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2014年 11月26日 ～ 29日

笹本さん 集中講義.

「一次元可積分確率過程」

Sasamoto

KPZ に関する概観
ランダム行列
TASEP
Schur process
KPZ equation
Duality and q-TASEP
Macdonald process.

Basics

what is the KPZ equation ?
The KPZ equation is not really well-defined

Explicit formulas

(1/1) Ballistic deposition の order は知られていない..

KPZ equation (Kardar Parisi - Zhang '86)

$$\partial_t h(x,t) = \frac{1}{2}\lambda\left(\partial_x h(x,t)\right)^2 + \nu\,\partial_x^2 h(x,t) + \sqrt{D}\,\eta(x,t)$$

$$\langle \eta(x,t)\,\eta'(x',t') \rangle =$$

$$\partial_t h(x,t) = \frac{1}{2}\left(\partial_x h(x,t)\right)^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta$$

$$\partial_t h(x,t) = \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t) \qquad \text{Edwards-Wilkinson}$$

noise

Add nonlinearity $(\partial_x h(x,t))^2 \Rightarrow$ KPZ

$$\partial_t h = v\sqrt{1+(\partial_x h)^2} \simeq v + \frac{v}{2}(\partial_x h)^2 + \cdots$$

RG argument $\left(\beta = \frac{1}{3}\right)$

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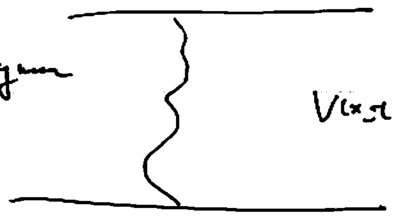
Cole–Hopf transformation

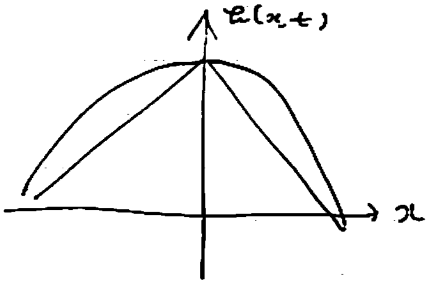
$$Z(x,t) = \exp\left(h(x,t)\right)$$

$$\Rightarrow \quad \partial_t Z(x,t) = \frac{1}{2}\partial_x^2 Z(x,t) + \eta(x,t)\, Z(x,t)$$

directed polymer



$V(x,t) \rightarrow \eta(x,t)$
randomize



narrow edge

$$\delta_0 = \lim_{\delta \downarrow 0} c_\delta\, e^{-|x|/\delta}$$

$$\eta(x,t) = \text{“}\frac{dB(x,t)}{dt}\text{”}$$

$$dZ(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial t^2}\, dt + Z(x,t) \times dB(x,t)$$

$$dB(x,t)\, dB(x',t) = \delta(x-x')\, dt.$$

Stratonovich

$$Z(x,t) \circ dB(x,t) = Z(x,t) \cdot dB(x,t) + \frac{1}{2}\, dZ(x,t)\, dB(x,t)$$

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$$dZ(x,t) = \frac{1}{2}\partial_x^2 Z(x,t)\,dt + Z(x,t)\,dB(x,t)$$

well-defined

Cole–Hopf sol. $h = \log Z(t)$ (Bertini–Jacod...)

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) - \infty + \eta(x,t)$$

§2. Explicit formula. $Z(x,0) = \delta(x)$ (narrow wedge)

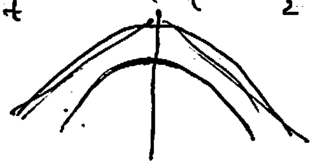
$$\left\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \right\rangle = \det(1 - K_{s,t})$$

where $\gamma_t = (t/2)^{1/3}$ and $K_{s,t}$

$$K_{s,t}(x,y) = \int_{-\infty}^{\infty} d\lambda \, \frac{Ai(x+\lambda)\,Ai(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}$$

Thm

$$h(x,t) = -\frac{x^2}{2t} - \frac{1}{12}\gamma_t^3 + \gamma_t \xi_t \qquad \left(\gamma_t = \left(\frac{t}{2}\right)^{1/3}\right)$$



$$F_{\xi_t}(s) = \mathbb{P}(\xi_t \le s) = 1 - \int_{-\infty}^{\infty} \exp\left(-e^{\gamma_t(s-u)}\right) \left\{ \det\left(1 - P_u(B_t - P_{Ai})P_u\right) - \det\left(1 - P_u B_t P_u\right) \right\} du$$

$P_{Ai}(x,y) = Ai(x)\,Ai(y)$, P_u = proj. onto $[u,\infty)$

$$B_t(x,y) = \int_{-\infty}^{\infty} \frac{Ai(x+\lambda)\,Ai(y+\lambda)}{e^{\gamma_t\lambda}-1}$$

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$$\left(\underline{\text{GUE}} \quad P(H)\,dH \propto e^{-\mathrm{Tr}\,H^2}\,dH \atop 2^{-1}\Delta(x)^2 \prod e^{-x_i^2} \right)$$

GUE TW-dist.

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{x_{max} - \sqrt{2N}}{2^{-1/2} N^{-1/6}} \le s \right) = F_2(s) = \det(I - P_s K_2 P_s)$$

$$K_2(x,y) = \int_0^\infty d\lambda\, Ai(x+\lambda)\, Ai(y+\lambda)$$

Dotsenko – Le Doussal – Calabrese

$$Z(x,t) = \mathbb{E}\left[e^{\int_0^t \eta(b(s),\, t-s)\, ds}\, Z(b(t), 0) \right]$$

$$\Downarrow$$

$$\langle (Z_\beta(x,t))^N \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

$$H_N = -\frac{1}{2}\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^N \delta(x_j - x_k)$$

attractive. δ- Bose gas.

$$\left\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \right\rangle = \sum_{N=0}^\infty \frac{(-e^{-\gamma_t s})^N}{N!} \langle Z^N(0,t)\rangle e^{N\frac{\gamma_t^3}{12}}$$

$$\langle Z^N \rangle \sim e^{N^3} \; !$$

水

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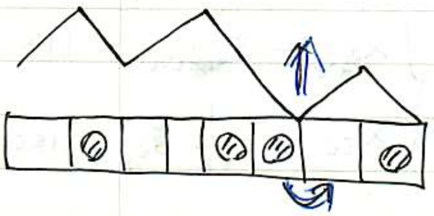
TASEP or ASEP

$N(x, t)$ = Integrated current at $(x, x+1)$ up to time t

$\Updownarrow$

height



ASEP ~~phase~~.

$\Uparrow$

ASEP

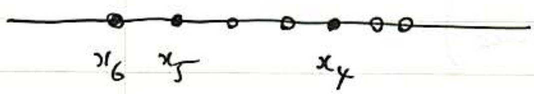
ASEPを左の 1=1 対応で 界面と思う.

ASEP $\Longrightarrow$ KPZ (weakly asymmetric limit)

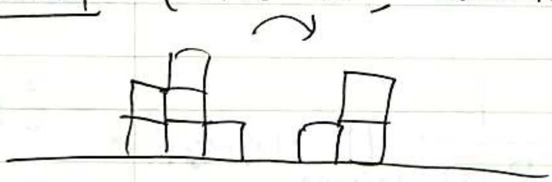
q-TASEP

Borodin − Corwin A particle i hops with rate $1 - q^{x_{i-1} - x_i - 1}$

$q \to 1$ で KPZ



q-TAZRP (1998 TS, Wadati)



q − Boson operators

Flat case (replica) (Le Doussal, Calabrese)

Multiple case (replica) (Dotsenko)

Arndt − Heinzel − Rittenberg. (1998)

Halpin − Healy (2012)

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1. Random matrices

1.1. Gaussian ensembles

$\mathcal{H}_N^{(\beta)}$ = the set of $\left(\begin{array}{c}\text{real sym.} \\ \text{hermitian}\end{array}\right)$ matrices of size N.

