Cobordism of knots associated with complex hypersurface singularities

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§1. An Example

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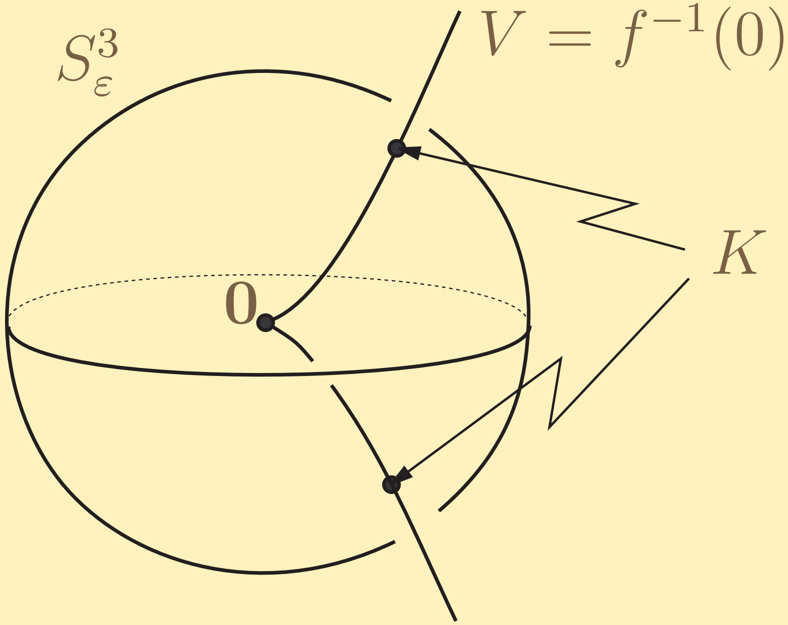

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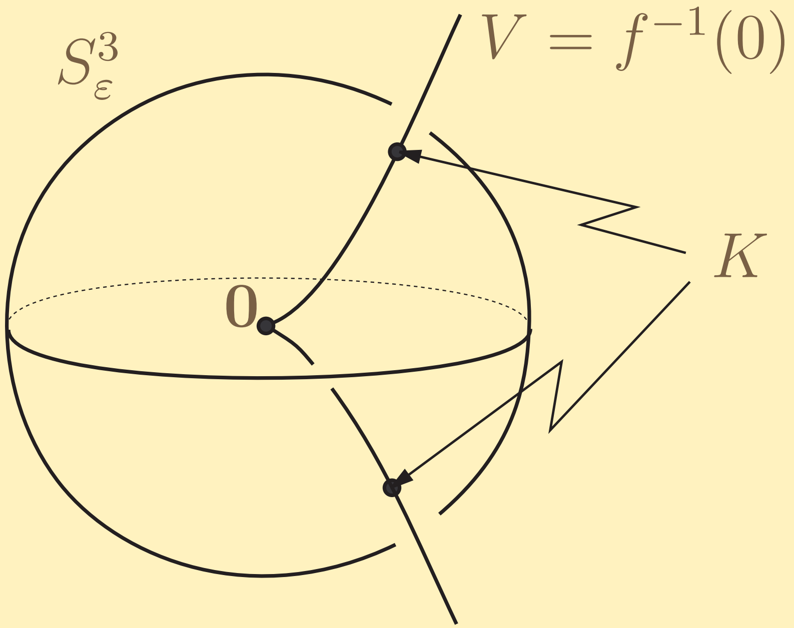

K is a knot in S^3 !

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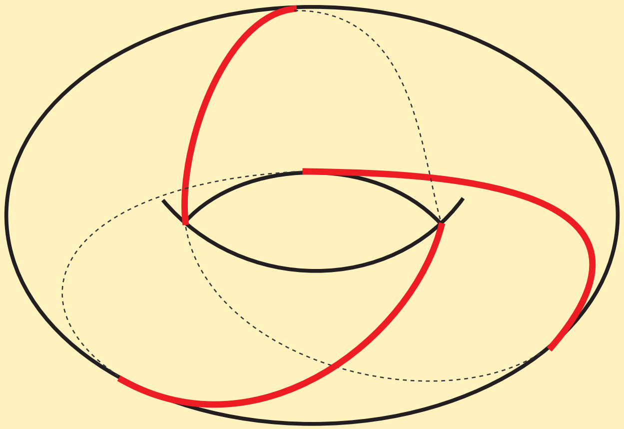
This is a trefoil knot!



