Elimination of Definite Fold and Broken Lefschetz Fibrations

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Kyushu University

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§1. Introduction

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(3) A singularity that has the normal form

$$\boxed{(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 - x_4^2)}$$

is called an **indefinite fold singularity** (or a **round singularity**).

Definition 1.2 (Auroux, Donaldson and Katzarkov 2005, etc.)
Let $f : M \longrightarrow \Sigma$ be a smooth map.
(1) f is a **broken Lefschetz fibration** (**BLF**, for short) if it has at most Lefschetz and indefinite fold singularities.

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Remark 1.4 Sometimes we impose the condition that $f|_{Z(f)}$ should be an embedding into Σ (e.g. Gay–Kirby).

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$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.

Base Diagrams for Folds

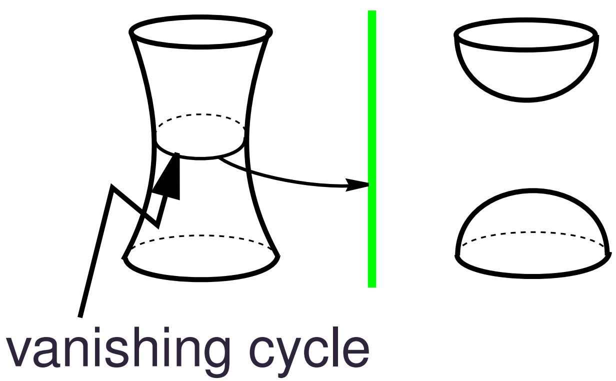

Figure 1: **Indefinite fold**

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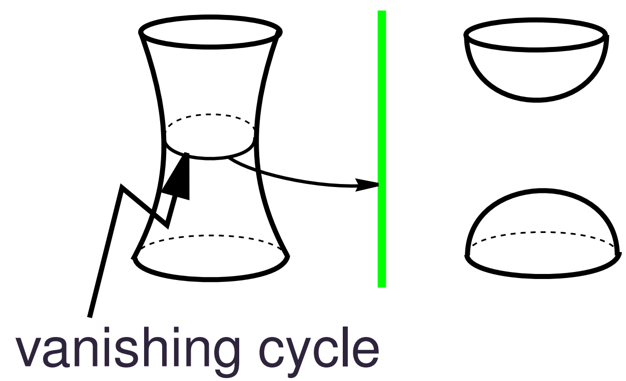

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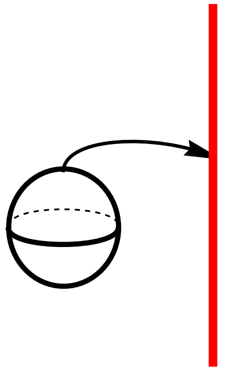
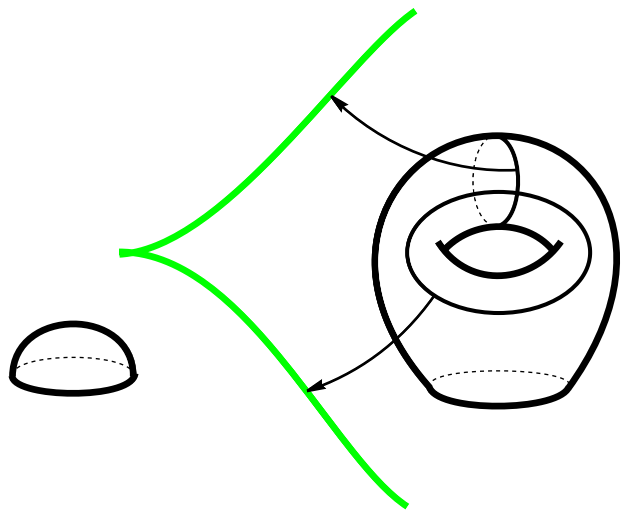


Figure 2: **Definite fold**

Base Diagrams for Cusps



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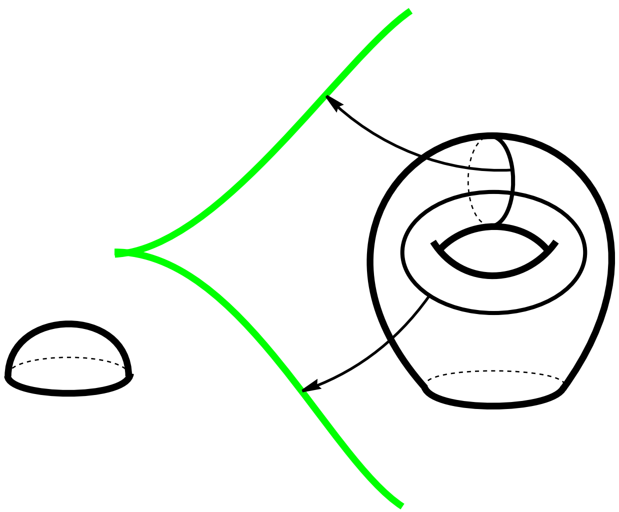