$\beta = 1$, $\beta = 2$

$\{\xi_{ij}\}_{i,j=1}^{\infty}$ i.i.d. $N(0,1)$

def. GOE $\quad X \in \mathcal{H}_N^{(1)}$
$$\begin{cases} X_{ii} = \xi_{ii} & (1 \le i \le N) \\ X_{ij} = \dfrac{1}{\sqrt{2}}\xi_{ij} & (1 \le i < j \le N) \end{cases}$$

GUE $\quad X \in \mathcal{H}_N^{(2)}$
$$\begin{cases} X_{ii} = \dfrac{1}{\sqrt{2}}\xi_{ii} & (1 \le i \le N) \\ X_{ij} = \dfrac{1}{2}(\xi_{ij} + i\,\eta_{ij}) & (1 \le i < j \le N) \end{cases}$$

[Rem: $P_\beta(dX) \propto e^{-\frac{\beta}{2}\mathrm{Tr}(X^2)}$ とすると 上と一致.]

x_i, $1 \le i \le N$ (eigenvalues)

Prop (Joint distribution)
The density is
$$P_{N\beta}(x_1, \cdots, x_N) = \frac{1}{\underbrace{Z_N(\beta)}_{\text{normalization}}} \prod_{1 \le i < j \le N} |x_i - x_j|^\beta e^{-\frac{\beta}{2}\sum_{i=1}^N x_i^2}$$
$(\beta = 1, 2)$

∵) For GOE,
$$X = O \Lambda\, {}^tO \qquad \Lambda = \mathrm{diag}(x_1, \cdots, x_N),\ {}^tOO = I$$

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$$\begin{cases} {}^tO\,dX\,O = \delta O\cdot\Lambda - \Lambda\cdot\delta O + d\Lambda \\ \delta O = {}^tO\,dO \end{cases}$$

$$\mathrm{Tr}(dX)^2 = 2\,\mathrm{Tr}(\delta O\cdot\Lambda\cdot\delta O\cdot\Lambda)$$

$$- 2\,\mathrm{Tr}(\delta O\cdot\Lambda^2\cdot\delta O) + \mathrm{Tr}(d\Lambda)^2$$

$$= 2\sum_{k<l}\delta O_{kl}\,\delta O_{kl}\,(\Lambda_k-\Lambda_l)^2 + \sum_k (d\Lambda_k)^2 \quad //$$

例. GUE の場合にやってみよ.

② Largest e.v. dist

$$\mathbb{P}_{N\beta}[x_{max}\le u] = \frac{1}{Z_{N\beta}}\int_{(-\infty,u]^N}\prod_{i<j}|x_i-x_j|^\beta\, e^{-\sum_{i=1}^N x_i^2}\,dx_1\cdots dx_N$$

x_{max} ↑ largest eigenvalue

For GUE, ($\beta=2$),

$$\mathbb{P}_{N2}[x_{max}\le u] = \frac{1}{Z_{N2}}\int \Delta(x)\cdot\Delta(x)\left(\prod_{i<j} 1_{(-\infty,u]}(x_i)\, e^{-x_i^2}\right) dx_1\cdots dx_N$$

$$\Delta(x) = \prod_{i<j}(x_j - x_i)$$

1.2. Fredholm determinant

μ : a measure on $\mathbb{R}$ s.t. $\int_{\mathbb{R}} \mu(dx) < \infty$ $\left(\text{mainly } \mu(dx) = \rho(x)\,dx\right)$

Kernel K : a measurable fun. $K : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{C}$ s.t.

$$\|K\| = \sup_{(x,y)\in\mathbb{R}^2} |K(x,y)| < \infty$$

<u>Def.</u> The Fredholm determinant associated with the Kernel is defined as

$$\det(I-K)_{L^2(\mathbb{R},\mu)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\det K(x_i, x_j)\right)_{1\le i,j\le N} d\mu(x_1)\cdots d\mu(x_N)$$

Let us consider a function in the form of a product of determinants.

$$Q_N(x_1, \dots, x_N) = \frac{1}{Z_N} \det\left(f_i(x_j)\right)_{i,j=1}^{N} \cdot \det\left(g_i(x_j)\right)_{i,j=1}^{N}$$

where f_i, g_i $(1\le i\le N)$ $\mathbb{R} \longrightarrow \mathbb{C}$

s.t. $\max_i \sup_{x\in\mathbb{R}} \left(|f_i(x)|, |g_i(x)|\right) < \infty$

and $\int f_i(x)\, g_j(x)\, d\mu(x) = \delta_{ij}$.

<u>Prop.</u> For $A \in \mathcal{B}(\mathbb{R})$

$$\int_{A^N} Q_N(x_1, \dots, x_N)\, d\mu(x_1)\cdots d\mu(x_N) = \det\left(I - K\right)_{L^2(\mathbb{R}, 1_{A^c} d\mu)}$$

where $K(x,y) = \sum_{i=1}^{N} f_i(x)\, g_i(y)$

Prop $\mathbb{P}_{N2}[x_{max} \le u] = \det(I-K)_{L^2(u,\infty)}$

where $K_{N2}(x,y) = \sum_{n=0}^{N-1} \frac{H_n(x)H_n(y)}{\sqrt{2}\, n!\, 2^n} e^{-\frac{x^2+y^2}{2}}$

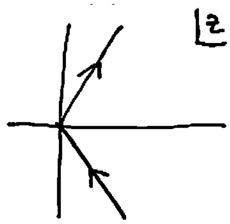
and $\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = \sqrt{2}\, n!\, 2^n \delta_{nm}$

1.3. Tracy-Widom distribution

Def (Airy kernel)

$$K_2(x,y) = \frac{A_i(x)A_i'(y) - A_i(y)A_i'(x)}{x-y}$$

$$A_i(x) = \frac{1}{2\pi i}\int_C e^{\frac{z^3}{3} - xz} dz$$



GUE TW dist

$$F_2(s) = \det(I-K_2)_{L^2(s,\infty)}$$

Thm. $\lim_{N\to\infty} \mathbb{P}_{N2}\left[\left(x_{max} - \sqrt{2N}\right)\sqrt{2}\, N^{1/6} \le s\right] = F_2(s)$

(Similary for $\beta = 1$)

1.4. GUE Dyson's BM

Time dependent GUE $\qquad N(0,1) \longrightarrow \text{BM}$

$$\{ B_{ij}, \tilde{B}_{ij} \}_{1 \le i < j \le N} \quad : \text{i.i.d. std BM}$$

Def. (Time-dep. GUE $X : [0,\infty) \longrightarrow \mathcal{H}_N^{(2)}$)

$$X_{jj} = B_{jj} \qquad (1 \le j \le N)$$

$$X_{ij} = \frac{1}{\sqrt{2}} \left(B_{ij} + \sqrt{-1}\, \tilde{B}_{ij} \right) \qquad (1 \le i < j \le N)$$

Thm. (Dyson) $\{X_i(t)\}$ satisfies

$$dX_i = dW_i + \sum_{j \ne i} \frac{1}{X_i - X_j} dt$$

where W_i, $1 \le i \le N$ are i.i.d. std. BMs.

$$X(t) = \begin{pmatrix} x_1(t) & & 0 \\ & & \\ 0 & & x_N(t) \end{pmatrix} \Longrightarrow X(t+dt) = \begin{pmatrix} x_1(t) + dB_{11}(t) & & * \\ & & \\ * & & \end{pmatrix}$$

問. $N=2$ の場合 check せよ.

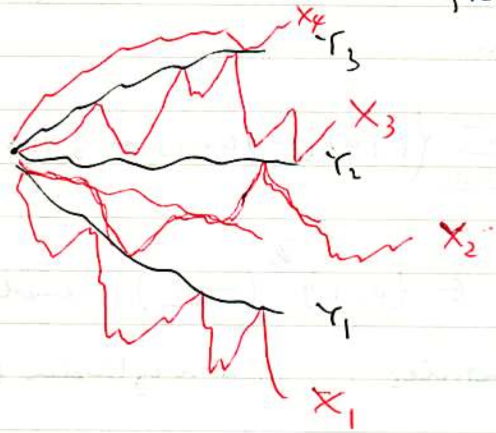
Gelfand - Tzetlin

1.5. BM on GT

Y_j $(1 \le j \le N)$ Dyson's BM with N particles starting from the origin.

i.e. $dY_j = dB_j + \sum_{k \neq j} \frac{1}{Y_j - Y_k} dt$

X_j $(1 \le j \le N+1)$ BM but reflected by Y_{j-1} and Y_j.



$$dX_j = d\tilde{B}_j + dL_j^- - dL_j^+$$

local time $X_j = Y_{j-1}$, $X_j = Y_j$.

Thm. (Warren)

X_j $(1 \le j \le N+1)$ is DBM with $N+1$ particles.

$x_3^{(3)}$ $x_3^{(2)}$ $x_3^{(1)}$

$x_2^{(2)}$ $x_1^{(2)}$

$x_1^{(1)}$

$$x_i^{(k+1)} \geq x_i^{(k)} \geq x_{i+1}^{(k+1)}$$

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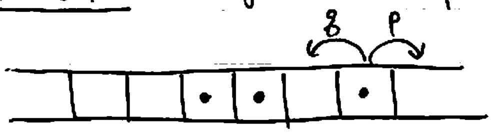
2. TASEP

2.1. ASEP asymmetric simple exclusion process

Informally



$p, q \geq 0 \quad pq \neq 0$

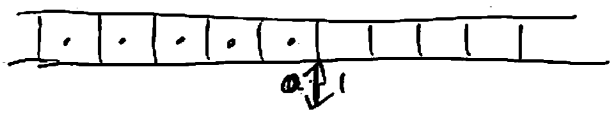
(rescaling of time $\Rightarrow$ sometimes $p+q=1$, $p=1$)

More ~~In~~formally, the process can be constructed using the generator

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} \{p\eta(x)(1-\eta(x+1)) + q\eta(x+1)(1-\eta(x))\}\{f(\eta^{x,x+1}) - f(\eta)\}$$

where $\eta = (\eta_x)_{x \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} (=\Omega)$ denotes a configuration of particles. f is a cylinder function on Ω and $\eta^{x,x+1}$ denotes a configuration s.t. $\eta(x)$ and $\eta(x+1)$ are swapped in η.