§2. Milnor's Fibration Theorem

$f = f(z_1, z_2, \ldots, z_{n+1})$ complex polynomial with $f(\mathbf{0}) = 0$
s.t. $\mathbf{0}$ is an **isolated critical point** of f, i.e.,

$$\frac{\partial f}{\partial z_1}(z) = \cdots = \frac{\partial f}{\partial z_{n+1}}(z) = 0 \iff z = \mathbf{0}$$

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$V = f^{-1}(0) \subset \mathbf{C}^{n+1}$: complex hypersurface
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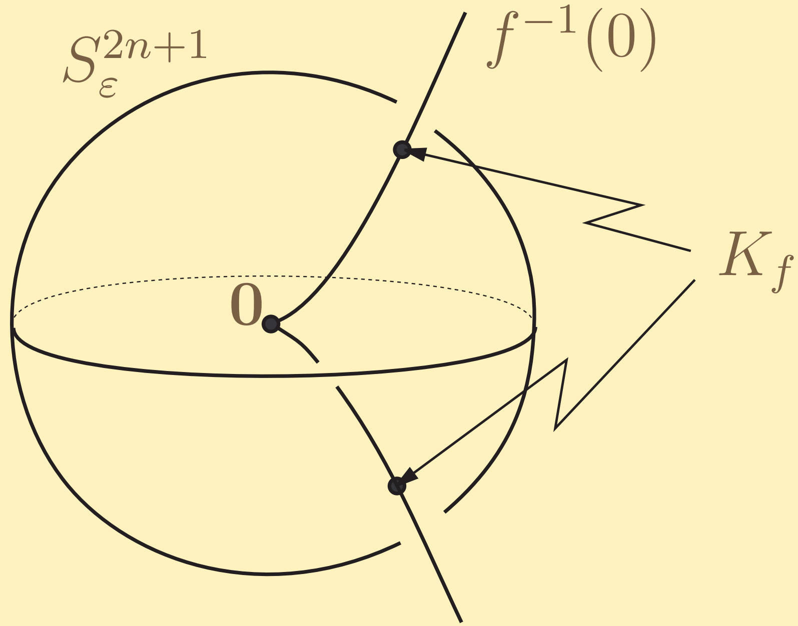
K_f is a $(2n-1)$-dim. smooth closed manifold embedded in S_ε^{2n+1}.

Cone structure theorem



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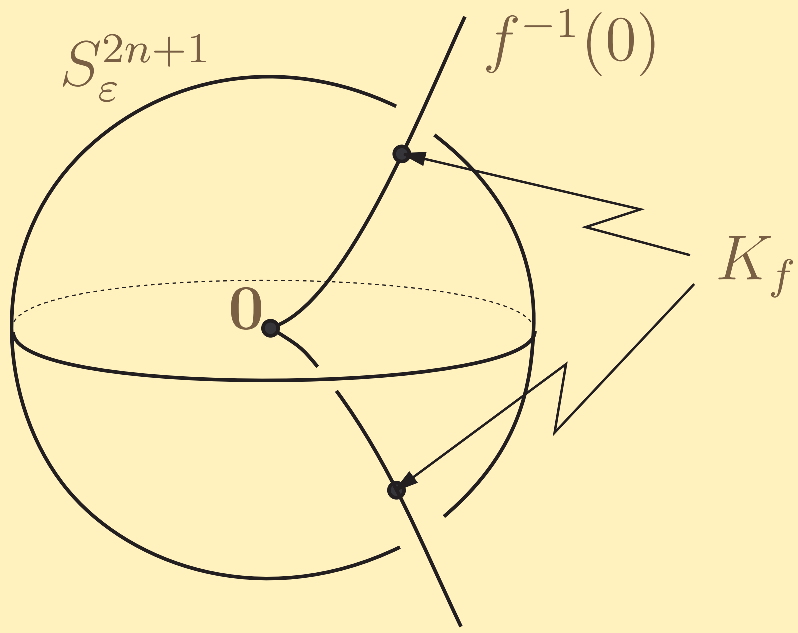