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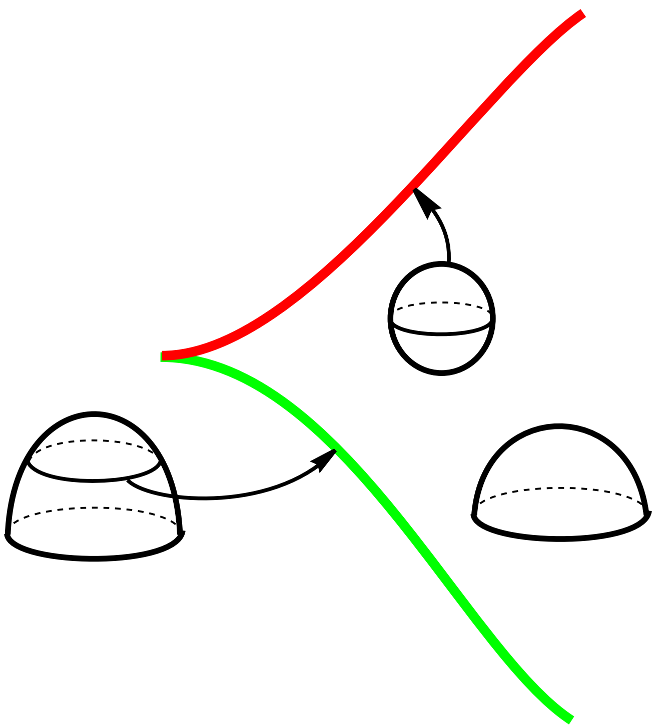

Figure 4: Definite cusp

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$\boxed{\text{Levine (1965)}}$ Every smooth map $M \to \Sigma$ is homotopic to an excellent map without a cusp if $\chi(M)$ is even, and with exactly one cusp if $\chi(M)$ is odd.

§2. Elimination of Definite Fold

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In other words, we can eliminate **definite fold singularities** by homotopy.

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For Step 3, we need the following "moves".

Modifying Excellent Maps

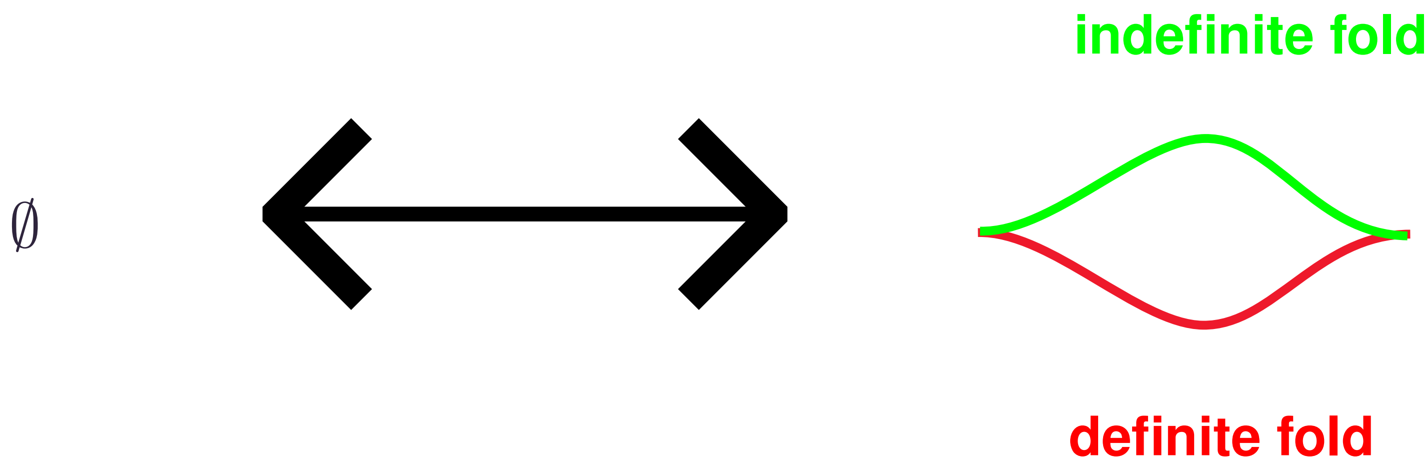

Figure 5: Birth

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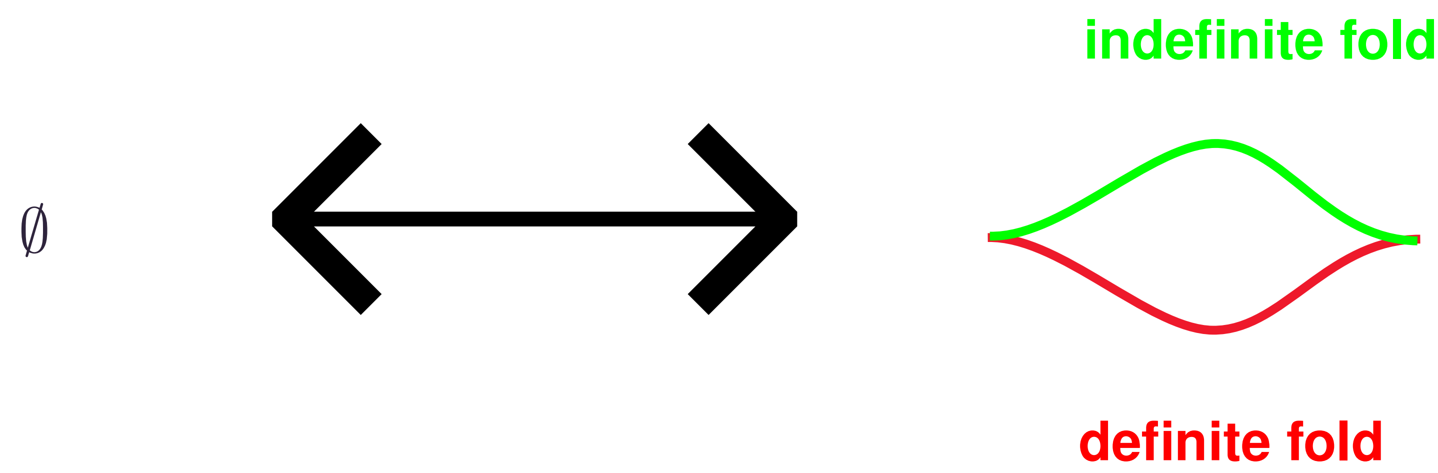

