NEW EXAMPLES OF NEUWIRTH–STALLINGS PAIRS AND NON-TRIVIAL REAL MILNOR FIBRATIONS

R. ARAÚJO DOS SANTOS, M.A.B. HOHLENWERGER, O. SAEKI AND T.O. SOUZA

Abstract. We use topology of configuration spaces to give a characterization of Neuwirth–Stallings pairs (S^5, K) with $\dim K = 2$. As a consequence, we construct polynomial map germs $(\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ with an isolated singularity at the origin such that their Milnor fibers are not diffeomorphic to a disk, thus putting an end to Milnor's non-triviality question. Furthermore, for a polynomial map germ $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ or $(\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$, $n \geqslant 3$, with an isolated singularity at the origin, we study the conditions under which the associated Milnor fiber has the homotopy type of a bouquet of spheres. We then construct, for every pair (n, p) with $n/2 \geqslant p \geqslant 2$, a new example of a polynomial map germ $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with an isolated singularity at the origin such that its Milnor fiber has the homotopy type of a bouquet of a positive number of spheres.

1. Introduction

In the book "Singular points of complex hypersurfaces" [16], John Milnor studied the topology of complex polynomial function germs in terms of the associated locally trivial fiber bundles. He also showed the existence of such structures for real polynomial map germs with an isolated singularity as follows.

Theorem 1.1 ([16, Theorem 11.2]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n \geqslant p \geqslant 2$, be a polynomial map germ with an isolated singularity at the origin. Then, there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leqslant \varepsilon_0$, the complement of an open tubular neighborhood of the link $K = f^{-1}(0) \cap S^{n-1}_\varepsilon$ in S^{n-1}_ε is the total space of a smooth fiber bundle over the sphere S^{p-1}, with each fiber F_f being a smooth compact $(n-p)$-dimensional manifold bounded by a copy of K, where S^{n-1}_ε denotes the sphere in $\mathbb{R}^n$ with radius ε centered at the origin.*

By using the conical structure of the singularity, Milnor proved that the diffeomorphism type of the link does not change for all $\varepsilon > 0$ small enough. Moreover, since the origin is an isolated singularity, we have that $0 \in \mathbb{R}^p$ is a regular value of $f|S^{n-1}_\varepsilon : S^{n-1}_\varepsilon \to \mathbb{R}^p$. Therefore, if the link $K = f^{-1}(0) \cap S^{n-1}_\varepsilon$ is not empty, then it is a smooth $(n-p-1)$-dimensional submanifold of the sphere with trivial normal bundle. It also implies that, for each fixed ε one can find a small enough δ, $0 < \delta \ll \varepsilon$, and a closed disk D^p_δ centered at the origin in $\mathbb{R}^p$ with radius δ, such

Date: June 5, 2014.

2000 *Mathematics Subject Classification.* Primary 32S55; Secondary 57R45, 58K05.

Key words and phrases. Neuwirth–Stallings pair, higher open book structure, configuration space, real Milnor fiber, real polynomial map germ.

that the restriction map $f : f^{-1}(D^p_\delta) \cap S^{n-1}_\varepsilon \to D^p_\delta$ is a smooth trivial fiber bundle, which implies the triviality of the fibration

$$f : f^{-1}(D^p_\delta \setminus \{0\}) \cap S^{n-1}_\varepsilon \to D^p_\delta \setminus \{0\}.$$

By composing this with the radial projection $\pi : D^p_\delta \setminus \{0\} \to S^{p-1}_\delta$ onto the boundary sphere and scaling to the unit sphere, one finds that the bundle structure on a neighborhood of the link K is given by

$$\frac{f}{\|f\|} : f^{-1}(D^p_\delta \setminus \{0\}) \cap S^{n-1}_\varepsilon \to S^{p-1}.$$

Now, one can glue this fiber bundle with that given in Theorem 1.1 along the common boundary $f^{-1}(S^{p-1}_\delta) \cap S^{n-1}_\varepsilon$ in a smooth way, so that we get a smooth locally trivial fiber bundle

$$S^{n-1}_\varepsilon \setminus K \to S^{p-1}.$$

Remark 1.2. Following Milnor's proof of Theorem 1.1, one sees that, although no precise information about the bundle projection above was provided, in the real settings in general one cannot expect that it is given by $f/\|f\|$ outside a neighborhood of the link. See Milnor's example [16, p. 99] or the following example adapted from Milnor's one. Consider $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by

$$f(x, y, z) = (x, x^2 + yx^2 + y^3 + yz^2).$$

It is easy to see that the singular point set is given by $\Sigma(f) = \{(0, 0, 0)\}$, and for all $\varepsilon > 0$ small enough, we have $K = \{(0, 0, \varepsilon), (0, 0, -\varepsilon)\}$. However, $f/\|f\|$ does have singular points and therefore it is not a submersion.

Definition 1.3 (Looijenga [15])**.** Let $K = K^{n-p-1}$ be an oriented submanifold of dimension $n - p - 1$ of the oriented sphere S^{n-1} with trivial normal bundle, or let $K = \emptyset$. Suppose that for some trivialization $c : N(K) \to K \times D^p$ of a tubular neighborhood $N(K)$ of K, the fiber bundle defined by the composition

$$N(K) \setminus K \xrightarrow{c} K \times (D^p \setminus \{0\}) \xrightarrow{\pi} S^{p-1},$$

with the last projection being given by $\pi(x, y) = y/\|y\|$, extends to a smooth fiber bundle $S^{n-1} \setminus K \to S^{p-1}$. Then, the pair (S^{n-1}, K^{n-p-1}) is called a *Neuwirth–Stallings pair*, or an *NS-pair* for short.

According to Theorem 1.1 and the previous discussion, for all $\varepsilon > 0$ sufficiently small, the pair $(S^{n-1}_\varepsilon, f^{-1}(0) \cap S^{n-1}_\varepsilon)$ is an NS-pair. In this case Looijenga called it the *NS-pair associated to the singularity.*

More recently, several generalizations of such a structure have been obtained. For instance, in [4] the authors considered a real analytic map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with non-isolated singularities at the origin and introduced a condition which also ensures that the pair $(S^{n-1}_\varepsilon, f^{-1}(0) \cap S^{n-1}_\varepsilon)$ is an NS-pair with the bundle projection given by $f/\|f\| : S^{n-1}_\varepsilon \setminus K \to S^{p-1}$, where the link $K = f^{-1}(0) \cap S^{n-1}_\varepsilon$ is a smooth manifold. It was called a *higher open book structure* of the sphere S^{n-1}_ε. In [1, 2] it was shown an extension of such structures for spheres of small and big enough radii (at infinity), but allowing singularity in the "binding" K. In this case, it was called a *singular open book structure* of the sphere.

As pointed out by Milnor in [16, p. 100], the hypothesis of Theorem 1.1 is so strong that examples are difficult to find, and he posed the following question.

Problem 1.4. For which dimensions $n \geqslant p \geqslant 2$ do non-trivial examples exist ?

Milnor did not exactly specify what "trivial" means here: however, he proposed to say that a real polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is *trivial* if the fiber F_f of the bundle given in Theorem 1.1 is diffeomorphic to a closed disk D^{n-p}. In particular, this implies that the fibers of the associated NS-pair are diffeomorphic to the $(n-p)$-dimensional open disk.

Remark 1.5. For a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin, it follows from [16, Appendix B] that the fibers of the associated Milnor fibration are diffeomorphic to a $2n$-dimensional disk if and only if 0 is a non-singular point of f; in fact, the function germ f is trivial if and only if the Milnor number $\mu_f = \deg_0(\nabla f(z))$ is equal to zero, where $\deg_0(\nabla f(z))$ stands for the topological degree of the map

$$\varepsilon \frac{\nabla f}{\|\nabla f\|} : S_\varepsilon^{2n+1} \to S_\varepsilon^{2n+1}$$

for all $\varepsilon > 0$ small enough, and

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_{n+1}} \right).$$

In [7] Church and Lamotke used results of Looijenga [15] and answered the above question in the following way.

