-dimensional case

Theorem 1.5 (Saito, 1961) M^n: **orientable**
$f : M^n \to \mathbf{R}^n$ *special generic map*
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Theorem 1.6 (Blank–Curley, 1985)
$f : M^n \to \mathbf{R}^n$ generic,
*$\exists$**immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ s.t. $f = \pi \circ \eta$*
$\Longleftrightarrow$ $\mathrm{rk}\, df \geq n-1$, and $[\overline{\{\text{cusps}\}}]^* + \underline{w_1(\nu)} = 0$ in $H^1(\overline{\{\text{folds}\}}; \mathbf{Z}_2)$,
where ν is the normal line bundle of $\{\text{folds}\}$ in M^n.

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Definition 1.8 $f : M^n \to N^p$ is a **special generic map** (**SGM**, for short) if it has **only definite fold singularities**.

Examples

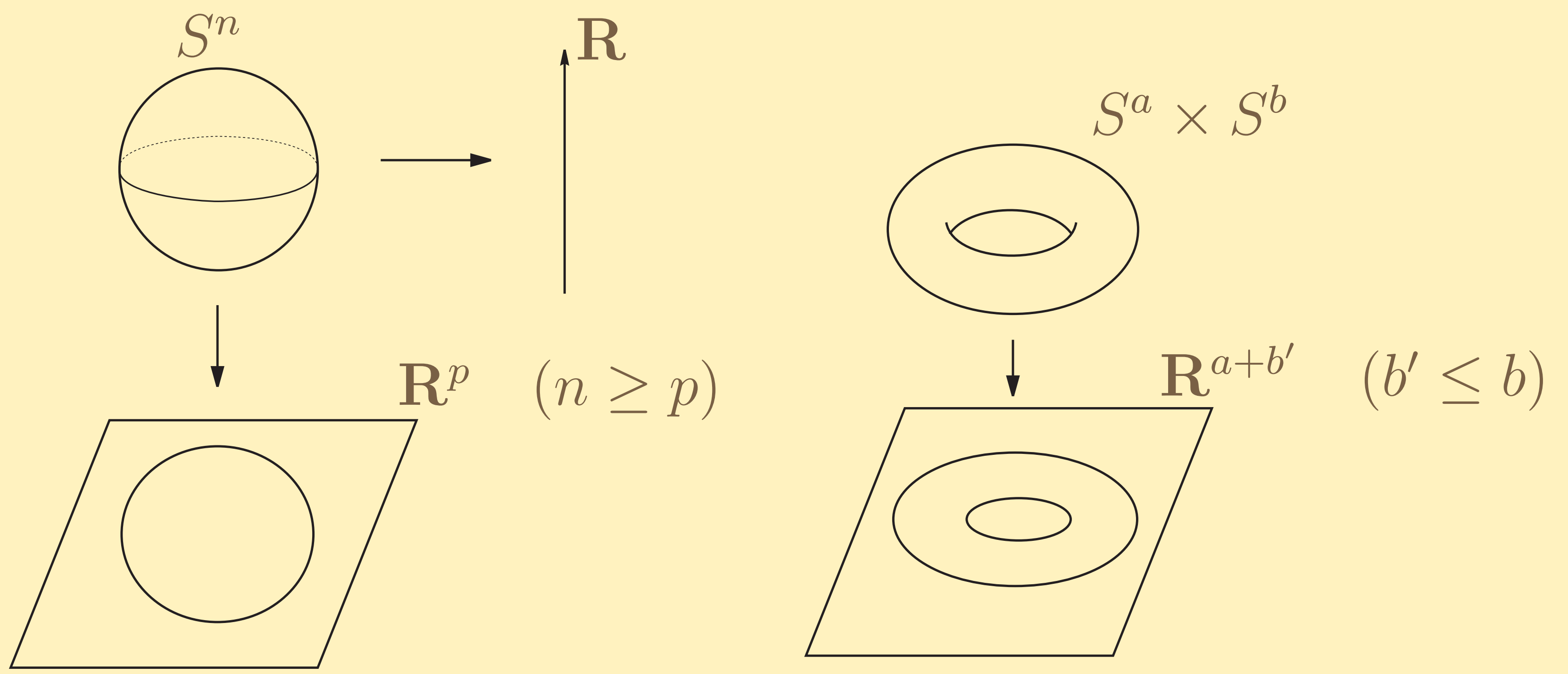

Figure 1: Examples of special generic maps

§2. Desingularizing Special Generic Functions

Special generic functions

Theorem 2.1 (Reeb, Smale, Cerf et al)

