Determinantal Martingales and Interacting Particle Systems *

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Abstract

Determinantal process is a dynamical extension of a determinantal point process such that any spatio-temporal correlation function is given by a determinant specified by a single continuous function called the correlation kernel. Noncolliding diffusion processes are important examples of determinantal processes. In the present lecture, we introduce determinantal martingales and show that if the interacting particle system (IPS) has determinantal-martingale representation, then it becomes a determinantal process. From this point of view, the reason why noncolliding diffusion processes and noncolliding random walk are determinantal is simply explained by the fact that the harmonic transform with the Vandermonde determinant provides a proper determinantal martingale. Recently O'Connell introduced an interesting IPS, which can be regarded as a stochastic version of a quantum Toda lattice. It is a geometric lifting of the noncolliding Brownian motion and is not determinantal, but Borodin and Corwin discovered a determinant formula for a special observable for it. We also discuss this new topic from the present view-point of determinantal martingale.

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1 Introduction

1.1 Determinantal martingale

Let $V(t), t \in \mathcal{T}$ be a Markov process in a state space $S \subset \mathbb{R}$, where the set of time is continuous $\mathcal{T} = [0, \infty)$ or discrete $\mathcal{T} = \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$. The probability space is denoted as $(\Omega, \mathcal{F}, \mathrm{P})$, where the expectation is written as E. We introduce a filtration $\{\mathcal{F}(t) : t \in \mathcal{T}\}$ defined by $\mathcal{F}(t) = \sigma(V(s), s \in [0, t] \cap \mathcal{T})$. When $V(t), t \in [0, \infty)$ is a continuous time Markov process, provided it has right-continuous sample paths, the transition probability density is given by

$$p(t, y|x) = \frac{\partial}{\partial y} \mathrm{P}(V(t) \leq y | V(0) = x), \quad x, y \in S, \tag{1.1}$$

and when $V(t), t \in \mathbb{N}_0$ is a Markov chain, the transition probability is given by

$$p(t, y|x) = \mathrm{P}(V(t) = y | V(0) = x), \quad x, y \in S. \tag{1.2}$$

The process $M(t), t \in \mathcal{T}$ on $(\Omega, \mathcal{F}, \mathrm{P})$ adapted to $\{\mathcal{F}(t) : t \in \mathcal{T}\}$, is said to be a *martingale* if, for every $s < t, s, t \in \mathcal{T}$,

$$\mathrm{E}[M(t)|\mathcal{F}(s)] = M(s) \quad \text{a.s.} \quad \mathrm{P}. \tag{1.3}$$

Let $\mathbb{N} = \{1, 2, \dots\}$. For a Markov process $V(t), t \in [0, \infty)$, if there exists a nondecreasing sequence $\{T_n : n \in \mathbb{N}\}$ of stopping times of $\{\mathcal{F}(t) : t \in [0, \infty)\}$, such that $M_n(t) \equiv M(t \wedge T_n), t \in \mathcal{T}$ is a martingale for every $n \in \mathbb{N}$ and $\mathrm{P}\left[\lim_{n \to \infty} T_n = \infty\right] = 1$, then we say $M(t), t \in \mathcal{T}$ is a *local martingale*.

For $N \in \mathbb{N}$, we put

$$\mathbb{W}_N = \{\boldsymbol{x} = (x_1, \dots, x_N) \in S^N : x_1 < x_2 < \cdots < x_N\}. \tag{1.4}$$

For $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N$, we define a measure ξ by a sum of point masses concentrated on u_j's, $1 \leq j \leq N$,

$$\xi(\cdot) = \sum_{j=1}^{N} \delta_{u_j}(\cdot). \tag{1.5}$$

Depending on ξ, we assume that there is a one-parameter family of maps

$$\mathcal{M}_\xi^u(\cdot, \cdot) : \mathcal{T} \times \mathbb{R} \mapsto \mathbb{R} \tag{1.6}$$

with a parameter $u \in \mathbb{C}$, such that

$$\begin{aligned}
&(i) && \mathcal{M}_\xi^u(\cdot, V(\cdot)) \text{ is a local martingale,} \\
&(ii) && \mathcal{M}_\xi^{u_k}(\cdot, x) : 1 \leq k \leq N \quad \text{are linearly independent functions of } x, \\
&(iii) && \mathcal{M}_\xi^{u_k}(0, u_j) = \delta_{jk}, \quad u_j, u_k \in \operatorname{supp} \xi = \{u_1, \dots, u_N\}.
\end{aligned} \tag{1.7}$$

Let $\{V_j(t), t \in \mathcal{T} : 1 \leq j \leq N\}$ be a collection of N independent copies of $V(t), t \in \mathcal{T}$. We consider the N-component vector-valued process $\boldsymbol{V}(t) = (V_1(t), \dots, V_N(t))$, $t \in \mathcal{T}$, for which the initial values are fixed to be $V_j(0) = u_j \in S, 1 \le j \le N$ and the probability space is denoted by $(\Omega, \mathcal{F}, \mathrm{P}_{\boldsymbol{u}})$ with expectation $\mathrm{E}_{\boldsymbol{u}}, \boldsymbol{u} = (u_1, \dots, u_N)$.

For $n \in \mathbb{N}$, let $\mathbb{I}_n = \{1, 2, \dots, n\}$. Let $\boldsymbol{x} = (x_1, \dots, x_N) \in S^N$ and $1 \le N' \le N$. The cardinality of a finite set A is denoted by $\sharp A$. We write $\mathbb{J} \subset \mathbb{I}_N, \sharp\mathbb{J} = N'$, if $\mathbb{J} = \{j_1, \dots, j_{N'}\}, 1 \le j_1 < \dots < j_{N'} \le N$, and put $\boldsymbol{x}_{\mathbb{J}} = (x_{j_1}, \dots, x_{j_{N'}})$. In particular, we write $\boldsymbol{x}_{N'} = \boldsymbol{x}_{\mathbb{I}_{N'}}, 1 \le N' \le N$. (By definition $\boldsymbol{x}_N = \boldsymbol{x}$.) Suppose $\boldsymbol{u} \in \mathbb{W}_N$ and $\xi(\cdot) = \sum_{j=1}^N \delta_{u_j}(\cdot)$. For $\mathbb{J} \subset \mathbb{I}_N, 1 \le \sharp\mathbb{J} \le N$, consider a determinant of local martingales

$$\mathcal{D}_\xi(t, \boldsymbol{V}_{\mathbb{J}}(t)) = \det_{j,k \in \mathbb{J}} \Big[\mathcal{M}_\xi^{u_k}(t, V_j(t)) \Big], \quad t \in \mathcal{T}. \tag{1.8}$$

We call (1.8) a *determinantal martingale* [34].

Let $t \in \mathcal{T}, t \le T \in \mathcal{T}$. In this lecture we study the following expectation for an $\mathcal{F}(t)$-measurable function F of $\boldsymbol{V}(\cdot)$, which is symmetric at each time, weighted by the determinantal martingale,

$$\mathrm{E}_{\boldsymbol{u}} \Big[F(\boldsymbol{V}(\cdot)) \mathcal{D}_\xi(T, \boldsymbol{V}(T)) \Big], \quad \boldsymbol{u} \in S^N. \tag{1.9}$$

By the assumptions (1.7) for the map (1.6), we can prove the following.

Lemma 1.1 *Assume that $\xi(\cdot) = \sum_{j=1}^N \delta_{u_j}(\cdot)$ with $\boldsymbol{u} \in \mathbb{W}_N$. Let $1 \le N' \le N$. For $t \in \mathcal{T}, t \le T < \infty$ and a measurable function $F_{N'}$ on $S^{N'}$,*

$$\begin{aligned} &\sum_{\mathbb{J} \subset \mathbb{I}_N, \sharp\mathbb{J} = N'} \mathrm{E}_{\boldsymbol{u}} \left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t)) \mathcal{D}_\xi(T, \boldsymbol{V}(T)) \right] \\ &\qquad = \int_{\mathbb{W}_{N'}} \xi^{\otimes N'}(d\boldsymbol{v}) \mathrm{E}_{\boldsymbol{v}} \left[F_{N'}(\boldsymbol{V}_{N'}(t)) \mathcal{D}_\xi(T, \boldsymbol{V}_{N'}(T)) \right]. \end{aligned} \tag{1.10}$$

This shows the *reducibility* of the determinantal martingale in the sense that, if we observe a symmetric function depending on N' variables, $N' \le N$, then the size of determinantal martingale can be reduced from N to N'. Proof is given in Section 2.1.

For an integer $M \in \mathbb{N}$, consider a sequence of times $0 < t_1 < \dots < t_M \le T \in \mathcal{T}$, $t_m \in \mathcal{T}, 1 \le m \le M$, and a sequence of measurable functions $\boldsymbol{\chi} = (\chi_{t_1}, \dots, \chi_{t_M})$. Then given an integral kernel

$$\mathbf{K}(s, x; t, y); \quad (s, x), (t, y) \in \mathcal{T} \times S, \tag{1.11}$$

the *Fredholm determinant* is defined as

$$\begin{aligned} &\mathop{\mathrm{Det}}_{\substack{(s,t) \in \{t_1, \dots, t_M\}, \\ (x,y) \in S^2}} \Big[\delta_{st} \delta_x(y) + \mathbf{K}(s, x; t, y) \chi_t(y) \Big] \\ &= \sum_{\substack{N_m \ge 0, \\ 1 \le m \le M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}} \prod_{m=1}^M \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}\left(x_j^{(m)}\right) \right\} \det_{\substack{1 \le j \le N_m, 1 \le k \le N_n, \\ 1 \le m, n \le M}} \left[\mathbf{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)}) \right], \end{aligned} \tag{1.12}$$

where $\boldsymbol{x}_{N_m}^{(m)}$ denotes $(x_1^{(m)}, \dots, x_{N_m}^{(m)})$ and $d\boldsymbol{x}_{N_m}^{(m)} = \prod_{j=1}^{N_m} dx_j^{(m)}$, $1 \le m \le M$. Let $\mathbf{1}(\omega)$ be the indicator of ω; $\mathbf{1}(\omega) = 1$ if ω is satisfied, and $\mathbf{1}(\omega) = 0$ otherwise. The reducibility of determinantal martingales (Lemma 1.1) implies the following identity.

Lemma 1.2 *Let* $\boldsymbol{u} \in \mathbb{W}_N$ *and* $\xi = \sum_{j=1}^N \delta_{u_j}$. *For* $t \in \mathcal{T}, 0 \le t \le T \in \mathcal{T}$,

$$\begin{aligned} &\mathrm{E}_{\boldsymbol{u}}\left[\prod_{m=1}^M \prod_{j=1}^N \{1+\chi_{t_m}(V_j(t_m))\}\mathcal{D}_\xi(T, \boldsymbol{V}(T))\right] \\ &\quad = \mathop{\mathrm{Det}}_{\substack{(s,t)\in\{t_1,\dots,t_M\},\\ (x,y)\in S^2}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}_\xi(s,x;t,y)\chi_t(y)\Big], \end{aligned} \tag{1.13}$$

where

$$\mathbb{K}_\xi(s,x;t,y) = \int_S \xi(dv) p(s,x|v)\mathcal{M}_\xi^v(t,y) - \mathbf{1}(s>t)p(s-t,x|y). \tag{1.14}$$

Proof is given in Section 2.2.

1.2 Determinantal-martingale representations (DMR)

Let $\mathfrak{M}$ be the space of nonnegative integer-valued Radon measures on S. Any element ξ of $\mathfrak{M}$ can be represented as $\xi(\cdot) = \sum_{j\in\mathbb{I}} \delta_{x_j}(\cdot)$ with a countable index set $\mathbb{I}$, in which a sequence of points in S, $\boldsymbol{x} = (x_j)_{j\in\mathbb{I}}$ satisfying $\xi(K) = \sharp\{x_j, x_j \in K\} < \infty$ for any compact subset $K \subset S$. In this lecture, we consider interacting particle systems as $\mathfrak{M}$-valued processes and write them as

$$\Xi(t,\cdot) = \sum_{j=1}^N \delta_{X_j(t)}(\cdot), \quad t \in \mathcal{T}. \tag{1.15}$$

The probability law of $\Xi(t,\cdot)$ starting from a fixed configuration $\xi \in \mathfrak{M}$ is denoted by $\mathbb{P}_\xi$ and the process specified by the initial configuration is expressed by $(\Xi(t), \mathbb{P}_\xi)$. The expectations w.r.t.$\mathbb{P}_\xi$ is denoted by $\mathbb{E}_\xi$. We introduce a filtration $\{\mathcal{F}(t)\}_{t\in\mathcal{T}}$ defined by $\mathcal{F}(t) = \sigma(\Xi(s), s \in \mathcal{T} \cap [0,t])$. Let $\mathrm{C}_0(S)$ be the set of all continuous real-valued functions with compact supports on S. We set

$$\mathfrak{M}_0 = \{\xi \in \mathfrak{M} : \xi(\{x\}) \le 1 \text{ for any } x \in S\}, \tag{1.16}$$

which denotes a collection of configurations without any multiple points.

For any integer $M \in \mathbb{N}$, a sequence of times $\boldsymbol{t} = (t_1, \dots, t_M) \in \mathcal{T}^M$ with $0 < t_1 < \cdots < t_M \le T \in \mathcal{T}$, $t_m \in \mathcal{T}, 1 \le m \le M$, and a sequence of functions $\boldsymbol{f} = (f_{t_1}, \dots, f_{t_M}) \in \mathrm{C}_0(S)^M$, the *moment generating function* of multitime distribution of $(\Xi(t), \mathbb{P}_\xi)$ is defined by

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \equiv \mathbb{E}_\xi\left[\exp\left\{\sum_{m=1}^M \int_S f_{t_m}(x)\Xi(t_m, dx)\right\}\right]. \tag{1.17}$$

It is expanded w.r.t. $\chi_{t_m}(\cdot) = e^{f_{t_m}(\cdot)} - 1, 1 \le m \le M$ as

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \sum_{\substack{N_m\ge 0,\\ 1\le m\le M}} \int_{\prod_{m=1}^M \mathbb{W}_{N_m}} \prod_{m=1}^M \left\{ d\boldsymbol{x}_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}\left(x_j^{(m)}\right)\right\} \rho_\xi\Big(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\Big), \tag{1.18}$$

and it defines the *spatio-temporal correlation functions* $\rho_\xi(\cdot)$ for the process $(\Xi(t), \mathbb{P}_\xi)$.

We introduce the following definitions.

Definition 1.3 *Let $t \in \mathcal{T}$, $0 < t \leq T \in \mathcal{T}$. For an $\mathcal{F}(t)$-measurable bounded function F, if $\mathbb{E}_\xi[F(\Xi(\cdot))]$ is expressed by a Fredholm determinant, $\mathbb{E}_\xi[F(\Xi(\cdot))]$ is said to be Fredholm determinantal (F-determinantal, for short). If the moment generating function (1.17) is Fredholm-determinantal, we say the process $(\Xi(t), \mathbb{P}_\xi)$ is determinantal. In this case, all spatio-temporal correlation functions are given by determinants as*

$$\rho_\xi\Big(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\Big) = \det_{\substack{1 \leq j \leq N_m, 1 \leq k \leq N_n, \\ 1 \leq m,n \leq M}} \Big[\mathbb{K}_\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})\Big], \tag{1.19}$$

$0 < t_1 < \cdots < t_M \leq T \in \mathcal{T}$, $t_m \in \mathcal{T}, 1 \leq m \leq M$, $1 \leq N_m \leq N$, $\boldsymbol{x}_{N_m}^{(m)} \in S^{N_m}, 1 \leq m \leq M \in \mathbb{N}$, and the integral kernel $\mathbb{K}_\xi$ depending on ξ is called the correlation kernel.

Definition 1.4 *Let $t \in \mathcal{T}, 0 < t \leq T \in \mathcal{T}$. If there exists $\mathcal{M}_\xi^u(\cdot,\cdot), u \in S$ defining $\mathcal{D}_\xi(\cdot,\cdot)$ by (1.8) such that the following equality holds for an $\mathcal{F}(t)$-measurable bounded function F,*

$$\mathbb{E}_\xi[F(\Xi(\cdot))] = \mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^N \delta_{V_j(\cdot)}\right) \mathcal{D}_\xi(T, \boldsymbol{V}(T))\right], \tag{1.20}$$

then we say $(\Xi(t), \mathbb{P}_\xi)$ has a determinantal-martingale representation (DMR, for short) for F. If $(\Xi(t), \mathbb{P}_\xi)$ has DMR for any $\mathcal{F}(t)$-measurable bounded function, $t \in \mathcal{T}, 0 < t \leq T \in \mathcal{T}$, it is said to be having DMR.

Lemma 1.2 gives the following statement.

Proposition 1.5 *If $(\Xi(t), \mathbb{P}_\xi)$ has DMR for F, then $\mathbb{E}_\xi[F(\Xi(\cdot))]$ is F-determinantal. If $(\Xi(t), \mathbb{P}_\xi)$ has DMR, then it is determinantal.*

In the present lectures, we will prove the following.

1. The noncolliding Brownian motion (BM) and the noncolliding squared Bessel process with parameter $\nu > -1$ (BESQ$^{(\nu)}$) have DMR for $\xi \in \mathfrak{M}_0, \xi(S) < \infty$ (Theorem 5.4). Then they are both determinantal (Corollaries 5.5 and 5.6).