Step initial condition



$N(t) = \#$(particles on $x \geq 1$)

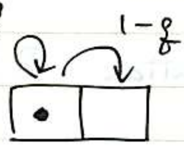
Thm. (Johansson 2000)

$$\mathbb{P}\left(N(t) > N\right) = \frac{1}{Z_N} \int_{[0,t]^N} \prod_{i<j} (x_i - x_j)^2 \prod_{j=1}^{N} \left(e^{-x_j} dx_j\right)$$

Thm.

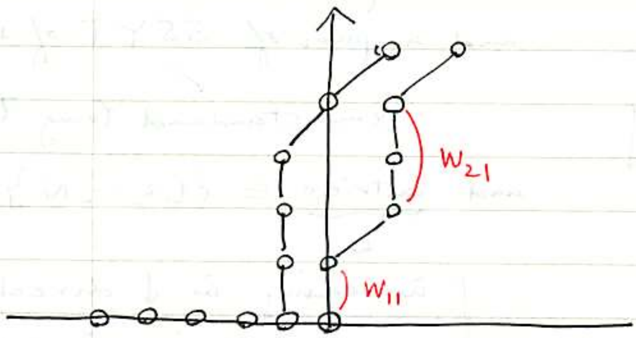
$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N(t) - \frac{t}{4}}{2^{-4/3}\, t^{1/3}} \geq -s \right) = F_2(s)$$

Main argument of the proof.



$(0<q<1)$

Discrete time TASEP



$w = \{w_{ij}\}_{i,j \geq 1}$

waiting time

w_{ij} = waiting time for i-th hop of j-th particle

(after it becomes possible)

i.i.d. geometric

$$\mathbb{P}(w_{ij} = n) = (1-q)\cdot q^n$$

$$W = \begin{bmatrix} w_{11} & w_{12} & \\ w_{21} & w_{22} & \\ & & \end{bmatrix}$$

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path

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$$G(M,N) = \max_{\pi} \Big(\sum_{(i,j)\in\pi} w_{ij} \Big)$$

π : down ~~—~~ right path from $(1,1)$ to (M,N)

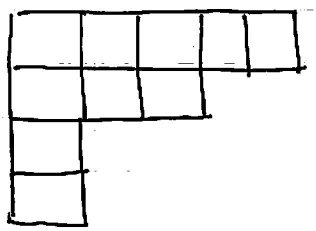
$G(M,N) + M + N - 1 =$ time at which Nth particle has made M-th hop.

$$\boxed{\textbf{Prop} \quad \mathbb{P}(N(t) \geq N) = \mathbb{P}\big(G(N,N) + 2N - 1 \leq t\big)}$$

Generalization $\mathbb{P}(w_{ij} = n) = (1 - a_i b_j)(a_i b_j)^n \quad n = 0, 1, 2 \cdots$

RSK correspondence $=$ 1:1 map between $N\times N$-matrix and a pair of SSYT of some shape

SSYT: semi standard Young tableaux

YD $=$ 

and entries $\in \{1, 2, \cdots, N\}$

$\downarrow$

- increasing in $\downarrow$ direction
- non-decreasing in $\longrightarrow$ direction

partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$, $\lambda_i \in \mathbb{Z}_+$, $\lambda_1 \geq \lambda_2 \geq \cdots$

In addition, $G(N,N) = \lambda_1$

The meas. on λ is $\quad = \frac{1}{Z} S_\lambda(a) S_\lambda(b)$

$$\underbrace{\sum_{\substack{T = \text{SSYT} \\ \text{of shape } \lambda}} a_1^{\#(1 \text{ in } T)} \cdots a_N^{\#(N \text{ in } T)}}_{S_\lambda(a_1, a_2, \cdots a_N)} \times \sum_{\substack{\tilde{T} = \text{SSYT} \\ \text{of shape } \lambda}} b_1^{\#(1 \text{ in } \tilde{T})} \cdots b_N^{\#(N \text{ in } \tilde{T})}$$

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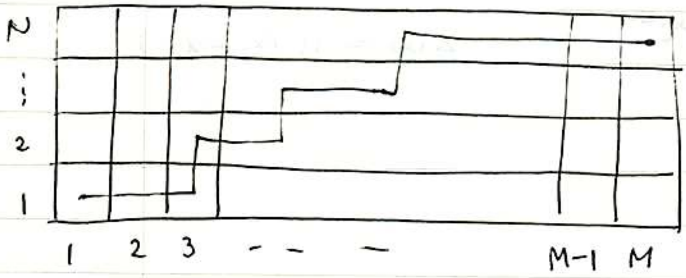
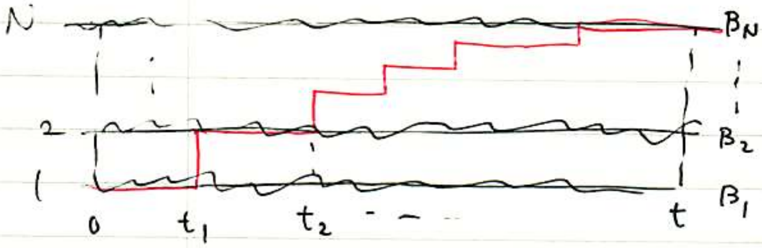
2.3. zero temperature directed polymer



$G(M, N)$: max

$(M \to \infty)$



i.i.d. BM.

$$E = \sum_{i=1}^{N} \left(B_i(t_i) - B_i(t_{i-1}) \right) \qquad t_0 := 0$$

$$\boxed{\max_{0 = t_0 < t_1 < \cdots < t_N \le t} E} \overset{d}{=} x_{max}^{GUE} \qquad \text{(Baryshnikov)}$$

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復習

ランダム行列 Dyson BM

GUE

$$\Delta(x)^2 \prod_i e^{-x_i^2} \qquad \Delta(x) = \prod (x_j - x_i)$$

$$\downarrow$$

$$\det f \cdot \det g$$

TASEP

$$\frac{1}{Z} s_\lambda(a)\, s_\lambda(b)$$ ← Schur measure

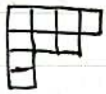
3. Schur process

3.1. Schur function

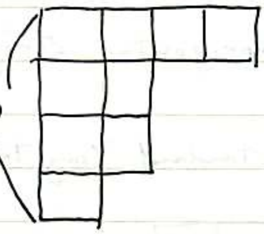
partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N, 0, \cdots)$ $\quad \lambda_j \in \mathbb{Z}_+$

$(\lambda_1 \geq \lambda_2 \geq \cdots)$ $\quad 1 \leq j \leq N$

ex. $\lambda = (4, 2, 2, 1) \longleftrightarrow \lambda' = (4, 3, 1, 1)$



$\lambda =$ $l(\lambda)$



$\leftarrow \lambda_1$

$\leftarrow \lambda_2$

$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N$

skew Young diagram

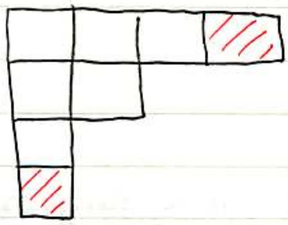
λ, μ -- Young diagram s.t. $\lambda \supset \mu$ $(\Leftrightarrow \lambda_i \geq \mu_i \; (\forall i))$

$\rightarrow \lambda/\mu = \lambda - \mu \qquad |\lambda/\mu| = |\lambda| - |\mu|$

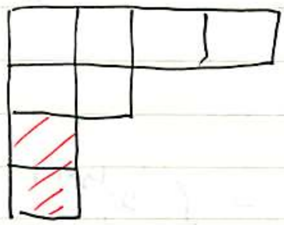
λ/μ is a horizontal strip

$\underset{\text{def}}{\Longleftrightarrow}$ In each column, λ/μ has at most one box.

$\lambda/\mu =$



horizontal



not horizontal

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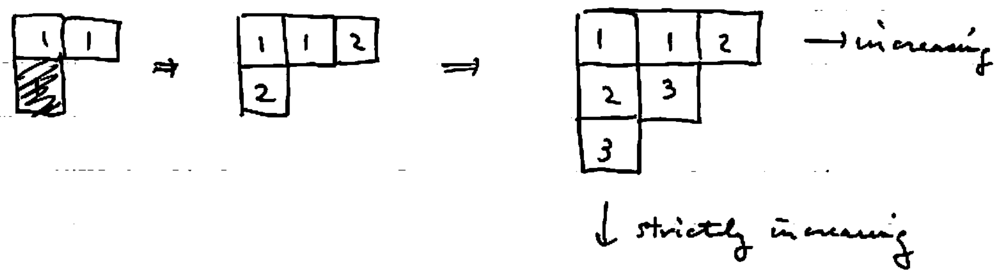
column-strict (skew) Young diagram tableux is a sequence of partitions

$$\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(M)} = \lambda$$

s.t. $\lambda^{(i)} / \lambda^{(i-1)}$ $(1 \le {}^{\forall} i \le M)$ is a horizontal strip.

