定値折り目特異点の消去と特異レフシェッツ束

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§1. Broken Lefschetz Fibrations

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(2) A singularity that has the normal form

$$\boxed{(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 - x_4^2)}$$

is called an **indefinite fold singularity**.

Definition 1.2 (Auroux–Donaldson–Katzarkov, 2005, etc.)
Let $f : M^4 \to \Sigma^2$ be a C^∞ map.
f is a **broken Lefschetz fibration** (**BLF**, for short) if
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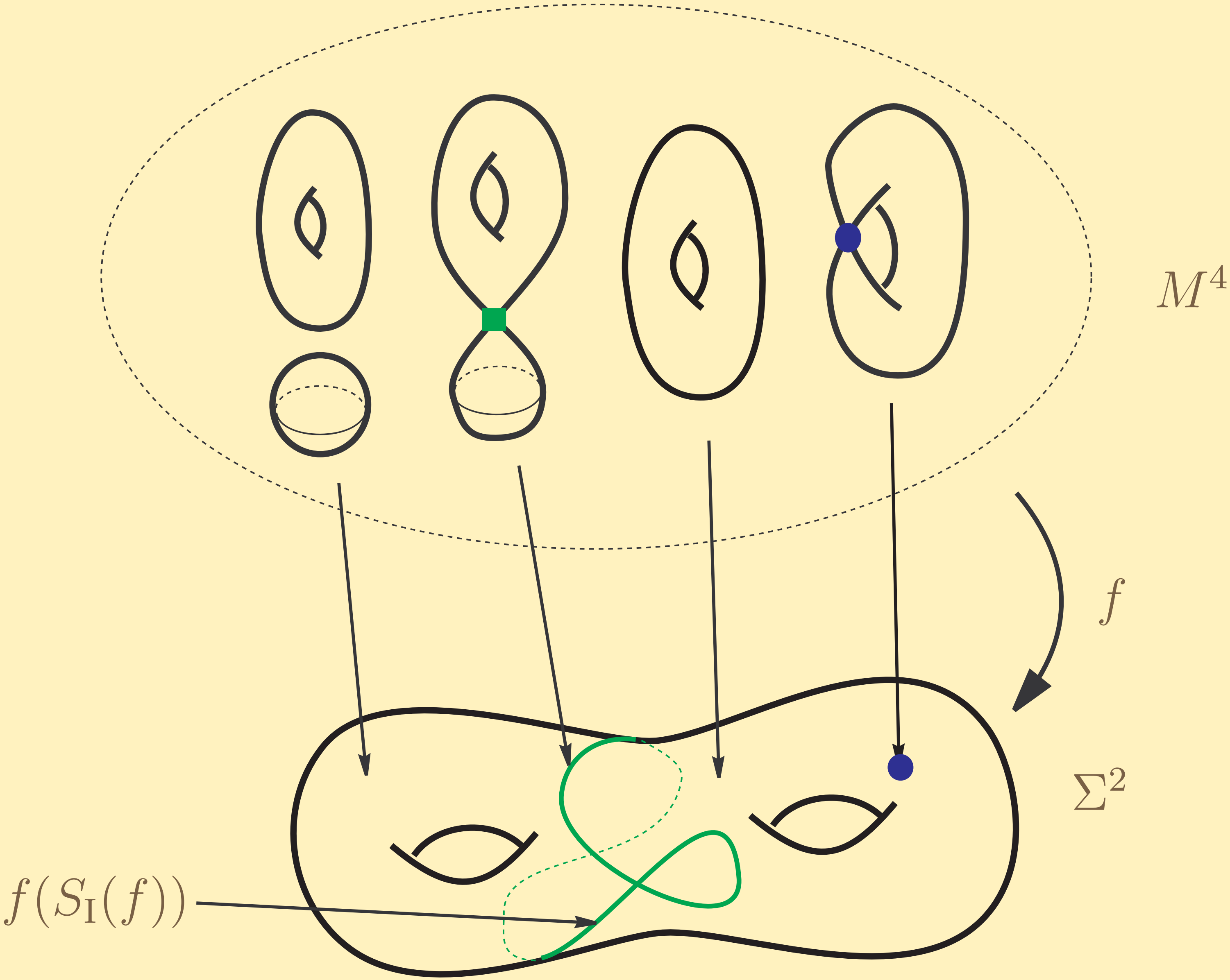
A usual **Lefschetz fibration** (**LF**, for short) is a special case of a BLF.
(LF $\Longleftrightarrow$ BLF with $S_{\mathrm{I}}(f) = \emptyset$)

Fibers of a BLF



Near-Symplectic Structures

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Near-symplectic structure: $\omega \in \Omega^2(M^4)$, $d\omega = 0$, $\omega^2 \geq 0$,

ω vanishes along a 1-dim. submanifold “transversely”.

Theorem 1.3 (ADK, 2005) *M^4: closed oriented 4-manifold,*
$Z \subset M^4$: 1-dim. closed submanifold
Then, the following two are equivalent.