Figure 5: Birth

Figure 6: Merge

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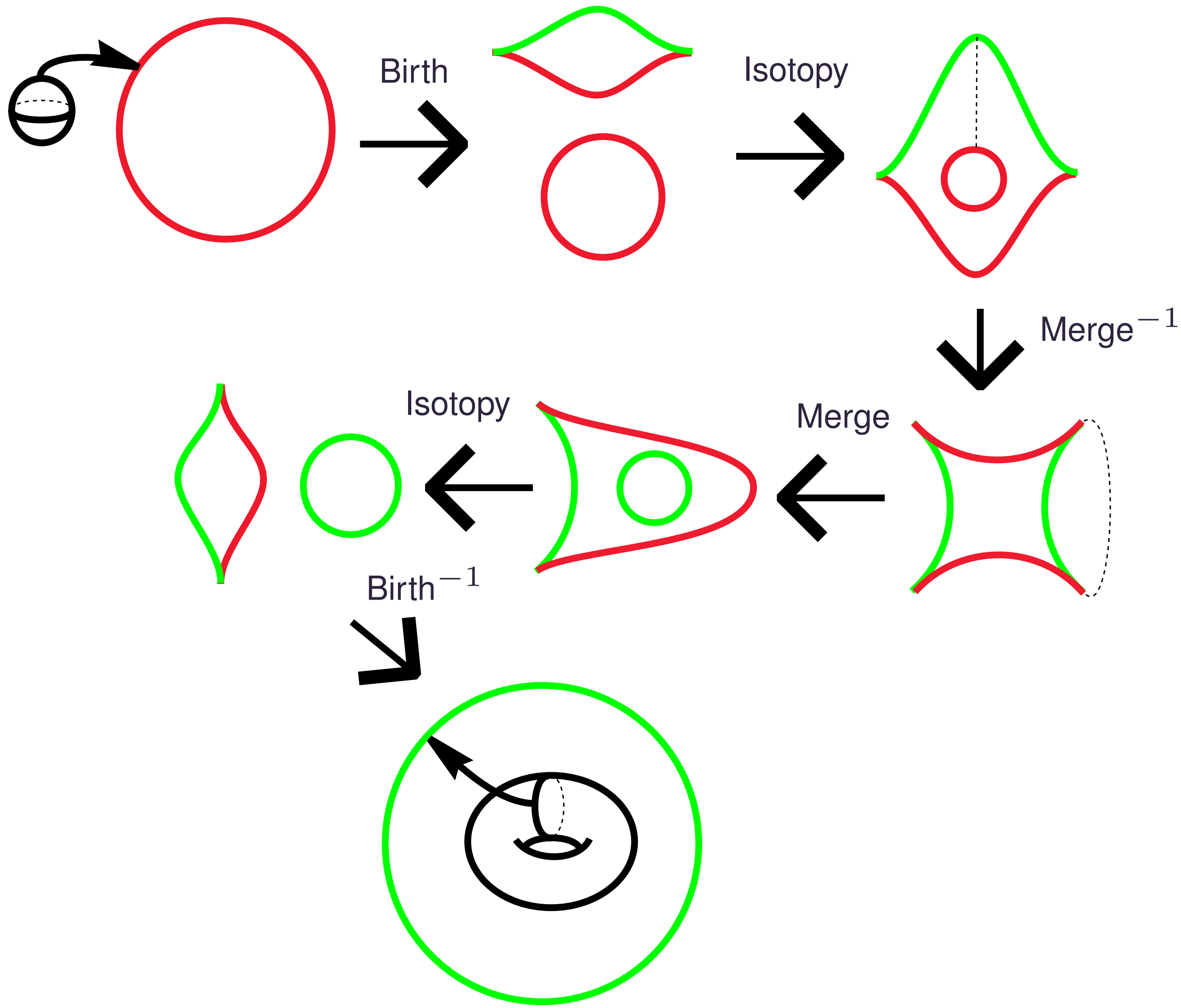