If $\mu = \phi$, this is equivalent to semi-standard Young tableux.



Def. (the (skew) Schur function for a skew YD λ/μ)

$$S_{\lambda/\mu}(x_1, \cdots, x_N) = \sum_{\substack{T = \text{column strict} \\ \text{tableux of shape } \lambda/\mu}} x^T$$

$$x^T = \prod_{i=1}^{N} x_i^{|\lambda^{(i)}/\lambda^{(i-1)}|}$$

When $N=1$,

$$S_{\lambda/\mu}(x_1) = \begin{cases} x_1^{|\lambda/\mu|} & \text{If } \lambda/\mu \text{ is a horizontal strip} \\ & \text{(i.e. } \mu_i \le \lambda_i \le \mu_{i-1}) \\ 0 & \text{otherwise} \end{cases}$$

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$$S_{\lambda'/\mu'}(x_1) = \begin{cases} x_1^{|\lambda'/\mu'|} & \text{if } \lambda'/\mu' \text{ is a horizontal strip.} \\ & \text{i.e. } (\lambda_j = \mu_j \text{ or } \lambda_j = \mu_j + 1) \\ & \text{and } \lambda_j \geq \mu_j \\ 0 & \text{otherwise} \end{cases}$$

Involution: ω s.t. $\omega S_{\lambda/\mu}(x) = S_{\lambda'/\mu'}(x)$

Some relations for Schur functions

Write $x = (x_1, \dots, x_N)$, $y = (y_1, \dots, y_M)$

Define $\Pi(x, y) = \prod_{i=1}^{N} \prod_{j=1}^{M} (1 - x_i y_j)^{-1}$

(cf. Macdonald)

$$\sum_{\mu} S_{\mu/\lambda}(x) S_{\mu/\nu}(y) = \sum_{\tau} S_{\nu/\tau}(x) S_{\lambda/\tau}(y) \, \Pi(x, y)$$

$$\sum_{\lambda} S_{\lambda}(x) S_{\mu/\lambda}(y) = S_{\mu}(x, y)$$

$$= S_{\lambda}(x_1, \dots, x_n, y_1, \dots)$$

By applying ω_y,

$$\sum_{\mu} S_{\mu/\lambda}(x) S_{\mu'/\nu'}(y) = \sum_{\tau} S_{\nu/\tau}(x) S_{\lambda'/\tau'}(y) \underbrace{\omega_y \Pi(x, y)}_{= \prod_{i=1}^{N} \prod_{j=1}^{M} (1 + x_i y_j)}$$

$\omega_x \omega_y$

$$\sum_{\lambda} S_{\lambda'}(x) S_{\mu'/\lambda'}(y) = S_{\mu'}(x, y)$$

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3.2. A dynamics on Gelfand–Tsetlin pattern

$$GT_N = \left\{ \lambda_i^{(j)},\quad 1 \le i \le j \le N \;\middle|\; \begin{array}{l} \lambda_i^{(j)} \le N \\ \lambda_i^{(j-1)} \le \lambda_i^{(j)} \le \lambda_{i-1}^{(j-1)} \end{array} \right\}$$

$$\begin{array}{lccccc} \lambda^{(3)} = & \lambda_3^{(3)} & & \lambda_2^{(3)} & & \lambda_1^{(3)} \\ \lambda^{(2)} = & & \lambda_2^{(2)} & & \lambda_1^{(2)} & \\ \lambda^{(1)} = & & & \lambda_1^{(1)} & & \end{array} \qquad \begin{array}{ccc} & \lambda_i^{(j)} & \\ \lambda_i^{(j-1)} & & \lambda_{i-1}^{(j-1)} \end{array}$$

Each $\lambda^{(j)} = \{ \lambda_i^{(j)} \quad 1 \le i \le j \}$ is a YD.

Initial condition $\quad \lambda_i^{(j)}(0) = 0 \quad (\forall i, j)$

Dynamics order $\quad \lambda^{(1)} \longrightarrow \lambda^{(2)} \longrightarrow \cdots \longrightarrow \lambda^{(N)}$

Basically

$$\lambda_i^{(j)}(t+1) = \lambda_i^{(j)}(t) + \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases} \qquad p \in [0,1)$$

If $\lambda_i^{(j-1)}(t+1) = \lambda_i^{(j)}(t)$, then $\lambda_i^{(j)}(t+1) = \lambda_i^{(j)}(t)$.

(blocking)

If $\lambda_i^{(j-1)}(t+1) = \lambda_i^{(j)}(t)+1$, then $\lambda_i^{(j)}(t+1) = \lambda_i^{(j)}(t)+1$ (pushing)

$$\begin{array}{ccccc} \lambda_3^{(3)} & & \lambda_2^{(3)} & & \lambda_1^{(3)} \\ & \lambda_2^{(2)} & & \lambda_1^{(2)} & \\ & & \lambda_1^{(1)} & & \end{array}$$

$\longrightarrow$ ここのみを見ると"大体"TASEP

3.3. Schur process

Prop: One step transition probability for this dynamics is given by

$$G(\underline{\mu}, \underline{\lambda}) = \mathbb{P}\left(\underline{\lambda}(t+1) = \underline{\lambda} \mid \underline{\lambda}(t) = \underline{\mu}\right)$$

$$= \prod_{j=1}^{N} \frac{S_{\lambda^{(j)}/\lambda^{(j-1)}}(1)\, S_{\lambda^{(j)\prime}/\mu^{(j)\prime}}(r)}{\Delta(\lambda^{(j-1)}, \mu^{(j)})}$$

with $r = \frac{p}{1-p}$ and

$$\Delta(\lambda, \mu) = \sum_{\nu} S_{\nu/\lambda}(1)\, S_{\nu'/\mu'}(r)$$

Ex. (N=2)

$$\mathbb{P}\left(\underline{\lambda}(t+1) = \begin{matrix}\lambda_2^{(2)} & \lambda_1^{(2)} \\ & \lambda_1^{(1)}\end{matrix} \;\middle|\; \underline{\lambda}(t) = \begin{matrix}\mu_2^{(2)} & \mu_1^{(2)} \\ & \mu_1^{(1)}\end{matrix}\right)$$

$$\overset{?}{=} \frac{S_{\lambda^{(2)}/\lambda^{(1)}}(1)\, S_{\lambda^{(2)\prime}/\mu^{(2)\prime}}(r)}{\Delta(\lambda^{(1)}, \mu^{(2)})} \times \frac{S_{\lambda^{(1)}}(1)\, S_{\lambda^{(1)\prime}/\mu^{(1)\prime}}}{\Delta(\phi, \mu^{(1)})}$$

$$\left((\mu_2^{(2)}, \mu_1^{(2)}) \Longrightarrow (\lambda_2^{(2)}, \lambda_1^{(2)})\right) \quad \left(\mu_1^{(1)} \Longrightarrow \lambda_1^{(1)}\right)$$

$\mu_1^{(1)} \Longrightarrow \lambda_1^{(1)}$

$$S_{\lambda^{(1)}}(1) = \begin{cases} 1 & \lambda_i^{(1)} = 0 \quad (i \geq 2) \\ 0 & \text{otherwise} \end{cases}$$

$$S_{\lambda^{(1)\prime}/\mu^{(1)\prime}}(r) = \begin{cases} r & \lambda_1^{(1)} = \mu_1^{(1)} + 1, \quad \mu_i^{(1)} \leq \lambda_i^{(1)} \quad (i \geq 2) \\ 1 & \lambda_1^{(1)} = \mu_1^{(1)}, \quad \mu_i^{(1)} \leq \lambda_i^{(1)} \quad (i \geq 2) \\ 0 & \text{otherwise} \end{cases}$$

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Borodin-Ferrari

$$\Delta(\phi, \mu^{(1)}) = \begin{cases} 1+r & \mu_i^{(1)} = 0 \ (i \geq 2) \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{S_{\lambda^{(1)}}(1)\, S_{\lambda^{(1)\prime}/\mu^{(1)\prime}}(r)}{\Delta(\phi, \mu^{(1)})} = \begin{cases} \frac{r}{1+r} = p & \text{if } \lambda_1^{(1)} = \mu_1^{(1)} + 1,\ \mu_i^{(1)} = \lambda_i^{(1)} = 0 \ (i \geq 2) \\ \frac{1}{1+r} = 1-p & \text{if } \lambda_1^{(1)} = \mu_1^{(1)},\ \mu_i^{(1)} = \lambda_i^{(1)} = 0 \ (i \geq 2) \\ 0 & \end{cases}$$

Thm. The prob. that at time t the config. is $\underline{\lambda}$ is

$$P_{1,r}(\underline{\lambda}, t) = \frac{S_{\lambda^{(1)}}(1)\, S_{\lambda^{(2)}/\lambda^{(1)}}(1) \cdots S_{\lambda^{(N)}/\lambda^{(N-1)}}(1)\, S_{\lambda^{(N)}}(\underbrace{r, \ldots, r}_{t})}{(1+r)^{tN}}$$

Here we check

$$\sum_{\underline{\mu}} P_{1,r}(\underline{\mu}, t)\, G(\underline{\mu}, \underline{\lambda}) = P_{1,r}(\underline{\lambda}, t+1)$$

for $N=2$.