(1) $\exists$**near-symplectic form** *ω on M^4 with zero locus Z.*

(2) $\exists$**broken Lefschetz pencil** (**BLP**) *f over S^2 with $S_{\mathrm{I}}(f) = Z$ s.t. there is an $h \in H^2(M^4; \mathbf{R})$ satisfying $h(C) > 0$ for every component C of every fiber of f.*

Furthermore, if (2) holds, then a **deformation class of near-symplectic forms** *that restrict to a volume form on each fiber away from Z is canonically associated to f.*

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$\exists$BLP $\Longrightarrow$ $\exists$BLF on a blown up 4-manifold

BLF is a special case of a BLP (BLF $=$ BLP without base points).

A Remark

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Not every 4-manifold admits a symplectic structure.
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In fact, there are a variety of such structures on a given 4-manifold M^4.

§2. Singularities of Generic Maps

Definite Fold and Cusp

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Definition 2.5 (1) A singularity that has the normal form

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$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^3 - 3x_1x_2 + x_3^2 \pm x_4^2)$$

is called a **cusp**.

Base Diagrams for Folds

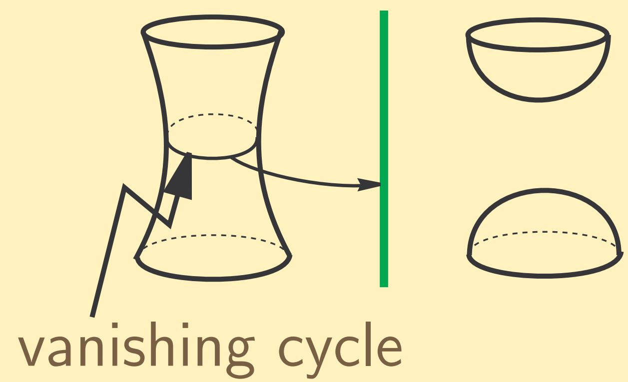

Figure 1: **Indefinite fold**

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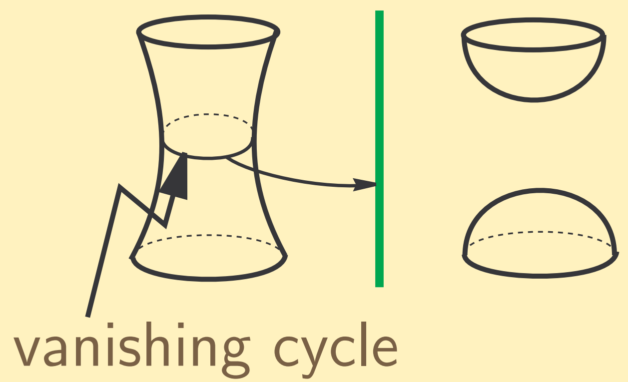

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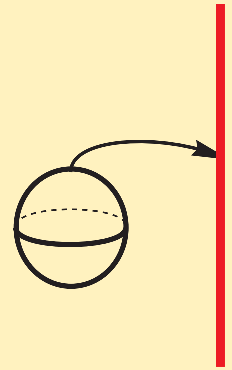

Figure 2: **Definite fold**

Base Diagrams for Cusps

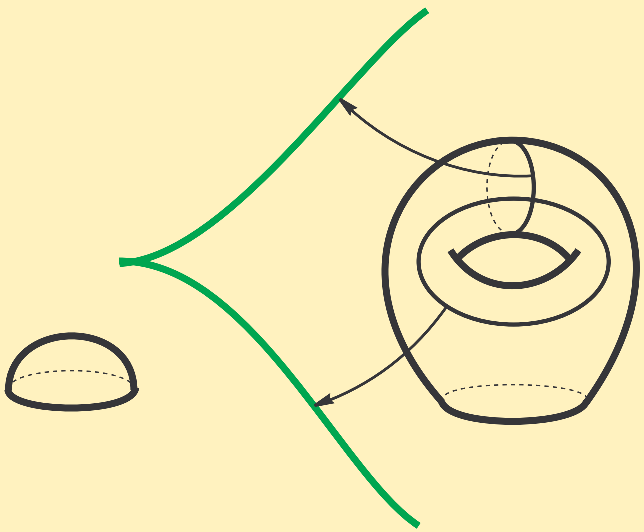

Figure 3: Indefinite cusp

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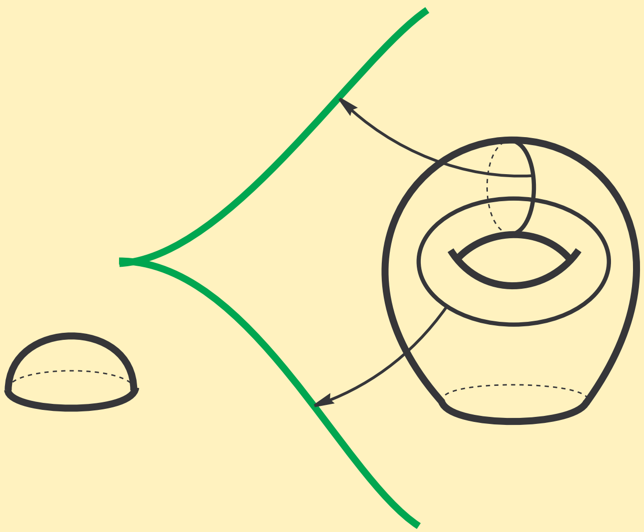