2. The simple and symmetric noncolliding random walk (RW) has DMR for $\xi \in \mathfrak{M}, \xi(\mathbb{Z}) \in \mathbb{N}$ (Theorem 6.1). Then it is determinantal (Corollary 6.2).

3. The O'Connell process has a *variation* of DMR for a special quantity, which will be denoted as
$$\Theta^a(X_1^a(\cdot) - h), \quad a > 0, \quad h \in \mathbb{R}. \tag{1.21}$$
Then its expectation is F-determinantal (Proposition 7.3).

1.3 Complex-process representations (CPR)

Let $W(t), t \in \mathcal{T}$ be a Markov process in $\widetilde{S}$ started at 0 defined independently from $V(t), t \in \mathcal{T}$ on the probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathrm{P}}_0)$, where the expectation is denoted as $\check{\mathrm{E}}_0$. We introduce a complex process

$$Z(t) = V(t) + iW(t), \quad t \in \mathcal{T}, \tag{1.22}$$

where $i = \sqrt{-1}$. We consider a possibility that there exists a one-parameter family of functions $\varphi_\xi^u : \mathbb{C} \to \mathbb{C}$ with the parameter $u \in \mathbb{C}$ and $\xi = \sum_{j=1}^N \delta_{u_j}, \boldsymbol{u} \in \mathbb{W}_N$ such that the equality

$$\mathcal{M}_\xi^u(t, V(t)) = \check{\mathrm{E}}_0[\varphi_\xi^u(Z(t))], \quad t \in \mathcal{T}, t \le T < \infty \tag{1.23}$$

hold. In this case, we set a collection of N independent copies of $W(\cdot)$ and denote the probability space as $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathrm{P}}_{\mathbf{0}})$, where $\mathbf{0}$ denotes the zero in S^N. Define the space $(\Omega, \mathcal{F}, \mathbf{P}_{\boldsymbol{u}})$ as a product of $(\Omega, \mathcal{F}, \mathrm{P}_{\boldsymbol{u}})$ and $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathrm{P}}_{\mathbf{0}})$, which is the probability space for the N-component complex vector-valued process $\boldsymbol{Z}(t) = (Z_1(t), \cdots, Z_N(t))$ with $Z_j(t) = V_j(t) + iW_j(t), 1 \le j \le N$, $t \in \mathcal{T}$. Then the determinantal martingale (1.8) is written as

$$\begin{aligned} \mathcal{D}_\xi(t, \boldsymbol{V}_{\mathbb{J}}(t)) &= \det_{j,k \in \mathbb{J}} \left[\check{\mathrm{E}}_0[\varphi_\xi^{u_k}(Z_j(t))]\right] \\ &= \check{\mathrm{E}}_{\mathbf{0}} \left[\det_{j,k \in \mathbb{J}}[\varphi_\xi^{u_k}(Z_j(t))]\right], \end{aligned} \tag{1.24}$$

for $\mathbb{J} \in \mathbb{I}_N$ and the DMR (1.20) is rewritten as

$$\mathbb{E}_\xi[F(\Xi(\cdot))] = \mathbf{E}_{\boldsymbol{u}} \left[F\left(\sum_{j=1}^N \delta_{\Re Z_j(\cdot)}\right) \det_{1 \le j,k \le N}[\varphi_\xi^{u_k}(Z_j(T))\right], \quad t \in \mathcal{T}, t \le T < \infty. \tag{1.25}$$

We call (1.25) the *complex-process representation* (CPR, for short) for $\mathbb{E}_\xi[F(\Xi(\cdot))]$.

We will prove the following.

4. The noncolliding BM and the noncolliding BES$^{(\nu)}$ with $\nu = 2n+1, n \in \mathbb{N}_0$ have CPRs (Corollaries 5.7, 5.8).

5. The noncolliding RW has CPR (Theorem 6.1).

6. The O'Connell process has *a variation* of CPR for $\Theta^a(X_1^a(t) - h), a > 0, h \in \mathbb{R}$. (Proposition 7.2)

1.4 Infinite particle systems

Under some conditions on initial configuration $\xi \in \mathfrak{M}_0$ for interacting particle system $(\Xi(t), \mathbb{P}_\xi)$, the map $\mathcal{M}_\xi^u(\cdot, \cdot), u \in \mathbb{C}$ will be well-defined even for infinite particle limits, $\xi(S) = N \to \infty$. Assume $M \in \mathbb{N}$, $0 < t_1 < \cdots < t_M \le T < \infty$, $\phi_m \in \mathrm{C}_0(S), 1 \le m \le M$

and $G = G(\{x_m\}_{m=1}^M)$ is a polynomial on S^M. For $0 < t \leq T < \infty$, if an $\mathcal{F}(t)$-measurable function $F(\Xi(\cdot))$ is represented as

$$F(\Xi(\cdot)) = G\left(\left\{\int_S \phi_m(x)\Xi(t_m, dx)\right\}_{m=1}^M\right),$$

we say F is polynomial. By the reducibility of determinantal martingale (Lemma 1.1), if the degree of the polynomial F is $n \in \mathbb{N}$, $\mathcal{D}_\xi(T, \boldsymbol{V}(T))$ in the DMR (1.20) can be expressed by using $\{\mathcal{D}_\xi(T, \boldsymbol{V}_{\mathbb{J}}(T)) : \mathbb{J} \subset \mathbb{N}, \sharp\mathbb{J} \leq n\}$. In this sense, the present DMRs (and CPRs) are valid for $(\Xi(t), \mathbb{P}_\xi)$ also in the case $\xi(S) = \infty$. We will discuss the following.

7. Some sufficient conditions for ξ are given such that the DMRs (and CPRs) are valid for the interacting particle systems $(\Xi(t), \mathbb{P}_\xi)$ with infinite numbers of particles, and infinite-dimensional determinantal processes are well-defined (Section 5.9).

1.5 Systems started at initial configurations with multiple points

The basic of the present theory is for the deterministic initial configuration with no multiple point, $\xi \in \mathfrak{M}_0$. But we will also discuss the following case.

8. The map $\mathcal{M}_\xi^u(\cdot, \cdot), u \in \mathbb{C}$ can be extended for the system started at configurations with multiple points (Section 5.8).

2 Proofs of Lemmas 1.1 and 1.2

For a finite set $\mathbb{J}$, we write the collection of all permutations of elements in $\mathbb{J}$ as $\mathcal{S}(\mathbb{J})$. In particular, for $\mathbb{I}_n = \{1, 2, \dots, n\}, n \in N$, we express $\mathcal{S}(\mathbb{I}_n)$ simply by $\mathcal{S}_n$. For $\boldsymbol{x} = (x_1, \dots, x_N) \in S^n$, $\sigma \in \mathcal{S}_n$, we put $\sigma(\boldsymbol{x}) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. For an $n \times n$ matrix $B = (B_{jk})_{1 \le j,k \le n}$, the determinant is defined by

$$\begin{aligned} \det B &= \det_{1 \le j,k \le n} [B_{jk}] \\ &= \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} B_{j\sigma(j)}. \end{aligned} \tag{2.1}$$

Any permutation $\sigma \in \mathcal{S}_n$ can be decomposed into a product of cycles. Let the number of cycles in the decomposition be $\ell(\sigma)$ and express σ by

$$\sigma = \mathsf{c}_1 \mathsf{c}_2 \cdots \mathsf{c}_{\ell(\sigma)}, \tag{2.2}$$

where c_λ denotes a cyclic permutation

$$\mathsf{c}_\lambda = (c_\lambda(1) c_\lambda(2) \cdots c_\lambda(q_\lambda)), \quad 1 \le q_\lambda \le n, \quad 1 \le \lambda \le \ell(\sigma). \tag{2.3}$$

For each $1 \le \lambda \le \ell(\sigma)$, we write the set of entries $\{c_\lambda(j)\}_{j=1}^{q_\lambda}$ of c_λ simply as $\{\mathsf{c}_\lambda\}$, in which the periodicity $c_\lambda(j + q_\lambda) = c_\lambda(j), 1 \le j \le q_\lambda$ is assumed. By definition, for each $1 \le \lambda \le \ell(\sigma)$, $c_\lambda(j), 1 \le j \le q_\lambda$ are distinct indices chosen from $\mathbb{I}_n$, $\{\mathsf{c}_\lambda\} \cap \{\mathsf{c}_{\lambda'}\} = \emptyset$ for $1 \le \lambda \ne \lambda' \le \ell(\sigma)$, and $\sum_{\lambda=1}^{\ell(\sigma)} q_\lambda = n$. The determinant (2.1) is also given by

$$\det B = \sum_{\sigma \in \mathcal{S}_n} (-1)^{n - \ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{j=1}^{q_\lambda} B_{c_\lambda(j) c_\lambda(j+1)}. \tag{2.4}$$

2.1 Proof of Lemma 1.1

By definition of determinant (2.1), (1.8), and independence of $V_j(\cdot), 1 \le j \le N$, the LHS of (1.10) is equal to

$$\begin{aligned} &\sum_{\mathbb{J} \subset \mathbb{I}_N, \sharp \mathbb{J} = N'} \mathrm{E}_{\boldsymbol{u}} \left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t)) \det_{j,k \in \mathbb{I}_N} [\mathcal{M}_\xi^{u_k}(T, V_j(T))] \right] \\ &= \sum_{\mathbb{J} \subset \mathbb{I}_N, \sharp \mathbb{J} = N'} \mathrm{E}_{\boldsymbol{u}} \left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t)) \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \prod_{j=1}^{N} \mathcal{M}_\xi^{u_{\sigma(j)}}(T, V_j(T)) \right] \\ &= \sum_{\mathbb{J} \subset \mathbb{I}_N, \sharp \mathbb{J} = N'} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \mathrm{E}_{\boldsymbol{u}} \left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t)) \prod_{j \in \mathbb{J}} \mathcal{M}_\xi^{u_{\sigma(j)}}(T, V_j(T)) \prod_{k \in \mathbb{I}_N \setminus \mathbb{J}} \mathcal{M}_\xi^{u_{\sigma(k)}}(T, V_k(T)) \right] \\ &= \sum_{\mathbb{J} \subset \mathbb{I}_N, \sharp \mathbb{J} = N'} \sum_{\sigma \in \mathcal{S}_N} \mathrm{sgn}(\sigma) \mathrm{E}_{\boldsymbol{u}} \left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t)) \prod_{j \in \mathbb{J}} \mathcal{M}_\xi^{u_{\sigma(j)}}(T, V_j(T)) \right] \\ &\qquad \times \prod_{k \in \mathbb{I}_N \setminus \mathbb{J}} \mathrm{E}_{\boldsymbol{u}} \left[\mathcal{M}_\xi^{u_{\sigma(k)}}(T, V_k(T)) \right]. \end{aligned} \tag{2.5}$$

By the martingale property of $\mathcal{M}_\xi^u(\cdot, V(\cdot))$ and the condition (iii) of (1.7),

$$\begin{aligned}\prod_{k\in\mathbb{I}_N\setminus\mathbb{J}} \mathrm{E}_{\boldsymbol{u}}\left[\mathcal{M}_\xi^{u_{\sigma(k)}}(T, V_k(T))\right] &= \prod_{k\in\mathbb{I}_N\setminus\mathbb{J}} \mathrm{E}_{\boldsymbol{u}}\left[\mathcal{M}_\xi^{u_{\sigma(k)}}(0, V_k(0))\right] \\ &= \prod_{k\in\mathbb{I}_N\setminus\mathbb{J}} \mathcal{M}_\xi^{u_{\sigma(k)}}(0, u_k) \\ &= \prod_{k\in\mathbb{I}_N\setminus\mathbb{J}} \delta_{k\sigma(k)}. \end{aligned} \tag{2.6}$$

Then (2.5) is equal to

$$\begin{aligned}&\sum_{\mathbb{J}\subset\mathbb{I}_N, \sharp\mathbb{J}=N'} \sum_{\sigma\in\mathcal{S}(\mathbb{J})} \mathrm{sgn}(\sigma)\mathrm{E}_{\boldsymbol{u}}\left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t))\prod_{j\in\mathbb{J}}\mathcal{M}_\xi^{u_{\sigma(j)}}(T, V_j(T))\right] \\ &= \sum_{\mathbb{J}\subset\mathbb{I}_N, \sharp\mathbb{J}=N'} \mathrm{E}_{\boldsymbol{u}}\left[F_{N'}(\boldsymbol{V}_{\mathbb{J}}(t)) \det_{j,k\in\mathbb{J}}[\mathcal{M}_\xi^{u_k}(T, V_j(T))]\right] \\ &= \int_{\mathbb{W}_{N'}} \xi^{\otimes N'}(d\boldsymbol{v})\mathrm{E}_{\boldsymbol{v}}\left[F_{N'}(\boldsymbol{V}_{N'}(t)) \det_{j,k\in\mathbb{I}_{N'}}[\mathcal{M}_\xi^{u_k}(T, V_j(T))]\right], \end{aligned} \tag{2.7}$$

where equivalence of $V_j(\cdot), 1 \le j \le N$ in probability law is used. This is the RHS of (1.10) and the proof is completed. ∎

2.2 Proof of Lemma 1.2

By performing binomial expansion of $\prod_{m=1}^M \prod_{j=1}^N \{1 + \chi_{t_m}(V_j(t_m))\}$ and by Lemma 1.1, the LHS of (1.13) gives

$$\sum_{\substack{N_m\ge 0,\\ 1\le m\le M}} \sum_{1\le p\le N} \sum_{\substack{\sharp\mathbb{J}_m=N_m,\\ 1\le m\le M,\\ \bigcup_{m=1}^M \mathbb{J}_m=\mathbb{I}_p}} \int_{\mathbb{W}_p} \xi^{\otimes p}(d\boldsymbol{v})\mathrm{E}_{\boldsymbol{v}}\left[\prod_{m=1}^M \prod_{j_m\in\mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m))\mathcal{D}_\xi(T, \boldsymbol{V}_p(T))\right]. \tag{2.8}$$

On the other hand, the RHS of (1.13) has an expansion according to (1.12). Then, for proof of Lemma 1.2, it is enough to show that the following equality is established for any $M \in \mathbb{N}, (N_1, \dots, N_M) \in \mathbb{N}^M$

$$\begin{aligned}&\int_{\prod_{m=1}^M \mathbb{W}_{N_m}} \prod_{m=1}^M \left\{d\boldsymbol{x}_{N_m}^{(m)} \prod_{j=1}^{N_m} \chi_{t_m}\left(x_j^{(m)}\right)\right\} \det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \left[\mathbb{K}_\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})\right] \\ &= \sum_{1\le p\le N} \sum_{\substack{\sharp\mathbb{J}_m=N_m,\\ 1\le m\le M,\\ \bigcup_{m=1}^M \mathbb{J}_m=\mathbb{I}_p}} \int_{\mathbb{W}_p} \xi^{\otimes p}(d\boldsymbol{v})\mathrm{E}_{\boldsymbol{v}}\left[\prod_{m=1}^M \prod_{j_m\in\mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{j,k\in\mathbb{I}_p} \left[\mathcal{M}_\xi^{v_k}(T, V_j(T))\right]\right]. \end{aligned} \tag{2.9}$$

Here we will prove (2.9) by fixing $M \in \mathbb{N}$, $(N_1, \dots, N_M) \in \mathbb{N}^M$.