Indeed,

$$\sum_{\mu^{(1)}, \mu^{(2)}} P_{1,r}(\mu^{(1)}, \mu^{(2)}, t)\, G(\mu^{(1)}, \mu^{(2)}, \lambda^{(1)}, \lambda^{(2)})$$

$$= \sum_{\mu^{(1)}, \mu^{(2)}} \frac{\boxed{S_{\mu^{(1)}}(1)}\, S_{\mu^{(2)}/\mu^{(1)}}(1)\, S_{\mu^{(2)}}(\overbrace{r, \ldots, r}^{t})}{(1+r)^{2t}} \cdot \frac{S_{\lambda^{(2)}/\lambda^{(1)}}(1)\, S_{\lambda^{(2)\prime}/\mu^{(2)\prime}}(r)\, S_{\lambda^{(1)}}(1)\, S_{\lambda^{(1)\prime}/\mu^{(1)\prime}}(r)}{\Delta(\lambda^{(1)}, \mu^{(2)}) \cdot \boxed{\Delta(\phi, \mu^{(1)})}}$$

$$= \sum_{\mu^{(2)}} \frac{S_{\mu^{(2)}}(\overbrace{r,\dots,r}^{t})}{(1+r)^{2t}} \cdot \frac{S_{\lambda^{(2)}/\lambda^{(1)}}(1)\, S_{\lambda^{(2)}/\mu^{(2)}}(r)\, S_{\lambda^{(1)}}(1)}{(1+r^2)}$$

$$= \frac{S_{\lambda^{(1)}}(1)\, S_{\lambda^{(2)}/\lambda^{(1)}}(1)\, S_{\lambda^{(1)}}(\overbrace{r,\dots,r}^{t+1})}{(1+r)^{2(t+1)}}$$

One can consider a similar process on GT_N s.t. prob. is given by.

$$P_{a,b}(\underline{\lambda}, t) = \frac{1}{2} S_{\lambda^{(1)}}(a_1)\, S_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots S_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N) \times S_{\lambda}(b_1,\dots,b_t)$$

Schur process. ($a=1$, $b=r$ のときは M.C. の prob.)

$\Downarrow$ ここで $\lambda^{(N)}$ だけ見ると

$$\frac{1}{2} S_{\lambda}(a)\, S_{\lambda}(b) \qquad (\text{Schur meas.})$$

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4. KPZ equation

$$\frac{\partial h}{\partial t} = \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \nu \frac{\partial^2 h}{\partial x^2} + \sqrt{D}\,\eta(t,x)$$

η = Gaussian white noise with $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

4.1. Scaling

Set $\alpha = \dfrac{\lambda D^{1/2}}{(2\nu)^{3/2}}$.

For $\tilde{x} = \alpha^2 x$, $\tilde{t} = 2\nu\alpha^4 t$, $\tilde{h} = \dfrac{\lambda}{2\nu} h$

$$\Rightarrow \quad \frac{\partial \tilde{h}}{\partial t} = \frac{1}{2}\left(\frac{\partial \tilde{h}}{\partial x}\right)^2 + \frac{1}{2}\frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} + \eta(\tilde{x},\tilde{t})$$

4.2. Cole-Hopf solution

We can "define"

$$h(x,t) = \log Z(x,t)$$

where $Z(x,t)$ is the solution to SHE

$$dZ = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z\, dt + Z \cdot dB(x,t)$$

One can modify the noise as

$$\langle \eta_\kappa(x,t)\,\eta_\kappa(x',t')\rangle = C_\kappa(x-x')\,\delta(t-t')$$

SHE

$$\frac{\partial Z_\kappa}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z_\kappa + \eta_\kappa Z_\kappa \qquad C_\kappa(x) \xrightarrow[\kappa\to\infty]{} \delta_0(x)$$

Then $\displaystyle\lim_{\kappa\to\infty} Z_\kappa = Z$

For $h_\kappa = \log Z_\kappa$, $\quad \dfrac{\partial h_\kappa}{\partial t} = \dfrac{1}{2}\left(\dfrac{\partial h_\kappa}{\partial x}\right)^2 + \dfrac{1}{2}\dfrac{\partial^2 h_\kappa}{\partial x^2} + \eta_\kappa - \dfrac{1}{2}C_\kappa(0)$

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4.3. Feynman – Kac Formula

4.3.1. usual formula.

The solution to

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} u + V(x)\cdot u \quad , \quad u(x,0) = f(x).$$

is written as

$$u(x,t) = E_x\left[f(b_t) \exp \int_0^t V(b_s)\, ds \right]$$

b_s = std. BM s.t. $b_0 = x$.

4.3.2. SHE case

Bertini –

$$Z_K(x,t) = E\left[\exp \int_0^t \eta_K(b(s), t-s)\, ds \times Z_K(b(t), 0)\right] \times e^{-\frac{1}{2} C_K(0) t}$$

$Z(x,t) := \lim_{K \to \infty} Z_K(x,t)$ で定義.

4.4. Replica method

Random Ising model

random

$$H = -\sum_{\langle ij \rangle} J_{ij} \, s_i s_j \qquad s_i \in \{-1, 1\}$$

partition function

$$Z = \sum_{s_i \in \{\pm 1\}} e^{-\beta H}$$

Free energy $\quad F = \log Z.$

$$\downarrow$$

$$\langle \log Z \rangle$$

average over $J = (J_{ij})$

In some cases, it is easier to compute $\langle Z^N \rangle$

⑩ Replica trick

$$\langle \log Z \rangle = \lim_{N \downarrow 0} \frac{\langle Z^N \rangle - 1}{N} \qquad \cdots$$

Moment problem

For a prob. density $p(x)$, $x \in \mathbb{R}$ the n-th moment

$$m_n = \int_{\mathbb{R}} t^n \, p(t) \, dt \qquad n = 0, 1, \cdots.$$

Is it possible to recover $p(t)$ from $\{m_n\}$?

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Generating function

$$g(\zeta) = \sum_{n=0}^{\infty} \frac{m_n}{\zeta^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\zeta^{n+1}} \int_{\mathbb{R}} t^n p(t)\, dt$$

$$\doteqdot \int_{\mathbb{R}} \frac{1}{\zeta - t} p(t)\, dt.$$

$$p(x) = \frac{1}{2\pi i} \left\{ \lim_{\varepsilon \downarrow 0} \left(g(x+i\varepsilon) - g(x-i\varepsilon) \right) \right.$$

But, in general, $p(t)$ is not determined by $\{m_n\}$.

Consider a r.v. $h \sim N(0,1)$, put $Z = e^{h}$.

Z is distributed as log normal

$$\mathbb{P}\left(Z \in dz \right) = \frac{1}{\sqrt{2\pi}\, z} e^{-\frac{(\log z)^2}{2}} 1_{z>0}\, dz$$

It is easy to see that

$$\langle Z^n \rangle = \int_0^{\infty} z^n \frac{1}{\sqrt{2\pi}\, z} e^{-\frac{(\log z)^2}{2}} dz = e^{\frac{n^2}{2}}$$

For any a s.t $|a| \leq 1$

$$\frac{1}{\sqrt{2\pi}\, z} e^{-\frac{(\log z)^2}{2}} \left(1 + a \sin(2\pi \log z) \right) 1_{z>0}$$

gives the same moment $e^{n^2/2}$.