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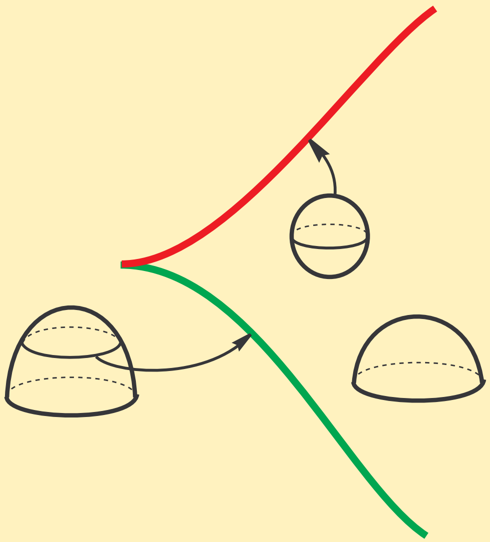

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Excellent Map

Facts.

$\boxed{\text{Whitney (1955)}}$ Every C^∞ map $M^4 \to \Sigma^2$ is homotopic to (actually, approximated by) a C^∞ map with at most definite fold, indefinite fold, definite cusp, and indefinite cusp singularities.

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Levine (1965) [Cusps can be eliminated in pairs.]
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- **Excellent maps** may have definite folds and cusps, but have no Lefschetz critical point.
- **BLFs** may have Lefschetz critical points, but have no definite fold or cusp.

§3. Elimination of Definite Fold

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In other words, we can eliminate **definite fold singularities** by homotopy.

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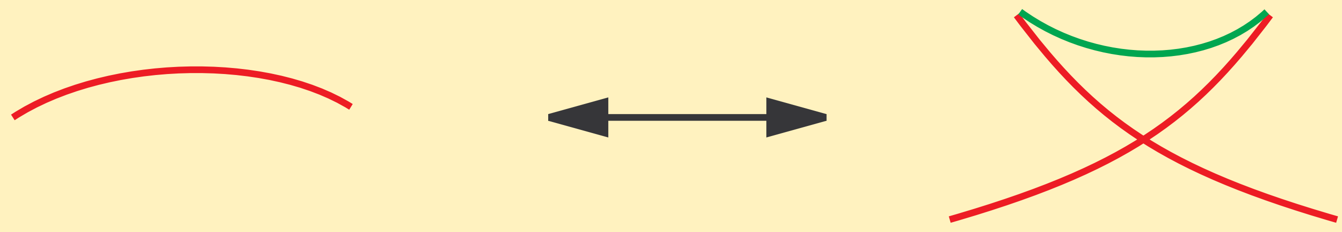
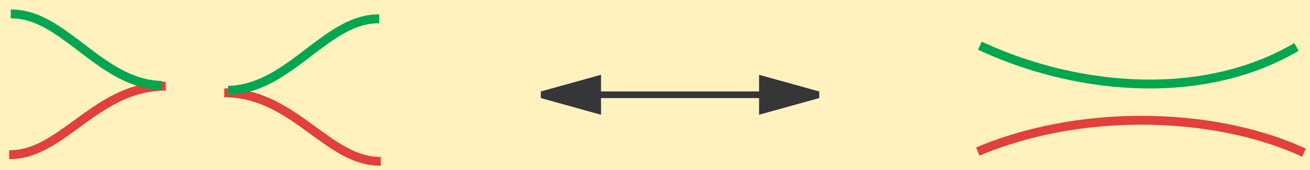
Theorem 3.2 (S., 1995) *$g : M^4 \to \Sigma^2$ a C^∞ map*
$L \subset M^4$: a non-empty closed 1-dim. submanifold
$\exists$excellent map $f : M^4 \to \Sigma^2$ homotopic to g s.t. $S(f) = L$
$\Longleftrightarrow [L]_2 = 0$ in $H_1(M^4; \mathbf{Z}_2)$

Moves for Excellent Maps



Flip



Merge

Figure 5: Moves for modifying the **definite fold locus**

Proof (continued)

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For Step 3, we need the following additional move.