Let $\mathbb{I}^{(1)} = \mathbb{I}_{N_1}$ and $\mathbb{I}^{(m)} = \mathbb{I}_{\sum_{j=1}^m N_j} \setminus \mathbb{I}_{\sum_{j=1}^{m-1} N_j}, 2 \leq m \leq M$. Put $n = \sum_{m=1}^M N_m$ and $\tau_j = \sum_{m=1}^M t_m \mathbf{1}(j \in \mathbb{I}^{(m)}), 1 \leq j \leq n$. Then the integrand in the LHS of (2.9) is simply written as

$$\prod_{j=1}^n \chi_{\tau_j}(x_j) \det_{1 \leq j,k \leq n} [\mathbb{K}_\xi(\tau_j, x_j; \tau_k, x_k)], \tag{2.10}$$

and the integral $\int_{\prod_{m=1}^M \mathbb{W}_{N_m}} \prod_{m=1}^M d\boldsymbol{x}_{N_m}^{(m)}(\cdot)$ can be replaced by $\{\prod_{m=1}^M N_m!\}^{-1} \int_{S^n} d\boldsymbol{x}\,(\cdot)$. By the formula (2.4) of determinant, (2.10) is expressed as

$$\begin{aligned}
&\sum_{\sigma \in \mathcal{S}_n} (-1)^{n-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{j=1}^{q_\lambda} \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \mathbb{K}_\xi(\tau_{c_\lambda(j)}, x_{c_\lambda(j)}; \tau_{c_\lambda(j+1)}, x_{c_\lambda(j+1)}) \\
&= \sum_{\sigma \in \mathcal{S}_n} (-1)^{n-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{j=1}^{q_\lambda} \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \Big\{ \mathcal{G}_{\tau_{c_\lambda(j)}, \tau_{c_\lambda(j+1)}}(x_{c_\lambda(j)}, x_{c_\lambda(j+1)}) \\
&\qquad - \mathbf{1}(\tau_{c_\lambda(j)} > \tau_{c_\lambda(j+1)}) p(\tau_{c_\lambda(j)} - \tau_{c_\lambda(i+1)}, x_{c_\lambda(j)} | x_{c_\lambda(j+1)}) \Big\},
\end{aligned} \tag{2.11}$$

where with (1.14) we have set

$$\begin{aligned}
\mathcal{G}_{s,t}(x,y) &\equiv \mathbb{K}_\xi(s, x; t, y) + \mathbf{1}(s > t) p(s-t, x|y) \\
&= \int_S \xi(dv) p(s, x|v) \mathcal{M}_\xi^v(t, y), \quad (s,t) \in \mathcal{T}^2, \quad (x,y) \in S^2.
\end{aligned} \tag{2.12}$$

We will perform binomial expansions in (2.11). In order to show the result, we introduce the following notations. For each cyclic permutation c_λ, we consider a subset of $\{\mathsf{c}_\lambda\}$,

$$\mathcal{C}(\mathsf{c}_\lambda) = \Big\{ c_\lambda(j) \in \{\mathsf{c}_\lambda\} : \tau_{c_\lambda(j)} > \tau_{c_\lambda(j+1)} \Big\}.$$

Choose $\mathbf{M}_\lambda$ such that $\{\mathsf{c}_\lambda\} \setminus \mathcal{C}(\mathsf{c}_\lambda) \subset \mathbf{M}_\lambda \subset \{\mathsf{c}_\lambda\}$, and define $\mathbf{M}_\lambda^{\mathrm{c}} = \{\mathsf{c}_\lambda\} \setminus \mathbf{M}_\lambda$. (Then $\mathbf{M}_\lambda^{\mathrm{c}} \subset \mathcal{C}(\mathsf{c}_\lambda)$.) Therefore if we put

$$\begin{aligned}
G(\mathsf{c}_\lambda, \mathbf{M}_\lambda) &= \int_{S^{\{\mathsf{c}_\lambda\}}} \prod_{j=1}^{q_\lambda} \Big\{ dx_{c_\lambda(j)}\ \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) p(\tau_{c_\lambda(j)} - \tau_{c_\lambda(i+1)}, x_{c_\lambda(j)} | x_{c_\lambda(j+1)})^{\mathbf{1}(c_\lambda(j) \in \mathbf{M}_\lambda^{\mathrm{c}})} \\
&\qquad \times \mathcal{G}_{\tau_{c_\lambda(j)}, \tau_{c_\lambda(j+1)}}(x_{c_\lambda(j)}, x_{c_\lambda(j+1)})^{\mathbf{1}(c_\lambda(j) \in \mathbf{M}_\lambda)} \Big\},
\end{aligned} \tag{2.13}$$

the LHS of (2.9) is expanded as

$$\frac{1}{\prod_{m=1}^M N_m!} \sum_{\sigma \in \mathcal{S}_n} (-1)^{n-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \sum_{\substack{\mathbf{M}_\lambda : \\ \{\mathsf{c}_\lambda\} \setminus \mathcal{C}(\mathsf{c}_\lambda) \subset \mathbf{M}_\lambda \subset \{\mathsf{c}_\lambda\}}} (-1)^{\sharp \mathbf{M}_\lambda^{\mathrm{c}}} G(\mathsf{c}_\lambda, \mathbf{M}_\lambda). \tag{2.14}$$

Using only the entries of $\mathbf{M}_\lambda$, we can define a subcycle $\widehat{\mathsf{c}_\lambda}$ of c_λ uniquely as follows. For each $1 \le j \le q_\lambda$ with $c_\lambda(j) \in \mathbf{M}_\lambda$, we define

$$\begin{aligned} \overline{j} &= \min\{k > j : c_\lambda(k) \in \mathbf{M}_\lambda\} \\ \underline{j} &= \max\{k < j : c_\lambda(k) \in \mathbf{M}_\lambda\}. \end{aligned} \tag{2.15}$$

Since c_λ is a cyclic permutation, $\widehat{q_\lambda} \equiv \sharp\mathbf{M}_\lambda \ge 1$. Let $j_1 = \min\{1 \le i \le q_\lambda : c_\lambda(j) \in \mathbf{M}_\lambda\}$. If $\widehat{q_\lambda} \ge 2$, define $j_{k+1} = \overline{j}_k, 1 \le k \le \widehat{q_\lambda} - 1$. Then $\widehat{\mathsf{c}_\lambda} = (\widehat{c_\lambda}(1)\widehat{c_\lambda}(2)\cdots\widehat{c_\lambda}(\widehat{q_\lambda})) \equiv (c_\lambda(j_1)c_\lambda(j_2)\cdots c(j_{\widehat{q_\lambda}}))$.

Moreover, we decompose the set $\mathbf{M}_\lambda$ into M subsets, $\mathbf{M}_\lambda = \bigcup_{m=1}^M \mathbf{J}_m^\lambda$, by letting

$$\mathbf{J}_m^\lambda = \mathbf{J}_m^\lambda(\mathsf{c}_\lambda, \mathbf{M}_\lambda) = \left\{ c_\lambda(j) \in \mathbf{M}_\lambda : \underline{j} <^\exists k \le j, \text{ s.t. } c_\lambda(k) \in \mathbb{I}^{(m)} \right\}, \quad 1 \le m \le M. \tag{2.16}$$

By definition, if $c_\lambda(j) \in \mathbf{M}_\lambda$ and $\underline{j} \le j-2$, then for all k, s.t. $\underline{j} < k < j$, we see $k \in \mathbf{M}_\lambda^{\mathrm{c}} \subset \mathcal{C}(\mathsf{c}_\lambda)$ and thus $\tau_{c_\lambda(k)} > \tau_{c_\lambda(k+1)} \ge \tau_{c_\lambda(j)}$. Then in general $\mathbf{J}_m^\lambda \cap \mathbf{J}_{m'}^\lambda \ne \emptyset, m \ne m'$, and $\mathbf{J}_1^\lambda = \mathbb{I}_{N_1} \cap \mathbf{M}_\lambda = \mathbb{I}_{N_1} \cap \{\mathsf{c}_\lambda\}$, $\mathbf{J}_m^\lambda \subset \mathbb{I}_{\sum_{k=1}^m N_k}$ for $2 \le m \le M$, $\mathbf{J}_m^\lambda \cap \mathbb{I}^{(k)} \subset \mathbf{J}_k^\lambda$ for $1 \le k < m \le M$.

Now we prove the following lemma.

Lemma 2.1 *The quantity (2.13) is equal to*

$$\int_{S^{\mathbf{M}_\lambda}} \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(j)})\ \mathrm{E}_{\boldsymbol{v}} \left[\prod_{m=1}^M \prod_{j_m\in\mathbf{J}_m^\lambda} \chi_{t_m}(V_{j_m}(t_m)) \prod_{j=1}^{\widehat{q_\lambda}} \mathcal{M}_\xi^{v_{\widehat{c_\lambda}(j)}}(T, V_{\widehat{c_\lambda}(j+1)}(T)) \right]. \tag{2.17}$$

Proof of Lemma 2.1. We note that if we set

$$\begin{aligned} &F(\{x_{c_\lambda(k)} : c_\lambda(k) \in \mathbf{M}_\lambda^{\mathrm{c}}\}) \\ &\quad = \int_{S^{\mathbf{M}_\lambda}} \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \left\{ dx_{c_\lambda(j)}\ \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \mathcal{G}_{\tau_{c_\lambda(j)},\tau_{c_\lambda(j+1)}}(x_{c_\lambda(j)}, x_{c_\lambda(j+1)}) \right\} \\ &\qquad \times \prod_{k:c_\lambda(k)\in\mathbf{M}_\lambda^{\mathrm{c}}} p(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)}|x_{c_\lambda(k+1)}), \end{aligned} \tag{2.18}$$

which is the integral only over $S^{\mathbf{M}_\lambda}$, then (2.13) is obtained by performing the integral of it also over $S^{\mathbf{M}_\lambda^{\mathrm{c}}} = S^{\{\mathsf{c}_\lambda\}} \setminus S^{\mathbf{M}_\lambda}$,

$$G(\mathsf{c}_\lambda, \mathbf{M}_\lambda) = \int_{S^{\mathbf{M}_\lambda^{\mathrm{c}}}} \prod_{k:c_\lambda(k)\in\mathbf{M}_\lambda^{\mathrm{c}}} \left\{ dx_{c_\lambda(k)} \chi_{\tau_{c_\lambda(k)}}(x_{c_\lambda(k)}) \right\} F(\{x_{c_\lambda(k)} : c_\lambda(k) \in \mathbf{M}_\lambda^{\mathrm{c}}\}). \tag{2.19}$$

In (2.18), use the definition (2.12) for $\mathcal{G}_{\tau_{c_\lambda(j)},\tau_{c_\lambda(j+1)}}(x_{c_\lambda(j)}, x_{c_\lambda(j+1)})$ by putting the integral variables to be $v = v_{c_\lambda(j)}$. We obtain

$$
\begin{aligned}
& F(\{x_{c_\lambda(k)} : c_\lambda(k) \in \mathbf{M}_\lambda^{\mathrm{c}}\}) \\
= & \int_{S^{\mathbf{M}_\lambda}} \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(j)}) \int_{S^{\mathbf{M}_\lambda}} \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \left\{ dx_{c_\lambda(j)} p(\tau_{c_\lambda(j)}, x_{c_\lambda(j)} | v_{c_\lambda(j)}) \chi_{\tau_{c_\lambda(j)}}(x_{c_\lambda(j)}) \right\} \\
\times & \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, x_{c_\lambda(j+1)}) \prod_{k:c_\lambda(k)\in\mathbf{M}_\lambda^{\mathrm{c}}} p(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | x_{c_\lambda(k+1)}) \\
= & \int_{S^{\mathbf{M}_\lambda}} \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(j)}) \mathrm{E}_{\boldsymbol{v}} \Bigg[\prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \Bigg\{ \chi_{\tau_{c_\lambda(j)}}(V_{c_\lambda(j)}(\tau_{c_\lambda(j)})) \\
& \qquad \times \mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, V_{c_\lambda(j+1)}(\tau_{c_\lambda(j+1)}))^{\mathbf{1}(c_\lambda(j+1)\in\mathbf{M}_\lambda)} \\
& \qquad \times \mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, x_{c_\lambda(j+1)})^{\mathbf{1}(c_\lambda(j+1)\in\mathbf{M}_\lambda^{\mathrm{c}})} \Bigg\} \\
& \times \prod_{k:c_\lambda(k)\in\mathbf{M}_\lambda^{\mathrm{c}}} \Bigg\{ p(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | V_{c_\lambda(k+1)}(\tau_{c_\lambda(k+1)}))^{\mathbf{1}(c_\lambda(k+1)\in\mathbf{M}_\lambda)} \\
& \qquad \times p(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | x_{c_\lambda(k+1)})^{\mathbf{1}(c_\lambda(k+1)\in\mathbf{M}_\lambda^{\mathrm{c}})} \Bigg\} \Bigg].
\end{aligned}
$$

Using Fubini's theorem, (2.19) is given by

$$
\begin{aligned}
& \int_{S^{\mathbf{M}_\lambda}} \prod_{j:c_\lambda(i)\in\mathbf{M}_\lambda} \xi(dv_{c_\lambda(j)}) \mathrm{E}_{\boldsymbol{v}} \Bigg[\prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda} \chi_{\tau_{c_\lambda(j)}}(V_{c_\lambda(j)}(\tau_{c_\lambda(j)})) \\
& \qquad \times \prod_{j:c_\lambda(j),c_\lambda(j+1)\in\mathbf{M}_\lambda} \mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, V_{c_\lambda(j+1)}(\tau_{c_\lambda(j+1)})) \\
& \quad \times \int_{S^{\mathbf{M}_\lambda^{\mathrm{c}}}} \prod_{k:c_\lambda(k)\in\mathbf{M}_\lambda^{\mathrm{c}}} \left\{ dx_{c_\lambda(k)} \chi_{\tau_{c_\lambda(k)}}(x_{c_\lambda(k)}) \right\} \\
& \qquad \times \prod_{k:c_\lambda(k)\in\mathbf{M}_\lambda^{\mathrm{c}},c_\lambda(k+1)\in\mathbf{M}_\lambda} p(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | V_{c_\lambda(k+1)}(\tau_{c_\lambda(k+1)})) \\
& \qquad \times \prod_{k:c_\lambda(k),c_\lambda(k+1)\in\mathbf{M}_\lambda^{\mathrm{c}}} p(\tau_{c_\lambda(k)} - \tau_{c_\lambda(k+1)}, x_{c_\lambda(k)} | x_{c_\lambda(k+1)}) \\
& \qquad \times \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda,c_\lambda(j+1)\in\mathbf{M}_\lambda^{\mathrm{c}}} \mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, x_{c_\lambda(j+1)}) \Bigg].
\end{aligned} \tag{2.20}
$$

We perform integration over $x_{c_\lambda(k)}$'s for $c_\lambda(k) \in \mathbf{M}_\lambda^{\mathrm{c}}$ before taking the expectation $\mathrm{E}_{\boldsymbol{v}}$. That is, integrals over $x_{c_\lambda(k)}$'s with indices in intervals $\underline{j} < k < j$ for all j, s.t. $c_\lambda(j) \in \mathbf{M}_\lambda$ are done. For each j, s.t. $c_\lambda(j) \in \mathbf{M}_\lambda$, if $\underline{j} < j-1$,

$$\chi_{\tau_{c_\lambda(j)}}(V_{c_\lambda(j)}(\tau_{c_\lambda(j)}))\Big\{\prod_{k=\underline{j}+1}^{j-1}\int_S dx_{c_\lambda(k)}\chi_{\tau_{c_\lambda(k)}}(x_{c_\lambda(k)})\Big\}$$
$$\times p(\tau_{c_\lambda(j-1)} - \tau_{c_\lambda(j)}, x_{c_\lambda(j-1)}|V_{c_\lambda(j)}(\tau_{c_\lambda(j)}))$$
$$\times \prod_{l=\underline{j}+2}^{j-1} p(\tau_{c_\lambda(l-1)} - \tau_{c_\lambda(l)}, x_{c_\lambda(l-1)}|x_{c_\lambda(l)})\mathcal{M}_\xi^{v_{c_\lambda(\underline{j})}}(\tau_{c_\lambda(\underline{j}+1)}, x_{c_\lambda(\underline{j}+1)})$$

coincides with the conditional expectation of

$$\prod_{k=\underline{j}+1}^{j}\chi_{\tau_{c_\lambda(k)}}(V_{c_\lambda(j)}(\tau_{c_\lambda(k)}))\mathcal{M}_\xi^{v_{c_\lambda(\underline{j})}}(\tau_{c_\lambda(\underline{j}+1)}, V_{c_\lambda(j)}(\tau_{c_\lambda(\underline{j}+1)}))$$

w.r.t. $\mathrm{E}_{\boldsymbol{v}}[\,\cdot\,|V_{c_\lambda(j)}(\tau_{c_\lambda(j)})]$. Since

$$\prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda}\mathcal{M}_\xi^{v_{c_\lambda(\underline{j})}}(\tau_{c_\lambda(\underline{j}+1)}, V_{c_\lambda(j)}(\tau_{c_\lambda(\underline{j}+1)})) = \prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda}\mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, V_{c_\lambda(\bar{j})}(\tau_{c_\lambda(j+1)})),$$

(2.20) is equal to

$$\int_{S^{\mathbf{M}_\lambda}}\prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda}\xi(dv_{c_\lambda(j)})$$
$$\times\mathrm{E}_{\boldsymbol{v}}\left[\prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda}\left\{\prod_{k=\underline{j}+1}^{j}\chi_{\tau_{c_\lambda(k)}}(V_{c_\lambda(j)}(\tau_{c_\lambda(k)}))\mathcal{M}_\xi^{v_{c_\lambda(j)}}(\tau_{c_\lambda(j+1)}, V_{c_\lambda(\bar{j})}(\tau_{c_\lambda(j+1)}))\right\}\right].$$

Then, by definition (2.16), we arrive at the expression (2.17) of $G(\mathsf{c}_\lambda, \mathbf{M}_\lambda)$, if we use the martingale property of $\mathcal{M}_\xi^u$. ∎

Let $\mathbf{M} \equiv \bigcup_{\lambda=1}^{\ell(\sigma)}\mathbf{M}_\lambda$. Since $n - \sum_{\lambda=1}^{\ell(\sigma)}\sharp\mathbf{M}_\lambda^{\mathrm{c}} = \sharp\mathbf{M}$, the LHS of (2.9), which is written above as (2.14) with Lemma 2.1, becomes now

$$\frac{1}{\prod_{m=1}^M N_m!}\sum_{\sigma\in\mathcal{S}_n}\sum_{\substack{\mathbf{M}:\\ \mathbb{I}_n\setminus\bigcup_{\lambda=1}^{\ell(\sigma)}\mathcal{C}(\mathsf{c}_\lambda)\subset\mathbf{M}\subset\mathbb{I}_n}}(-1)^{\sharp\mathbf{M}-\ell(\sigma)}\int_{S^{\mathbf{M}}}\prod_{\lambda=1}^{\ell(\sigma)}\prod_{j:c_\lambda(j)\in\mathbf{M}_\lambda}\xi(dv_{c_\lambda(j)})$$
$$\times\mathrm{E}_{\boldsymbol{v}}\left[\prod_{\lambda=1}^{\ell(\sigma)}\left\{\prod_{m=1}^{M}\prod_{j_m\in\mathbf{J}_m^\lambda}\chi_{t_m}(V_{j_m}(t_m))\prod_{j=1}^{\widehat{q_\lambda}}\mathcal{M}_\xi^{v_{\widehat{c_\lambda}(j)}}(T, V_{\widehat{c_\lambda}(j+1)}(T))\right\}\right]. \qquad (2.21)$$