Theorem 2.1 (Milnor, 1968)
$(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2}) \approx \mathrm{Cone}(S_\varepsilon^{2n+1}, K_f)$ *(homeo.)*

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(1) $\varphi_f = f/|f| : S_\varepsilon^{2n+1} \setminus K_f \to S^1$ *is a locally trivial fibration.*

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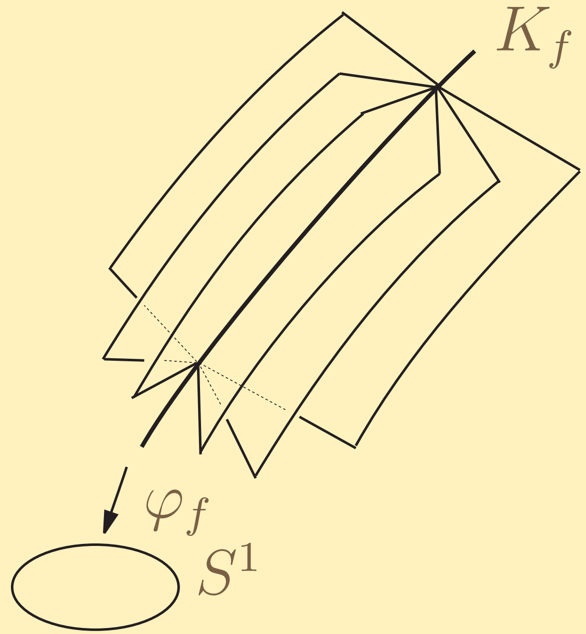

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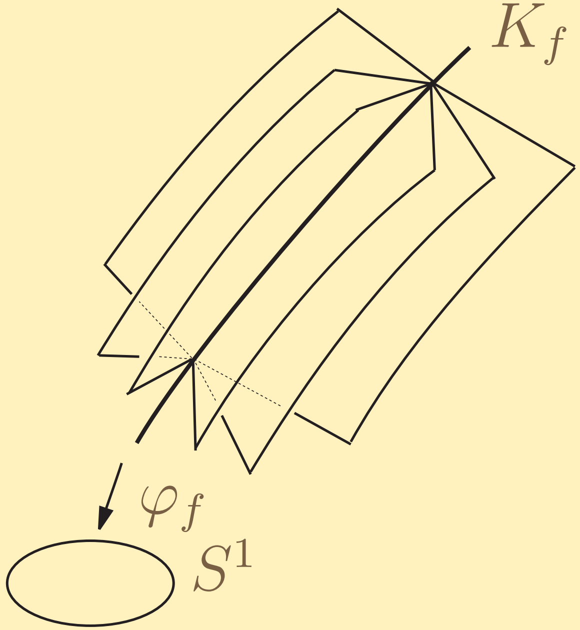

K_f is a fibered knot, but K_f may not be a sphere.

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$n = 1$: fibered link in S^3
$n = 2$: connected 3-manifold in S^5 with simply connected fibers
$n = 3$: simply connected 5-manifold in S^7 with 2-connected fibers
$\vdots$

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Proposition 2.4 *The isotopy class of the link K_f is completely determined by the components and their linking numbers.*

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We denote by $\Delta_f(t)$ the **characteristic polynomial** of

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It is known that $\Delta_f(t)$ coincides with the **Alexander polynomial** of K_f.

§3. Classification

The **Seifert form** associated with f is the bilinear form

$$L_f : H_n(F_f;\mathbf{Z}) \times H_n(F_f;\mathbf{Z}) \to \mathbf{Z} \quad \text{define by}$$

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Theorem 3.1 (Durfee, Kato, 1974) *For $n \geq 3$, two algebraic knots K_f and K_g are isotopic $\Longleftrightarrow$ the Seifert forms L_f and L_g are isomorphic.*

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For $n = 1, 2$, the above theorem does not hold.