Theorem 1.6. (a) *For $0 \leqslant n-p \leqslant 2$, non-trivial examples occur precisely for the dimensions $(n,p) \in \{(2,2),(4,3),(4,2)\}$.*

(b) *For $n-p \geqslant 4$, non-trivial examples occur for all (n,p).*

(c) *For $n-p=3$, non-trivial examples occur for $(5,2)$ and $(8,5)$. Moreover, if the 3-dimensional Poincaré Conjecture is false, then there are non-trivial examples for all (n,p). If the Poincaré Conjecture is true, then all examples are trivial except $(5,2)$, $(8,5)$ and possibly $(6,3)$.*

Since the Poincaré Conjecture has been proved to be true, we have that for $n-p=3$ the map f can be non-trivial only if $(n,p) \in \{(6,3),(8,5),(5,2)\}$. Therefore, Problem 1.4 has been open uniquely for the dimension pair $(6,3)$.

In [19] the authors used an extension of Milnor-Khimshiashvili's formula proved in [3] (see Theorem 5.3 of the present paper) for real isolated singularity map germs to show a manageable characterization of Church-Lamotke's results when the Milnor fiber is 3-dimensional, as follows.

Theorem 1.7. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$, be a polynomial map germ with an isolated singularity at the origin, and suppose $n-p=3$. Denote by $\deg_0(\nabla f_1)$ the topological degree of the mapping*

$$\varepsilon \frac{\nabla f_1}{\|\nabla f_1\|} : S_\varepsilon^{n-1} \to S_\varepsilon^{n-1},$$

where

$$\nabla f_1 = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_n} \right).$$

(a) *If the pair $(n,p) = (6,3)$, then the following three are equivalent.*

(i) *f is trivial.*
(ii) $\deg_0(\nabla f_1) = 0$.
(iii) *The link K is connected.*

(b) *If the pair $(n,p) = (8,5)$, then the following three are equivalent.*

(i) *f is trivial.*
(ii) $\deg_0(\nabla f_1) = 0$.
(iii) *The link K is not empty.*

(c) *If the pair $(n,p) = (5,2)$, then the following two are equivalent.*

(i) *f is trivial.*
(ii) $\pi_1(F_f) = 1$, *i.e. the Milnor fiber F_f is simply connected.*

In this paper we aim to give a characterization of NS-pairs (S^5, K) with $\dim K = 2$, and use it to prove the existence of non-trivial real polynomial map germs $(\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ with an isolated singularity at the origin, putting an end to Problem 1.4 posed by Milnor. For this, we will use tools from configuration spaces and a construction by Funar in [10, Section 2.7]. More precisely, we first classify fiber bundles $E^5 \to S^2$ with fiber the 3-sphere with the interiors of a disjoint union of 3-disks removed, such that the boundary fibrations are trivial. We will show that the isomorphism classes of such bundles are in one-to-one correspondence with the second homotopy group of a certain configuration space, and that its elements correspond to a skew-symmetric integer matrix. Then, we show that a given fiber bundle $E^5 \to S^2$ is associated with an NS-pair (S^5, K) if and only if the skew-symmetric matrix is unimodular. As a consequence, we see that the number of boundary components of a fiber is always odd. Furthermore, this allows us to construct a lot of non-trivial NS-pairs (S^5, K), and then the Looijenga construction [15] leads to non-trivial polynomial map germs with an isolated singularity.

Our second aim in this paper is to introduce necessary and sufficient conditions under which the Milnor fiber in the pairs of dimensions $(2n, n)$ and $(2n+1, n)$, $n \geqslant 3$, is, up to homotopy, a bouquet (or a wedge) of spheres. As applications, we give examples of polynomial map germs $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n/2 \geqslant p \geqslant 2$, such that the associated Milnor fiber is a bouquet of a non-zero number of spheres.

Throughout the paper, the (co)homology groups are with integer coefficients unless otherwise specified. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

2. Classification of bundles

Let (S^5, K^2) be an NS-pair, where K^2 is a closed 2-dimensional manifold embedded in the 5-dimensional sphere S^5. We have the associated fibration $\pi : S^5 \setminus \operatorname{Int} N(K^2) \to S^2$, where $N(K^2)$ denotes a closed tubular neighborhood of K^2 in S^5, and we denote by F its fiber, which is a compact 3-dimensional manifold bounded by a copy of K^2. Since S^5 does not fiber over S^2, we have $K^2 \neq \emptyset$. Furthermore, we have the homotopy exact sequence

$$\pi_2(S^5 \setminus \operatorname{Int} N(K^2)) \to \pi_2(S^2) \to \pi_1(F) \to \pi_1(S^5 \setminus \operatorname{Int} N(K^2)).$$

Since π is trivial on the boundary, it has a section, so that the homomorphism $\pi_2(S^5 \setminus \operatorname{Int} N(K^2)) \to \pi_2(S^2)$ is surjective. Furthermore, $S^5 \setminus \operatorname{Int} N(K^2)$ is simply connected. Therefore, the compact 3-dimensional manifold F is also simply connected.

Then, by a standard argument, we see that $K^2 \cong \partial F$ consists of some copies of S^2 and that F is homotopy equivalent to a 3-dimensional sphere with some points removed. Then, by the solution to the Poincaré Conjecture, we see that F is diffeomorphic to $S^3_{(k+1)}$ for some non-negative integer k, where $S^3_{(k+1)}$ denotes the 3-sphere with the interiors of $k+1$ disjoint 3-balls removed. Therefore, π is a smooth fiber bundle with fiber $S^3_{(k+1)}$ such that it is trivial on the boundary. In this section, we classify such fiber bundles.

Let $\operatorname{Diff}(S^3)$ be the topological group of diffeomorphisms of S^3. By the solution to the Smale Conjecture by Hatcher [12], we have that $\operatorname{Diff}(S^3)$ is homotopy equivalent to the orthogonal group $O(4)$.

Let us denote by B^3 the 3-dimensional closed ball and for a non-negative integer k, we denote by $\cup^{k+1}B^3$ the disjoint union of $k+1$ copies of B^3. We sometimes regard $\cup^{k+1}B^3$ to be "standardly" embedded in S^3, and we denote by $j_{k+1} : \cup^{k+1}B^3 \to S^3$ the inclusion map.

We denote by $\operatorname{Emb}(\cup^{k+1}B^3, S^3)$ the space of all smooth embeddings of $\cup^{k+1}B^3$ into S^3, not necessarily the standard one, and by $\operatorname{Diff}(S^3, \cup^{k+1}B^3)$ the subspace of $\operatorname{Diff}(S^3)$ consisting of those diffeomorphisms which restrict to the inclusion map j_{k+1} on $\cup^{k+1}B^3$. Furthermore, we denote by $\operatorname{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ the topological group of diffeomorphisms of $S^3_{(k+1)}$ which restrict to the identity on the boundary. Note that $S^3_{(k+1)} = S^3 \setminus \cup^{k+1} \operatorname{Int} B^3$.

The lemma below follows from [6, Proposition 1, p. 120].

Lemma 2.1. *The canonical map* $\operatorname{Diff}(S^3, \cup^{k+1}B^3) \to \operatorname{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ *induces isomorphisms*

$$\pi_i(\operatorname{Diff}(S^3, \cup^{k+1}B^3)) \to \pi_i(\operatorname{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)}))$$

for all i.

Now consider the natural map

$$\varphi : \operatorname{Diff}(S^3) \to \operatorname{Emb}(\cup^{k+1}B^3, S^3)$$

that sends each diffeomorphism of S^3 to its restriction to $\cup^{k+1}B^3$. The following is a consequence of the Cerf–Palais fibration theorem (see [6, Appendice], [17]).

Lemma 2.2. *The natural map φ as above is the projection of a locally trivial fiber bundle with fiber* $\operatorname{Diff}(S^3, \cup^{k+1}B^3)$.