We define

$$\widehat{\sigma} \equiv \widehat{\mathsf{c}_1}\widehat{\mathsf{c}_2}\cdots\widehat{\mathsf{c}_{\ell(\sigma)}}$$

and

$$\mathbf{J}_m \equiv \bigcup_{\lambda=1}^{\ell(\sigma)} \mathbf{J}_m^\lambda, \quad 1 \le m \le M.$$

Note that $\ell(\widehat{\sigma}) = \ell(\sigma)$. The obtained $(\mathbf{J}_m)_{m=1}^M$'s form a collection of series of index sets satisfying the following conditions, which we write as $\mathcal{J}(\{N_m\}_{m=1}^M)$:

$$\begin{aligned}
&\mathbf{J}_1 = \mathbb{I}_{N_1}, \quad \mathbf{J}_m \subset \mathbb{I}_{\sum_{k=1}^m N_k} \quad \text{for} \quad 2 \le m \le M, \\
&\mathbf{J}_m \cap \mathbb{I}^{(k)} \subset \mathbf{J}_k \quad \text{for} \quad 1 \le k < m \le M, \quad \text{and} \\
&\sharp \mathbf{J}_m = N_m \quad \text{for} \quad 1 \le m \le M.
\end{aligned} \tag{2.22}$$

For each $(\mathbf{J}_m)_{m=1}^M \in \mathcal{J}(\{N_m\}_{m=1}^M)$, we put

$$A_1 = 0 \quad \text{and} \quad A_m = \sharp\left(\mathbf{J}_m \cap \mathbb{I}_{\sum_{k=1}^{m-1} N_k}\right) = \sharp\left(\mathbf{J}_m \cap \bigcup_{k=1}^{m-1} \mathbf{J}_k\right), \quad 2 \le m \le M.$$

Then, if we put $\mathbf{M} = \bigcup_{m=1}^M \mathbf{J}_m$, $\sharp\mathbf{M} = \sum_{m=1}^M (N_m - A_m)$, which means that from the original index set $\mathbb{I}_n = \bigcup_{m=1}^M \mathbb{I}^{(m)}$ with $\sharp\mathbb{I}^{(m)} = N_m, 1 \le m \le M$, we obtain a subset $\mathbf{M}$ by eliminating A_m elements at each level $1 \le m \le M$. By this reduction, we obtain $\widehat{\sigma} \in \mathcal{S}(\mathbf{M})$ from $\sigma \in \mathcal{S}_n$. It implies that, for all $\widehat{\sigma} \in \mathcal{S}(\mathbf{M})$, the number of σ's in $\mathcal{S}_n$ which give the same $\widehat{\sigma}$ and $(\mathbf{J}_m)_{m=1}^M$ by this reduction is given by $\prod_{m=1}^M A_m!$, where $0! \equiv 1$. Then (2.21) is equal to

$$\begin{aligned}
&\sum_{\substack{\mathbf{M}: \\ \max_m\{N_m\} \le \sharp\mathbf{M} \le N}} \sum_{\substack{(\mathbf{J}_m)_{m=1}^M \subset \mathcal{J}(\{N_m\}_{m=1}^M): \\ \bigcup_{m=1}^M \mathbf{J}_m = \mathbf{M}}} \frac{\prod_{m=1}^M A_m!}{\prod_{m=1}^M N_m!} \sum_{\widehat{\sigma} \in \mathcal{S}(\mathbf{M})} (-1)^{\sharp\mathbf{M} - \ell(\widehat{\sigma})} \\
&\qquad \times \sharp\mathbf{M}! \int_{\mathbb{W}_{\sharp\mathbf{M}}} \xi^{\otimes\mathbf{M}}(d\boldsymbol{v}) \mathrm{E}_{\boldsymbol{v}} \left[\prod_{m=1}^M \prod_{j_m \in \mathbf{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \prod_{\lambda=1}^{\ell(\widehat{\sigma})} \prod_{j=1}^{\widehat{q_\lambda}} \mathcal{M}_\xi^{v_{\widehat{c_\lambda}(j)}}(T, V_{\widehat{c_\lambda}(j+1)}(T)) \right] \\
&= \sum_{\substack{\mathbf{M}: \\ \max_m\{N_m\} \le \sharp\mathbf{M} \le N}} \sum_{\substack{(\mathbf{J}_m)_{m=1}^M \subset \mathcal{J}(\{N_m\}_{m=1}^M): \\ \bigcup_{m=1}^M \mathbf{J}_m = \mathbf{M}}} \sharp\mathbf{M}! \prod_{m=1}^M \frac{A_m!}{N_m!} \\
&\qquad \times \int_{\mathbb{W}_{\sharp\mathbf{M}}} \xi^{\otimes\mathbf{M}}(d\boldsymbol{v}) \mathrm{E}_{\boldsymbol{v}} \left[\prod_{m=1}^M \prod_{j_m \in \mathbf{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{j,k \in \mathbf{M}} \left[\mathcal{M}_\xi^{v_k}(T, V_j(T)) \right] \right].
\end{aligned} \tag{2.23}$$

Assume $1 \le p \le N$, $0 \le A_m \le N_m, 2 \le m \le M$ and set $A_1 = 0$. Consider

$$\Lambda_1 = \Bigg\{ (\mathbf{J}_m)_{m=1}^M \subset \mathcal{J}(\{N_m\}_{m=1}^M) : \sharp \left(\bigcup_{m=1}^M \mathbf{J}_m \right) = p, \\ \sharp \left(\mathbf{J}_m \cap \bigcup_{k=1}^{m-1} \mathbf{J}_k \right) = A_m, 2 \le m \le M \Bigg\},$$

$$\Lambda_2 = \Bigg\{ (\mathbb{J}_m)_{m=1}^M : \sharp \mathbb{J}_m = N_m, 1 \le m \le M, \bigcup_{m=1}^M \mathbb{J}_m = \mathbb{I}_p, \\ \sharp \left(\mathbb{J}_m \cap \bigcup_{k=1}^{m-1} \mathbb{J}_k \right) = A_m, 2 \le m \le M \Bigg\}.$$

Since $V_j(\cdot)$'s are i.i.d. in $\mathrm{P}_{\boldsymbol{v}}$, the integral in (2.23) has the same value for all $(\mathbf{J}_m)_{m=1}^M \in \Lambda_1$ with $\bigcup_{m=1}^M \mathbf{J}_m = \mathbf{M}$ and it is also equal to

$$\int_{\mathbb{W}_p^{\mathbb{A}}} \xi^{\otimes p}(d\boldsymbol{v}) \mathrm{E}_{\boldsymbol{v}} \left[\prod_{m=1}^M \prod_{j_m \in \mathbb{J}_m} \chi_{t_m}(V_{j_m}(t_m)) \det_{j,k \in \mathbb{I}_p} \left[\mathcal{M}_\xi^{v_k}(T, V_j(T)) \right] \right]$$

for all $(\mathbb{J}_m)_{m=1}^M \in \Lambda_2$.

In Λ_1, for each $2 \le m \le M$, A_m elements in $\mathbf{J}_m$ are chosen from $\bigcup_{k=1}^{m-1} \mathbf{J}_k$, in which $\sharp(\bigcup_{k=1}^{m-1} \mathbf{J}_k) = \sum_{k=1}^{m-1}(N_k - A_k)$, and the remaining $N_m - A_m$ elements in $\mathbf{J}_m$ are from $\mathbb{I}^{(m)}$ with $\sharp \mathbb{I}^{(m)} = N_m$. Then

$$\sharp \Lambda_1 = \prod_{m=2}^M \binom{\sum_{k=1}^{m-1}(N_k - A_k)}{A_m} \binom{N_m}{N_m - A_m}.$$

In Λ_2, on the other hand, N_1 elements in $\mathbb{J}_1$ is chosen from $\mathbb{I}_p$, and then for each $2 \le m \le M$, A_m elements in $\mathbb{J}_m$ are chosen from $\bigcup_{k=1}^{m-1} \mathbb{J}_k$ with $\sharp(\bigcup_{k=1}^{m-1} \mathbb{J}_k) = \sum_{k=1}^{m-1}(N_k - A_k)$ and the remaining $N_m - A_m$ elements in $\mathbb{J}_m$ are from $\mathbb{I}_p \setminus \bigcup_{k=1}^{m-1} \mathbb{J}_k$ with $\sharp(\mathbb{I}_p \setminus \bigcup_{k=1}^{m-1} \mathbb{J}_k) = p - \sum_{k=1}^{m-1}(N_k - A_k)$. Then

$$\sharp \Lambda_2 = \binom{p}{N_1} \prod_{m=2}^M \binom{\sum_{k=1}^{m-1}(N_k - A_k)}{A_m} \binom{p - \sum_{k=1}^{m-1}(N_k - A_k)}{N_m - A_m}.$$

Since $\sum_{m=1}^M (N_m - A_m) = p$, we see $\sharp \Lambda_2 / \sharp \Lambda_1 = p! \prod_{m=1}^M A_m!/N_m!$. Then (2.23) is equal to the RHS of (2.9) and the proof is completed. ∎

3 Polynomial Martingales

For $n \in \mathbb{N}_0$, here we consider the monic polynomials of degrees n with time-dependent coefficients,

$$m_n(t,x) = x^n + \sum_{j=0}^{n-1} c_n^{(j)}(t) x^j, \quad t \geq 0 \tag{3.1}$$

satisfying the conditions such that

$$m_n(0,x) = x^n, \tag{3.2}$$

and that, if we replace x by the Markov process $V(t), t \in \mathcal{T}$, then they are local martingales. Such polynomials $\{m_n(t,v)\}_{n \in \mathbb{N}_0}$ are called the *polynomial martingales* associated with the process $V(\cdot)$.

3.1 Brownian motion (BM) and Hermite polynomials

Let $V(t) = B(t), t \in \mathcal{T} = [0,\infty)$, the one-dimensional standard Brownian motion (BM) on $S = \mathbb{R}$. The transition probability density is given by

$$p(t,y|x) = \begin{cases} \dfrac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, & t > 0, x, y \in \mathbb{R} \\ \\ \delta(y-x), & t = 0, x, y \in \mathbb{R}. \end{cases} \tag{3.3}$$

For $n \in \mathbb{N}_0$, the Hermite polynomials of degrees $n \in \mathbb{N}_0$ are given by

$$H_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \frac{n!}{j!(n-2j)!} (2x)^{n-2j}, \quad n \in \mathbb{N}_0, \tag{3.4}$$

which solve the differential equation

$$y'' - 2xy' + 2ny = 0. \tag{3.5}$$

The following is proved.

Lemma 3.1 *The polynomials of $B(t), t \in [0,\infty)$,*

$$m_n(t,B(t)) = \left(\frac{t}{2}\right)^{n/2} H_n\left(\frac{B(t)}{\sqrt{2t}}\right), \quad t \geq 0, \quad n \in \mathbb{N}_0, \tag{3.6}$$

are all local martingales.

Proof. $m_0(t, B(t)) \equiv 1$. By Itô's formula, for $n \geq 1$,

$$\begin{aligned} dm_n(t, B(t)) &= \left[\frac{n}{2} \frac{t^{n/2-1}}{2^{n/2}} H_n \left(\frac{B(t)}{\sqrt{2t}} \right) + \left(\frac{t}{2} \right)^{n/2} H_n' \left(\frac{B(t)}{\sqrt{2t}} \right) \left(-\frac{B(t)}{(2t)^{3/2}} \right) \right] dt \\ &\quad + \left(\frac{t}{2} \right)^{n/2} H_n' \left(\frac{B(t)}{\sqrt{2t}} \right) \frac{1}{\sqrt{2t}} dB(t) + \frac{1}{2} \left(\frac{t}{2} \right)^{n/2} H_n'' \left(\frac{B(t)}{\sqrt{2t}} \right) \frac{1}{2t} dt \\ &= \frac{1}{2} \left(\frac{t}{2} \right)^{(n-1)/2} H_n' \left(\frac{B(t)}{\sqrt{2t}} \right) dB(t) + A_n(t) dt. \end{aligned}$$

Here we find

$$A_n(t) = \frac{t^{n/2-1}}{2^{n/2+2}} \left[H_n'' \left(\frac{B(t)}{\sqrt{2t}} \right) - \sqrt{\frac{2}{t}} B(t) H_n' \left(\frac{B(t)}{\sqrt{2t}} \right) + 2n H_n \left(\frac{B(t)}{\sqrt{2t}} \right) \right] = 0,$$

for (3.5). Then $m_n(t, B(t))$ are given by stochastic integrals

$$m_n(t, B(t)) = \frac{1}{2^{(n+1)/2}} \int_0^t s^{(n-1)/2} H_n' \left(\frac{B(s)}{\sqrt{2s}} \right) dB(s), \quad t \geq 0, \quad n \geq 1. \tag{3.7}$$

The proof is thus completed. ∎

We call the polynomials

$$m_n(t, x) = \left(\frac{t}{2} \right)^{n/2} H_n \left(\frac{x}{\sqrt{2t}} \right), \quad t \geq 0, \quad n \in \mathbb{N}_0, \tag{3.8}$$

the *polynomial martingales associated with* $B(t), t \in [0, \infty)$.

Remark 1. It is obvious that

$$G_\alpha^{\mathrm{BM}}(t, B(t)) = e^{\alpha B(t) - t\alpha^2/2} \tag{3.9}$$

is martingale for any $\alpha \in \mathbb{C}$. It is known that

$$G_\alpha^{\mathrm{BM}}(t, x) = \sum_{n=0}^{\infty} \left(\frac{t}{2} \right)^{n/2} H_n \left(\frac{x}{\sqrt{2t}} \right) \frac{\alpha^n}{n!}, \tag{3.10}$$

where $H_n, n \in \mathbb{N}_0$ are the Hermite polynomials given by (3.4). Then Lemma 3.1 is immediately concluded, if we confirm that $m_n(t, x), n \in \mathbb{N}_0$ are monic.

3.2 Squared Bessel Processes (BESQ$^{(\nu)}$) and Laguerre polynomials

Let $V(t) = R^{(\nu)}(t), t \in \mathcal{T} = [0, \infty)$, $\nu > -1$, the squared Bessel process with index $\nu > -1$ (BESQ$^{(\nu)}$) on $S = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. For $\nu = D/2-1, D \in \mathbb{N}$, $R^{(\nu)}(\cdot)$ can be defined as a sum of squares of D independent BMs, $B_j(\cdot), 1 \leq j \leq D$ such as $R^{(\nu)}(t) = \sum_{j=1}^{D} B_j(t)^2, t \geq 0$. For general $\nu > -1$, it is given by the solution of the stochastic differential equation (SDE),

$$R^{(\nu)}(t) = \int_0^t 2\sqrt{R^{(\nu)}(s)}dB(s) + 2(\nu+1)t, \quad t \geq 0, \tag{3.11}$$

where $B(\cdot)$ is a BM, and, if $-1 < \nu < 0$, a reflection wall is put at the origin. The transition probability density is given by

$$p^{(\nu)}(t, y|x) = \begin{cases} \dfrac{1}{2t}\left(\dfrac{y}{x}\right)^{\nu/2} \exp\left(-\dfrac{x+y}{2t}\right) I_\nu\left(\dfrac{\sqrt{xy}}{t}\right), & t > 0, x > 0, y \in \mathbb{R}_+, \\ \dfrac{y^\nu}{(2t)^{\nu+1}\Gamma(\nu+1)} e^{-y/2t}, & t > 0, x = 0, y \in \mathbb{R}_+, \\ \delta(y-x), & t = 0, x, y \in \mathbb{R}_+, \end{cases} \tag{3.12}$$

if $-1 < \nu < 0$, the origin is assumed to be reflecting. Here $I_\nu(x)$ is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n+1+\nu)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{3.13}$$

with the Gamma function $\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du$, $\Re u > 0$.

For $n \in \mathbb{N}_0$, the Laguerre polynomial of degree $n \in \mathbb{N}_0$ with index $\nu > -1$ is given by

$$L_n^{(\nu)}(x) = \sum_{j=0}^{n} (-1)^j \frac{\Gamma(n+\nu+1)}{\Gamma(\nu+j+1)(n-j)!j!} x^j, \quad n \in \mathbb{N}_0, \tag{3.14}$$

which solve the differential equation

$$xy'' + (\nu+1-x)y' + ny = 0. \tag{3.15}$$

The following is derived.

Lemma 3.2 *For $\nu > -1$,*

$$m_n(t, x) = (-1)^n n!(2t)^n L_n^{(\nu)}\left(\frac{x}{2t}\right), \quad t \geq 0, \quad n \in \mathbb{N}_0 \tag{3.16}$$

are the polynomial martingales associated with the BESQ$^{(\nu)}$, $R^{(\nu)}(t), \nu > -1, t \in [0, \infty)$.