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$$m_n = e^{\frac{n^2}{2}} \text{ a.s.}$$

$$g(\xi) = \sum_{n\geq 0}^{\infty} \frac{1}{\xi^{n+1}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{nx - \frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \frac{dx}{\xi - e^x}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dz}{z} e^{-\frac{(\log z)^2}{2}} \frac{1}{\xi - z}$$

$$\therefore \frac{\mathbb{P}(z + dz)}{dz} = \frac{1}{2\pi i} \left(g(z + i\varepsilon) - g(z - i\varepsilon) \right)$$

$$= \frac{1}{\sqrt{2\pi} z} e^{-\frac{(\log z)^2}{2}} dz.$$

KPZ

$$\langle Z^n \rangle \sim e^{n^3/3} = \int_{\mathbb{R}} A_i(x) e^{nx} dx$$

4.5. δ - Bose gas

For KPZ,

$$\left\langle e^{-e^{h(x,t)+\frac{t}{24}-\gamma_t s}}\right\rangle = \sum_{N=0}^{\infty}\frac{\left(-e^{-\gamma_t s+\frac{t}{24}}\right)^N}{N!}\left\langle Z(x,t)^N\right\rangle$$

Using Feynman-Kac representation,

$$Z_\kappa(x,t) = E_x\left[\exp\left(\int_0^t \eta_\kappa(b(s),t-s)\,ds\right)\cdot Z_\kappa(b(t),0)\right]\times e^{-\frac{1}{2}C_\kappa(0)t}$$

$$=\left\langle E_{x_1}\cdots E_{x_N}\left[\prod_{i=1}^N\left\{\exp\int_0^t \eta_\kappa(b_i(s_i),t-s_i)\,ds_i\cdot Z_\kappa(b_i(t),0)\,e^{-\frac{1}{2}C_\kappa(0)t}\right\}\right]\right\rangle$$

$$\left(\left\langle\prod_{i=1}^N \exp\int_0^t \eta_\kappa(b_i(s_i),t-s_i)\right\rangle = e^{\frac{1}{2}\sum_{j,k=1}^N C_\kappa(x_j-x_k)}\right)$$

$$E_{x_1}\cdots E_{x_N}\left[e^{\int ds\,\frac{1}{2}\sum_{j\neq k}C_\kappa(b_j(s)-b_k(s))}\prod_{i=1}^N Z(b_i(t),0)\right]$$

$$=\left\langle x_1\cdots x_N\right| e^{-H_{N,\kappa}t}\left|0,0,\cdots,0\right\rangle$$

Therefore, $G_N(x_1\cdots x_N;t)$ satisfies

$$\begin{cases}\dfrac{\partial G_N(x_1\cdots x_N;t)}{\partial t} = -H_{N,\kappa}G_N(x_1\cdots x_N,t)\\ G_N(x_1\cdots x_N;0) = \cancel{\prod_{i=1}^N\delta(x_i)}\left\langle\prod_{i=1}^N Z(x_i,0)\right\rangle\end{cases}$$

where $H_{N,\kappa} = -\frac{1}{2}\sum_{j=1}^N\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}C_\kappa(x_j-x_k)$

As $k \to \infty$,

$$G_N(x_1, \dots, x_N ; t) = \left\langle \prod_{i=1}^{N} Z(x_i, t) \right\rangle$$

satisfies

$$\frac{\partial G_N}{\partial t} = -H_N G_N \quad , \quad H_N = \frac{-1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k} \delta(x_j - x_k)$$

$$G_N(\, , t=0) = \left\langle \prod_{i=1}^{N} Z(x_i, t) \right\rangle$$

↳ wedge $\prod_{i=1}^{N} \delta(x_i)$

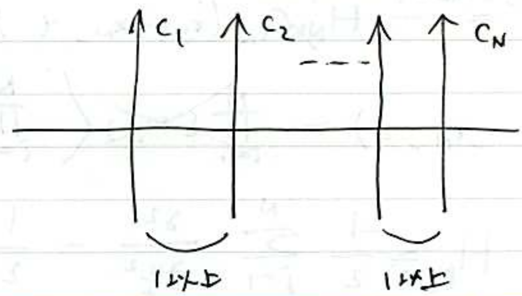
Hamiltonian of (attaractive) Bose gas.

$$\frac{\partial G}{\partial t} = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} G \qquad x_1 < x_2 < \dots < x_N$$

$$\left(\partial_{x_i} - \partial_{x_{i+1}} - 1 \right) G \Big|_{x_{i+1} = x_i + 0} = 0 \qquad (1 \le i \le N-1)$$

For wedge $Z(x, 0) = \delta(x)$

$$\left\langle \prod_{i=1}^{N} Z(x_i, t) \right\rangle = \left(\frac{1}{2\pi i} \right)^N \int_{C_1} \cdots \int_{C_N} \prod_{1 \le A < B \le N} \frac{z_A - z_B}{z_A - z_B - 1} e^{\frac{1}{2} \sum_{j=1}^{N} z_j^2 t + \sum_{j=1}^{N} x_j z_j} dz_1 \cdots dz_N$$



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$$H_N = -\frac{1}{2}\sum_j \frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}\delta(x_j - x_k)$$

wedge $\quad \left\langle \prod_i Z(x_i, 0) \right\rangle = \prod_i \delta(x_i)$

$$\left\langle \prod_i Z(x_i, t)\right\rangle = \int \cdots \int \prod_{A<B} \frac{z_A - z_B}{z_A - z_B - 1} \prod_j e^{\frac{1}{2} z_j^2 t + x_j z_j} dz_j$$

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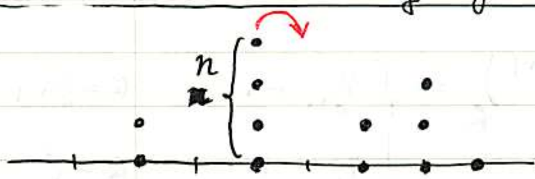
5. qTASEP and duality
qTAZRP

5.1. Process — totally asymmetric zero range process



only jump to the right

jump rate $\quad [n] = \dfrac{1-q^n}{1-q} \quad (0 \le q < 1)$

$$[n]! = [n]\cdot[n-1] \cdots\cdots [1]$$

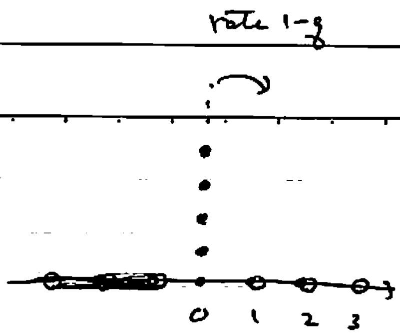
$$(\alpha; q)_n = (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})$$

Rem. $\quad q \to 1 \quad$ indep.

$\quad\quad q = 0 \quad$ rate 1 $\quad \left(\begin{array}{c}\text{equivalent to} \\ \text{TASEP}\end{array}\right)$

Stationary meas. at 1-site

$$\mathbb{P}(\eta_j = n) = \left(\frac{\alpha}{1-q}; q\right)_\infty \frac{\alpha^n}{[n]!}$$



Initial condition

$$\eta_0 = +\infty, \quad \eta_j = 0 \quad \text{if } \cancel{j\neq 0} \ j \geq 1$$

Height function

$$h_j = \sum_{k\geq j}^{\infty} \eta_k \quad (j \geq 1)$$

Generator

$$Lf(\eta) = \sum_{j\in\mathbb{Z}} [\eta_j] \left(f(\eta^{j,j+1}) - f(\eta) \right)$$

$$(\eta^{j,j+1})_k = \begin{cases} \eta_{j+1} + 1 & k = j+1 \\ \eta_j - 1 & k = j \\ \eta_k & k \neq j, j+1 \end{cases}$$

5.2. Result

Thm.

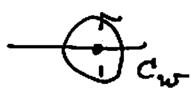
$$\left\langle \frac{1}{(\zeta q^{h_N}; q)_\infty} \right\rangle = \det(I + K_\zeta)_{L^2(C_w)},$$

where the kernel

$$K_\zeta(w, w') = \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w,w'}(q^s)\, ds$$

$$g_{w,w'}(x) = \frac{1}{xw - w'} \left(\frac{(xw; q)_\infty}{(w; q)_\infty} \right)^N e^{tw(x-1)}$$

C_w -- small contour around 1



$$z_j = q^{h_j}$$

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KPZ のとき

$$\left\langle e^{-e^{h+\frac{t}{24}-\gamma_t s}} \right\rangle = \det(I - K_{t,s})$$

$$K_{ts}(x,y) = \int_{\mathbb{R}} \frac{A_i(x+\lambda)\, A_i(y+\lambda)}{e^{\gamma_t(\lambda - s)}+1}\, d\lambda$$

$$\begin{tikzcd}
q\text{-TASEP} \arrow[d, "q\to 1"'] \arrow[dr] & \\
\text{KPZ} & \text{O'Connell} \arrow[l]
\end{tikzcd}$$

$$dZ_j = (Z_{j-1} - Z_j)\,dt + Z_j\, dB_j$$

<u>Rem. 1</u> By the q-binomial thm,

$$\frac{1}{(x;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_\infty}$$

LHS is written as

$$\sum_{n=0}^{\infty} \frac{\zeta^n}{(q;q)_\infty} \langle q^{n h_N} \rangle \qquad z_N = q^{h_N}$$

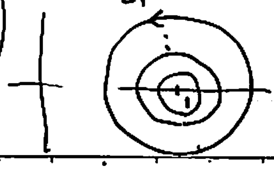
The theorem is based on the following formula for $\langle q^{n h_N} \rangle$