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$\exists$**special generic function** $M^n \to \mathbf{R}$

$\Longleftrightarrow$

(1) $M^n \approx S^n$ *(homeomorphic)* $\quad (n \neq 4)$

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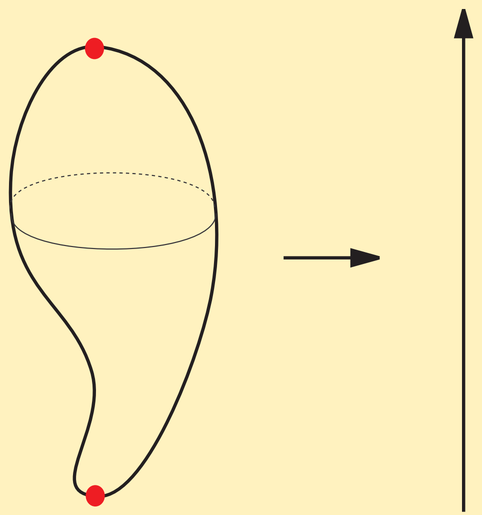

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Two immersions are **regularly homotopic** if they are in the same connected component of the space $\{$immersions $M^k \to \mathbf{R}^\ell\}$.

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For $\forall$diffeomorphism $\varphi : S^{n-1} \to S^{n-1}$ preserving the orientation, the immersions i and $i \circ \varphi$ are **regularly homotopic**.

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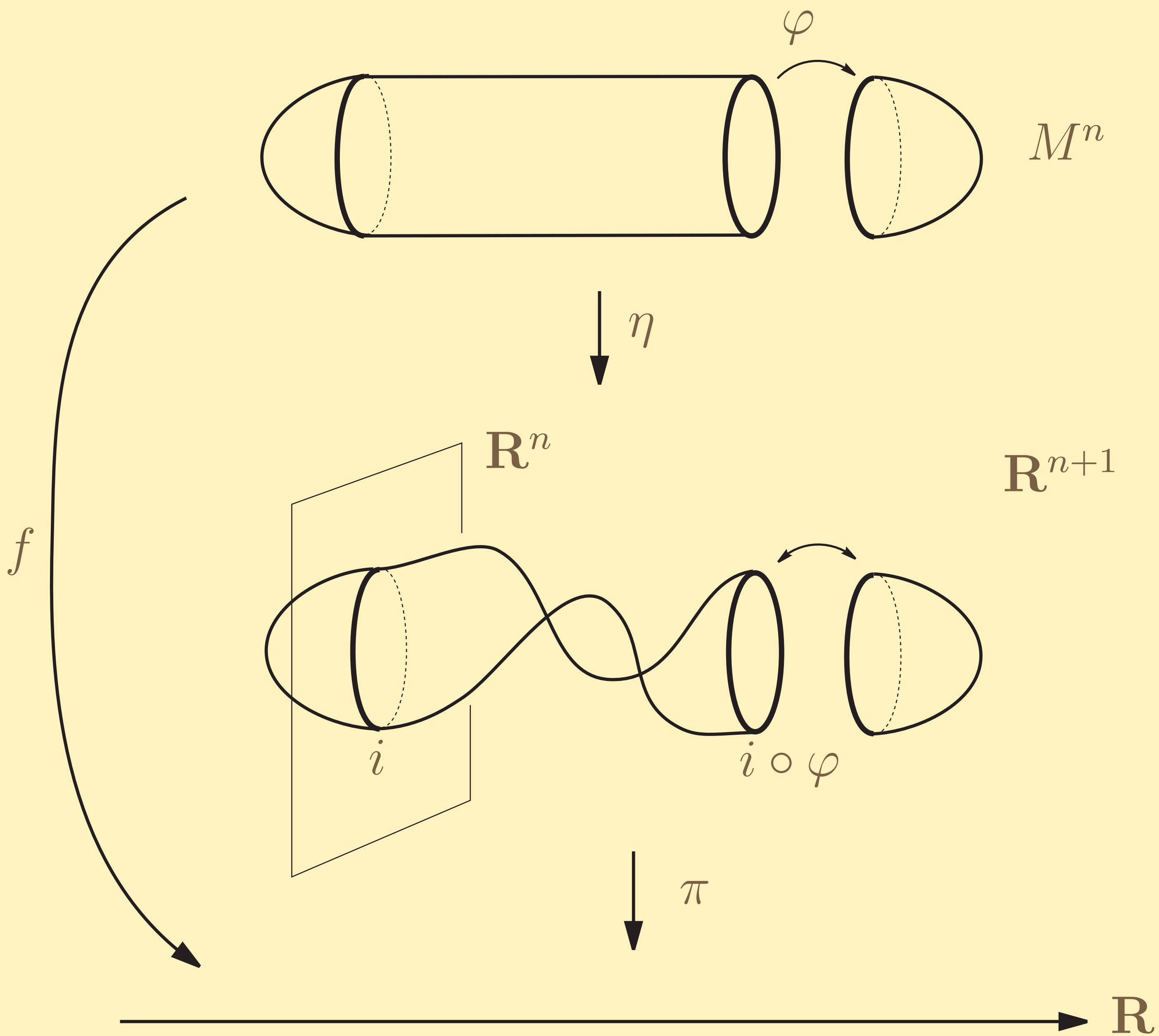
This is proved by using the **Smale–Hirsch Theory**.