Proof. $m_0(t, R^{(\nu)}(t)) \equiv 1$. By (3.11), the quadratic variation of BESQ$^{(\nu)}$ is $\langle R^{(\nu)} \rangle_t =$

$4\int_0^t R^{(\nu)}(s)ds, t \geq 0, \nu > -1$. Then, for $n \geq 1$, Itô's formula gives

$$\begin{aligned} dm_n(t, R^{(\nu)}) &= (-1)^n n! \left[n 2^n t^{n-1} L_n^{(\nu)}\left(\frac{R^{(\nu)}(t)}{2t}\right) + (2t)^n L_n^{(\nu)\prime}\left(\frac{R^{(\nu)}(t)}{2t}\right)\left(-\frac{R^{(\nu)}(t)}{2t^2}\right)\right] \\ &\quad + (-1)^n n!(2t)^n L_n^{(\nu)\prime}\left(\frac{R^{(\nu)}(t)}{2t}\right)\frac{1}{2t}\left\{2\sqrt{R^{(\nu)}(t)}dB(t) + 2(\nu+1)dt\right\} \\ &\quad + \frac{1}{2}(-1)^n n!(2t)^n L_n^{(\nu)\prime\prime}\left(\frac{R^{(\nu)}(t)}{2t}\right)\frac{1}{(2t)^2} 4R^{(\nu)}(t)dt \\ &= (-1)^n n! 2^n t^{n-1}\sqrt{R^{(\nu)}(t)} L_n^{(\nu)\prime}\left(\frac{R^{(\nu)}(t)}{2t}\right) dB(t) + A_n^{(\nu)}(t)dt \end{aligned}$$

with

$$\begin{aligned} A_n^{(\nu)}(t) &= (-1)^n n! 2^n t^{n-1}\left[\frac{R^{(\nu)}(t)}{2t} L_n^{(\nu)\prime\prime}\left(\frac{R^{(\nu)}(t)}{2t}\right)\right. \\ &\quad \left. + \left(\nu + 1 - \frac{R^{(\nu)}(t)}{2t}\right) L_n^{(\nu)\prime}\left(\frac{R^{(\nu)}(t)}{2t}\right) + n L_n^{(\nu)}\left(\frac{R^{(\nu)}(t)}{2t}\right)\right]. \end{aligned}$$

For (3.15), $A_n^{(\nu)}(t) = 0, n \geq 1$. Then $m_n(t, R^{(\nu)}(t)), \nu > -1$ are given by stochastic integrals

$$m_n(t, R^{(\nu)}(t)) = (-1)^n n! 2^n \int_0^t s^{n-1}\sqrt{R^{(\nu)}(s)} L_n^{(\nu)\prime}\left(\frac{R^{(\nu)}(s)}{2s}\right) dB(s), \quad t \geq 0, \quad n \geq 1. \tag{3.17}$$

The proof is thus completed. ▮

Remark 2. For $\alpha \in \mathbb{C}, t \geq 0, x \in \mathbb{R}, \nu > -1$, let

$$G_\alpha^{(\nu)}(t, x) = \frac{e^{\alpha x/(1+2t\alpha)}}{(1+2t\alpha)^{\nu+1}}. \tag{3.18}$$

By Itô's formula, we can see

$$dG_\alpha^{(\nu)}(t, R^{(\nu)}(t)) = \frac{2\alpha\sqrt{R^{(\nu)}(t)}}{1+2t\alpha} G_\alpha^{(\nu)}(t, R^{(\nu)}(t)) dB(t), \tag{3.19}$$

that is, $G_\alpha^{(\nu)}(t, R^{(\nu)}(t))$ is a local martingale for any $\alpha \in \mathbb{C}$. It is known that

$$G_\alpha^{(\nu)}(t, x) = \sum_{n=0}^\infty (-1)^n n!(2t)^n L_n^{(\nu)}\left(\frac{x}{2t}\right)\frac{\alpha^n}{n!}. \tag{3.20}$$

Then Lemma 3.2 is obtained.

3.3 Random walk (RW) and Fujita's polynomials

Let $\mathbb{Z}$ be a set of all integers and $N \in \mathbb{N} \equiv \{1, 2, \dots\}$. Let $V(t), t \in \mathbb{N}_0$ be a one-dimensional, simple and symmetric RW on $S = \mathbb{Z}$ starting from 0 at time $t = 0$,

$$V(t) = \zeta(1) + \zeta(2) + \cdots + \zeta(t), \quad t \in \mathbb{N}, \tag{3.21}$$

where $\{\zeta(t) : t \in \mathbb{N}\}$ are i.i.d. with

$$\mathrm{P}[\zeta(1) = 1] = \frac{1}{2}, \quad \mathrm{P}[\zeta(1) = -1] = \frac{1}{2}. \tag{3.22}$$

The following *discrete Itô's formula* was given by Fujita [20, 22, 21].

Lemma 3.3 *For any $f : \mathbb{N}_0 \times \mathbb{Z} \to \mathbb{R}$ and any $t \in \mathbb{N}_0$,*

$$\begin{aligned} & f(t+1, V(t+1)) - f(t, V(t)) \\ & \quad = \frac{1}{2}\Big[f(t+1, V(t)+1) - f(t+1, V(t)-1)\Big](V(t+1) - V(t)) \\ & \quad\quad + \frac{1}{2}\Big[f(t+1, V(t)+1) - 2f(t+1, V(t)) + f(t+1, V(t)-1)\Big] \\ & \quad\quad + f(t+1, V(t)) - f(t, V(t)). \end{aligned} \tag{3.23}$$

We perform the Esscher transform with parameter $\alpha \in \mathbb{R}$, $V(\cdot) \to \widetilde{V}_\alpha(\cdot)$ as

$$\widetilde{V}_\alpha(t) = \frac{e^{\alpha V(t)}}{\mathrm{E}[e^{\alpha V(t)}]}, \quad t \in \mathbb{N}_0.$$

By (3.22), $\mathrm{E}[e^{\alpha\zeta(1)}] = (e^\alpha + e^{-\alpha})/2 = \cosh\alpha$, then we have

$$\widetilde{V}_\alpha(t) = G_\alpha(t, V(t)) \tag{3.24}$$

with

$$G_\alpha(t, x) = \frac{e^{\alpha x}}{(\cosh\alpha)^t}, \quad t \in \mathbb{N}_0, \quad x \in \mathbb{Z}. \tag{3.25}$$

If we set $f = G_\alpha$ in (3.23), the second and third terms in the RHS vanish. Then

$$\begin{aligned} & G_\alpha(t+1, V(t+1)) - G_\alpha(t, V(t)) \\ & \quad = \frac{1}{2}\Big[G_\alpha(t+1, V(t)+1) - G_\alpha(t+1, V(t)-1)\Big]\zeta(t+1), \end{aligned}$$

which implies that $G_\alpha(t, V(t))$ is $\{\zeta(1), \zeta(2), \dots, \zeta(t)\}$-martingale for any $\alpha \in \mathbb{R}$ [20, 22, 21]. From now on, we simply say '$G_\alpha(t, V(t))$ is martingale' in such a situation.

Expansion of (3.25) with respect to α around $\alpha = 0$

$$G_\alpha(t, x) = \sum_{n=0}^{\infty} m_n(t, x)\frac{\alpha^n}{n!}, \tag{3.26}$$

determines a series of monic polynomials of degrees n studied by Fujita in [20, 21]

$$m_n(t,x) = x^n + \sum_{j=1}^{n-1} c_n^{(j)} x^j, \quad n \in \mathbb{N}_0, \tag{3.27}$$

such that

$$\begin{aligned}&c_n^{(j)}(0) = 0, \quad 1 \leq j \leq n-1, \quad \text{and}\\ &m_n(t, V(t)) \text{ is martingale}, \quad t \in \mathbb{N}_0.\end{aligned}$$

For example,

$$\begin{aligned}
m_0(t,x) &= 1,\\
m_1(t,x) &= x,\\
m_2(t,x) &= x^2 - t,\\
m_3(t,x) &= x^3 - 3tx,\\
m_4(t,x) &= x^4 - 6tx^2 + (3t+2)t,\\
m_5(t,x) &= x^5 - 10tx^3 + 5(3t+2)tx.
\end{aligned}$$

They satisfy the recurrence relations

$$m_n(t,x) = \frac{1}{2}[m_n(t+1, x+1) + m_n(t+1, x-1)], \quad n \in \mathbb{N}_0.$$

We call $m_n(t,x), n \in \mathbb{N}$, *Fujita's polynomials* and $m_n(t, V(t)), n \in \mathbb{N}_0$, *Fujita's polynomial martingales* for the simple and symmetric RW [20, 21].

Remark 3. The Esscher transform with parameter α for BM, $B(t), t \geq 0$ is given by

$$\widetilde{B}_\alpha(t) = G_\alpha^{\text{BM}}(t, B(t))$$

with

$$G_\alpha^{\text{BM}}(t,x) = \frac{e^{\alpha x}}{\mathrm{E}[e^{\alpha B(t)}]} = \frac{e^{\alpha x}}{\displaystyle\int_{-\infty}^{\infty} dx e^{\alpha x} p^{\text{BM}}(t, x|0)} = e^{\alpha x - \alpha^2 t/2}, \tag{3.28}$$

where $p^{\text{BM}}(t, y|x)$ is the transition probability density of BM (3.3). This is nothing but (3.9).

4 Integral Transforms and Complex-Process Representations (CPR) for Polynomial Martingales

For each set of polynomial martingales $\{m_n(t,x) : n \in \mathbb{N}_0\}$ associated with the Markov process $V(t)$, here we want to determine the integral transform of an integrable function f of the form

$$\mathsf{M}[f(W)|(t,x)] = \int_S dw\, q(t,w|x) f(w), \tag{4.1}$$

such that it satisfies the equalities

$$m_n(t,x) = \mathsf{M}\left[(cW)^n\right|(t,x)], \quad \forall n \in \mathbb{N}_0, \quad \forall t \in \mathcal{T} \tag{4.2}$$

with some constant $c \in \mathbb{C}$. If so, given any polynomial f,

$$\mathsf{M}[f(cW)|(t,V(t))], t \geq 0 \text{ is a local martingale, and} \tag{4.3}$$

$$\mathsf{M}[f(cW)|(0,V(0))] = f(V(0)). \tag{4.4}$$

Note that by setting $f \equiv 1$ in (4.4) we have $\mathsf{M}[1|(t,V(t))] \equiv 1, t \geq 0$.

4.1 BM

For BM, $V(t) = B(t), t \in [0,\infty)$, we set

$$c = i. \tag{4.5}$$

and

$$\begin{aligned} q(t,y|x) &= p(t,y|c^{-1}x) \\ &= \begin{cases} \dfrac{1}{\sqrt{2\pi t}} e^{-(ix+y)^2/2t}, & t > 0, x, y \in \mathbb{R} \\ \\ \delta(y-x), & t = 0, x, y \in \mathbb{R}, \end{cases} \end{aligned} \tag{4.6}$$

Then we can prove the following.

Lemma 4.1 *With (4.5) and (4.6), (4.2) are satisfied.*

Proof. Since $p(t,\cdot|x)$ solves the diffusion equation, $q(t,\cdot|x) = p(t,\cdot|c^{-1}x)$ satisfies $\partial q/\partial t = c^{-2}(1/2)\partial^2 q/\partial x^2$. Then Itô's formula implies

$$\begin{aligned} d\mathsf{M}[f(W)|(t,B(t))] &= \int_{\mathbb{R}} dw\, dq(t,w|B(t)) f(w) \\ &= \left[\int_{\mathbb{R}} dw \left\{\frac{\partial q}{\partial t}(t,w|B(t)) + \frac{1}{2}\frac{\partial^2 q}{\partial x^2}(t,w|B(t))\right\} f(w)\right] dt \\ &\quad + \left\{\int_{\mathbb{R}} dw\, \frac{\partial q}{\partial x}(t,w|B(t)) f(w)\right\} dB(t) \\ &= \left[\int_{\mathbb{R}} dw\, \frac{\partial q}{\partial x}(t,w|B(t)) f(w)\right] dB(t), \end{aligned}$$

if $c^{-2} = -1 \Leftrightarrow c = \pm i$. Therefore, the assignment (4.5) of the value c guarantees that $\mathsf{M}[f(W)|(t, B(t))]$ is a local martingale. The Hermite polynomials have the following integral representations, $n \in \mathbb{N}_0, x \in \mathbb{R}$ (for instance, see Eq.(6.1.4) in [1]),

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, e^{-(ix+u)^2} (iu)^n \tag{4.7}$$

$$= \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, e^{-u^2} (x + iu)^n. \tag{4.8}$$

The formula (4.7) gives (4.2) with appropriate change of variables. ∎

Remark 4. The function $G^{\mathrm{BM}}_{\alpha}(t, B(t)), t \geq 0$ given by (3.9) is martingale for any $\alpha \in \mathbb{R}$. Then its Fourier transform with respect to α,

$$q(t, w|B(t)) = \frac{1}{2\pi} \int_{\mathbb{R}} d\alpha\, e^{-i\alpha w} G^{\mathrm{BM}}_{\alpha}(t, B(t)) \tag{4.9}$$

is also martingale for any $w \in \mathbb{R}$. We find that

$$q(t, w|x) = \frac{1}{2\pi} \int_{\mathbb{R}} d\alpha\, e^{-i\alpha w} e^{\alpha x - t\alpha^2/2}$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-(w+ix)^2/2t}, \tag{4.10}$$

which is equal to $p(t, w|c^{-1}x)$ with $c = i$ as mentioned as (4.5) and (4.6) above. By the Fourier reverse transform of (4.9), we have

$$G^{\mathrm{BM}}_{\alpha}(t, B(t)) = \int_{\mathbb{R}} dw\, q(t, w|B(t)) e^{i\alpha w}. \tag{4.11}$$

Expansion of the both sides with respect to α gives

$$\sum_{n=0}^{\infty} m_n(t, B(t)) \frac{\alpha^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{R}} dw\, q(t, w|B(t)) (iw)^n \frac{\alpha^n}{n!}, \tag{4.12}$$

which implies

$$\mathsf{M}[(iW)^n|(t, B(t))] \equiv \int_{\mathbb{R}} dw\, q(t, w|B(t)) (iw)^n$$

$$= m_n(t, B(t)), \qquad n \in \mathbb{N}_0, \tag{4.13}$$

where $m_n(t, x)$ is given by (3.8).

4.2 BESQ$^{(\nu)}$

For BESQ$^{(\nu)}$, $V(t) = R^{(\nu)}(t), \nu > -1, t \in [0, \infty)$, we set

$$c = -1. \tag{4.14}$$

and

$$q^{(\nu)}(t, y|x) = p^{(\nu)}(t, y|c^{-1}x)$$
$$= \begin{cases} \dfrac{1}{2t}\left(\dfrac{y}{x}\right)^{\nu/2} \exp\left(-\dfrac{(-x)+y}{2t}\right) J_\nu\left(\dfrac{\sqrt{xy}}{t}\right), & t > 0, x > 0, y \in \mathbb{R}_+, \\ \dfrac{y^\nu}{(2t)^{\nu+1}\Gamma(\nu+1)} e^{-y/2t}, & t > 0, x = 0, y \in \mathbb{R}_+, \\ \delta(y - x), & t = 0, x, y \in \mathbb{R}_+, \end{cases} \tag{4.15}$$

where $J_\nu(x)$ is the Bessel function defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\nu)} \left(\frac{z}{2}\right)^{2n+\nu}. \tag{4.16}$$

As usual we define z^ν to be $\exp(\nu \log z)$, where the argument of z is given by its principal value;

$$z^\nu = \exp\left[\nu\Big\{\log|z| + \sqrt{-1}\arg(z)\Big\}\right], \quad -\pi < \arg(z) \leq \pi.$$

In order to obtain (4.15) from (3.12), we have used the relation

$$I_\nu(z) = \begin{cases} e^{-\nu\pi i/2} J_\nu(iz), & -\pi < \arg(z) \leq \pi/2, \\ e^{3\nu\pi i/2} J_\nu(iz), & \pi/2 < \arg(z) \leq \pi. \end{cases}$$

Then we can prove the following.