Theorem 3.3 (S, 1999) *For every $k \geq 2$, there exist simple fibered 3-knots $K_1, K_2, \ldots, K_k$ s.t.*
(1) they are all diffeomorphic as abstract 3-manifolds,
(2) their Seifert forms are all isomorphic,
(3) K_i and K_j are not isotopic if $i \neq j$.

For $a_1, a_2, \ldots, a_{n+1} \geq 2$, set

$$f(z_1, z_2, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}},$$

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Theorem 3.4 (Yoshinaga–Suzuki, 1978)
For two Brieskorn–Pham type polynomials f and g, the following three are equivanent.

(1) *K_f and K_g are isotopic.*

(2) *f and g have the same set of exponents.*

(3) $\Delta_f(t) = \Delta_g(t)$.

§4. Cobordism

Definition 4.1 Two oriented $(2n-1)$-knots K_0 and K_1 in S^{2n+1} are **cobordant** if $\exists X (\cong K_0 \times [0,1]) \subset S^{2n+1} \times [0,1]$, a properly embedded oriented $2n$-dim. submanifold, such that

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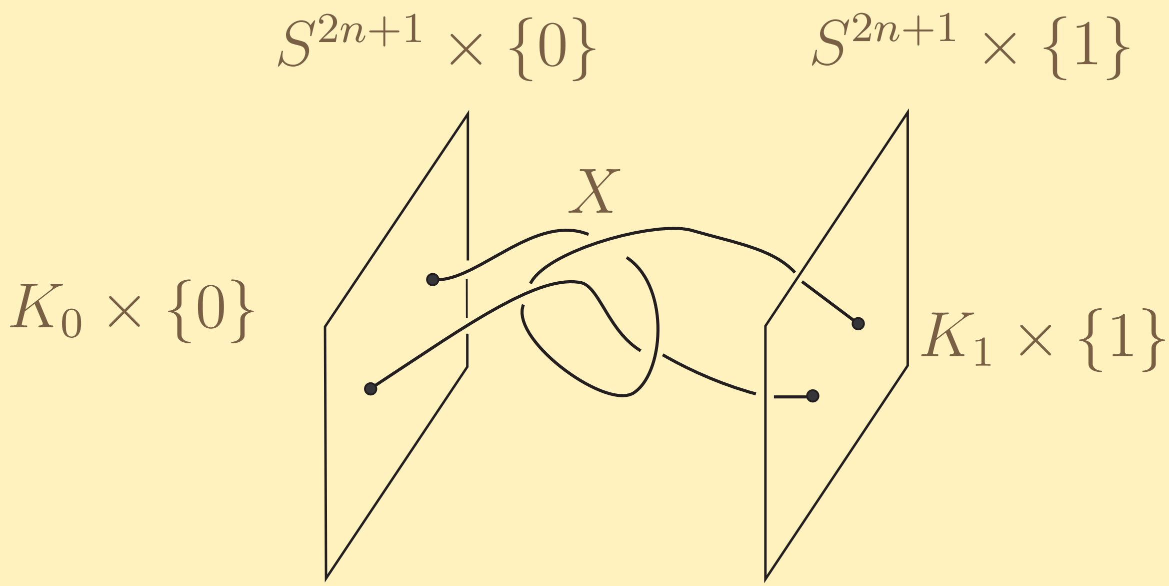

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Today's Topic: Problem 4.2 for weighted homogeneous polynomials (in particular, Brieskorn polynomials).

§5. Results

Known results

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It has long been conjectured that cobordant algebraic knots would be isotopic for all n.

Case where $n = 1$ and the polynomials are irreducible at $\mathbf{0}$.

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For algebraic knots K_f and K_g in S^3_ε, the following three are equivalent.
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du Bois–Michel, 1993
Examples of two algebraic (spherical) knots that are cobordant, but are not isotopic, $n \geq 3$.

Let $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms defined on free $\mathbf{Z}$-modules of finite ranks.

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Definition 5.2 Suppose $m = \operatorname{rank} G$ is even.
A direct summand $M \subset G$ is called a **metabolizer**
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Algebraic cobordism

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Theorem 5.3 (Blanlœil–Michel, 1997) *For $n \geq 3$, two algebraic knots K_f and K_g are cobordant $\Longleftrightarrow$ Seifert forms L_f and L_g are algebraically cobordant.*

In fact, for (possibly non-simple) fibered knots, we have the following.