<u>Prop.</u>

$$\langle q^{n h_N(t)} \rangle = (-1)^n q^{\frac{n(n-1)}{2}} \int \frac{dz_1}{2\pi i} \cdots \int \frac{dz_n}{2\pi i} \prod_{k<l} \frac{z_k - z_l}{z_k - q z_l} \times \prod_{k=1}^{n} (1-z_k)^{-N} \frac{e^{-t z_k}}{z_k}$$

$\left(\begin{array}{l} \gamma_k = \text{contains } \{q\gamma_l\}_{l>k} \text{ and } 1 \\ \text{but not } 0 \end{array} \right)$

z_1



prop $\longrightarrow$ Thm

prop. For a meromorphic function $f(z)$, $\mathbb{A}$: a fixed set of singularities of f,

Assume $q^m \mathbb{A} \cap \mathbb{A} = \phi$ $(\forall m \geq 1)$

$$\frac{(-1)^n}{2\pi i} q^{\frac{n(n-1)}{2}} \int \cdots \int \prod_{1 \leq A < B \leq n} \frac{z_A - z_B}{z_A - q z_B} \frac{f(z_1) \cdots f(z_n)}{z_1 \cdots z_n} dz_1 \cdots dz_n$$

$$= (q;q)_n \sum_{\substack{\lambda \vdash n \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{1}{m_1! m_2! \cdots} \frac{1}{(2\pi i)^{\ell(\lambda)}} \int \cdots \int \det \left[\frac{1}{w_i q^{\lambda_i} - w_j} \right]_{i,j=1}^{\ell(\lambda)}$$

$$\times \prod_{j=1}^{\ell(\lambda)} f(w_j) f(q w_j) \cdots f(q^{\lambda_j - 1} w_j) \, dw_j$$

z_j contains $\{q z_k\}_{k>j}$, $\mathbb{A}$ but not 0

w_j contains $\mathbb{A}$

For qTAZRP one applies this prop with

$$f(z) = \left(\frac{1}{1-z}\right)^N e^{-tz}$$

Then one can proceed as

$$\left\langle \frac{1}{(\zeta q^{x_N}; q)_\infty} \right\rangle = \sum_{n=0}^{\infty} \sum_{L=0}^{\infty} \frac{1}{m_1! m_2! \cdots} \left(\frac{1}{2\pi i}\right)^L \int \cdots$$

$$\begin{cases} \sum m_i = L \\ \sum i m_i = n \end{cases}$$

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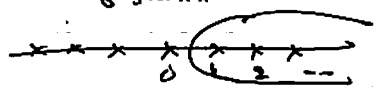
$$= \sum_{L=0}^{\infty} \frac{1}{L!} \sum_{m_1,\dots,m_L\geq 1} \int\cdots\int \det\left[\frac{1}{q^{m_i}w_i - w_j}\right]_{i,j=1}^{L} \prod (1-q)^{m_j} \zeta^{m_j}$$

$$\times f(w_j)\cdots f(q^{m_j-1} w_j) \frac{dw_j}{2\pi i}$$

$$= \det(I+K)_{L^2(\mathbb{Z}_{\geq 0}\times C_w)}$$

$$\left(\begin{aligned} K(n_1,w_1;n_2,w_2) &= \frac{1}{q^{n_1}w_1 - w_2}\, \zeta^{n_1} f(w_1)\cdots f(q^{n_1-1}w_1) \\ &= \zeta^{n_1} g_{w_1,w_2}(q^{n_1}) \end{aligned} \right)$$

pole of $\frac{1}{\sin \pi x}$



$$K(w_1,w_2) = \sum_{n\geq 1}^{\infty} \zeta^n g_{w_1,w_2}(q^n)$$

$$= \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} \Gamma(-s)\Gamma(1+s)(-\zeta)^s g_{w_1w_2}(q^s)\, ds$$

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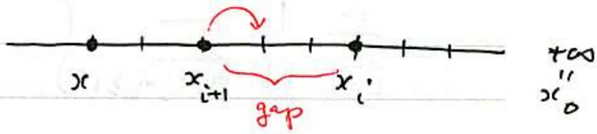
5.3. Duality

rate $[x_i - x_{i+1} - 1]$



q-TASEP $\xrightarrow[q \downarrow 0]{}$ TASEP

$\frac{1-q^{gap}}{1-q}$

$\eta_j(t) = x_j(t) - x_{j+1}(t) - 1$ is q-TAZRP

step init. cond. for q-TASEP

$$x_j(0) = -j, \quad j \geq 1$$

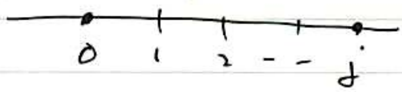
corresponds to $\eta_0(0) = +\infty$ and $\eta_j(0) = 0 \quad (j \geq 1)$

Moreover, $x_j(t) + j = \sum_{k=j}^{\infty} \eta_k(t) = l_j(t)$

Slightly different version of the q TAZRP

There ~~hopping~~ is no hopping once a particle enters 0

rate $[\xi_j]$



Define

$$D(x, \xi) = \prod_{j=0}^{n} q^{(x_j + j)\xi_j} = \begin{cases} 0 & \text{if } \xi_0 > 0 \\ \prod_{j=1}^{n} q^{(x_j + j)\xi_j} & \text{if } \xi_0 = 0 \end{cases}$$

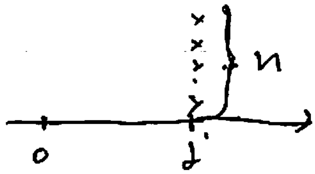
x: q-TASEP, ξ: q-TAZRP

Prop (duality) $(\forall n)$

$$\boxed{E^x\left[D(x(t), \xi)\right] = E^\xi\left[D(x, \xi(t))\right]}$$

$x(0) = x$, $\xi(0) = \xi$

Cor.



$$\langle q^{n(x_j+j)}\rangle = E^{\xi(0)}\left[1_{\xi_0(t)=0} \right]$$

$$= \mathrm{Prob}\left[\begin{array}{l}\text{starting from no particle has} \\ \text{arrived } 0 \text{ at time } t\end{array}\right]$$

$$= \sum_{1\le y_1\le y_2\le \dots \le y_N} P(y;t \mid y_0; t)$$

$$\begin{cases} y = (y_1, \dots, y_N) \\ y_0 = (j, \dots, j) \end{cases}$$

transition prob. for n particles

q-TAZRP (3)

$$P(y;t \mid y_0, 0) = \int \Psi^r_z(y)\, \Psi^l_z(y_0)\, e^{\Lambda_z t}\, d\mu(z)$$

Λ_z = eigenvalue $\quad \Lambda_z = -\sum_{i=1}^{n} z_i$

$\Psi^{r,l}_z(y)$: right, left eigenfunction of L_n

z = label $\qquad$ q-TASEP

$d\mu(z)$ = (spectral) measure.

By Bethe ansatz for $z = (z_1, \dots, z_n)$

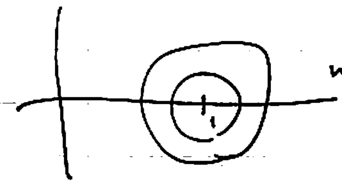
$$\Psi^l_z(y) = \sum_{\sigma\in S_n} \prod_{1\le B<A\le n} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - \cancel{q} z_{\sigma(B)}} \prod_{j=1}^{n} (1-z_{\sigma(j)})^{-y_j}$$

$$\Psi^r_z(y) = \frac{1}{c_q(y)} \sum_{\sigma\in S_n} \prod_{1\le B<A\le n} \frac{z_{\sigma(A)} - q^{-1} z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} \prod_{j=1}^{n} (1-z_{\sigma(j)})^{y_j}$$

where $C_q(y)$ is appeared for diagonal $y_i = y_j$.