Lemma 4.2 *With (4.14) and (4.15), (4.2) are satisfied.*

Proof. Since $q^{(\nu)}(t, \cdot|x) = p^{(\nu)}(t, \cdot|c^{-1}x)$ satisfies

$$\frac{\partial q^{(\nu)}}{\partial t} = c^{-1}\left\{2x\frac{\partial^2 q^{(\nu)}}{\partial x^2} + 2(\nu+1)\frac{\partial q^{(\nu)}}{\partial x}\right\},$$

$\mathsf{M}^{(\nu)}[f(W)|(t, R^{(\nu)}(t))]$ is a local martingale, if $c^{-1} = -1 \Leftrightarrow c = -1$. Therefore, the assignment (4.14) of the value c guarantees that $\mathsf{M}[f(W)|(t, R^{(\nu)}(t))]$ is a local martingale. The integral representations of the Laguerre polynomials in terms of Bessel functions (for instance, see Eq.(6.2.15) in [1]),

$$L_n^{(\nu)}(x) = \frac{1}{n!}\frac{e^x}{x^{\nu/2}} \int_0^\infty du\, e^{-u} u^{n+\nu/2} J_\nu(2\sqrt{xu}), \quad n \in \mathbb{N}_0, \nu > -1, x \in \mathbb{R}_+, \tag{4.17}$$

give (4.2). ▮

Remark 5. The following integral formula is established,

$$\frac{e^x}{x^{\nu/2}} \int_0^\infty du\, u^{\nu/2} e^{-(1-\alpha)u} J_\nu(2\sqrt{xu}) = \frac{e^{-x\alpha/(1-\alpha)}}{(1-\alpha)^{\nu+1}}. \tag{4.18}$$

It gives an integral representation for $G_\alpha^{(\nu)}(t,x)$ studied in Remark 2,

$$G_\alpha^{(\nu)}(t,x) = \int_0^\infty dw\, q^{(\nu)}(t,w|x) e^{-\alpha w}, \tag{4.19}$$

where $q^{(\nu)}$ is given by (4.15).

4.3 CPR for BM

The integral formula (4.8) implies

$$m_n(t,x) = \check{\mathrm{E}}_0[(x+iW(t))^n], \quad n \in \mathbb{N}_0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \tag{4.20}$$

where $\check{\mathrm{E}}_0$ denotes the expectation of BM, $W(t), t \geq 0$. It is independent from $B(t), t \geq 0$ and started at $W(0) = 0$. Then, if we consider the complex BM,

$$Z(t) = B(t) + iW(t), \quad t \geq 0, \tag{4.21}$$

then

$$m_n(t, B(t)) = \mathsf{M}[(iW)^n | (t, B(t))] = \check{\mathrm{E}}_0[Z(t)^n], \quad t \geq 0, \quad n \in \mathbb{N}. \tag{4.22}$$

As a matter of course, the map $z \to z^n, z \in \mathbb{C}, n \in \mathbb{N}$ are analytic, and then $Z(t)^n, t \geq 0, n \in \mathbb{N}_0$ are conformal maps of $Z(t), t \geq 0$. Since the probability distribution of the complex BM is conformal invariant, $Z(t)^n, t \geq 0, n \in \mathbb{N}$ are time changes of $Z(t)$. In other words, $Z(\cdot)^n, n \in \mathbb{N}$ are *conformal local martingales* (see Section V.2 of [61]). Since $B(\cdot) = \Re Z(\cdot)$ and $W(\cdot) = \Im Z(\cdot)$ are independent one-dimensional standard BM's, $m_n(\cdot, B(\cdot)), n \in \mathbb{N}$, which are obtained by taking the average over the imaginary parts of $Z(\cdot)^n$ as (4.22) are also local martingales.

4.4 CPR for Bessel processes (BES$^{(\nu)}$)

The *Bessel process* with index ν (BES$^{(\nu)}$), $\widetilde{R}^{(\nu)}(t), t \geq 0$, is defined by

$$\widetilde{R}^{(\nu)}(t) \equiv \sqrt{R^{(\nu)}(t)}, \quad t \geq 0, \quad \nu > -1, \tag{4.23}$$

where $R^{(\nu)}(t), t \geq 0$ is BESQ$^{(\nu)}$. It solves the SDE

$$d\widetilde{R}^{(\nu)}(t) = dB(t) + \frac{2\nu+1}{2} \frac{dt}{\widetilde{R}^{(\nu)}(t)}, \quad t \geq 0. \tag{4.24}$$

The transition probability density is obtained from (3.12) as

$$\widetilde{p}^{(\nu)}(t,y|x) = p^{(\nu)}(t,y^2|x^2)2y$$
$$= \begin{cases} \dfrac{1}{t}\dfrac{y^{\nu+1}}{x^\nu}\exp\left(-\dfrac{x^2+y^2}{2t}\right) I_\nu\left(\dfrac{xy}{t}\right), & t>0, x>0, y\in\mathbb{R}_+, \\ \dfrac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)} e^{-y^2/2t}, & t>0, x=0, y\in\mathbb{R}_+, \\ \delta(y-x), & t=0, x,y\in\mathbb{R}_+. \end{cases} \quad (4.25)$$

Corresponding to this, the integral transform (4.1) for BESQ$^{(\nu)}$, which is denoted as $\mathsf{M}^{(\nu)}[\cdot|\cdot]$, is converted into that for BES$^{(\nu)}$ expressed as $\widetilde{\mathsf{M}}^{(\nu)}[\cdot|\cdot]$ so that the following relation holds,

$$\mathsf{M}^{(\nu)}[f(-W)|(t,R^{(\nu)}(t))] = \widetilde{\mathsf{M}}^{(\nu)}[\widetilde{f}(iW)|(t,\widetilde{R}^{(\nu)}(t))], \quad t\geq 0, \quad (4.26)$$

where f and $\widetilde{f}$ are polynomials with the relation $\widetilde{f}(z)=f(z^2), z\in\mathbb{C}$. For it, we set

$$\widetilde{q}^{(\nu)}(t,y|x) = q^{(\nu)}(t,y^2|x^2)2y$$
$$= \begin{cases} \dfrac{1}{t}\dfrac{y^{\nu+1}}{x^\nu}\exp\left(-\dfrac{(-x^2)+y^2}{2t}\right) J_\nu\left(\dfrac{xy}{t}\right), & t>0, x>0, y\in\mathbb{R}_+, \\ \dfrac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)} e^{-y^2/2t}, & t>0, x=0, y\in\mathbb{R}_+, \\ \delta(y-x), & t=0, x,y\in\mathbb{R}_+. \end{cases} \quad (4.27)$$

and define the integral transform for BES$^{(\nu)}, \nu>-1$ by

$$\widetilde{\mathsf{M}}^{(\nu)}[f(W)|(t,x)] = \int_{\mathbb{R}_+} dw\, \widetilde{q}^{(\nu)}(t,w|x)f(w), \quad (t,x)\in[0,\infty)\times\mathbb{R}_+ \quad (4.28)$$

for an integrable function f. Then the relation (4.26) is satisfied.

For $m\in\mathbb{N}_0$, the Bessel functions have the following expansions by the trigonometric functions,

$$J_{2m+1/2}(x) = (-1)^m\sqrt{\frac{2}{\pi x}}\left[\sin x\sum_{k=0}^{m}\frac{(-1)^k(2m+2k)!}{(2k)!(2m-2k)!}(2x)^{-2k} + \cos x\sum_{k=0}^{m-1}\frac{(-1)^k(2m+2k+1)!}{(2k+1)!(2m-2k-1)!}(2x)^{-(2k+1)}\right],$$
$$J_{2m+3/2}(x) = (-1)^m\sqrt{\frac{2}{\pi x}}\left[-\cos x\sum_{k=0}^{m}\frac{(-1)^k(2m+2k+1)!}{(2k)!(2m-2k+1)!}(2x)^{-2k} + \sin x\sum_{k=0}^{m}\frac{(-1)^k(2m+2k+2)!}{(2k+1)!(2m-2k)!}(2x)^{-(2k+1)}\right]. \quad (4.29)$$

They are obtained from Eq.(4.6.12) in [1]. For example, if we set $m = 0$ in (4.29), we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right).$$

Assume that $\widetilde{f}(z)$ is a polynomial of z^2, and thus

$$\widetilde{f}(-z) = \widetilde{f}(z), \quad z \in \mathbb{C}. \tag{4.30}$$

Then (4.28) with (4.27) gives

$$\begin{aligned} \widetilde{\mathsf{M}}^{(1/2)} \left[\widetilde{f}(iW) \Big| (t,x) \right] &= \int_{\mathbb{R}_+} dw \sqrt{\frac{2}{\pi t}} \frac{w}{x} e^{-(-x^2+w^2)/2t} \sin(xw/t) \widetilde{f}(iw) \\ &= \frac{1}{\sqrt{2\pi t}} \frac{1}{ix} \int_{\mathbb{R}_+} dw\, w \left\{ e^{-(w-ix)^2/2t} - e^{-(w+ix)^2/2t} \right\} \widetilde{f}(iw), \end{aligned}$$

and

$$\begin{aligned} &\widetilde{\mathsf{M}}^{(3/2)} \left[\widetilde{f}(iW) \Big| (t,x) \right] \\ &= \int_{\mathbb{R}_+} dw \sqrt{\frac{2}{\pi t}} \frac{w}{x} e^{-(-x^2+w^2)/2t} \left\{ \frac{t}{xw} \sin(xw/t) - \cos(xw/t) \right\} \widetilde{f}(iw) \\ &= \frac{1}{\sqrt{2\pi t}} \left[\frac{t}{ix^3} \int_{\mathbb{R}_+} dw\, w \left\{ e^{-(w-ix)^2/2t} - e^{-(w+ix)^2/2t} \right\} \right. \\ &\qquad \left. - \frac{1}{x^2} \int_{\mathbb{R}_+} dw\, w^2 \left\{ e^{-(w-ix)^2/2t} + e^{-(w+ix)^2/2t} \right\} \right] \widetilde{f}(iw). \end{aligned}$$

By the assumption (4.30), they are rewritten as

$$\begin{aligned} \widetilde{\mathsf{M}}^{(1/2)} \left[\widetilde{f}(iW) \Big| (t,x) \right] &= \frac{1}{\sqrt{2\pi t}} \frac{1}{(-i)x} \int_{\mathbb{R}} dw\, w e^{-(w+ix)^2/2t} \widetilde{f}(iw) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty+ix}^{\infty+ix} du\, \frac{x+iu}{x} e^{-u^2/2t} \widetilde{f}(x+iu), \\ \widetilde{\mathsf{M}}^{(3/2)} \left[\widetilde{f}(iW) \Big| (t,x) \right] &= \frac{1}{\sqrt{2\pi t}} \left[\frac{t}{(-i)x^3} \int_{\mathbb{R}} dw\, w e^{-(w+ix)^2/2t} \widetilde{f}(iw) \right. \\ &\qquad \left. + \frac{1}{(-i)^2 x^2} \int_{\mathbb{R}} dw\, w^2 e^{-(w+ix)^2/2t} \widetilde{f}(iw) \right] \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty+ix}^{\infty+ix} du \left\{ \frac{t(x+iu)}{x^3} + \frac{(x+iu)^2}{x^2} \right\} e^{-u^2/2t} \widetilde{f}(x+iu), \end{aligned}$$

where we have changed the integral variables by $u = w + ix$. Since the integrands are entire, $\int_{-\infty+ix}^{\infty+ix} du\,(\cdot)$ can be replaced by $\int_{\mathbb{R}} du\,(\cdot)$. Then we have the following expressions for the martingales (4.26) with $\nu = 1/2$ and $3/2$

$$\widetilde{\mathsf{M}}^{(1/2)}\left[\widetilde{f}(iW)\Big|(t,\widetilde{R}^{(1/2)}(t))\right] = \check{\mathrm{E}}_0\left[\frac{Z^{(1/2)}(t)}{\Re Z^{(1/2)}(t)}\widetilde{f}(Z^{(1/2)}(t))\right], \quad t \geq 0,$$

$$\widetilde{\mathsf{M}}^{(3/2)}\left[\widetilde{f}(iW)\Big|(t,\widetilde{R}^{(1/2)}(t))\right]$$
$$= \check{\mathrm{E}}_0\left[\left\{\frac{tZ^{(3/2)}(t)}{(\Re Z^{(3/2)}(t))^3} + \frac{(Z^{(3/2)}(t))^2}{(\Re Z^{(3/2)}(t))^2}\right\}\widetilde{f}(Z^{(3/2)}(t))\right], \quad t \geq 0, \qquad (4.31)$$

where $Z^{(\nu)}(t) = \widetilde{R}^{(\nu)}(t) + iW(t), \nu = 1/2, 3/2$. These calculations are generalized as follows.

Lemma 4.3 *Let $\widetilde{f}(z)$ be a polynomial of z^2. Then $\widetilde{\mathsf{M}}^{(n+1/2)}[\widetilde{f}(iW)|(\cdot,\widetilde{R}^{(n+1/2)}(\cdot))]$, $n \in \mathbb{N}_0$, are local martingales, and*

$$\widetilde{\mathsf{M}}^{(n+1/2)}\left[\widetilde{f}(iW)\Big|(t,\widetilde{R}^{(n+1/2)}(t))\right] = \check{\mathrm{E}}_0\left[Q_t^{(n+1/2)}(Z^{(n+1/2)}(t))\widetilde{f}(Z^{(n+1/2)}(t))\right], \quad t \geq 0, \qquad (4.32)$$

where

$$Z^{(n+1/2)}(t) = \widetilde{R}^{(n+1/2)}(t) + iW(t), \quad t \geq 0, \qquad (4.33)$$

and

$$Q_t^{(n+1/2)}(z) = \left(\frac{t}{2}\right)^n \frac{z}{(\Re z)^{2n+1}} \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!k!}\left(\frac{2(\Re z)z}{t}\right)^k, \quad z \in \mathbb{C}. \qquad (4.34)$$

Note that the equalities (4.22) with (3.6) hold even if we replace the complex BM, (4.21), by the present complex diffusion, (4.33), since the imaginary parts are the same;

$$\check{\mathrm{E}}_0[(Z^{(n+1/2)}(t))^k] = m_k(t,\widetilde{R}^{(n+1/2)}(t)), \quad t \geq 0, \quad n,k \in \mathbb{N}_0. \qquad (4.35)$$

Then for monomials $\widetilde{f}(z) = z^{2\ell}$, (4.32) gives the following. For $n, \ell \in \mathbb{N}_0$,

$$\widetilde{\mathsf{M}}^{(n+1/2)}\left[(iW)^{2\ell}\Big|(t,\widetilde{R}^{(n+1/2)}(t))\right]$$
$$= \left(\frac{t}{2}\right)^n \frac{1}{(\widetilde{R}^{(n+1/2)}(t))^{2n+1}} \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!k!}\left(\frac{2\widetilde{R}^{(n+1/2)}(t)}{t}\right)^k \check{\mathrm{E}}_0[(Z^{n+1/2}(t))^{2\ell+k+1}]$$
$$= \frac{1}{2^{2n+1}}\left(\frac{t}{2}\right)^{\ell}\left(\frac{\widetilde{R}^{(n+1/2)}(t)}{\sqrt{2t}}\right)^{-(2n+1)}$$
$$\times \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!k!}\left(2\frac{\widetilde{R}^{(n+1/2)}(t)}{\sqrt{2t}}\right)^k H_{2\ell+k+1}\left(\frac{\widetilde{R}^{(n+1/2)}(t)}{\sqrt{2t}}\right). \qquad (4.36)$$

4.5 CPR for RW

For $t > 0$, let $\eta_\ell(t), \ell \in \mathbb{N}$ be a series of i.i.d. random variables with the Gamma(t) probability density

$$\mathrm{P}^\Gamma(\cdot \in dx) = \frac{1}{\Gamma(t)} x^{t-1} e^{-t} dx, \quad x > 0, \tag{4.37}$$

where $\Gamma(t)$ is the Gamma function. When $t \in \mathbb{N}$,

$$\eta_\ell(t) \stackrel{\mathrm{d}}{=} \varepsilon_\ell^{(1)} + \cdots + \varepsilon_\ell^{(t)}, \quad \ell \in \mathbb{N}, \tag{4.38}$$

where $\varepsilon_\ell^{(k)}, k = 1, 2, \dots, t$ are independent random variables with standard exponential distribution $\mathrm{Prob}(\varepsilon_\ell^{(k)} \geq x) = e^{-x}, x \geq 0$. We consider a random variable

$$C(t) = \frac{2}{\pi^2} \sum_{\ell \in \mathbb{N}} \frac{\eta_\ell(t)}{(\ell - 1/2)^2}, \tag{4.39}$$

since it is known [7] that its Laplace transform is given by

$$\mathrm{E}^\Gamma[e^{-\lambda C(t)}] = \frac{1}{(\cosh \sqrt{2\lambda})^t}, \quad t > 0, \tag{4.40}$$

where E^Γ denotes the expectation with respect to $\eta_\ell(t), \ell \in \mathbb{N}$. In [7], it is shown that $C(t) \in [0, \infty)$ is infinitely divisible and its probability density $\mu_{C(t)}(\cdot)$, which is defined for integrable functions f as

$$\mathrm{E}^\Gamma[f(C(t))] = \int_0^\infty dc\, \mu_{C(t)}(c) f(c), \tag{4.41}$$

is explicitly given by

$$\mu_{C(t)}(c) = \frac{2^t}{\Gamma(t)} \sum_{\ell=0}^\infty (-1)^\ell \frac{\Gamma(\ell + t)}{\Gamma(\ell + 1)} \frac{(2\ell + t)}{\sqrt{2\pi c^3}} e^{-(2\ell+t)^2/2c}, \quad t > 0, x > 0. \tag{4.42}$$

The generating function of the polynomials (3.25) is thus written as

$$\begin{aligned} G_\alpha(t, x) &= \mathrm{E}^\Gamma\left[e^{\alpha x - \alpha^2 C(t)/2}\right] \\ &= \sum_{n=0}^\infty \frac{\alpha^n}{n!} \mathrm{E}^\Gamma\left[\left(\frac{C(t)}{2}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2C(t)}}\right)\right], \end{aligned} \tag{4.43}$$

where we used the formulas (3.9) and (3.10). It gives the relations

$$m_n(t, x) = \mathrm{E}^\Gamma\left[m_n^{\mathrm{BM}}(C(t), x)\right], \quad n \in \mathbb{N}_0, \quad t \in N_0, \tag{4.44}$$

that is, let $m_n^{\mathrm{BM}}(C(t), x)$ be a random time change $t \to C(t)$ of the polynomial for BM, then its average over $C(t)$ gives Fujita's polynomial for RW.