Therefore, we have the homotopy exact sequence:

$$\begin{aligned} & \pi_2(\operatorname{Diff}(S^3), \mathrm{id}) \to \pi_2(\operatorname{Emb}(\cup^{k+1}B^3, S^3), j_{k+1}) \to \pi_1(\operatorname{Diff}(S^3, \cup^{k+1}B^3), \mathrm{id}) \\ \to \quad & \pi_1(\operatorname{Diff}(S^3), \mathrm{id}) \to \pi_1(\operatorname{Emb}(\cup^{k+1}B^3, S^3), j_{k+1}) \to \cdots . \end{aligned} \tag{2.1}$$

Let $\mathbb{F}_{k+1}(S^3)$ be the configuration space of $k+1$ points in S^3. This space can be naturally identified with $\operatorname{Emb}(\{0, 1, \ldots, k\}, S^3)$.

Lemma 2.3. *The space* $\mathrm{Emb}(\cup^{k+1}B^3, S^3)$ *is homotopy equivalent to* $\mathbb{F}_{k+1}(S^3) \times O(3)^{k+1}$.

Proof. For a given embedding $\eta : \cup^{k+1}B^3 \to S^3$, we associate the element of $\mathbb{F}_{k+1}(S^3)$ which sends the i-th point to the η-image of the center of the i-th 3-ball. Furthermore, by associating the normalized differential of η at each center, we get an element of $O(3)^{k+1}$. (Note that the tangent bundle TS^3 of S^3 is trivial, and we fix its trivialization here.) Then, we can show that the map $\mathrm{Emb}(\cup^{k+1}B^3, S^3) \to \mathbb{F}_{k+1}(S^3) \times O(3)^{k+1}$ thus obtained gives a homotopy equivalence. (For example, see [6, Appendice, §5, Proposition 3].) □

Recall that $\mathrm{Diff}(S^3) \simeq O(4)$. Furthermore, $\mathbb{F}_{k+1}(S^3)$ is 1-connected (see [9], [10, Proof of Proposition 2.30]) and $\pi_2(O(3)) = 0$. Thus, the exact sequence (2.1) turns into

$$\begin{aligned} 0 \quad &\to \pi_2(\mathbb{F}_{k+1}(S^3)) \to \pi_1(\mathrm{Diff}(S^3, \cup^{k+1}B^3), \mathrm{id}) \to \pi_1(O(4), \mathrm{id}) \\ &\to \pi_1(O(3)^{k+1}, \mathrm{id}) \to \cdots . \end{aligned}$$

Note that $\pi_1(O(4), \mathrm{id}) \cong \mathbb{Z}_2$ and $\pi_1(O(3)^{k+1}, \mathrm{id}) \cong (\mathbb{Z}_2)^{k+1}$. By choosing the standard embedding j_{k+1} so that it is equivariant with respect to the natural $SO(2)$-actions, we see that the homomorphism $\pi_1(O(4), \mathrm{id}) \to \pi_1(O(3)^{k+1}, \mathrm{id})$ sends the generator $1 \in \mathbb{Z}_2$ to $(1, 1, \ldots, 1) \in (\mathbb{Z}_2)^{k+1}$. In particular, it is injective. Thus, we have that the boundary homomorphism

$$\pi_2(\mathbb{F}_{k+1}(S^3)) \to \pi_1(\mathrm{Diff}(S^3, \cup^{k+1}B^3), \mathrm{id}) \tag{2.2}$$

is an isomorphism.

By [9] and [10, Proof of Proposition 2.30], we have the following important result.

Lemma 2.4. *The homotopy group* $\pi_2(\mathbb{F}_{k+1}(S^3))$ *is isomorphic to* $\mathbb{Z}^{k(k-1)/2}$.

Note that, for a smooth fiber bundle $S^3_{(k+1)} \hookrightarrow E^5 \to S^2$ with structure group $\mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$, its characteristic map is an element of

$$\pi_1(\mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)}), \mathrm{id}),$$

which is isomorphic to $\pi_1(\mathrm{Diff}(S^3, \cup^{k+1}B^3), \mathrm{id}) \cong \mathbb{Z}^{k(k-1)/2}$ by Lemma 2.1, (2.2) and Lemma 2.4.

In fact, given such a smooth fiber bundle $\pi : E^5 \to S^2$, one can consider $S^2 = D^2_1 \cup D^2_2$, where D^2_i, $i = 1, 2$, denote the 2-dimensional closed disk. Since each D^2_i is contractible, the restriction $\pi : \pi^{-1}(D^2_i) \to D^2_i$ is a trivial fiber bundle with fiber $S^3_{(k+1)}$, and we have $\pi^{-1}(D^2_i) \cong S^3_{(k+1)} \times D^2_i$. Hence, we can recover the total space

$$E^5 = (S^3_{(k+1)} \times D^2_1) \cup_h (S^3_{(k+1)} \times D^2_2)$$

for some diffeomorphism $h : S^3_{(k+1)} \times \partial D^2_2 \to S^3_{(k+1)} \times \partial D^2_1$ defined by $h(x, t) = (\alpha(t)(x), t)$, where $\alpha : S^1 = \partial D^2 \to \mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ corresponds to the characteristic map. Therefore, the structure of the fiber bundle is completely determined by the homotopy class $[\alpha] \in \pi_1(\mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)}), \mathrm{id})$.

3. Characterization of NS-pairs

For a non-negative integer k, let

$$S^3_{(k+1)} \hookrightarrow E^5 \xrightarrow{\pi} S^2 \tag{3.1}$$

be a smooth fiber bundle such that its restriction to the boundary $\partial S^3_{(k+1)} \hookrightarrow \partial E^5 \to S^2$ is a trivial bundle. In this section, we characterize such fiber bundles that arise from an NS-pair (S^5, K) with $K \cong \cup^{k+1} S^2$.

We start by gluing the trivial bundle $\cup^k (B^3 \times S^2) \to S^2$ along the k boundary components to get the B^3-fibration

$$\widetilde{\pi} : Y = E^5 \cup (\cup^k (B^3 \times S^2)) \to S^2. \tag{3.2}$$

This fibration is trivial, since the structure group $\mathrm{Diff}(B^3, \partial B^3)$ is contractible by Hatcher's solution to the Smale Conjecture [12], where $\mathrm{Diff}(B^3, \partial B^3)$ denotes the space of those diffeomorphisms of B^3 which fix ∂B^3 pointwise. Therefore, the total space Y of the fibration (3.2) is diffeomorphic to $B^3 \times S^2$. Then, by gluing $B^3_0 \times S^2$ to $Y = B^3 \times S^2$ by the map $\partial B^3_0 \times S^2 \to \partial B^3 \times S^2$ given by $(x, y) \mapsto (y, x)$, we get the sphere S^5, where B^3_0 is a copy of the closed 3-dimensional ball. Set $S^2_0 = x_0 \times S^2$, where x_0 is the center of B^3_0, and we write $N(S^2_0) = B^3_0 \times S^2$, which is identified with the closed tubular neighborhood of S^2_0 in S^5.

To fix the notation we write $\cup^{k+1} B^3 = \cup_{i=0}^k B^3_i$, and denote by x_i the center of B^3_i, where we consider $\cup^k B^3 = \cup_{i=1}^k B^3_i$. We also write $S^2_i = x_i \times S^2$, $i = 1, 2, \ldots, k$. Note that the 2-spheres S^2_i, $i = 0, 1, \ldots, k$, are all embedded in S^5 in a standard way. Furthermore, each of S^2_i, $i = 1, 2, \ldots, k$, has linking number ± 1 with S^2_0. In the following discussions, we orient S^2_i, $i = 0, 1, 2, \ldots, k$, in such a way that the linking number of S^2_i with S^2_0 is equal to $+1$, $i = 1, 2, \ldots, k$.