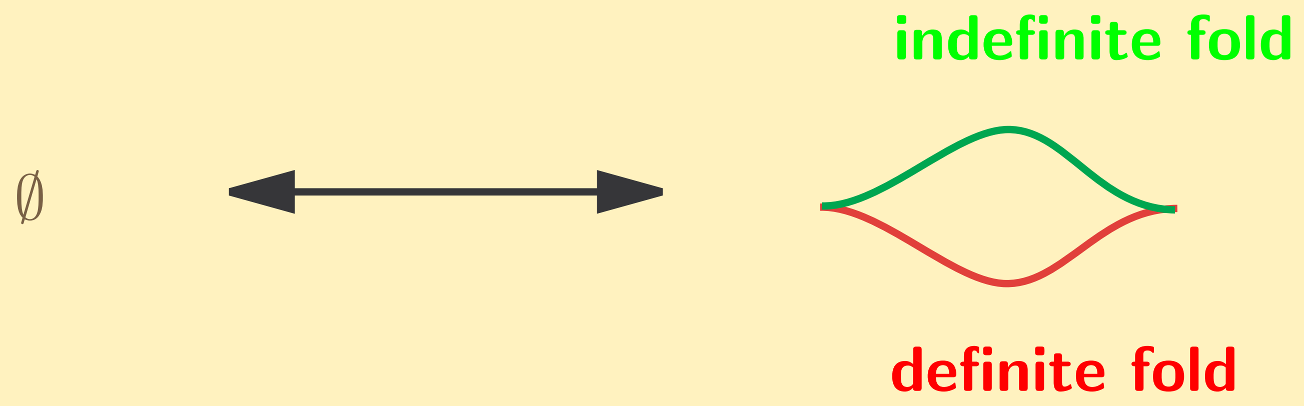

Figure 6: Birth

Definite to Indefinite

Step 3. Definite fold circle $\rightsquigarrow$ Indefinite one (Williams, 2010)

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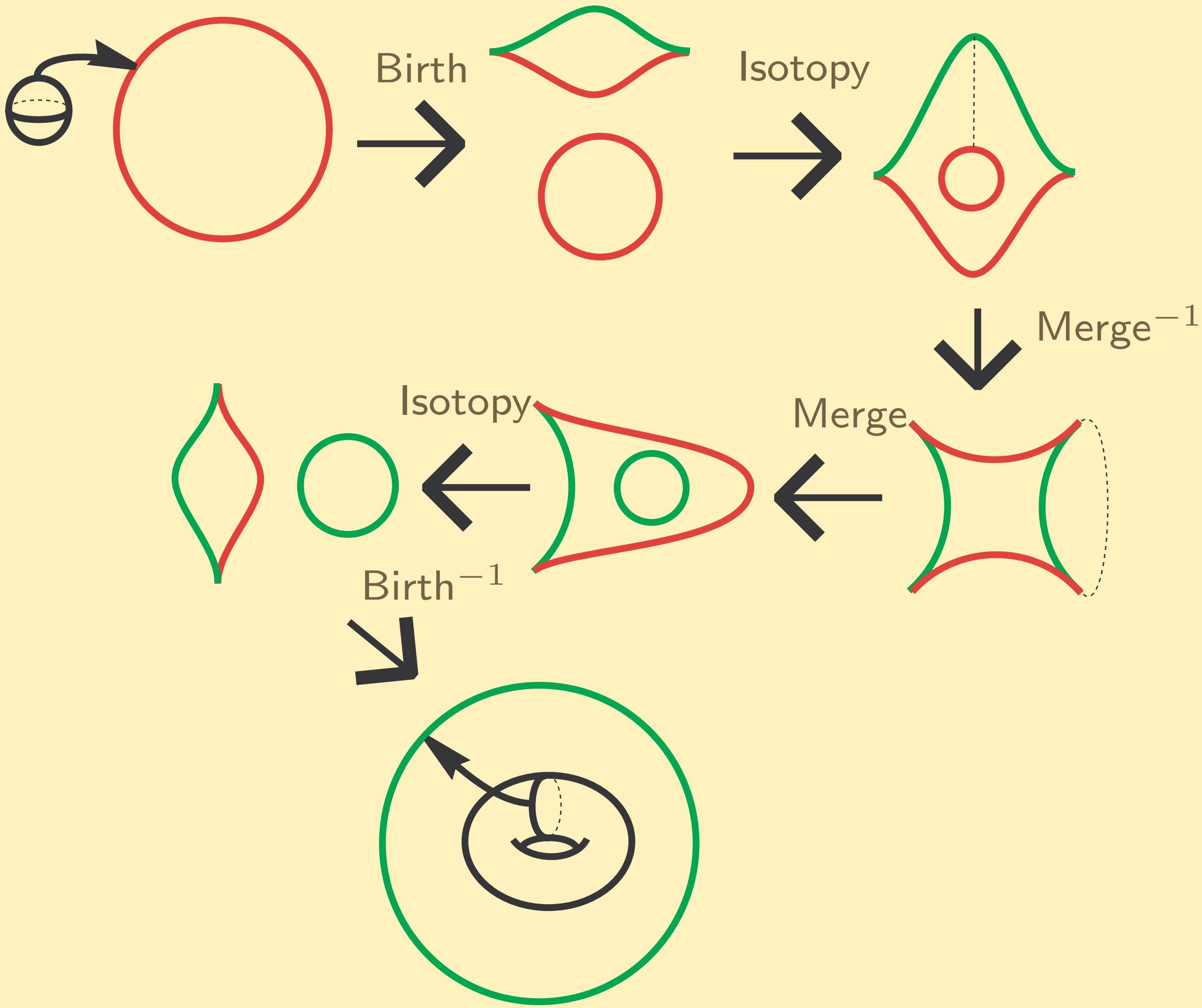

Q.E.D.