On the other hand, by (4.20), if we introduce a one-dimensional BM, $W(t), t \geq 0$ with $W(0) = 0$, and write the expectation with respect to $W(\cdot)$ as $\check{\mathrm{E}}$, we have the following expressions

$$m_n^{\mathrm{BM}}(t,x) = \check{\mathrm{E}}[(x+iW(t))^n], \quad n \in \mathbb{N}_0, \quad t \geq 0. \tag{4.45}$$

Combination of (4.44) and (4.45) gives

$$m_n(t,x) = \mathrm{E}^{\Gamma}\Big[\check{\mathrm{E}}\Big[(x+iW(C(t)))^n\Big]\Big], \quad n \in \mathbb{N}_0, \quad t \in \mathbb{N}_0. \tag{4.46}$$

Let

$$\begin{aligned} \widetilde{W}(t) &\stackrel{\mathrm{d}}{=} W(C(t)) \stackrel{\mathrm{d}}{=} \sqrt{\frac{C(t)}{t}} W(t), \quad t \in \mathbb{N}, \\ \widetilde{W}(0) &= 0. \end{aligned} \tag{4.47}$$

We regard $\widetilde{W}(t), t \in \mathbb{N}_0$ as a discrete-time process and the expectation w.r.t. this process is written as

$$\widetilde{\mathrm{E}}[f(\widetilde{W}(t))] = \mathrm{E}^{\Gamma}\Big[\check{\mathrm{E}}[f(W(C(t)))]\Big], \quad t \in \mathbb{N}_0 \tag{4.48}$$

for integrable functions f.

With RW, $V(t), t \in \mathbb{N}_0$, we consider a discrete-time complex process

$$Z(t) = V(t) + i\widetilde{W}(t), \quad t \in \mathbb{N}_0. \tag{4.49}$$

Note that $\Re Z(t) = V(t) \in \mathbb{Z}$ and $\Im Z(t) = \widetilde{W}(t) \in \mathbb{R}$. The above results are summarized as follows [35].

Lemma 4.4 *Fujita's polynomial martingales, $m_n(t, V(t)), n \in \mathbb{N}_0, t \in \mathbb{N}_0$, for the simple and symmetric RW have the following complex-process representations,*

$$m_n(t, V(t)) = \widetilde{\mathrm{E}}[Z(t)^n], \quad n \in \mathbb{N}_0, \quad t \in \mathbb{N}_0. \tag{4.50}$$

5 Noncolliding Diffusion Processes

5.1 Map for martingales

For a configuration

$$\xi(\cdot) = \sum_{j=1}^{N} \delta_{u_j}(\cdot) \in \mathfrak{M}_0, \tag{5.1}$$

we define a polynomial of $x \in \mathbb{C}$ with a parameter $u \in \mathbb{C}$ as

$$\Phi_\xi^u(x) = \prod_{r \in \mathrm{supp}\,\xi \cap \{u\}^{\mathrm{c}}} \frac{x-r}{u-r}. \tag{5.2}$$

with $\mathrm{supp}\,\xi = \{u_j : 1 \le j \le N\}$. Note that

$$\Phi_\xi^{u_k}(u_j) = \delta_{jk}, \quad 1 \le j, k \le N. \tag{5.3}$$

Then we have the following statement.

Proposition 5.1 *For the Markov process $V(t), t \in \mathcal{T}$, assume that the integral transform (4.1) satisfying (4.2) is obtained. Then with $\boldsymbol{u} \in \mathbb{W}_N$ and $\xi = \sum_{j=1}^N \delta_{u_j} \in \mathfrak{M}_0$, the map (1.6) satisfying (1.7) is given by*

$$\mathcal{M}_\xi^u(\cdot,\cdot) = \mathsf{M}[\Phi_\xi^u(cW)|(\cdot,\cdot)]. \tag{5.4}$$

Proof. By definition (5.2), $\Phi_\xi^u(x)$ is polynomial. Then $\mathcal{M}_\xi^u(\cdot, V(\cdot))$ is polynomial and local martingale by (4.3). By (4.4),

$$\mathcal{M}_\xi^u(0, V(0)) = \mathsf{M}[\Phi_\xi^u(cW)|(0, V(0))] = \Phi_\xi^u(V(0)).$$

Then

$$\mathcal{M}_\xi^{u_k}(0, u_j) = \Phi_\xi^{u_k}(u_j) = \delta_{jk}, \quad 1 \le j, k \le N$$

by (5.3). The proof is completed. ∎

The determinantal martingale (1.8) is now given as

$$\mathcal{D}_\xi(t, \boldsymbol{V}_{\mathbb{J}}(t)) = \det_{j,k \in \mathbb{J}} \Big[\mathsf{M}[\Phi_\xi^{u_k}(cW)|(t, V_j(t))]\Big], \quad \mathbb{J} \subset \mathbb{I}_N, \quad t \in \mathcal{T}. \tag{5.5}$$

The integral transform (4.1) is extended to the linear integral transform of functions of $\boldsymbol{x} \in S^N$ such that, if $F^{(k)}(\boldsymbol{x}) = \prod_{j=1}^N f_j^{(k)}(x_j)$ with integrable functions $f_j^{(k)}, 1 \le j \le N, k = 1, 2$, then

$$\mathsf{M}\left[F^{(k)}(\boldsymbol{W})\,\middle|\{(t_\ell, x_\ell)\}_{\ell=1}^N\right] = \prod_{j=1}^N \mathsf{M}\left[f_j^{(k)}(W_j)\middle|(t_j, x_j)\right], \quad k = 1, 2. \tag{5.6}$$

and

$$\mathsf{M}\Big[c_1 F^{(1)}(\boldsymbol{W}) + c_2 F^{(2)}(\boldsymbol{W}) \,\Big|\{(t_\ell, x_\ell)\}_{\ell=1}^N \Big]$$
$$= c_1 \mathsf{M}\Big[F^{(1)}(\boldsymbol{W}) \,\Big|\{(t_\ell, x_\ell)\}_{\ell=1}^N \Big] + c_2 \mathsf{M}\Big[F^{(2)}(\boldsymbol{W}) \,\Big|\{(t_\ell, x_\ell)\}_{\ell=1}^N \Big], \qquad (5.7)$$

$c_1, c_2 \in \mathbb{C}$, for $0 < t_j < \infty, 1 \leq j \leq N$, where $\boldsymbol{W} = (W_1, \dots, W_N) \in S^N$. In particular, if $t_\ell = t, 1 \leq {}^\forall \ell \leq N$, we write $\mathsf{M}[\cdot|\{(t_\ell, x_\ell)\}_{\ell=1}^N]$ simply as $\mathsf{M}[\cdot|(t, \boldsymbol{x})]$ with $\boldsymbol{x} = (x_1, \dots, x_N)$. Then, by the multilinearity of determinant, (5.5) is written as

$$\mathcal{D}_\xi(t, \boldsymbol{V}_{\mathbb{J}}(t)) = \mathsf{M}\left[\det_{j,k \in \mathbb{J}}\left[\Phi_\xi^{u_k}(cW_j)\right] \middle| (t, \boldsymbol{V}_{\mathbb{J}}(t))\right], \quad \mathbb{J} \subset \mathbb{I}_N, \quad t \in \mathcal{T}. \qquad (5.8)$$

5.2 Krattenthaler's determinant identity and h-transform

The following determinant identity was given as Lemma 2.2 in [47] and as Lemma 3 in [48] proved by Krattenthaler.

Lemma 5.2 *Let $X_1, \dots, X_N, A_2, \dots, A_N$ and $B_2, \dots, B_N$ be indeterminates. Then there holds*

$$\det_{1 \leq j,k \leq N}\Big[(X_j + A_N)(X_j + A_{N-1}) \cdots (X_j + A_{k+1})(X_j + B_k)(X_j + B_{k-1}) \cdots (X_j + B_2)\Big]$$
$$= \prod_{1 \leq j < k \leq N} (X_j - X_k) \prod_{2 \leq j \leq k \leq N} (B_j - A_k). \qquad (5.9)$$

The Vandermonde determinant is given as

$$h(\boldsymbol{x}) = \det_{1 \leq j,k \leq N}[x_j^{k-1}] = \prod_{1 \leq j < k \leq N} (x_k - x_j), \qquad (5.10)$$

for $\boldsymbol{x} = (x_1, \dots, x_N) \in S^N$. As a special case of (5.9), we obtain the following determinant identity.

Lemma 5.3 *Assume that $N \in \mathbb{N}, \boldsymbol{x} \in \mathbb{C}^N, \boldsymbol{u} \in \mathbb{W}_N$. Then*

$$\frac{h(\boldsymbol{x})}{h(\boldsymbol{u})} = \det_{1 \leq j,k \leq N}[\Phi_\xi^{u_k}(x_j)]. \qquad (5.11)$$

Proof. In the identity (5.9), set

$$X_j = x_j, \quad 1 \leq j \leq N,$$
$$A_j = -u_j, \quad B_j = -u_{j-1}, \quad 2 \leq j \leq N.$$

Then we find

$$H(\boldsymbol{u}, \boldsymbol{x}) \equiv \det_{1 \leq j,k \leq N}\left[\prod_{1 \leq \ell \leq N, \ell \neq j} (u_\ell - x_k)\right] = (-1)^{N(N-1)/2} h(\boldsymbol{u}) h(\boldsymbol{x}).$$

Since

$$\begin{aligned}\det_{1\le j,k\le N}[\Phi_\xi^{u_k}(x_j)] &= \frac{H(\boldsymbol{u},\boldsymbol{x})}{\prod_{1\le j\le N}\prod_{1\le k\le N:k\ne j}(u_k-u_j)} \\ &= \frac{H(\boldsymbol{u},\boldsymbol{x})}{(-1)^{N(N-1)/2}h(\boldsymbol{u})^2},\end{aligned}$$

the identity (5.11) is obtained as a special case of (5.9). ∎

Using the determinant identity (5.11) with (5.10), we see

$$\begin{aligned}\mathcal{D}_\xi(t,\boldsymbol{V}(t)) &= \mathsf{M}\left[\det_{1\le j,k\le N}[\Phi_\xi^{u_k}(cW_j)]\middle|(t,\boldsymbol{V}(t))\right] \\ &= \mathsf{M}\left[\frac{h(c\boldsymbol{W})}{h(\boldsymbol{u})}\middle|(t,\boldsymbol{V}(t))\right] \\ &= \mathsf{M}\left[\frac{1}{h(\boldsymbol{u})}\det_{1\le j,k\le N}[(cW_j)^{k-1}]\middle|(t,\boldsymbol{V}(t))\right] \\ &= \frac{1}{h(\boldsymbol{u})}\det_{1\le j,k\le N}\left[\mathsf{M}[(cW_j)^{k-1}|(t,V_j(t))]\right] \\ &= \frac{1}{h(\boldsymbol{u})}\det_{1\le j,k\le N}[m_{k-1}(t,V_j(t))]. \end{aligned} \tag{5.12}$$

By multilinearity of determinant, the Vandermonde determinant $\det_{1\le j,k\le N}[x_j^{k-1}]$ does not change by replacing x_j^{k-1} by any monic polynomial of x_j of degree $k-1$, $1\le j,k\le N$. Since $m_{k-1}(t,x_j)$ is a monic polynomial of x_j of degree $k-1$, (5.12) is equal to

$$\begin{aligned}\mathcal{D}_\xi(t,\boldsymbol{V}(t)) &= \frac{1}{h(\boldsymbol{u})}\det_{1\le j,k\le N}[(V_j(t))^{k-1}] \\ &= \frac{h(\boldsymbol{V}(t))}{h(\boldsymbol{u})}. \end{aligned} \tag{5.13}$$

This is the factor used for the harmonic transform (h-transform).

5.3 Noncolliding BM and noncolliding BESQ$^{(\nu)}$

For $N\in\mathbb{N}$, we consider N-particle systems of BM's, $\boldsymbol{X}(t)=(X_1(t),X_2(t),\dots,X_N(t))$, $t\ge 0$, and of BESQ$^{(\nu)}$ with index $\nu>-1$, $\boldsymbol{X}^{(\nu)}(t)=(X_1^{(\nu)}(t),X_2^{(\nu)}(t),\dots,X_N^{(\nu)}(t)), t\ge 0$, both *conditioned never to collide with each other particle*. The former process, which is called the *noncolliding BM*, solves the following set of SDEs

$$dX_j(t)=dB_j(t)+\sum_{\substack{1\le k\le N,\\ k\ne j}}\frac{dt}{X_j(t)-X_k(t)},\quad 1\le j\le N,\quad t\ge 0, \tag{5.14}$$

with independent one-dimensional standard BMs, $B_j(t), 1 \leq j \leq N, t \geq 0$ [13, 67, 24, 26, 40, 58, 59, 60]. The latter process, the *noncolliding BESQ*$^{(\nu)}$, does the following set of SDEs

$$
\begin{aligned}
dX_j^{(\nu)}(t) &= 2\sqrt{X_j^{(\nu)}(t)}d\check{B}_j(t) + 2(\nu+1)dt \\
&\quad +4X_j^{(\nu)}(t) \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} \frac{dt}{X_j^{(\nu)}(t) - X_k^{(\nu)}(t)}, \quad 1 \leq j \leq N, \quad t \geq 0, \qquad (5.15)
\end{aligned}
$$

where $\check{B}_j(t), 1 \leq j \leq N, t \geq 0$ are independent one-dimensional standard BMs different from $B_j(t), 1 \leq j \leq N, t \geq 0$, and, if $-1 < \nu < 0$, the reflection boundary condition is assumed at the origin [46, 41].

Consider subsets of $\mathbb{R}^N$, $\mathbb{W}_N^{\mathrm{A}} = \{\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 < \cdots < x_N\}$, and $\mathbb{W}_N^{+} = \{\boldsymbol{x} \in \mathbb{R}_+^N : x_1 < \cdots < x_N\}$. The former is called the Weyl chambers of types A_{N-1}. If we replace the condition $\boldsymbol{x} \in \mathbb{R}_+^N$ by $\boldsymbol{x} \in (0,\infty)^N$ for the latter, it will be the Weyl chamber of type C_N. It is proved that, provided $\boldsymbol{X}(0) \in \mathbb{W}_N^{\mathrm{A}}$ and $\boldsymbol{X}^{(\nu)}(0) \in \mathbb{W}_N^{+}$, then the SDEs (5.14) and (5.15) guarantee that with probability one $\boldsymbol{X}(t) \in \mathbb{W}_N^{\mathrm{A}}$, and $\boldsymbol{X}^{(\nu)}(t) \in \mathbb{W}_N^{+}, \forall t > 0$ [25]. That is, in both processes, at any positive time $t > 0$ there is no multiple point at which coincidence of particle positions $X_j(t) = X_k(t)$ or $X_j^{(\nu)}(t) = X_k^{(\nu)}(t)$ for $j \neq k$ occurs. It is the reason why these processes are called *noncolliding diffusion processes* [42]. In general, however, we can consider them starting from initial configurations with multiple points. In order to describe a general initial configuration we express it by a sum of delta measures in the form $\xi(\cdot) = \sum_{j=1}^N \delta_{x_j}(\cdot)$.

Let $\mathfrak{M}$ be the space of nonnegative integer-valued Radon measures on $\mathbb{R}$. For an element ξ of $\mathfrak{M}$, $\xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot)$ with a countable index set $\mathbb{I}$, we introduce the following operations.

(shift) with $u \in \mathbb{R}$, $\tau_u \xi(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i + u}(\cdot)$,

(dilatation) with $c > 0$, $c \circ \xi(\cdot) = \sum_{i \in \mathbb{I}} \delta_{c x_i}(\cdot)$,

(square) $\xi^{\langle 2 \rangle}(\cdot) = \sum_{i \in \mathbb{I}} \delta_{x_i^2}(\cdot)$.