For $y \in S^2$, we have $\cup_{i=1}^k (B^3_i \times y) \subset \widetilde{\pi}^{-1}(y) \cong B^3$. Therefore, to each $y \in S^2$ we can naturally associate an element of the k-point configuration space $\mathbb{F}_k(\mathrm{Int}\, B^3) \cong \mathbb{F}_k(\mathbb{R}^3)$. This defines a *classifying map* $c : S^2 \to \mathbb{F}_k(\mathbb{R}^3)$.

Then, we have the following.

Lemma 3.1. *The isomorphism classes of the fibrations as in* (3.1) *are in one-to-one correspondence with* $\pi_2(\mathbb{F}_k(\mathbb{R}^3)) \cong \mathbb{Z}^{k(k-1)/2}$. *The correspondence is given by associating the homotopy class of the classifying map* c.

Recall that according to [10, Lemma 2.31], a fiber bundle (3.1) corresponds to the element $(\mathrm{lk}(S^2_i, S^2_j))_{1 \leqslant i < j \leqslant k} \in \mathbb{Z}^{k(k-1)/2}$ in the above correspondence, where lk denotes the linking number in S^5, and we fix orientations of S^2_i, $i = 1, 2, \ldots, k$, and S^5.

Proof of Lemma 3.1. As has been seen in Section 2, the isomorphism classes of the bundles in question are in one-to-one correspondence with $\pi_2(\mathbb{F}_{k+1}(S^3)) \cong \pi_1(\mathrm{Diff}(S^3, \cup^{k+1} B^3), \mathrm{id})$. On the other hand, it is known that $\pi_2(\mathbb{F}_k(\mathbb{R}^3))$ is naturally isomorphic to $\pi_2(\mathbb{F}_{k+1}(S^3))$ (see [9, p. 38]).

Recall the locally trivial fiber bundle

$$\mathrm{Diff}(S^3, \cup^{k+1} B^3) \xrightarrow{\iota} \mathrm{Diff}(S^3) \xrightarrow{\varphi} \mathrm{Emb}(\cup^{k+1} B^3, S^3)$$

of Lemma 2.2, where ι is the natural inclusion map. For the homotopy class $[\alpha] \in \pi_1(\mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})) \cong \pi_1(\mathrm{Diff}(S^3, \cup^{k+1} B^3))$ of the characteristic map, its ι_*-image vanishes in $\pi_1(\mathrm{Diff}(S^3))$, so that there exists a continuous map $\widetilde{\alpha} : D^2 \to \mathrm{Diff}(S^3)$ which extends $\iota \circ \alpha : S^1 \to \mathrm{Diff}(S^3)$. Then, the homotopy class of $\varphi \circ \widetilde{\alpha} : D^2 \to \mathrm{Emb}(\cup^{k+1} B^3, S^3)$ in $\pi_2(\mathrm{Emb}(\cup^{k+1} B^3, S^3)) \cong \pi_2(\mathbb{F}_{k+1}(S^3)) \cong \pi_2(\mathbb{F}_k(\mathbb{R}^3))$ is the class corresponding to $[\alpha]$ by the isomorphism (2.2). By construction, this coincides with the homotopy class of the classifying map c. This completes the proof. □

Now, we have the following natural question.

Problem 3.2. Which elements of $\pi_2(\mathbb{F}_k(\mathbb{R}^3)) \cong \mathbb{Z}^{k(k-1)/2}$ correspond to an NS-pair?

We answer this question in our main result in this section, as follows.

Theorem 3.3. *The fiber bundle $S^3_{(k+1)} \hookrightarrow E^5 \xrightarrow{\pi} S^2$ as in* (3.1) *arises from an NS-pair if and only if* $\det\left(\mathrm{lk}(S^2_i, S^2_j)\right)_{1 \leqslant i,j \leqslant k} = \pm 1$, *where* $\mathrm{lk}(S^2_i, S^2_i) = 0$ *for all* $1 \leqslant i \leqslant k$ *by convention.*

Note that $\left(\mathrm{lk}(S^2_i, S^2_j)\right)_{1 \leqslant i,j \leqslant k}$ is a $k \times k$ skew-symmetric integer matrix.

The rest of this section is devoted to the proof of the above theorem.

For a fiber bundle (3.1), let F be its fiber. We fix the trivialization of the boundary fibration, and we write $\partial E^5 = \cup_{i=0}^k (K_i \times S^2)$, where $K_i \cong S^2$ are the boundary components of $F \cong S^3_{(k+1)}$ and are oriented in such a way that the cycle represented by K_0 is homologous to the sum of the cycles represented by K_i, $i = 1, 2, \ldots, k$. Let $X^5 = E^5 \cup (\cup_{i=0}^k (K_i \times B^3))$ be the closed 5-dimensional manifold obtained by gluing E^5 and $\cup_{i=0}^k (K_i \times B^3)$ along their boundaries in such a way that the natural projection

$$\cup_{i=0}^k (K_i \times (B^3 \setminus \{0\})) \to S^2$$

extends to a smooth fibration $X^5 \setminus K \to S^2$, where $K = \cup_{i=0}^k (K_i \times \{0\})$. Note that in this notation, K_i is identified with ∂B^3_i, $i = 1, 2, \ldots, k$, and K_0 is identified with $* \times S^2 \subset \partial B^3_0 \times S^2$. We warn the reader that the way that $K_i \times B^3$ are attached to E^5 is very different from that for the construction of $Y \subset S^5$ in (3.2).

The theorem is a consequence of Lemmas 3.4 and 3.5 below.

Lemma 3.4. *The fiber bundle π* (3.1) *arises from an NS-pair if and only if X^5 is homotopy equivalent to S^5.*

The above lemma is a consequence of the well-known fact that every homotopy 5-sphere is standard [18].

Since we see easily that X^5 is simply connected, it suffices to study the homology group $H_2(X^5)$. Now consider the following piece of the Mayer-Vietoris exact sequence:

$$H_2(\cup_{i=0}^k (K_i \times \partial B^3)) \xrightarrow{\rho} H_2(E^5) \oplus H_2(\cup_{i=0}^k (K_i \times B^3)) \to H_2(X^5) \to 0,$$

where the homomorphism $\rho = (i_{1*}, -i_{2*})$ is induced by the inclusions $i_1 : \cup_{i=0}^k (K_i \times \partial B^3) \to E^5$ and $i_2 : \cup_{i=0}^k (K_i \times \partial B^3) \to \cup_{i=0}^k (K_i \times B^3)$.

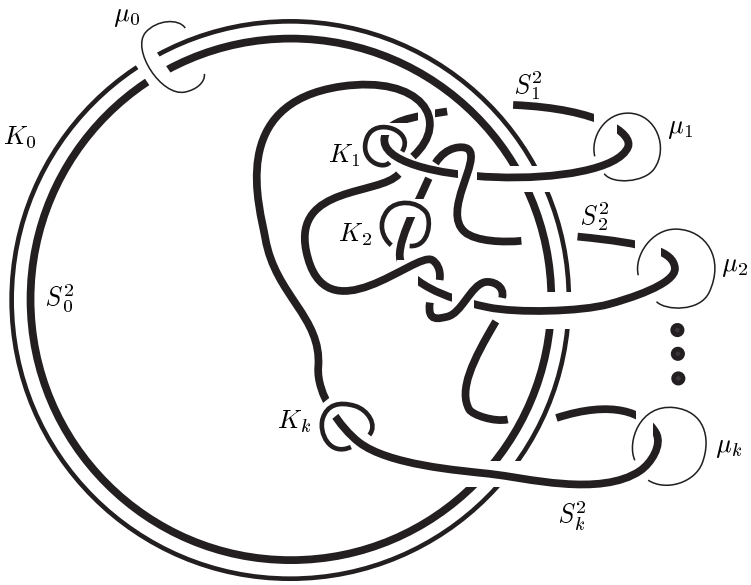

FIGURE 1. Situation in S^5

Figure 1 helps us to understand the images of the elements of $H_2(\cup_{i=0}^k(K_i \times \partial B^3))$ by the homomorphism ρ. Note that this depicts the situation in S^5 and not in X^5.