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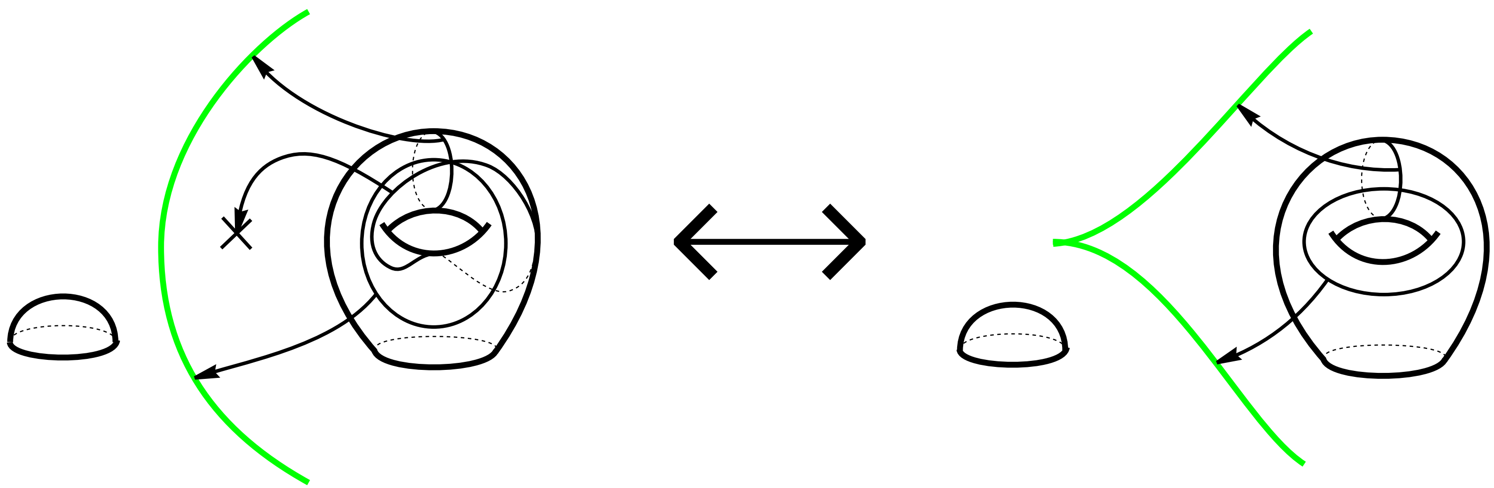

Figure 7: Sinking and Unsinking (Lekili 2009)

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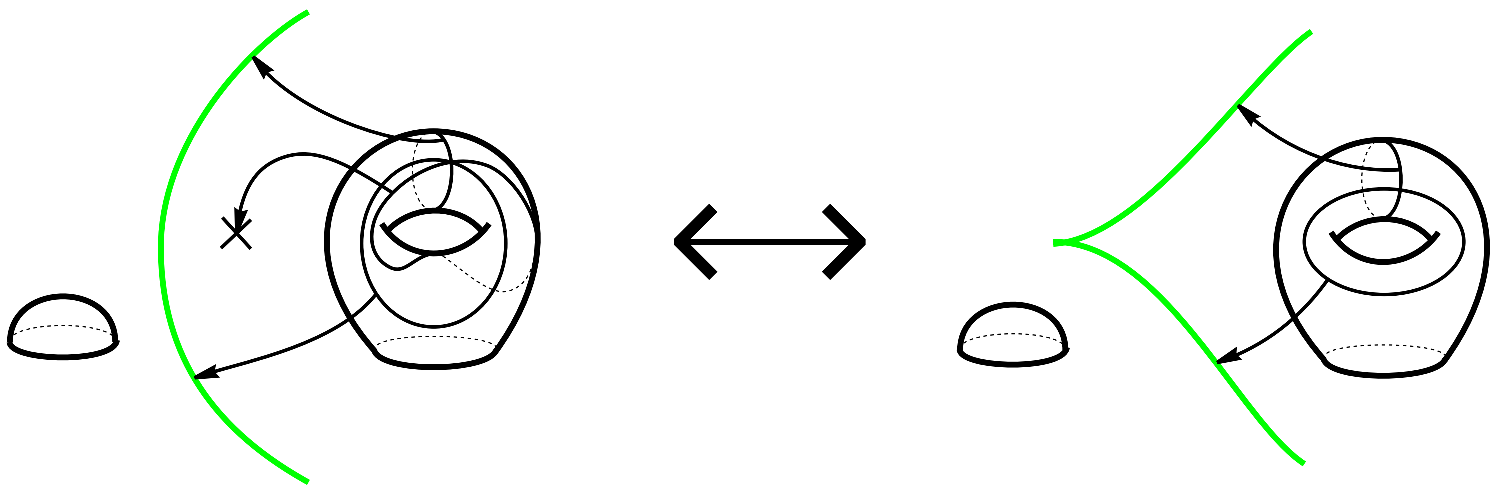

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Remark 2.3 For the existence of BLF (or ABLF), several proofs have been known (Auroux–Donaldson–Katzarkov, Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