Cobordism of fibered knots

In fact, for (possibly non-simple) fibered knots, we have the following.

Theorem 5.4 (Blanlœil–S., 2011)
K_0, K_1: fibered $(2n-1)$-knots, $(n-2)$-connected, $n \geq 3$.
K_0 and K_1 are cobordant
$\Longleftrightarrow$ their Seifert forms are algebraically cobordant.

Remark 5.5 At present, there is no efficient criterion for algebraic cobordism.
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Witt equivalence

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Lemma 5.6 *If two algebraic knots K_f and K_g are cobordant, then their Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.*

Weighted homogeneous poly.

Let f be a **weighted homogeneous polynomial** in $\mathbf{C}^{n+1}$, Let f be a **weighted homogeneous polynomial** in $\mathbf{C}^{n+1}$, i.e. $\exists(w_1, w_2, \ldots, w_{n+1}) \in \mathbf{Q}_{>0}^{n+1}$, called **weights**, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

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Brieskorn–Pham type polynomial $z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$
$\Longrightarrow$ weighted homogeneous of weights $(a_1, a_2, \ldots, a_{n+1})$

Set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1}.$$

$P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbf{Z}$ for some integer $m > 0$.

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Two non-degenerate weighted homogeneous polynomials f and g have the **same weights** if and only if $P_f(t) = P_g(t)$.

Theorem 5.7 *Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are* **Witt equivalent over $\mathbf{R}$** *iff*

$$P_f(t) \equiv P_g(t) \mod t+1.$$

The above theorem should be compared with the following.

Remark 5.8 *The Seifert forms L_f and L_g associated with non-degenerate weighted homogeneous polynomials f and g are* **isomorphic over $\mathbb{R}$** *iff*

$$P_f(t) \equiv P_g(t) \mod t^2 - 1.$$

Proposition 5.9 *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \textit{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham type polynomials.

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$$\prod_{j=1}^{n+1} \cot \frac{\pi\ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi\ell}{2b_j}$$

holds for all odd integers ℓ.

Theorem 5.10 *Suppose that for each of the Brieskorn–Pham type polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

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no exponent is a multiple of another one.
Then, the knots K_f and K_g are **cobordant** *iff*

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

up to order.

Case of two or three variables

Proposition 5.11 *Let f and g be weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w'_1, w'_2), respectively, with $w_j, w'_j \geq 2$.*
If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w'_j$, $j = 1, 2$, up to order.

Proposition 5.11 *Let f and g be weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w_1', w_2'), respectively, with $w_j, w_j' \geq 2$.*
If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w_j'$, $j = 1, 2$, up to order.

Proposition 5.12 *Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn–Pham type polynomials of three variables.*
If the Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$, then $a_j = b_j$, $j = 1, 2, 3$, up to order.

§6. Proofs

Proof of Theorem 5.7

Theorem 5.7 Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are **Witt equivalent over $\mathbf{R}$** iff

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Proof. For simplicity, we consider the case of n even.

Let $\Delta_f(t)$ be the **characteristic polynomial of the monodromy**

$$h_* : H_n(\operatorname{Int} F_f; \mathbf{C}) \to H_f(\operatorname{Int} F_f; \mathbf{C}),$$

where $h : \operatorname{Int} F_f \to \operatorname{Int} F_f$ is the characteristic diffeomorphism of the Milnor fibration $\varphi_f : S_\varepsilon^{2n+1} \setminus K_f \to S^1$.

Proof of Theorem 5.7 (Continued)

We have

$$H^n(F_f;\mathbf{C}) = \oplus_\lambda H^n(F_f;\mathbf{C})_\lambda,$$

where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f;\mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

Proof of Theorem 5.7 (Continued)

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where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f; \mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

The **intersection form** $S_f = L_f + L_f^T$ of F_f on $H^n(F_f; \mathbf{C})$ decomposes as the orthogonal direct sum of $(S_f)|_{H^n(F_f;\mathbf{C})_\lambda}$.

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Let $\mu(f)^+_\lambda$ (resp. $\mu(f)^-_\lambda$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F;\mathbf{C})_\lambda}$.