In order to describe the homomorphism ρ, let us fix bases of the homology groups. In the following, for a cycle z, we denote by $[z]$ the homology class represented by z. First, we have

$$H_2(\cup_{i=0}^k(K_i \times \partial B^3)) \cong \oplus_{i=0}^k H_2(K_i \times \partial B^3)$$

and each $H_2(K_i \times \partial B^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[K_i \times *]$ and $[y_i \times \partial B^3]$, where $y_i \in K_i$, $i = 0, 1, \ldots, k$. Furthermore, we have

$$H_2(\cup_{i=0}^k(K_i \times B^3)) \cong \oplus_{i=0}^k H_2(K_i \times B^3)$$

and each $H_2(K_i \times B^3) \cong \mathbb{Z}$ is generated by $\delta_i = [K_i \times *]$, $* \in B^3$, $i = 0, 1, \ldots, k$. On the other hand, we have

$$E^5 = S^5 \setminus (\cup_{i=0}^k \operatorname{Int} N(S_i^2)),$$

where $N(S_i^2) = B_i^3 \times S^2$ is the closed tubular neighborhood of S_i^2 in S^5, and $S_i^2 = x_i \times S^2$, $i = 1, 2, \ldots, k$, are so-called "Hopf duals" to S_0^2. Therefore, by Alexander duality we have

$$H_2(E^5) \cong H^2(\cup_{i=0}^k \operatorname{Int} N(S_i^2)).$$

Since $N(S_i^2) = B_i^3 \times S^2$, we can take the generators $\mu_i = [\partial B_i^3 \times *] \in H_2(E^5)$, $* \in S^2$, and $H_2(E^5)$ is freely generated by μ_i, $i = 0, 1, \ldots, k$. Here, we orient μ_i in such a way that the linking number of μ_i with S_i^2 is equal to $+1$. Observe that $\mu_i = i_{1*}([K_i \times *])$ for $i = 1, 2, \ldots, k$, and $\delta_i = i_{2*}([K_i \times *])$ for $i = 0, 1, \ldots, k$. Therefore, the images of the generators by the homomorphism ρ can be written as

follows.

$$\begin{aligned}
\rho([K_0 \times *]) &= \mu_1 + \mu_2 + \cdots + \mu_k - \delta_0, \\
\rho([K_i \times *]) &= \mu_i - \delta_i \quad (1 \leqslant i \leqslant k), \\
\rho([y_0 \times \partial B^3]) &= \mu_0 + 0, \\
\rho([y_i \times \partial B^3]) &= \sum_{0 \leqslant j \leqslant k, j \neq i} \mathrm{lk}(S_i^2, S_j^2)\mu_j + 0 \quad (1 \leqslant i \leqslant k).
\end{aligned}$$

Therefore, with respect to the above bases, the homomorphism ρ is represented by the following matrix:

$$R = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 0 & 0 & a_{11} & \cdots & a_{1k} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 1 & 0 & a_{k1} & \cdots & a_{kk} \\
 & & & & & & & \\
-1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0
\end{pmatrix},$$

where $a_{ij} = \mathrm{lk}(S_j^2, S_i^2)$, $i \neq j$, $1 \leqslant i, j \leqslant k$ and $a_{ii} = 0$. Observe that $a_{ij} = -a_{ji}$.

Lemma 3.5. *The* 5*-dimensional manifold* X^5 *is homotopy equivalent to* S^5 *if and only if* $\det R = \pm 1$.

Proof. If X^5 is homotopy equivalent to S^5, then its second homology group must vanish and therefore the homomorphism ρ must be an epimorphism, which implies that $\det R = \pm 1$.

Conversely, if ρ is an isomorphism, by the above Mayer-Vietoris exact sequence, we have $H_2(X^5) = 0$. Then by Poincaré duality, we see that X^5 has the homology of S^5. Then, a standard argument in algebraic topology shows that X^5 is homotopy equivalent to S^5.

This completes the proof of Lemma 3.5, and hence Theorem 3.3 has been proved. □

Since a skew-symmetric integer matrix has determinant ± 1 only if its size is even, we have the following.

Corollary 3.6. *If the fiber bundle* $S^3_{(k+1)} \hookrightarrow E^5 \to S^2$ *as in* (3.1) *arises from an NS-pair, then* k *must be even.*

Remark 3.7. In [15] Looijenga showed how to use the connected sum of NS-pairs to construct new ones. In fact, he proved that given an NS-pair (S^n, K^{n-p-1}) with fiber F, there exists a polynomial map germ $f : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{p+1}, 0)$ with an isolated singularity at the origin such that the associated NS-pair is isomorphic to the connected sum

$$(S^n, K^{n-p-1}) \sharp ((-1)^{n-1} S^n, (-1)^{n-p} K^{n-p-1})$$

with fiber being diffeomorphic to the interior of $\overline{F}\natural(-1)^{n-p}\overline{F}$, where "$\natural$" means the connected sum along the boundary. For further details the reader is referred to [15, p. 421].

The following proposition follows from the remark above and the previous result.

Proposition 3.8. *For every even integer $k \geqslant 0$, there exists an NS-pair (S^5, L_{k+1}) with L_{k+1} being diffeomorphic to the disjoint union of $k+1$ copies of S^2, and there exists a polynomial map germ $f : (\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ with an isolated critical point at 0 such that the associated NS-pair is isomorphic to $(S^5, L_{k+1}\sharp(-L_{k+1}))$. In particular, $L_{k+1}\sharp(-L_{k+1})$ consists of $2k+1$ connected components.*

Proof. First note that for each positive even integer k, there exists a skew-symmetric integer matrix of determinant ± 1. (For example, consider the direct sum of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and its copies.) Then, by the above argument, there exists an NS-pair (S^5, L_{k+1}) corresponding to that matrix. Now, one can just apply Looijenga's construction explained above in Remark 3.7. □

Corollary 3.9. *Given a real polynomial map germ as in Proposition 3.8 with $k > 0$, the fiber of the associated Milnor fibration is not diffeomorphic to a disk.*

This answers to Milnor's non-triviality question, Problem 1.4, for the dimension pair $(6, 3)$.

4. A generalization to higher dimensions

We can generalize the construction of Section 3 in higher dimensions as follows, in order to obtain new non-trivial examples of real polynomial map germs with an isolated singularity.

Let $n \geqslant 3$ be an integer. For a non-negative integer k, let $S^n_{(k+1)}$ denote the n-dimensional sphere S^n with the interior of the disjoint union of $k+1$ copies of the n-dimensional disks removed. In this section, we will construct a smooth fiber bundle

$$S^n_{(k+1)} \hookrightarrow E^{2n-1} \xrightarrow{\pi} S^{n-1}$$

such that the restriction to the boundary

$$\partial S^n_{(k+1)} \hookrightarrow \partial E^{2n-1} \xrightarrow{\pi} S^{n-1}$$

is a trivial bundle and that it arises from an NS-pair (S^{2n-1}, K^{n-1}).

Let $A = (a_{ij})$ be a $k \times k$ integer matrix which is $(-1)^n$-symmetric such that the diagonal entries all vanish. Let $S_0 \cong S^{n-1}$ be a trivially embedded oriented $(n-1)$-sphere in S^{2n-1}. Then, there exist mutually disjoint smoothly embedded oriented $(n-1)$-spheres S_i in S^{2n-1}, $i = 1, 2, \ldots, k$, such that

(1) S_i do not intersect S_0,
(2) S_i have linking number $+1$ with S_0,
(3) the linking number $\mathrm{lk}(S_i, S_j) = a_{ij}$, $i \neq j$.