§3. Lekili's Moves for BLF

Birth and Merge

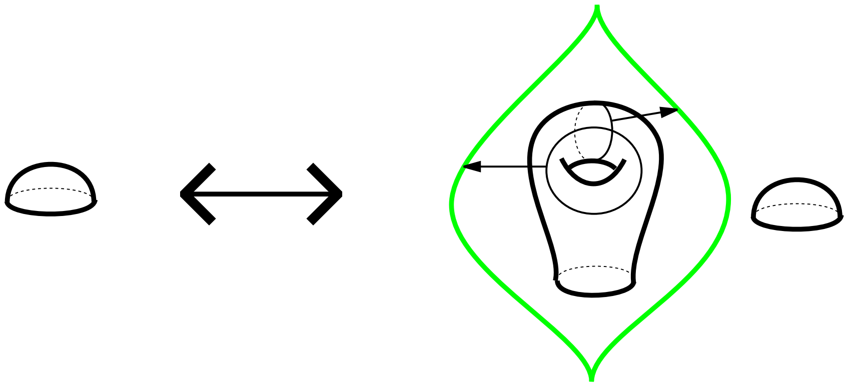
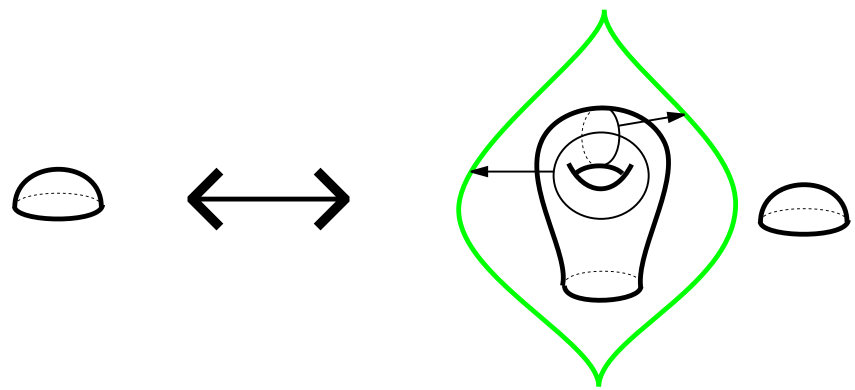


Figure 8: Birth



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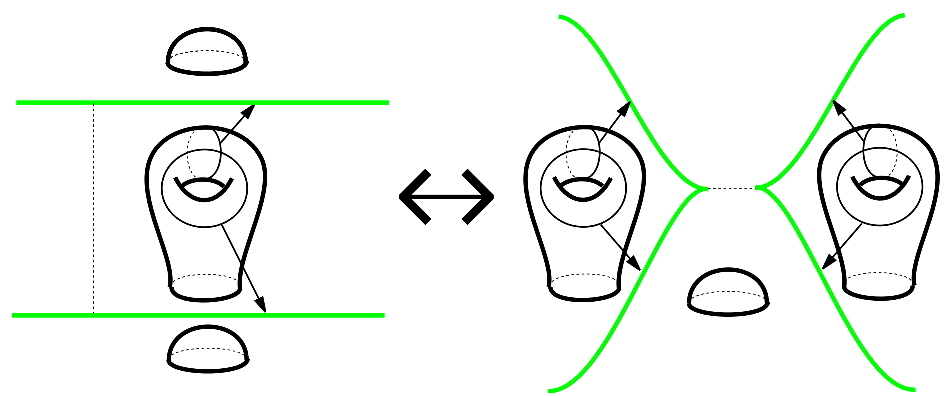


Figure 9: Merge

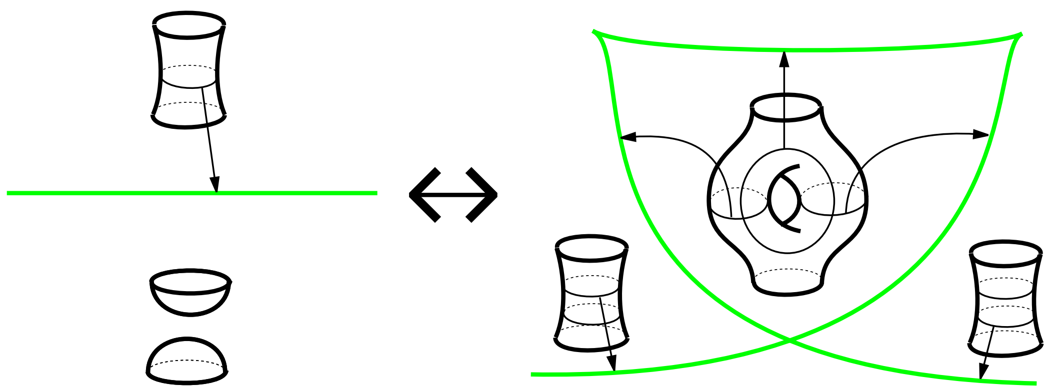
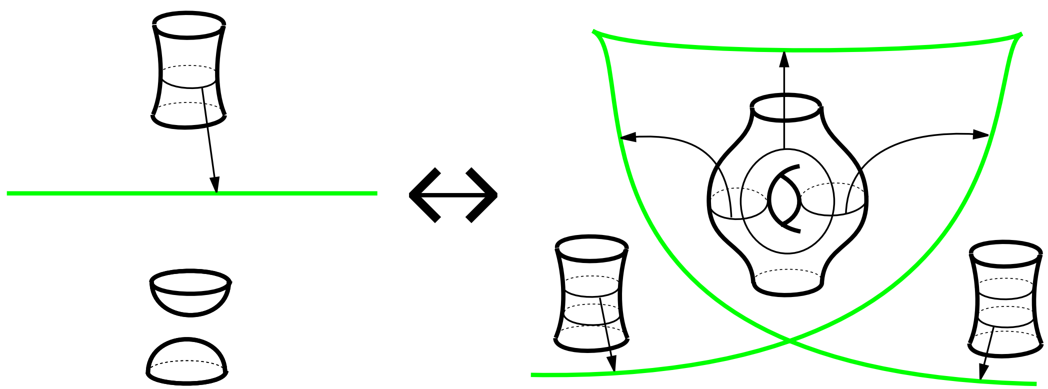