Thm. (Plancherel formula for n-particle qTASEP)

$$\int \frac{dz_1}{2\pi i}\cdots\int\frac{dz_n}{2\pi i}\prod_{1\le A<B\le n}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{n}(1-z_j)^{-y_j-1}\sum_{x} f(x)\,\Psi^{r}_{z}(x) = f(y)$$



nested contours

By shrinking contours,

Cor (completeness)

$$\sum_{\lambda\vdash n}\int\cdots\int d\mu_\lambda(w)\prod_{j=1}^{\ell(\lambda)}\frac{1}{(w_j;q)_{\lambda_j}}\,\Psi^{r}_{w(\lambda)}(x)\,\Psi^{\ell}_{w(\lambda)}(y)=\delta_{xy}$$

where $w(\lambda)=(w_1, qw_1,\dots, q^{\lambda_1-1}w_1, w_2,\dots, q^{\lambda_2-1}w_2,\dots, q^{\lambda_\ell-1}w_\ell)$

$$d\mu_\lambda(w)=\frac{(1-q)^n(-1)^n q^{-\frac{n^2}{2}}}{m_1!\,m_2!\cdots}\det\left[\frac{1}{w_i q^{\lambda_i}-w_j}\right]_{i,j=1}^{\ell(\lambda)}\prod_{j=1}^{\ell(\lambda)} w_j^{\lambda_j}\, q^{\frac{\lambda_j^2}{2}}\frac{dw_j}{2\pi i}$$

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6. Macdonald process

A generalization of Schur process
(skew)

6.1. Macdonald function / polynomial function

q, t : indeterminate

Define $\displaystyle f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty}$

$$\varphi_{\lambda/\mu} = \prod_{1\le i\le j\le \ell(\lambda)} \frac{f(q^{\lambda_i-\lambda_j} t^{j-i})\, f(q^{\mu_i-\mu_{j+1}} t^{j-i})}{f(q^{\lambda_i-\mu_j} t^{j-i})\, f(q^{\mu_i-\lambda_{j+1}} t^{j-i})}$$

$$\psi_{\lambda/\mu} = \prod_{1\le i\le j\le \ell(\lambda)} \frac{f(q^{\mu_i-\mu_j} t^{j-i})\, f(q^{\lambda_i-\lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i-\mu_j} t^{j-i})\, f(q^{\mu_i-\lambda_{j+1}} t^{j-i})}$$

The (skew) Macdonald $x = (x_1, \dots, x_N)$

$$P_{\lambda/\mu}(x) = \sum_T \psi_T x^T, \quad \psi_T = \prod_{i=1}^N \psi_{\lambda^{(i)}/\lambda^{(i-1)}}$$

$\| \ P_{\lambda/\mu}(x;q,t)$ column strict tableaux with $\lambda^{(0)}=\mu,\ \lambda^{(N)}=\lambda$

$$Q_{\lambda/\mu}(x) = \sum_{T \atop (\cdots)} \varphi_T x^T, \quad \varphi_T = \prod_{i=1}^N \varphi_{\lambda^{(i)}/\lambda^{(i-1)}}$$

Involution

$q = t \Rightarrow$ Schur

$\omega P_{\lambda/\mu}(x;q,t) = Q_{\lambda'/\mu'}(x;t,q)$

$\omega Q_{\lambda/\mu}(x;q,t) = P_{\lambda'/\mu'}(x;t,q)$

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When $N=1$,

$$P_{\lambda/\mu}(x_1) = \begin{cases} \psi_{\lambda/\mu} \; x_1^{|\lambda/\mu|} & \lambda/\mu \text{ : horizontal strip} \\ 0 & \text{otherwise} \end{cases}$$

$$Q_{\lambda/\mu}(x_1) = \begin{cases} \varphi_{\lambda/\mu} \; x_1^{|\lambda/\mu|} & \lambda/\mu = \text{h.s.} \\ 0 & \text{otherwise} \end{cases}$$

Some relations for Macdonald

Define $\quad \Pi(x\_y) := \prod_{i=1}^{N} \prod_{j=1}^{M} \dfrac{(t x_i y_j ; q)_\infty}{(x_i y_j ; q)_\infty} \qquad y = (y_1, \cdots, y_N)$

$$\omega_y \Pi(x\_y) = \prod_{i=1}^{N} \prod_{j=1}^{M} (1 + x_i y_j)$$

Then,

$$\sum_{\mu} P_{\mu/\lambda}(x) \, Q_{\mu/\nu}(y) = \sum_{\tau} P_{\nu/\tau}(x) \, Q_{\lambda/\tau}(y)$$

$$\sum_{\lambda} P_\lambda(x) \, P_{\mu/\lambda}(y) = P_\mu(x, y).$$

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6.2. Macdonald process

measure on GT_N

$$M_{a,b}(\lambda) = \frac{1}{Z} P_{\lambda^{(1)}}(a_1) P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N) \begin{cases} Q_{\lambda'}(b) \\ Q_{\lambda^{(N)}}(b) \end{cases}$$

$$\begin{cases} a = (a_1, a_2, \ldots, a_N) \\ b = (b_1, b_2, \ldots, b_M) \end{cases}$$

$$Z = \Pi(a; b)$$

$$\left(\,'a\ t\ z\ 12 \quad Z = \omega_b \Pi(a,b) \right)$$

<u>Marginal</u> for $\lambda = \lambda^{(N)}$ is the Macdonald measure.

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$$

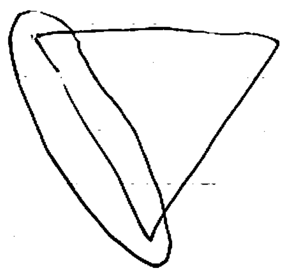
<u>Dynamics</u> $\quad G(\underline{\mu}, \underline{\lambda}) \quad (\Pi SS \longrightarrow \Pi PQ)$

soln.

<u>$t=0$</u> $\quad$ q-Whittaker meas. $\quad \begin{cases} a = (a_1, \ldots, a_N) & a_i = 1 \\ b = (b_1, \ldots, b_M) & b_i = r. \end{cases}$

$$Q_{\lambda'/\mu'}(x; q, t=0)$$

$$= \sum_T \psi'_T x^T$$



q-TASEP

$$\psi'_{\lambda/\mu} = \prod_{\substack{i \geq 1 \\ \lambda_i = \mu_i \\ \lambda_{i+1} = \mu_{i+1} + 1}} \left(1 - q^{\mu_i - \mu_{i+1}}\right)$$

B.C. discrete time q-TASEP

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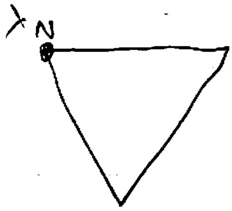
6.3. Expectations

Macdonald measure $\quad Z^{-1} P_\lambda(x) Q_\lambda(y) \qquad \left(\begin{aligned} Z &= \sum_\lambda P_\lambda(x) Q_\lambda(y) \\ &= \Pi(x,y) \end{aligned} \right)$

Want to compute



$$\langle \underset{F(\lambda)}{\cancel{q^{n(\lambda)}}} \rangle = \frac{1}{\Pi(x;y)} \sum_\lambda F(\lambda) P_\lambda(x) Q_\lambda(y)$$

If $\quad \underset{\substack{\uparrow \\ \text{op.}}}{D} P_\lambda = d_\lambda P_\lambda$,

$$\langle d_\lambda \rangle = \frac{1}{\Pi(x;y)} \sum_\lambda \underbrace{d_\lambda P_\lambda(x)}_{\substack{\parallel \\ DP_\lambda(x)}} Q_\lambda(y)$$

$$= \frac{D\,\Pi(x;y)}{\Pi(x;y)}$$

$$\langle d_\lambda^n \rangle = \frac{D^n\,\Pi(x;y)}{\Pi(x;y)}$$

Macdonald's difference operator

$$(D_N^r f)(x_1, x_2, \ldots, x_N)$$

$$:= \sum_{\substack{I \subset \{1,2,\ldots,N\} \\ |I| = r}} t^{\frac{r(r-1)}{2}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} f(x_1, \ldots, q x_i, \ldots, x_N)$$

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Prop.

$$(D_N^r P_\lambda)(x_1 \cdots x_N) = e_r(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \cdots q^{\lambda_N})\, P_\lambda(x_1 \cdots x_N)$$

where $e_r(x_1, \cdots x_N) = \sum_{1 \le i_1 < \cdots < i_r \le N} x_{i_1} \cdots x_{i_r}$

Then,

$$\langle e_r(q^{\lambda_1}t^{N-1}, \cdots, q^{\lambda_N}) \rangle = \frac{D_N^r \Pi(x; y)}{\Pi(x; y)}$$

$$= \frac{1}{(2\pi i)^r r!} \int \cdots \int \det\left[\frac{1}{tz_k - z_l}\right]_{k,l=1}^r \prod_{j=1}^r \left(\prod_{m=1}^r \frac{tz_j - x_m}{z_j - x_m}\right) \frac{f(qz_j)}{f(z_j)} dz_j$$

As $t \to 0$

$$\lim_{t \to 0} t^{-\frac{r(r-1)}{2}} e_r(q^{\lambda_1}t^{N-1}, \cdots q^{\lambda_N}) = q^{\lambda_N + \cdots + \lambda_{N-r+1}} \quad (= q^{\lambda_N})$$

For continuous time q-TASEP $f(z) = \left(\frac{1}{1-z}\right)^N e^{-tz}$ $\quad r = 1$

$$\Rightarrow \langle q^{n\lambda_N} \rangle = \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \int \cdots \int \prod_{1 \le A < B \le n} \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^n f(z_j) \frac{dz_j}{z_j}$$

$$\Rightarrow \left\langle \frac{1}{(\zeta q^{\lambda_N}; q)_\infty} \right\rangle = \det(I + K)$$