Proof of Theorem 2.2



Embedding lift

Theorem 2.4 $n \geq 2$

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Remark 2.5 When $n = 1$, the existence problem of an embedding lift has recently been solved by M. Yamamoto.

§3. Desingularizing SGM's into R^2

Theorem 3.1 (Burlet–de Rham, 1974; Porto–Furuya, 1990; S, 1993)

M^n: closed connected **orientable** $(n \geq 2)$
$\exists$special generic map $f : M^n \to \mathbf{R}^2$
$\Longleftrightarrow$ M^n is diffeomorphic to

$$\Sigma^n \sharp \left(\sharp_{i=1}^r (\Sigma_i^{n-1} \times S^1) \right)$$

for some homotopy spheres Σ^n and Σ_i^{n-1}
(for $n \leq 6$, they are standard spheres).

Theorem 3.2 M^n: **orientable**, $n \geq 2$.
$f : M^n \to \mathbf{R}^2$ *special generic map*
There always exists an **immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ *s.t.* $f = \pi \circ \eta$.

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Remark 3.3 The case $n = 2$ is a consequence of Haefliger's result.

Stein factorization

Definition 3.4 $f : M^n \to \mathbf{R}^p$ C^∞ map $(n > p)$
For $x, x' \in M^n$, define $x \sim_f x'$ if
(i) $f(x) = f(x')(= y)$, and
(ii) x and x' belong to the same connected component of $f^{-1}(y)$.

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$W_f = M^n / \sim_f$ quotient space, $q_f : M^n \to W_f$ quotient map

$\exists! \bar{f} : W_f \to \mathbf{R}^p$ that makes the diagram commutative:

$$\begin{tikzcd} M^n \arrow[rr, "f"] \arrow[dr, "q_f"'] & & \mathbf{R}^p \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

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The above diagram is called the **Stein factorization** of f.