Such embeddings do exist (for example, see [11]). Note that then

$$E^{2n-1} = S^{2n-1} \setminus \cup_{i=0}^{k} \operatorname{Int} N(S_i)$$

naturally fibers over S^{n-1} in such a way that the restriction to the boundary is trivial. (More precisely, consider the associated sub-fibration of the trivial fiber bundle $S^{2n-1} \setminus \operatorname{Int} N(S_0) \cong B^n \times S^{n-1} \to S^{n-1}$.) Then, by the same construction as in Section 3, we obtain an object (X^{2n-1}, K^{n-1}), where X^{2n-1} is a $(2n-1)$-dimensional smooth closed manifold, $X^{2n-1} \setminus \operatorname{Int} N(K^{n-1})$ is diffeomorphic to E^{2n-1}, and it fibers over S^{n-1} with fiber $S^n_{(k+1)}$ in such a way that the projection map is compatible with a trivialization of the closed tubular neighborhood $N(K^{n-1})$. Then, we have the following.

Lemma 4.1. *The manifold X^{2n-1} is a homotopy $(2n-1)$-sphere if and only if* $\det A = \pm 1$.

Proof. We see easily that E^{2n-1}, and hence X^{2n-1} is $(n-2)$-connected. Thus, X^{2n-1} is a homotopy $(2n-1)$-sphere if and only if $H_{n-1}(X^{2n-1})$ vanishes. Then, an argument using a Mayer-Vietoris exact sequence as in the previous section leads to the desired result. □

Combining this with the Looijenga construction (Remark 3.7), we have the following.

Corollary 4.2. *Let $n \geqslant 3$ be an integer. For every positive integer k with $k \equiv 1 \pmod 4$, there exists an NS-pair (S^{2n-1}, L_k) with L_k being diffeomorphic to the disjoint union of k copies of S^{n-1}, and there exists a polynomial map germ $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ with an isolated singularity at 0 such that the associated NS-pair is isomorphic to $(S^{2n-1}, L_k \sharp (-1)^n L_k)$. In particular, $L_k \sharp (-1)^n L_k$ consists of $2k-1$ connected components.*

Note that the associated Milnor fiber is diffeomorphic to $S^n_{(2k-1)}$ and is homotopy equivalent to the bouquet of $2k-2$ copies of the $(n-1)$-sphere.

Proof of Corollary 4.2. Set $\ell = (k-1)/2$, which is a nonnegative even integer. There exists an $\ell \times \ell$ $(-1)^n$-symmetric integer matrix A with determinant ± 1. By Haefliger [11], there exists an embedding of $\cup_{i=0}^{k} S_i^{n-1}$ into S^{2n-1} such that each component is embedded trivially, that S_i^{n-1} links with S_0 once for all $i > 0$, and that the linking matrix for $\cup_{i=1}^{\ell} S_i^{n-1}$ coincides with A, where each S_i^{n-1} is a copy of S^{n-1} and the linking number $\operatorname{lk}(S_i^{n-1}, S_i^{n-1}) = 0$, $i = 1, 2, \ldots, \ell$, by convention. Then by the above construction, we get the homotopy sphere X^{2n-1} in which the disjoint union of $\ell+1$ copies of the $(n-1)$-spheres is embedded. Then, the connected sum $X^{2n-1} \sharp (-X^{2n-1})$ is diffeomorphic to S^{2n-1} by [13], since $n \geqslant 3$. Therefore, by the connected sum construction, we get an NS-pair (S^{2n-1}, L_k), where L_k is diffeomorphic to the disjoint union of $2\ell + 1 = k$ copies of S^{n-1}.

Then, applying Looijenga's construction, we get the desired conclusion. □

5. Bouquet theorems for real isolated singularities

It is known that for a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin, the Milnor fiber F_f has the homotopy type of

a bouquet (or a wedge) of n-dimensional spheres. For real polynomial map germs with an isolated singularity, we cannot expect, in general, such a bouquet theorem, which can be seen as follows.

By Zeeman's twist spinning construction [20], one can construct an NS-pair (S^4, K^2) such that the fundamental group of the fiber is not a free group. Then Looijenga's construction leads to a non-trivial polynomial map germ $(\mathbb{R}^5, 0) \to (\mathbb{R}^2, 0)$ with an isolated singularity at the origin such that the Milnor fiber does not have a free fundamental group. Consequently, the Milnor fiber is not homotopy equivalent to a bouquet of spheres.

Remark 5.1. In the following, in order to get examples in higher dimensions, we use the spinning construction due to Artin [5]. For completeness, let us recall the construction. Let (S^m, K^k) be an NS-pair with $K^k \neq \emptyset$ and $\pi : S^m \setminus K^k \to S^{m-k-1}$ the associated fibration. We denote the fiber of the fibration $S^m \setminus \operatorname{Int} N(K^k) \to S^{m-k-1}$ by F^{k+1}, where $N(K^k)$ is the closed tubular neighborhood of K^k in S^m. We take a point $q \in K^k$ and a small m-disk neighborhood D in S^m such that $(D, D \cap K^k)$ is diffeomorphic to the standard disk pair (D^m, D^k) and that π restricted to $D \setminus (D \cap K^k)$ is equivalent to the standard fibration $D^m \setminus D^k \to S^{m-k-1}$. Then, we consider the quotient space of $(S^m \setminus \operatorname{Int} D, K^k \setminus (\operatorname{Int} D \cap K^k)) \times S^1$, where for each $x \in \partial D$, the points of the form (x, t) are identified to a point for all t. This kind of a construction is called the *spinning*. The resulting pair gives $(S^{m+1}, \widetilde{K}^{k+1})$, where $\widetilde{K}^{k+1}$ is a smoothly embedded submanifold of S^{m+1} of dimension $k+1$. By construction, there exists a fibration $\widetilde{\pi} : S^{m+1} \setminus \widetilde{K}^{k+1} \to S^{m-k-1}$ which restricts to π on $(S^m \setminus (\operatorname{Int} D \cup K^k)) \times \{t\}$ for each $t \in S^1$. It is straightforward to see that $(S^{m+1}, \widetilde{K}^{k+1})$ is an NS-pair. We call it the *spun* of the NS-pair (S^m, K^k). Note that the fiber $\widetilde{F}^{k+2}$ of the fibration $S^{m+1} \setminus \operatorname{Int} N(\widetilde{K}^{k+1}) \to S^{m-k-1}$ is diffeomorphic to the $(k+2)$-dimensional manifold obtained from $F^{k+1} \times S^1$ by identifying, for each $x \in \Delta^k$, the points of the form (x, t) to a point for all t, where Δ^k is a k-dimensional disk embedded in ∂F^{k+1} (near q). Note that the fundamental groups of $S^m \setminus K^k$ and $S^{m+1} \setminus \widetilde{K}^{k+1}$ are isomorphic, and that F^{k+1} and $\widetilde{F}^{k+2}$ also have isomorphic fundamental groups.

Let (S^4, K^2) be an NS-pair such that the fiber has non-free fundamental group. Then, applying once the spinning construction explained above to (S^4, K^2), one gets a non-trivial example in dimension $(6, 2)$ such that the Milnor fiber is not homotopy equivalent to a bouquet of spheres. Performing such procedures inductively one can construct examples in all pairs of dimensions $(n, 2)$, $n \geqslant 5$, such that the Milnor fiber is not homotopy equivalent to a bouquet of spheres.

In this section we give sufficient conditions to guarantee that the real Milnor fiber is homotopy equivalent to a bouquet of spheres of the same dimension, or of different dimensions.

Throughout this section we consider $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p \geqslant 2$, a polynomial map germ with an isolated singularity at the origin and the Milnor fibration (the "Milnor tube"),

$$f : f^{-1}(S^{p-1}_\delta) \cap D^n_\varepsilon \to S^{p-1}_\delta,$$

where $0 < \delta \ll \varepsilon \ll 1$. We denote by F_f its fiber and by $\beta_j = \operatorname{rank} H_j(F_f)$ its j-th Betti number.