Figure 10: Flip



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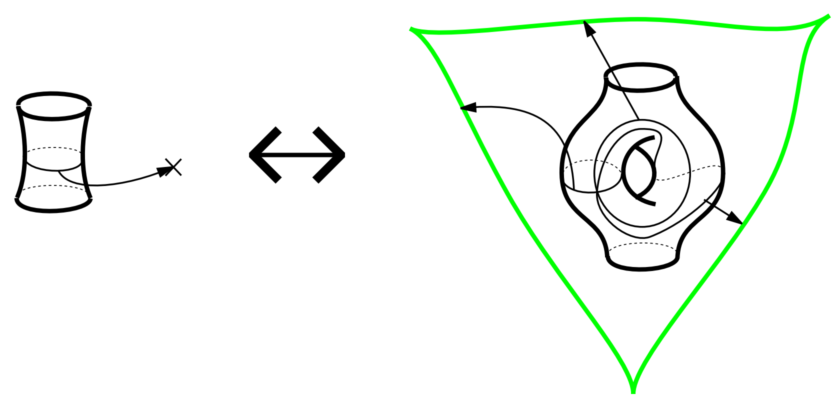

Figure 11: Wrinkle

One can convert each **achiral Lefschetz singularity** to one **circle of indefinite fold** and three **Lefschetz singularities** (Lekili 2009).

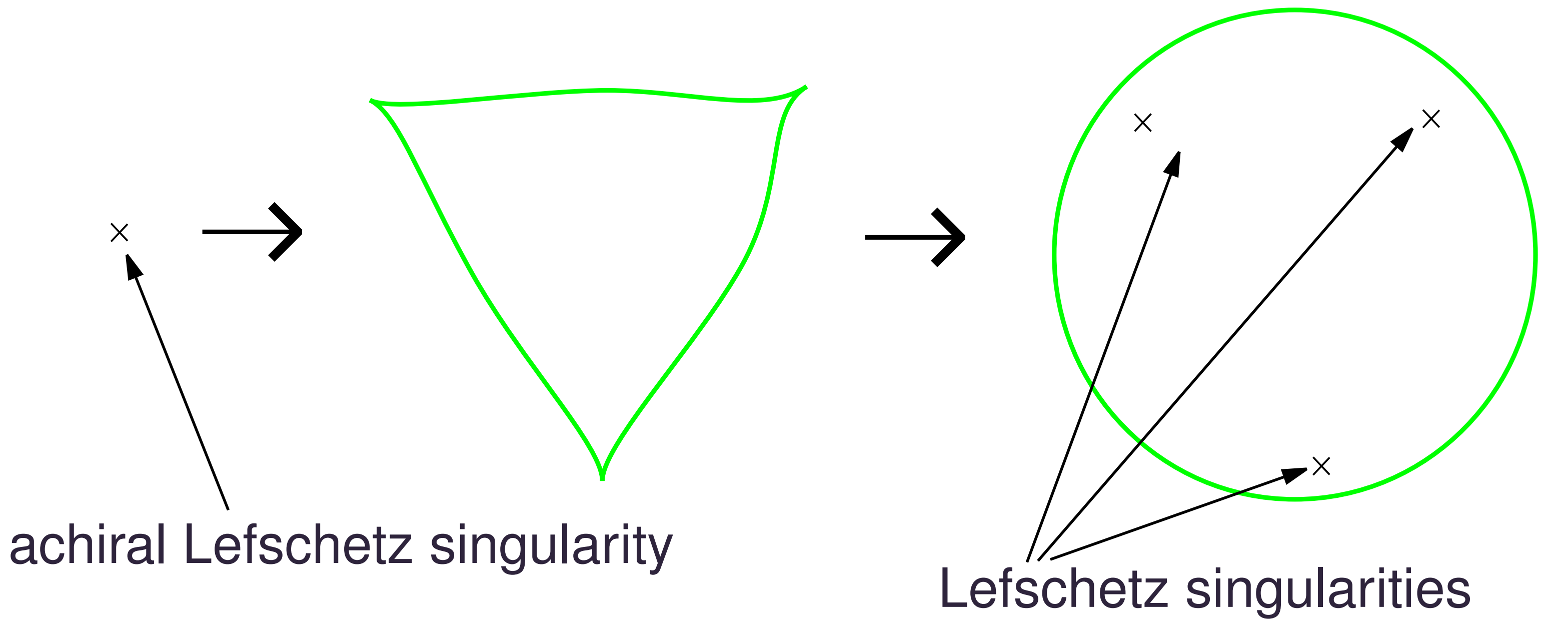

Figure 12: Removing an achiral Lefschetz singularity

Theorem 3.1 (Williams 2010) *If two BLFs $M \to \Sigma$ are homotopic, then one is obtained from the other by a finite sequence of Birth, Merge, Flip, Wrinkle, and Sink operations (and their inverses), together with "Isotopies".*

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Idea of Proof of Theorem 3.1
Each BLF can be homotoped to an **excellent map** without definite fold (by Wrinkle moves).
By singularity theory, the two excellent maps can be connected by a **generic 1-parameter family** $\{f_t\}$ of smooth maps.