Example

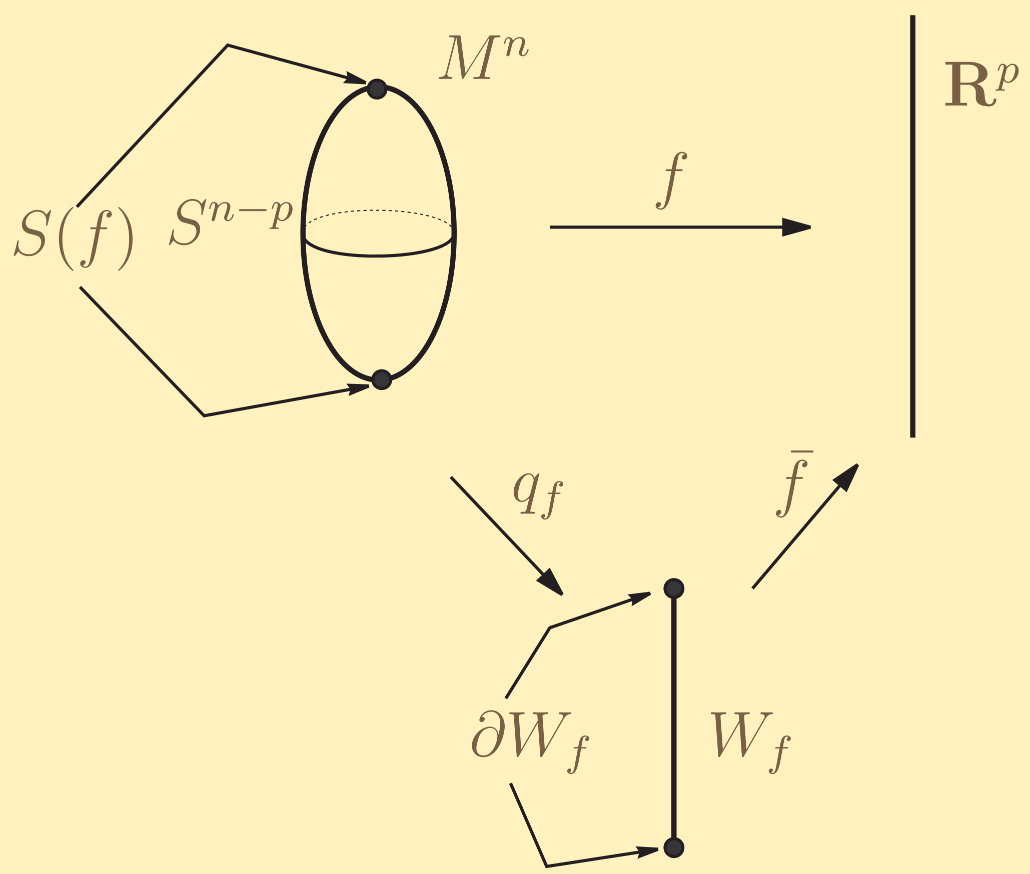

Figure 2: Stein factorization of a SGM

Proposition 3.5 $f : M^n \to \mathbf{R}^p$ *special generic map* $(n > p)$.

Fundamental properties

Proposition 3.5 $f : M^n \to \mathbf{R}^p$ *special generic map* $(n > p)$*.*

(1) *The singular point set* $S(f)$ *is a regular submanifold of* M^n *of dimension* $p-1$*,*

(2) W_f *has the structure of a smooth* p*-dim. manifold possibly with boundary such that* $\bar{f} : W_f \to \mathbf{R}^p$ *is an immersion.*

(3) $q_f|_{S(f)} : S(f) \to \partial W_f$ *is a diffeomorphism.*

(4) $q_f|_{M^n \setminus S(f)} : M^n \setminus S(f) \to \operatorname{Int} W_f$ *is a smooth* S^{n-p}*-bundle.*

Proof of Theorem 3.2

For a SGM $f : M \to \mathbf{R}^2$ $(p = 2)$, in order to construct an immersion lift $\eta : M^n \to \mathbf{R}^{n+1}$ of f, it is (almost) enough to construct a map

$$\widetilde{\eta} : M^n \to W_f \times \mathbf{R}^{n-1}$$

which is a lift of q_f and is an immersion into $\mathbf{R}^{n-1}$ on each S^{n-2}-fiber of the fibration $q_f : M^n \setminus S(f) \to \operatorname{Int} W_f$. Note that $\dim W_f = 2$.

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Outside of a 2-disk D in $\operatorname{Int} W_f$, we can construct such a lift $\widetilde{\eta}$ using lifts of special generic functions.
By construction, over ∂D, we have a family of **embeddings** $\eta_t : S^{n-2} \to \mathbf{R}^{n-1}$, $t \in \partial D$.

Proof of Theorem 3.2 (continued)

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This is possible if the following natural homomorphism is the zero map.

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where $\mathrm{Imm}^{\mathrm{TOP}}(S^{n-2}, \mathbf{R}^{n-1})$ denotes the space of locally flat topological immersions.

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For $n = 3, 4, 5$, we use some arguments on $\mathrm{Diff}(S^{n-2})$.

$\square$

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Turning the sphere $S^{n-2} \subset \mathbf{R}^{n-1}$ inside out (**sphere eversion**) is possible if and only if $n = 2, 4, 8$.