Consider $\pi : (\mathbb{R}^p, 0) \to (\mathbb{R}^{p-1}, 0)$, $p \geqslant 3$, the germ of the canonical projection. Clearly, the composition map germ $G = \pi \circ f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{p-1}, 0)$ also has an isolated singularity at the origin and thus we have two fibrations:

$$f : f^{-1}(S_\delta^{p-1}) \cap D_\varepsilon^n \to S_\delta^{p-1},$$

and

$$G : G^{-1}(S_\delta^{p-2}) \cap D_\varepsilon^n \to S_\delta^{p-2}.$$

In [8] it was shown the relationship between the Milnor fibers F_f and F_G. It is worth pointing out that the results in [8] hold in a more general setting which includes the case of non-isolated singularities. Nevertheless, in the special case of an isolated singularity, it provides a positive answer to a conjecture stated by Milnor in [16, p. 100] as follows:

Theorem 5.2 ([8]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p \geqslant 2$, be a polynomial map germ with an isolated singularity at the origin and set $G = \pi \circ f : (\mathbb{R}^n, 0) \to (\mathbb{R}^{p-1}, 0)$. Then, the Milnor fiber F_G of G is homeomorphic to $F_f \times [0, 1]$, where for $p = 2$ the Milnor fiber of G is, by definition, the intersection of a sufficiently small closed ball centered at the origin and the inverse image of a regular value sufficiently close to the origin. In particular, the Milnor fibers F_f and F_G have the same homotopy type.*

In [16, Chapter 11], Milnor provided information concerning the topology of the fiber F_f. It was proved in Lemma 11.4 that if $n < 2(p-1)$, then the Milnor fiber is necessarily contractible. It also follows from the first paragraph of the proof that for $n > p \geqslant 2$ in general, if the link is not empty, then the fiber F_f is $(p-2)$-connected, i.e., $\pi_i(F_f) = 0$, $i = 0, 1, \ldots, p-2$.

In [3] the authors proved formulae relating the Euler characteristic of the Milnor fiber and the topological degree of the gradient mapping of the coordinate functions, which extends Milnor's formula for complex function germs with an isolated singularity (see [16, p. 64]) and Khimshiashvili's formula [14] for isolated singularity real analytic function germs, as follows.

Theorem 5.3 ([3]). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p \geqslant 2$, be a polynomial map germ with an isolated singularity at the origin, and consider*

$$f(x) = (f_1(x), f_2(x), \ldots, f_p(x)),$$

an arbitrary representative of the germ. Denote by $\deg_0(\nabla f_i(x))$, for $i = 1, 2, \ldots, p$, the topological degree of the map $\varepsilon \dfrac{\nabla f_i}{\|\nabla f_i\|} : S_\varepsilon^{n-1} \to S_\varepsilon^{n-1}$, for $\varepsilon > 0$ small enough.

(i) *If n is even, then $\chi(F_f) = 1 - \deg_0 \nabla f_1$. Moreover, we have*

$$\deg_0 \nabla f_1 = \deg_0 \nabla f_2 = \cdots = \deg_0 \nabla f_p.$$

(ii) *If n is odd, then $\chi(F_f) = 1$. Moreover, we have $\deg_0 \nabla f_i = 0$ for $i = 1, 2, \ldots, p$.*

In particular, from item (ii) above it follows that if the source space is odd-dimensional, then the fiber can never be homotopy equivalent to a bouquet of a positive number of spheres of the same dimension.

In the following subsections, we consider the dimension pairs $(2n, n)$ and $(2n + 1, n)$, and study conditions for a Milnor fiber to have the homotopy type of a

bouquet of spheres. We also study the dimension pairs $(2n, p)$ and $(2n+1, p)$ with $2 \leqslant p \leqslant n$ using the composition with a projection.

5.1. **The case of** $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$**.** Consider $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$, $n \geqslant 2$, a polynomial map germ with an isolated singularity at the origin. Note that S^{2n-1} does not smoothly fiber over S^{n-1}. Hence, in this case F_f is an n-dimensional compact orientable manifold with non-empty boundary and $\pi_i(F_f) = 0$ for $i = 0, 1, \ldots, n-2$. Since $\partial F_f \neq \emptyset$, we have $H_n(F_f) = 0$. Moreover, since F_f is orientable, the homology $H_{n-1}(F_f)$ is torsion free. Then, by Theorem 5.3, item (i), we have $\beta_{n-1} = (-1)^n \deg_0(\nabla f_1)$.

Furthermore, in the special case $n = 2$, the fibers are compact connected surfaces with non-empty boundary, so that they have the homotopy type of a bouquet of 1-dimensional spheres (circles). Furthermore, for $n = 3$, we have seen in Section 2 that the fibers are diffeomorphic to $S^3_{(k+1)}$ for some non-negative integer k, and hence they are homotopy equivalent to a bouquet of 2-spheres. Therefore, we may assume that $n \geqslant 4$. Note that if $\deg_0(\nabla f_1) = 0$, then the Milnor fiber is contractible.

It follows from the Hurewicz theorem that the Hurewicz homomorphism

$$\rho_{n-1} : \pi_{n-1}(F_f) \to H_{n-1}(F_f) \cong \mathbb{Z}^{\beta_{n-1}}$$

is an isomorphism. Then, for each generator $\gamma_i \in H_{n-1}(F_f) \cong \mathbb{Z}^{\beta_{n-1}}$ there exists a continuous map $\varphi_i : S^{n-1} \to F_f$, $i = 1, 2, \ldots, \beta_{n-1}$, such that $\gamma_i = \rho_{n-1}([\varphi_i]) = (\varphi_i)_*([S^{n-1}])$, where $[S^{n-1}] \in H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ is the fundamental class (given by the natural orientation of S^{n-1}). Therefore, we have the continuous map

$$\varphi : \bigvee^{\beta_{n-1}} S^{n-1} \to F_f$$

obtained by the wedge of the maps $\varphi_i : S^{n-1} \to F_f$, for $i = 1, 2, \ldots, \beta_{n-1}$, which is a homotopy equivalence by the Whitehead theorem.

Thus we have proved the following:

Proposition 5.4. *Let $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ be a polynomial map germ with an isolated singularity at the origin, $n \geqslant 2$. Given $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, a representative of the germ f, we have the following.*

(i) $\beta_{n-1} = (-1)^n \deg_0(\nabla f_1)$.
(ii) *The Milnor fiber F_f has the homotopy type of a bouquet of $(n-1)$-dimensional spheres*

$$\bigvee^{\beta_{n-1}} S^{n-1},$$

where it means a point when $\beta_{n-1} = 0$.

For $n \geqslant 4$, it follows from Theorem 1.6, item (c), that in all pairs of dimensions $(2n, n)$ there exist non-trivial examples. However, these non-trivial examples due to Church–Lamotke [7] have contractible Milnor fibers (with non-simply connected links). On the other hand, according to our construction in Section 4 together with Theorem 5.2, we get the following.

Corollary 5.5. *For each pair of dimensions $(2n, p)$, $2 \leqslant p \leqslant n$, there exists a real isolated singularity polynomial map germ $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^p, 0)$ such that the Milnor*

fiber is, up to homotopy, a bouquet of $(n-1)$-dimensional spheres with the number of spheres equal to $|\deg_0(\nabla f_1)| > 0$, where $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$.