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We need to **eliminate the definite folds** appearing in the generic homotopy F.
Williams' idea: Remove the definite folds of the homotopy F by modifying it by "surgery" (not by homotopy).

Suppose that the generic homotopy F has no definite folds. Then, Lekili has shown that his moves (together with isotopies) generate F, by essentially using singularity theory. $\square$

§3. Isotopies

Bifurcations during Isotopies

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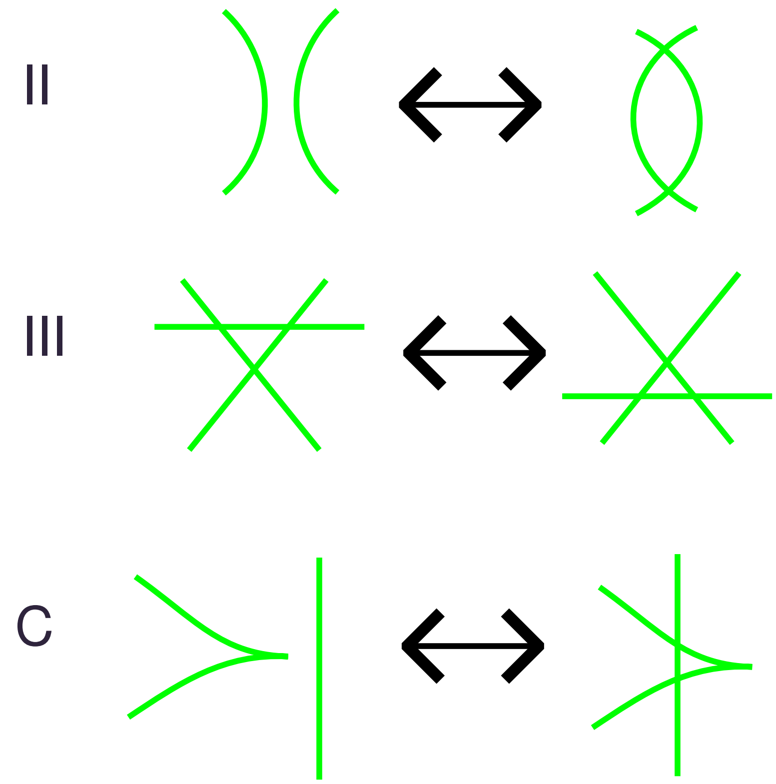

Figure 13: Moves involving isotopies

Y.K.S. Furuya, Sobre aplicações genéricas $M^4 \to \mathbf{R}^2$ (in Portuguese), PhD Thesis, University of São Paulo, 1986.

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Number of essential change types
II: 8 types
III: 13 types
C: 6 types

Example of Furuya's Move (1)

The integers indicate the genus of the corresponding fiber component.