Existence of BLF

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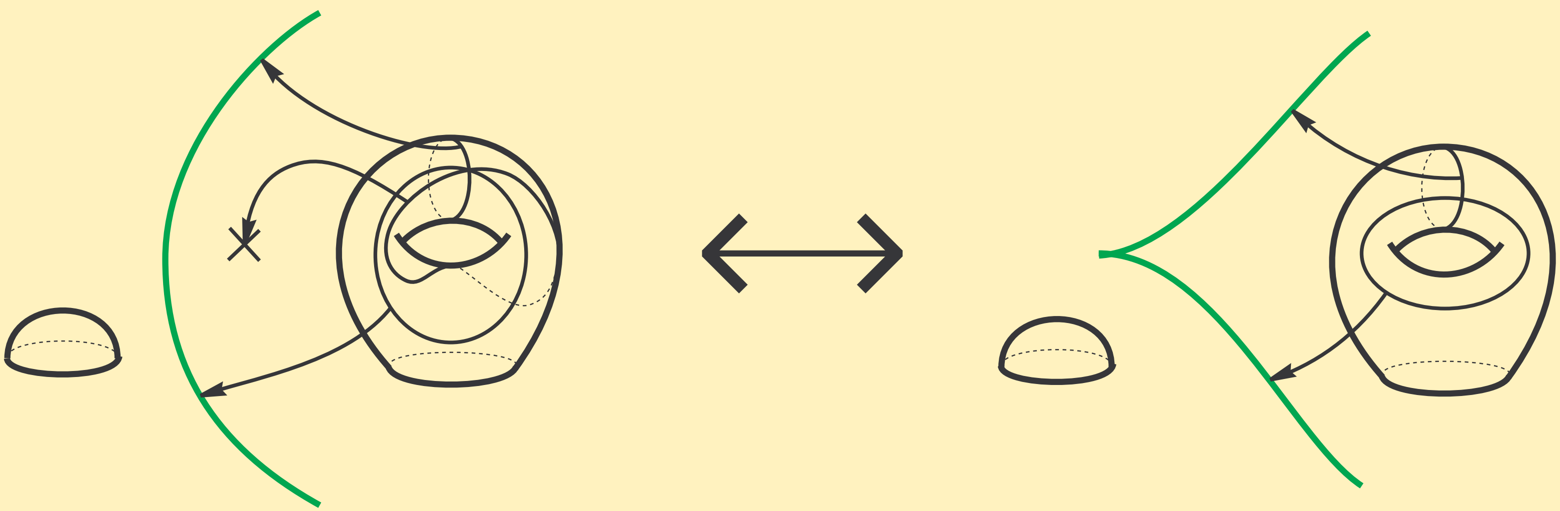

Figure 7: Sinking and Unsinking (Lekili, 2009)

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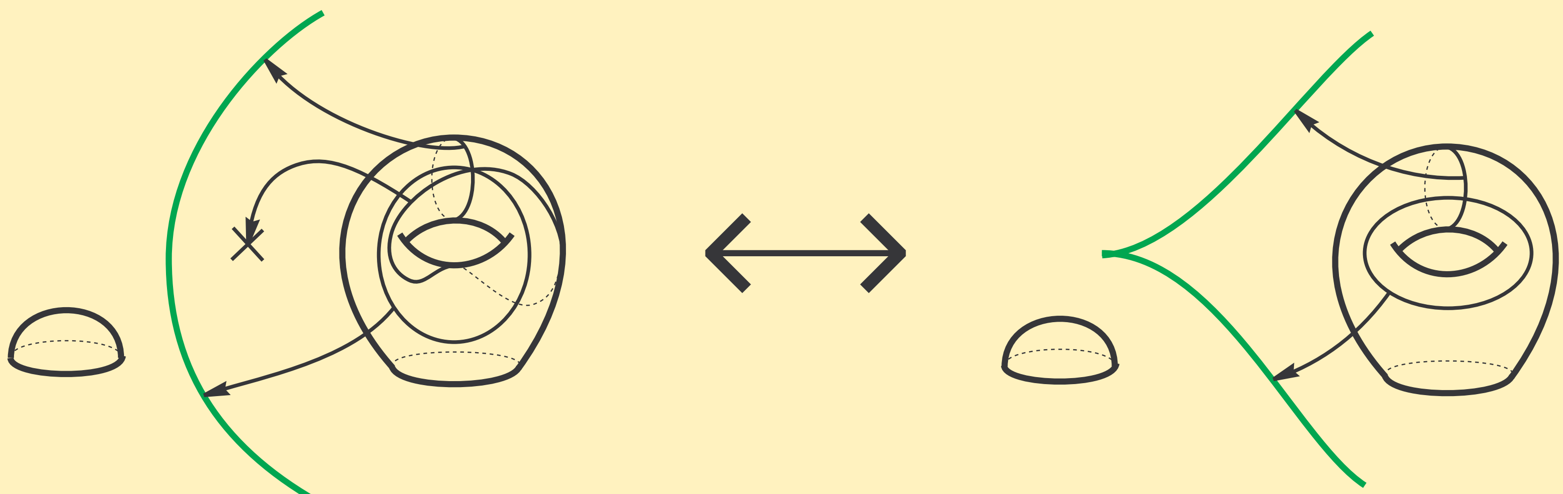