Embedding lift

Theorem 3.7 $f : M^n \to \mathbf{R}^2$ *special generic map,* $n \geq 3$
$\exists$**embedding** $\eta : M^n \to \mathbf{R}^{n+1}$ *s.t.* $f = \pi \circ \eta$
$\Longleftrightarrow \quad M \cong S^n$ *or* $\sharp^k(S^1 \times S^{n-1})$ *(diffeomorphic).*

Proof of $(\Leftarrow)$: The universal cover of $\sharp^k(S^1 \times S^{n-1})$ can be embedded in S^n. (Use the Schottky group argument. The free group of rank k can act on S^n as a Schottky group with totally disconnected limit set.)

Embedding lift

Theorem 3.7 *$f : M^n \to \mathbf{R}^2$ special generic map, $n \geq 3$*
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$(\Rightarrow)$: Standard argument.

§4. Further Results

Immersion lift

Theorem 4.1 *M^n: orientable, $(n, p) = (5, 3), (6, 3), (6, 4)$ or $(7, 4)$*
$f : M^n \to \mathbf{R}^p$ special generic map
*$\exists$**immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ s.t. $f = \pi \circ \eta$*
$\iff M^n$ is spin, i.e. $w_2(M^n) = 0$.

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Key to the proof:
The Stein factorization induces a smooth S^{n-p}-bundle

$$M^n \setminus S(f) \to \operatorname{Int} W_f.$$

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Key to the proof:
The Stein factorization induces a smooth S^{n-p}-bundle

$$M^n \setminus S(f) \to \mathrm{Int}\, W_f.$$

If $w_2(M^n) = 0$, then we can show that this is a trivial bundle.

Codimension -1 case

$f : M^n \to \mathbf{R}^p$ special generic map $(n > p)$
Orient $\mathbf{R}^p$. Then the quotient space W_f has the induced orientation.
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Theorem 4.2 M^n: *orientable,* $f : M^n \to \mathbf{R}^{n-1}$ *special generic*
$\exists$**immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ *s.t.* $f = \pi \circ \eta$
$\Longleftrightarrow [S(f)] = 0$ *in* $H_{n-2}(M^n; \mathbf{Z})$.

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If $[S(f)] = 0$, then we can show that this is a trivial bundle.

Summary

- Special generic function $M^n \to \mathbf{R}$ can always be desingularized by an immersion $M^n \to \mathbf{R}^{n+1}$.
  It can be desingularized by an embedding iff $M^n \cong S^n$ (diffeo.).
- Special generic map $f : M^n \to \mathbf{R}^2$ can always be desingularized by an immersion $M^n \to \mathbf{R}^{n+1}$ if M^n is orientable.
  It can be desingularized by an embedding iff $M^n \cong S^n$ or $\sharp^k(S^1 \times S^{n-1})$ (diffeomorphic).
  When M^n is non-orientable, f can be desingularized by an immersion iff $n = 2, 4, 8$ and $S(f)$ has an orientable nbhd.
- Special generic map $f : M^n \to \mathbf{R}^3$ with M^n orientable can be desingularized by an immersion $M^n \to \mathbf{R}^{n+1}$ iff M^n is spin for $n = 5$ and 6.
- Special generic map $f : M^n \to \mathbf{R}^{n-1}$ with M^n orientable can be desingularized by an immersion $M^n \to \mathbf{R}^{n+1}$ iff $[S(f)] = 0$ in $H_{n-2}(M^n; \mathbf{Z})$.

Thank you!

Embedding results

Theorem 4.3 M^n: *orientable,* $f : M^n \to \mathbf{R}^p$ *special generic map*
$(n, p) = (2, 1), (3, 2), (4, 3), (5, 3), (6, 3), (6, 4)$ *or* $(7, 4)$
$\Longrightarrow$ *∃regular homotopy of* **immersions** $\eta_t : M^n \to \mathbf{R}^{n+1}$, $t \in [0, 1]$,
with $f = \pi \circ \eta_0$ *s.t.* $f_t = \pi \circ \eta_t$ *is a special generic map,* $t \in [0, 1]$,
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Theorem 4.4 M^4: *orientable,* $\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
M^4 *can be embedded into* $\mathbf{R}^5$
$\Longleftrightarrow$ M^4 *is spin, i.e.* $w_2(M^4) = 0$.