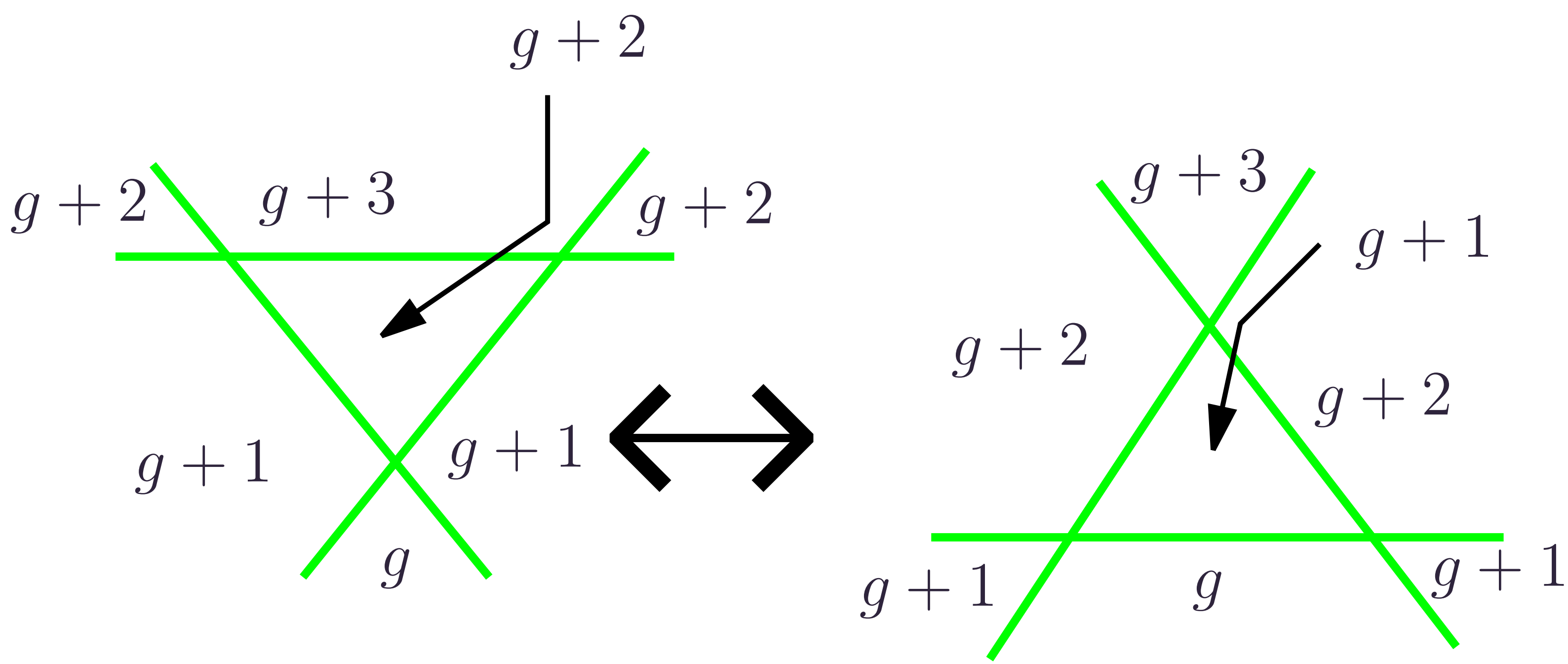

Figure 14: A type III move

Example of Furuya's Move (2)

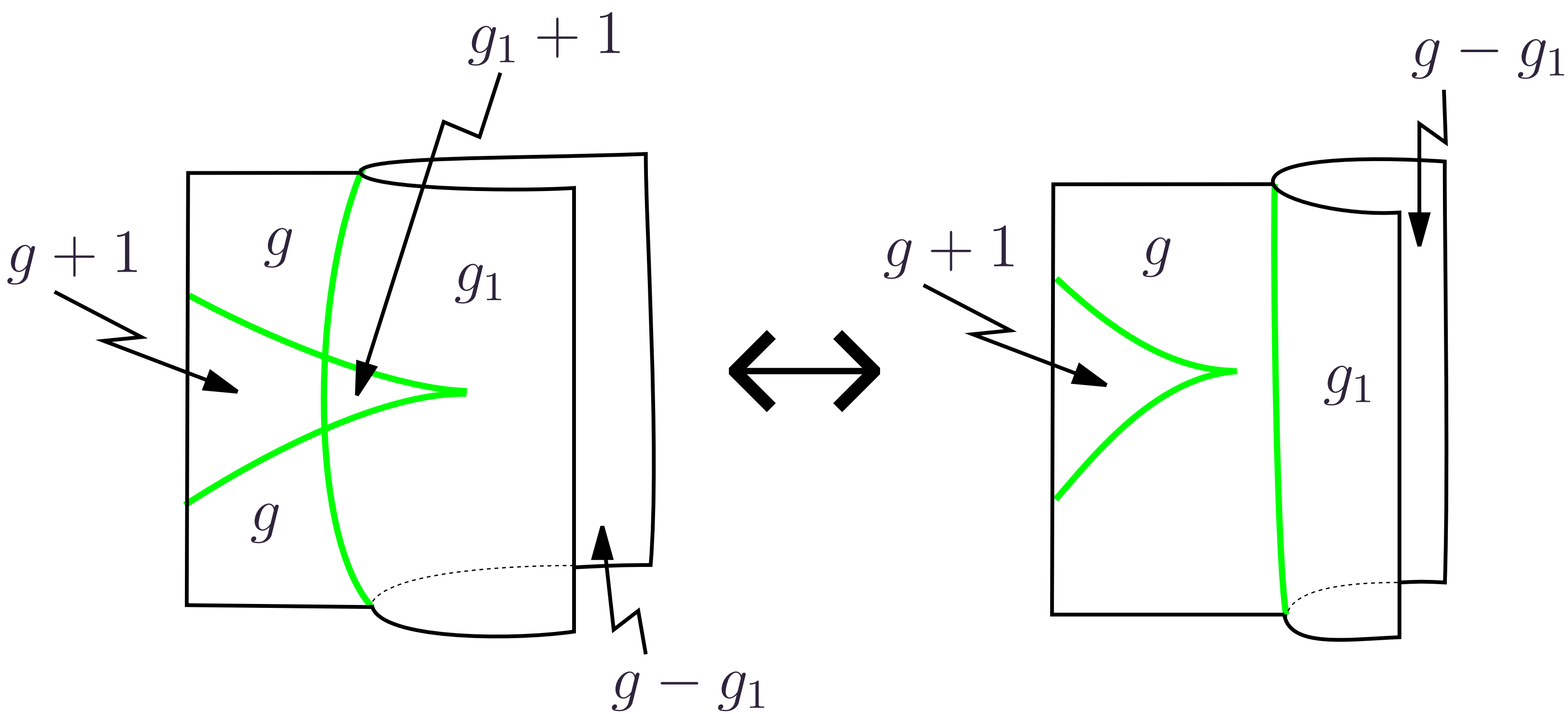

Figure 15: A type C move

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It is conjectured that **Lagrangian matching invariants equal the Seiberg–Witten invariants**.

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How about the class of ***fibrations with connected fibers****?*

Note.
These guarantee that if we start with a **near-symplectic BLF**, then we can perform the moves within the subclass of **near-symplectic BLFs**.

Thank you!