Figure 7: Sinking and Unsinking (Lekili, 2009)

Remark 3.4 For the existence of BLF, several proofs are known (Gay–Kirby, Baykur, Lekili, Akbulut–Karakurt).

We can also prove the following (cf. Lekili, 2009).

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Theorem 3.5 $g : M^4 \to S^2$ *a* C^∞ *map*
$L \subset M^4$: *a non-empty closed* 1-*dim. submanifold*
$\exists f : M^4 \to S^2$ *BLF homotopic to* g *s.t.* $S_{\mathrm{I}}(f) = L$
$\iff [L]_2 = 0$ *in* $H_1(M^4; \mathbf{Z}_2)$

Prescribed Indefinite Locus

We can also prove the following (cf. Lekili, 2009).

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Using similar techiniques in the context of near-symplectic structures (Perutz, 2006; Lekili, 2009), we can prove the following.

Theorem 3.6 M^4: *closed oriened* 4-*manifold with* $b_2^+(M^4) > 0$
$L \subset M^4$: *a non-empty closed* 1-*dim. submanifold*
$\exists$**near-symplectic structure** ω *whose zero locus coincides with* L
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Recent Result by Gay–Kirby

Theorem 3.7 (Gay–Kirby, 2011) *$g : M^4 \to \Sigma^2$ a C^∞ map*
$\exists f : M^4 \to \Sigma^2$ BLF homotopic to g
$\iff [\pi_1(\Sigma^2) : g_*\pi_1(M^4)] < +\infty$
*Furthermore, if $g_* : \pi_1(M^4) \to \pi_1(\Sigma^2)$ is surjective, then we can arrange so that $\forall$***fibers are connected***.*

Remark 3.8 **Fiber connectedness** is very important!

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Remark 3.8 **Fiber connectedness** is very important!
Recall the cohomological condition appearing in the ADK theorem on the existence and uniqueness of near-symplectic structures.

§4. Moves for BLFs

Lekili's Moves

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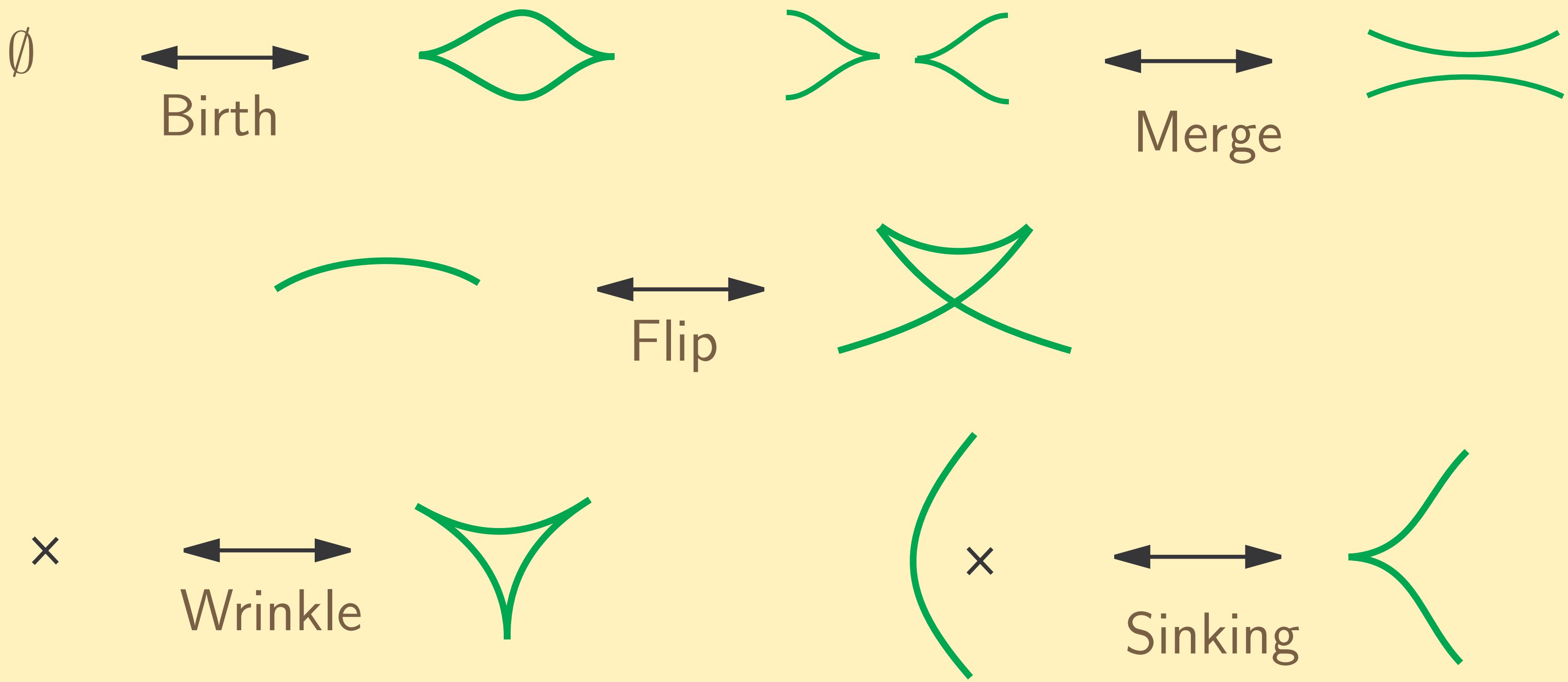