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Let $\mu(f)^+_\lambda$ (resp. $\mu(f)^-_\lambda$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F;\mathbf{C})_\lambda}$.
The integer

$$\sigma_\lambda(f) = \mu(f)^+_\lambda - \mu(f)^-_\lambda$$

is called the **equivariant signature** of f with respect to λ.

Lemma 6.1 (Steenbrink, 1977)

Set $P_f(t) = \sum c_\alpha t^\alpha$. Then we have

$$\sigma_\lambda(f) = \sum_{\substack{\lambda=\exp(-2\pi i\alpha) \\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi i\alpha), \\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, where $i = \sqrt{-1}$, and $\lfloor\alpha\rfloor$ is the largest integer not exceeding α.

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for $\lambda \neq 1$, *where* $i = \sqrt{-1}$, *and* $\lfloor\alpha\rfloor$ *is the largest integer not exceeding* α.

Remark 6.2 The equivariant signature for $\lambda = 1$ is always equal to zero.

Proof of Theorem 5.7 (Continued)

Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.
$\Longrightarrow \quad \sigma_\lambda(f) = \sigma_\lambda(g) \quad$ for all λ.

Proof of Theorem 5.7 (Continued)

Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.
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Set

$$P_f(t) = P_f^0(t) + P_f^1(t), \quad \text{where}$$

$$\begin{aligned} P_f^0(t) &= \sum_{\lfloor\alpha\rfloor\equiv 0 \pmod 2} c_\alpha t^\alpha, \\ P_f^1(t) &= \sum_{\lfloor\alpha\rfloor\equiv 1 \pmod 2} c_\alpha t^\alpha. \end{aligned}$$

We define $P_g^0(t)$ and $P_g^1(t)$ similarly.

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We define $P_g^0(t)$ and $P_g^1(t)$ similarly.
Since the equivariant signatures of f and g coincide, we have

$$\begin{aligned} tP_f^0(t) - P_f^1(t) &\equiv tP_g^0(t) - P_g^1(t) \mod t^2 - 1, \\ tP_f^1(t) - P_f^0(t) &\equiv tP_g^1(t) - P_g^0(t) \mod t^2 - 1. \end{aligned}$$

Proof of Theorem 5.7 (Continued)

Adding up these two congruences we have

$$(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2-1, \tag{1}$$

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Conversely, suppose that (2) holds.
$\Longrightarrow$ (1) holds.
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Proof of Theorem 5.7 (Continued)

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Conversely, suppose that (2) holds.
$\Longrightarrow$ (1) holds.
$\Longrightarrow$ f and g have the same equivariant signatures.

Then, we can prove that they are Witt equivalent over $\mathbf{R}$.

This completes the proof.

Proof of Proposition 5.9

Proposition 5.9 Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham type polynomials. Then, their Seifert forms are Witt equivalent over $\mathbf{R}$ iff

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integers ℓ.

Proof of Proposition 5.9 (Cont.)

Proof.

$P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some m.
Put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$.

Then, $P_f(t) \equiv P_g(t) \mod t+1$ holds
$\iff \quad Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$.

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Then, $P_f(t) \equiv P_g(t) \mod t+1$ holds
$\iff \quad Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$.

Note that ξ is of the form

$$\exp(\pi\sqrt{-1}\ell/m)$$

with ℓ odd and that

$$\frac{-1-\exp(\pi\sqrt{-1}\ell/a_j)}{\exp(\pi\sqrt{-1}\ell/a_j)-1} = \sqrt{-1}\cot\frac{\pi\ell}{2a_j}.$$

Then, we immediately get Proposition 5.9.

Proof of Theorem 5.10

Theorem 5.10 Suppose that for each of the Brieskorn–Pham type polynomials

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.
Then, the knots K_f and K_g are **cobordant** iff

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

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up to order.

This is a consequence of the "**Fox–Milnor type relation**" for the Alexander polynomials of cobordant algebraic knots.

Open problem

Problem 6.3 *Are the exponents cobordism invariants for Brieskorn–Pham type polynomials?*

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$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \quad \forall \text{odd integers } \ell$$

$\Longrightarrow \quad a_j = b_j$ up to order ?

Thank you!