5.2. **The case of $(\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$.** Consider now $f : (\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$, $n \geqslant 3$, a polynomial map germ with an isolated singularity at the origin. In this case, the Milnor fiber F_f is an $(n+1)$-dimensional compact orientable manifold with non-empty boundary and is $(n-2)$-connected. Then, $H_{n+1}(F_f) = 0$, $H_n(F_f)$ is torsion free and, by Theorem 5.3, $\beta_n = \beta_{n-1}$. Suppose that $H_{n-1}(F_f)$ is torsion free. Then, we have $H_{n-1}(F_f) \cong \mathbb{Z}^{\beta_{n-1}} \cong H_n(F_f)$. By Hurewicz theorem, the Hurewicz homomorphisms

$$\rho_{n-1} : \pi_{n-1}(F_f) \to H_{n-1}(F_f) \cong \mathbb{Z}^{\beta_{n-1}}$$

and

$$\rho_n : \pi_n(F_f) \to H_n(F_f) \cong \mathbb{Z}^{\beta_{n-1}}$$

are surjective. Then, by an argument similar to that used in the case $(2n, n)$, we can construct a homotopy equivalence

$$\varphi : \left(\bigvee^{\beta_{n-1}} S^{n-1} \right) \vee \left(\bigvee^{\beta_n} S^n \right) \to F_f.$$

Thus, we have proved the following result.

Proposition 5.6. *Let $f : (\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$, $n \geqslant 3$, be a real isolated singularity polynomial map germ. Then, the $(n-1)$-th homology $H_{n-1}(F_f)$ of the Milnor fiber is torsion free if and only if F_f has the homotopy type of a bouquet of spheres of the form*

$$\left(\bigvee^{\beta_{n-1}} S^{n-1} \right) \vee \left(\bigvee^{\beta_{n-1}} S^n \right) = \bigvee^{\beta_{n-1}} (S^{n-1} \vee S^n),$$

where it means a point when $\beta_{n-1} = 0$.

According to our construction in Section 4 together with Theorem 5.2 again, we get the following.

Corollary 5.7. *For each pair of dimensions $(2n+1, p)$, $2 \leqslant p \leqslant n$, there exists a real isolated singularity polynomial map germ $(\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^p, 0)$ such that the Milnor fiber is, up to homotopy, a bouquet of ℓ copies of the n-dimensional sphere and ℓ copies of the $(n-1)$-dimensional sphere with $\ell > 0$.*

Proof. For $n \geqslant 3$, this is a consequence of Proposition 5.6. For $n = 2$, we start with a non-trivial fibered knot (S^3, K). Then, its spun $(S^4, \widetilde{K})$ is a non-trivial fibered 2-knot, and its fiber is obtained by spinning a positive genus surface with boundary. Therefore, the fiber of $(S^4, \widetilde{K})$ has the homotopy type of a bouquet of a positive number of circles and 2-spheres. This completes the proof. □

5.3. **Application to k-stairs maps.** Given a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^q, 0)$, $n \geqslant q \geqslant 1$, with an isolated singularity at the origin, we say that a map germ $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $1 \leqslant q \leqslant p$, is a *$(p-q)$-stairs map for f* if there exist germs of polynomial functions $g_j : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $q+1 \leqslant j \leqslant p$, such that $F(x) = (f(x), g_{q+1}(x), g_{q+2}(x), \ldots, g_p(x))$ has an isolated singularity at the origin. If $p = q$, then by definition, we have $F(x) = f(x)$ and f is its own 0-stairs map.

Corollary 5.8. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n/2 \geqslant p \geqslant 2$, be a polynomial map germ with an isolated singularity at the origin. Then we have the following.*

(i) *If n is even and f admits a $(n/2 - p)$-stairs map, then the Milnor fiber is homotopy equivalent to a bouquet of $(n/2 - 1)$-dimensional spheres.*
(ii) *Suppose n is odd and $H_k(F_f)$ is torsion free for $k = (n-1)/2 - 1$, where F_f denotes the Milnor fiber. If f admits a $((n-1)/2 - p)$-stairs map, then the Milnor fiber is homotopy equivalent to a bouquet of k- and $(k+1)$-dimensional spheres, where the numbers of spheres are the same.*

Proof. Just apply Propositions 5.4, 5.6, and Theorem 5.2. □

We do not know whether or not the bouquet structure in the fiber characterizes the existence of such k-stairs maps for $k \geqslant 1$.

Acknowledgements

The authors would like to express their sincere gratitude to Edivaldo Lopes dos Santos and Denise de Mattos for stimulating discussions and invaluable comments.

The first author would like to thank very much the Fapesp grant 2013/23443-5 and CNPq/PQ-2 309819/2012-1. The second author would like to thank very much all Fapesp support given during the development of the PhD project, grant number 2012/12972-4. The third author has been supported in part by JSPS KAKENHI Grant Number 23244008, 23654028. The fourth author would like to thank CNPq/PDJ grant 502638/2012-5 during the post-doc in ICMC/USP.

References

[1] R. Araújo dos Santos, Y. Chen, and M. Tibăr, *Singular open book structures from real mappings*, Cent. Eur. J. Math. **11** (2013), 817–828.
[2] R. Araújo dos Santos, Y. Chen, and M. Tibăr, *Real polynomial maps and singular open books at infinity*, Accepted for publication in Scandinavica Math., preprint available on arxiv.org at http://arxiv.org/abs/1401.8286.
[3] R. Araújo dos Santos, D. Dreibelbis, and N. Dutertre, *Topology of the real Milnor fiber for isolated singularities*, Real and complex singularities, pp. 67–75, Contemp. Math., Vol. 569, Amer. Math. Soc., Providence, RI, 2012.
[4] R. Araújo dos Santos and M. Tibăr, *Real map germs and higher open book structures*, Geom. Dedicata **147** (2010), 177–185.
[5] E. Artin, *Zur Isotopie zweidimensionaler Flächen im R_4*, Abh. Math. Sem. Univ. Hamburg **4** (1925), 174–177.
[6] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ($\Gamma_4 = 0$), Lecture Notes in Mathematics, Vol. 53, Springer-Verlag, Berlin-New York, 1968.
[7] P.T. Church and K. Lamotke, *Non-trivial polynomial isolated singularities*, Nederl. Akad. Wetensch. Proc. Ser. A 78 = Indag. Math. **37** (1975), 149–154.
[8] N. Dutertre and R. Araújo dos Santos, *Topology of real Milnor fibration for non-isolated singularities*, Submitted for publication, preprint available on arxiv.org at http://arxiv.org/abs/1211.6233.
[9] E.R. Fadell and S.Y. Husseini, *Geometry and topology of configuration spaces*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001.
[10] L. Funar, *Global classification of isolated singularities in dimensions* $(4,3)$ *and* $(8,5)$, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 819–861.
[11] A. Haefliger, *Differentiable links*, Topology **1** (1962), 241–244.
[12] A.E. Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. (2) **117** (1983), 553–607.

[13] M.A. Kervaire and J.W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537.
[14] G.N. Khimshiashvili, *The local degree of a smooth mapping* (Russian), Sakharth. SSR Mecn. Akad. Moambe **85** (1977), 309–312.
[15] E. Looijenga, *A note on polynomial isolated singularities*, Nederl. Akad. Wetensch. Proc. Ser. A 74 = Indag. Math. **33** (1971), 418–421.
[16] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, Vol. 61, Princeton University Press, 1968.
[17] R.S. Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **34** (1960), 305–312.
[18] S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961), 391–406.
[19] T.O. Souza, M.A.B. Hohlenwerger, D. de Mattos, and R.N. Araújo dos Santos, *New characterization of trivial maps in* 3*-dimensional real Milnor fibers*, JP J. Geom. Topol. **12** (2012), 207–217.
[20] E.C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.

R. Araújo dos Santos, M.A.B. Hohlenwerger: Universidade de São Paulo, ICMC, Av. Trabalhador São Carlense, 400, Centro Postal Box 668, 13560-970, São Carlos, SP, Brazil

E-mail address: rnonato@icmc.usp.br, amelia@icmc.usp.br

O. Saeki: Institute of Mathematics for Industry, Kyushu University, Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan

E-mail address: saeki@imi.kyushu-u.ac.jp

T.O. Souza: Faculdade de Matemática, Universidade Federal de Uberlândia, Campus Santa Mônica - Bloco 1F - Sala 1F120, Av. João Naves de Avila, 2121, Uberlândia, MG, CEP: 38.408-100, Brazil

E-mail address: olitaciana@gmail.com