Figure 8: Lekili's moves

Uniqueness

Theorem 4.1 (Williams, 2010; Gay–Kirby, 2011)
If two BLFs $M^4 \to \Sigma^2$ are homotopic, then one is obtained from the other by a finite sequence of **Lekili's moves** (*Birth, Merge, Flip, Wrinkle, and Sink operations, and their inverses*)*, together with "Isotopies".*

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It is conjectured that **Lagrangian matching invariants equal the Seiberg–Witten invariants**.

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Find a sufficient sequence of moves that guarantees to stay within the class of fibrations **without null-homologous fiber components**.

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Note.
These guarantee that if we start with a **near-symplectic BLF**, then we can perform the moves within the subclass of **near-symplectic BLFs**.

An Answer

Theorem 4.3 (Gay–Kirby, 2011)
$f_0, f_1 : M^4 \to \Sigma^2$ **excellent maps** *without definite folds s.t. all the fibers are connected.*
$\Longrightarrow \exists$**generic homotopy** f_t *between* f_0 *and* f_1
s.t. $\forall$*fibers of* f_t *are connected.*

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cf. The proof that the **Kirby moves** are enough for converting one framed link diagram to another for a given 3-manifold.

§5. Simplified BLFs

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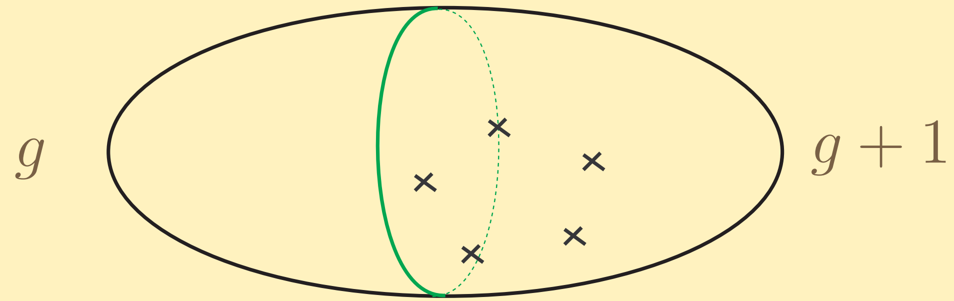

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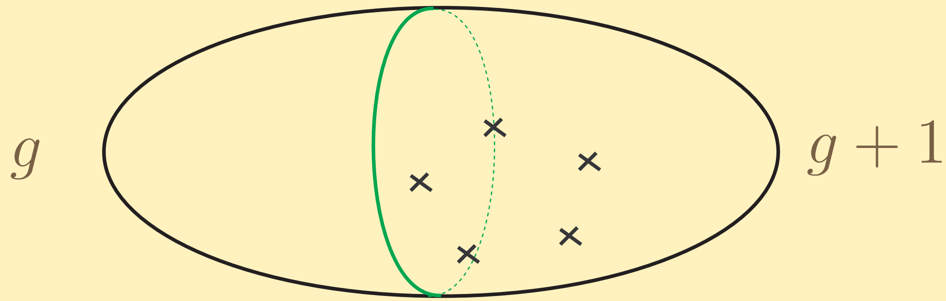

It is known that **every closed oriented 4-manifold admits a SBLF** (Gay–Kirby, etc.).

Surface Diagram

Williams (2010): Convert the Lefschetz singularities to cusps by Lekili's moves.

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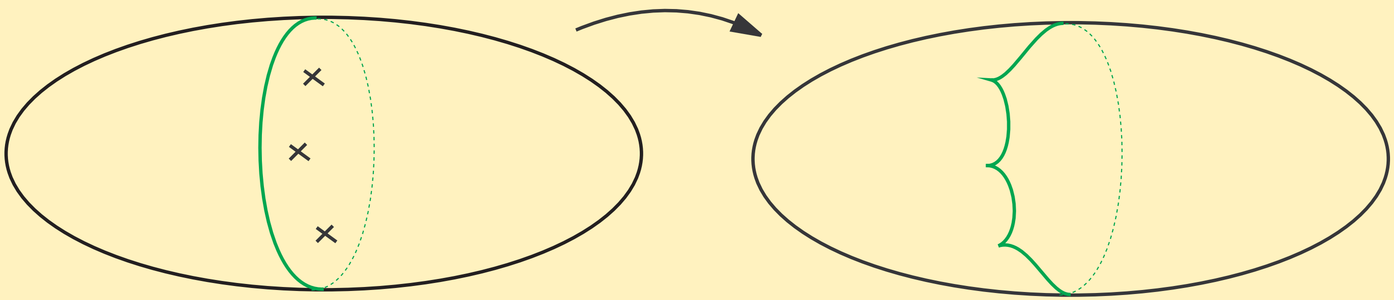

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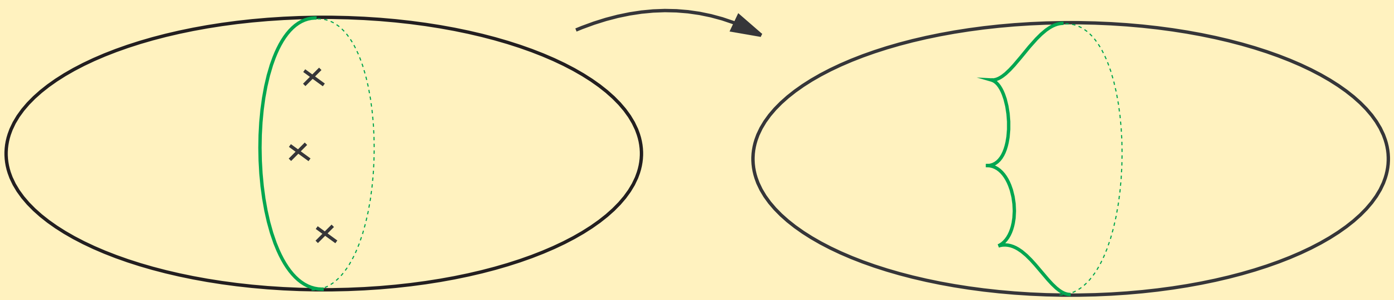

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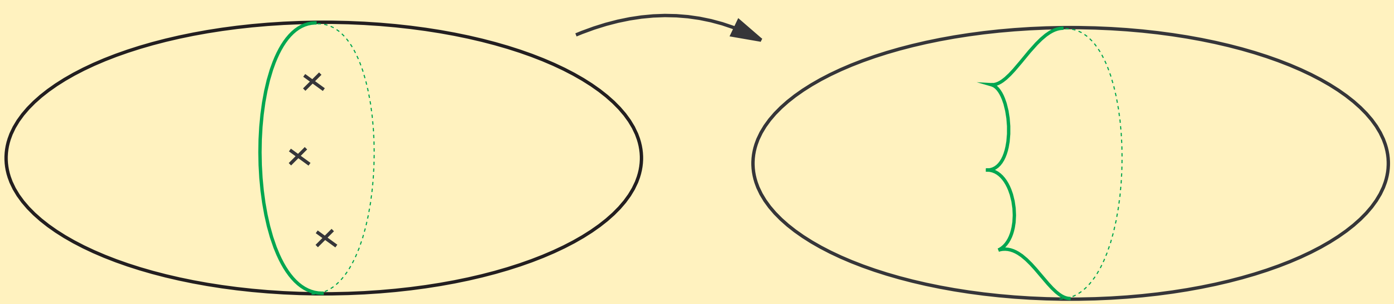

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Theorem 5.1 (Williams, 2011)
Surface diagram of a given closed oriented 4-manifold is unique up to certain moves, called stabilization, handleslide, multislide, and shift.

Summary

(1) Every closed oriented 4-manifold admits a lot of **BLFs**; when $b_2^+(M^4) > 0$, a lot of BLFs with associated **near-symplectic structures**.

(2) Two BLFs in a fixed homotopy class are related by **Lekili's moves**. They are also related in the class of BLFs with **connected fibers**. This would lead to prove the conjecture that the **Lagrangian matching invariant** defined for near-symplectic structures equals the **Seiberg-Witten invariant**.

(3) The **indefinite locus** of a BLF can be prescribed, and the **zero locus** of a near-symplectic structure as well.

(4) **Surface diagrams** arising from SBLFs may be useful to describe a given 4-manifold, like Heegaard diagrams or framed link diagrams for 3-manifolds.

Thank you for your attention !