Let $\mathfrak{M}^+ = \{(\xi \cap \mathbb{R}_+) : \xi \in \mathfrak{M}\}$. We consider the noncolliding BM and the noncolliding BESQ as $\mathfrak{M}$-valued and $\mathfrak{M}^+$-valued processes and write them as

$$
\Xi(t, \cdot) = \sum_{j=1}^N \delta_{X_j(t)}(\cdot), \quad \Xi^{(\nu)}(t, \cdot) = \sum_{j=1}^N \delta_{X_j^{(\nu)}(t)}(\cdot), \quad t \geq 0, \qquad (5.16)
$$

respectively [40, 41]. The probability law of $\Xi(t, \cdot)$ starting from a fixed configuration $\xi \in \mathfrak{M}$ is denoted by $\mathbb{P}_\xi$ and that of $\Xi^{(\nu)}(t, \cdot)$ from $\xi \in \mathfrak{M}^+$ by $\mathbb{P}_\xi^{(\nu)}$, and the noncolliding diffusion processes specified by initial configurations are expressed by $(\Xi(t), \mathbb{P}_\xi)$ and $(\Xi^{(\nu)}(t), \mathbb{P}_\xi^{(\nu)}), \nu >$

-1. The expectations w.r.t.$\mathbb{P}_\xi$ and $\mathbb{P}_\xi^{(\nu)}$ are denoted by $\mathbb{E}_\xi$ and $\mathbb{E}_\xi^{(\nu)}$, respectively. The set of $\hat{\mathfrak{M}}$-valued continuous functions defined on $[0,\infty)$ is denoted by $\mathrm{C}([0,\infty) \to \hat{\mathfrak{M}})$ for $\hat{\mathfrak{M}} = \mathfrak{M}$ or $\mathfrak{M}^+$. We introduce a filtration $\{\mathcal{F}(t)\}_{t\in[0,\infty)}$ on the space $\mathrm{C}([0,\infty) \to \hat{\mathfrak{M}})$ defined by $\mathcal{F}(t) = \sigma(\hat{\Xi}(s), s \in [0,t])$, where $\hat{\Xi}(\cdot) = \Xi(\cdot)$ for $\hat{\mathfrak{M}} = \mathfrak{M}$ and $\hat{\Xi}(\cdot) = \Xi^{(\nu)}(\cdot)$ for $\hat{\mathfrak{M}} = \mathfrak{M}^+$. Let $\mathrm{C}_0(S)$ be the set of all continuous real-valued functions with compact supports on $S = \mathbb{R}$ or $\mathbb{R}_+$. We set $\hat{\mathfrak{M}}_0 = \{\xi \in \hat{\mathfrak{M}} : \xi(\{x\}) \le 1 \text{ for any } x \in S\}$, which denotes collections of configurations without any multiple points.

5.4 H-transforms of absorbing processes

For the noncolliding BM, $\hat{\Xi}(\cdot) = \Xi(\cdot)$ (resp. $\mathrm{BESQ}^{(\nu)}, \nu > -1$, $\hat{\Xi}(\cdot) = \Xi^{(\nu)}(\cdot)$), we shall set $\hat{\boldsymbol{X}}(\cdot) = \boldsymbol{X}(\cdot)$ (resp. $\boldsymbol{X}^{(\nu)}(\cdot)$), $\hat{\mathbb{P}}_\xi = \mathbb{P}_\xi$ (resp. $\mathbb{P}_\xi^{(\nu)}$), $\hat{\mathbb{E}}_\xi = \mathbb{E}_\xi$ (resp. $\mathbb{E}_\xi^{(\nu)}$), and $S = \mathbb{R}$ (resp. $\mathbb{R}_+$).

Let $0 < t \le T < \infty$. For the noncolliding diffusion process $(\hat{\Xi}(t), \mathbb{P}_\xi)$, we consider the expectation for an $\mathcal{F}(t)$-measurable bounded function F,

$$\mathbb{E}_\xi[F(\hat{\Xi}(\cdot))].$$

It is sufficient to consider the case that F is given as $F\left(\hat{\Xi}(\cdot)\right) = \prod_{m=1}^M g_m(\hat{\boldsymbol{X}}(t_m))$ for an arbitrary $M \in \mathbb{N}$, $0 < t_1 < \cdots < t_M \le T < \infty$, with symmetric bounded measurable functions g_m on S^N, $1 \le m \le M$.

We can prove that the noncolliding BM is obtained as an h-transform of the absorbing BM, $\boldsymbol{B}(t) = (B_1(t), \dots, B_N(t)), t \ge 0$ in the Weyl chamber $\mathbb{W}_N^{\mathrm{A}}$ [24]. Similarly, the noncolliding $\mathrm{BESQ}^{(\nu)}$ is realized as an h-transform of the absorbing $\mathrm{BESQ}^{(\nu)}(t)$, $\boldsymbol{R}^{(\nu)}(t) = (R_1^{(\nu)}(t), \dots, R_N^{(\nu)}(t))$ in $\mathbb{W}_N^+$ [46]. For $\hat{\Xi}(\cdot) = \Xi(\cdot)$ (resp. $\hat{\Xi}(\cdot) = \Xi^{(\nu)}(\cdot)$), we set $\boldsymbol{V}(\cdot) = \boldsymbol{B}(\cdot)$ (resp. $\boldsymbol{V}(\cdot) = \boldsymbol{R}^{(\nu)}(\cdot)$), and $\mathbb{W}_N = \mathbb{W}_N^{\mathrm{A}}$ (resp. $\mathbb{W}_N^+$). Put

$$\tau = \inf\{t > 0 : \boldsymbol{V}(t) \notin \mathbb{W}_N\}. \tag{5.17}$$

Then under $\xi = \sum_{j=1}^N \delta_{u_j}$, the equality

$$\mathbb{E}_\xi\left[\prod_{m=1}^M g_m(\hat{\boldsymbol{X}}(t_m))\right] = \mathrm{E}_{\boldsymbol{u}}\left[\mathbf{1}(\tau > t_M) \prod_{m=1}^M g_m(\boldsymbol{V}(t_m)) \frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right] \tag{5.18}$$

is established.

5.5 DMR for noncolliding diffusion processes

Now we can prove the following theorem.

Theorem 5.4 *The noncolliding BM and the noncolliding $\mathrm{BESQ}^{(\nu)}$ with $\nu > -1$ have DMR for any $\xi \in \mathfrak{M}_0$, $\xi(S) < \infty$.*

Proof. We introduce the stopping times

$$\tau_{jk} = \inf\{t > 0 : \hat{V}_j(t) = \hat{V}_k(t)\}, \quad 1 \le j < k \le N. \tag{5.19}$$

Let $\sigma_{jk} \in \mathcal{S}_N$ be the permutation of $(j,k), 1 \le j,k \le N$. Note that in a configuration $\boldsymbol{u}'$ if $u'_j = u'_k, j \neq k$, then $\sigma_{jk}(\boldsymbol{u}') = \boldsymbol{u}'$, and the processes $\boldsymbol{V}(t)$ and $\sigma_{jk}(\boldsymbol{V}(t))$ are identical in distribution under the probability measure $\mathrm{P}_{\boldsymbol{u}'}$. By the strong Markov property of the process $\boldsymbol{V}(t)$ and by the fact that h is anti-symmetric and $g_m, 1 \le m \le M$ are symmetric,

$$\mathrm{E}_{\boldsymbol{u}}\left[\mathbf{1}(\tau = \tau_{jk} < t_M)\prod_{m=1}^{M} g_m(\boldsymbol{V}(t_m))\frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right] = 0.$$

Since $\mathrm{P}_{\boldsymbol{u}}(\tau_{jk} = \tau_{j'k'}) = 0$ if $(j,k) \neq (j',k')$, and

$$\tau = \min_{1 \le j < k \le N} \tau_{jk},$$

$$\mathrm{E}_{\boldsymbol{u}}\left[\mathbf{1}(\tau < t_M)\prod_{m=1}^{M} g_m(\boldsymbol{V}(t_m))\frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right] = 0.$$

Hence, (5.18) equals

$$\mathrm{E}_{\boldsymbol{u}}\left[\prod_{m=1}^{M} g_m(\boldsymbol{V}(t_m))\frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right]. \tag{5.20}$$

By the equality (5.13), the theorem is concluded. ∎

Then by Proposition 1.5, we will immediately conclude the following.

Corollary 5.5 *The noncolliding BM is determinantal for any $\xi \in \mathfrak{M}_0$, $\xi(\mathbb{R}) < \infty$. The correlation kernel is given by*

$$\begin{aligned}\mathbb{K}_\xi(s,x;t,y) &= \int_{\mathbb{R}} \xi(dv) p(s,x|v)\mathcal{M}^v_\xi(t,y) - \mathbf{1}(s>t)p(s-t,x|y), \\ &\qquad (s,x),(t,y) \in [0,\infty)\times\mathbb{R},\end{aligned} \tag{5.21}$$

where

$$\mathcal{M}^v_\xi(t,y) = \int_{\mathbb{R}} dw \, \frac{1}{\sqrt{2\pi t}} e^{-(iy+w)^2/2t}\Phi^v_\xi(iw). \tag{5.22}$$

Corollary 5.6 *The noncolliding BESQ$^{(\nu)}$, $\nu > -1$ is determinantal for any $\xi \in \mathfrak{M}^+_0$, $\xi(\mathbb{R}_+) < \infty$. The correlation kernel is given by*

$$\begin{aligned}\mathbb{K}^{(\nu)}_\xi(s,x;t,y) &= \int_{\mathbb{R}} \xi(dv) p^{(\nu)}(s,x|v)\mathcal{M}^v_\xi(t,y) - \mathbf{1}(s>t)p^{(\nu)}(s-t,x|y), \\ &\qquad (s,x),(t,y) \in [0,\infty)\times\mathbb{R}_+,\end{aligned} \tag{5.23}$$

where

$$\mathcal{M}^v_\xi(t,y) = \int_{\mathbb{R}_+} dw \, \frac{1}{2t}\left(\frac{w}{y}\right)^{\nu/2} e^{(y-w)/2t} J_\nu\left(\frac{\sqrt{wy}}{t}\right)\Phi^v_\xi(-w). \tag{5.24}$$

5.6 CPR for noncolliding BM

Let $W_j(t), t \geq 0, 1 \leq j \leq N$ be a collection of independent BM's on a probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathrm{P}}_{\mathbf{0}})$, where the expectation w.r.t.$\check{\mathrm{P}}_{\mathbf{0}}$ is written as $\check{\mathrm{E}}_{\mathbf{0}}$. Note that they are independent from $B_j(t), t \geq 0, 1 \leq j \leq N$, and $W_j(0) = 0, 1 \leq j \leq N$. From the equality (4.22). we have the following. Introduce a set of independent complex BM's

$$Z_j(t) = B_j(t) + iW_j(t), \quad 1 \leq j \leq N, \quad t \geq 0, \tag{5.25}$$

then

$$\mathsf{M}\left[\prod_{j=1}^{N} f_j(iW_j)\middle| \{(t_j, V_j(t_j))\}_{j=1}^{N}\right] = \check{\mathrm{E}}_{\mathbf{0}}\left[\prod_{j=1}^{N} f_j(Z_j(t_j))\right] \tag{5.26}$$

for polynomials f_j's, and then the determinantal martingale (1.8) is written as

$$\mathcal{D}_{\xi}^{\boldsymbol{u}}(T, \boldsymbol{V}(T)) = \check{\mathrm{E}}_{\mathbf{0}}\left[\det_{1\leq j,k\leq N}\left[\Phi_{\xi}^{u_k}(Z_j(T))\right]\right]. \tag{5.27}$$

Then Theorem 5.4 is transformed into the following.

Let $Z_j(t), 1 \leq j \leq N, t \geq 0$ be a set of independent complex BM's given by (5.25). If they start at $Z_j(0) = u_j \in \mathbb{R}, 1 \leq j \leq N$, the probability space is denoted by $(\Omega, \mathcal{F}, \mathbf{P}_{\boldsymbol{u}})$ with $\boldsymbol{u} = (u_1, \dots, u_N)$. The space $(\Omega, \mathcal{F}, \mathbf{P}_{\boldsymbol{u}})$ is a product of two probability spaces $(\Omega, \mathcal{F}, \mathrm{P}_{\boldsymbol{u}})$ for $B_j(\cdot) = \Re Z_j(\cdot)$ and $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathrm{P}}_{\mathbf{0}})$ for $W_j(\cdot) = \Im Z_j(\cdot), 1 \leq j \leq N$. The expectation w.r.t.$\mathbf{P}_{\boldsymbol{u}}$ is denoted by $\mathbf{E}_{\boldsymbol{u}}$.

Corollary 5.7 *The noncolliding BM has CPR (1.25) with the complex BMs (5.25) and*

$$\varphi_{\xi}^{u}(\cdot) = \Phi_{\xi}^{u}(\cdot), \quad u \in \mathbb{C}, \quad \xi \in \mathfrak{M}_0. \tag{5.28}$$

This result was given as Theorem 1.1 in [43], where the present CPR is called the *complex BM representation.*

5.7 CPR for noncolliding BES$^{(\nu)}$

We introduce a set of independent complex diffusions

$$Z_j^{(\nu)}(t) = \widetilde{R}_j^{(\nu)}(t) + iW_j(t), \quad 1 \leq j \leq N, \quad t \geq 0, \tag{5.29}$$

where $W_j(t), 1 \leq j \leq N$ are independent BM's on the probability space $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathrm{P}}_{\mathbf{0}})$. For BES and BESQ with odd dimensions, $D = 2n + 3, n \in \mathbb{N}_0$, the indices are half-odds $\nu = D/2 - 1 = n + 1/2, n \in \mathbb{N}_0$. Let $\widetilde{f}(z)$ be a polynomial of $z^2, z \in \mathbb{C}$. Then Lemma 4.3 gives the following. For $n \in \mathbb{N}_0$,

$$\begin{aligned}\widetilde{\mathsf{M}}^{(n+1/2)}&\left[\prod_{j=1}^{N} \widetilde{f}_j(iW_j)\middle| \left\{(t_j, \widetilde{R}_j^{(n+1/2)}(t_j))\right\}_{j=1}^{N}\right] \\ &= \check{\mathrm{E}}_{\mathbf{0}}\left[\prod_{j=1}^{N} Q_{t_j}^{(n+1/2)}(Z_j^{(n+1/2)}(t_j))\widetilde{f}_j(Z_j^{(n+1/2)}(t_j))\right],\end{aligned} \tag{5.30}$$

where $Q_t^{(n+1/2)}$ is given by (4.34). Then we have

$$\mathcal{D}_\xi^{(n+1/2),\boldsymbol{u}}(T, \boldsymbol{V}^{(n+1/2)}(T)) = \check{\mathrm{E}}\left[\det_{1\leq j,k\leq N}\left[Q_T^{(n+1/2)}(Z_j^{(n+1/2)}(T))\widetilde{\Phi}_\xi^{u_k}(Z_j^{(n+1/2)}(T))\right]\right], \quad (5.31)$$

where

$$\widetilde{\Phi}_\xi^v(z) = \prod_{r\in\mathrm{supp}\,\xi\cap\{v,-v\}^c} \frac{z^2-r^2}{v^2-r^2}, \quad z, v \in \mathbb{C}. \quad (5.32)$$

Then Theorem 5.4 is transformed into the following.

Corollary 5.8 *The noncolliding BES$^{(n+1/2)}, n \in \mathbb{N}_0$ has CPR (1.25) with the complex BMs (5.25) and*

$$\varphi_\xi^u(\cdot) = Q_t^{(n+1/2)}(\cdot)\widetilde{\Phi}_\xi^u(\cdot), \quad u \in \mathbb{C}, \quad \xi \in \mathfrak{M}_{+,0}. \quad (5.33)$$

5.8 Martingales for configurations with multiple points

We generalize the function (5.2) as following. Depending on the transition probability density of a process, $p(s,x|v)$, $0 < s < \infty, x, v \in S$ we put

$$\phi_\xi^u((s,x);z,\zeta) = \frac{p(s,x|\zeta)}{p(s,x|u)}\frac{1}{z-\zeta}\prod_{r\in\mathrm{supp}\,\xi}\left(\frac{z-r}{\zeta-r}\right)^{\xi(\{r\})}, \quad z, \zeta \in \mathbb{C}. \quad (5.34)$$

Let $C(\delta_u)$ be a closed contour on the complex plane $\mathbb{C}$ encircling a point u on S once in the positive direction and set

$$\begin{aligned}\Phi_\xi^u((s,x);z) &= \frac{1}{2\pi i}\oint_{C(\delta_u)} d\zeta\, \phi_\xi^u((s,x);z,\zeta) \\ &= \mathrm{Res}\left[\phi_\xi^u((s,x);z,\zeta);\zeta=u\right]. \end{aligned} \quad (5.35)$$

This function is defined for any finite configuration ξ, in which there can be multiple points in general. (If there is no multiple point, (5.35) is reduced to (5.2).) Since (5.35) is a polynomial with respect to z, we can extend $\mathcal{M}_\xi^u(t,y)$ to

$$\mathcal{M}_\xi^u((s,x)|(t,y)) = \mathsf{M}\left[\Phi_\xi^u((s,x);cW)\Big|(t,y)\right], \quad (s,x),(t,y) \in [0,\infty)\times S. \quad (5.36)$$

Let

$$\xi_{\mathrm{s}}(\cdot) = \sum_{u\in\mathrm{supp}\,\xi}\delta_u(\cdot). \quad (5.37)$$

Then the correlation kernel (1.14) is generalized to

$$\begin{aligned}\mathbb{K}_\xi(s,x;t,y) &= \int_S \xi_{\mathrm{s}}(dv)p(s,x|v)\mathcal{M}_\xi^v((s,x)|(t,y)) - \mathbf{1}(s>t)p(s-t,x|y), \\ &\qquad (s,x),(t,y)\in[0,\infty)\times S. \end{aligned} \quad (5.38)$$

By definition of the present martingales $\mathcal{M}_{\xi}^{\cdot}((\cdot,\cdot)|(\cdot,\cdot))$, it is written by double integral as

$$\mathbb{K}_{\xi}(s,x;t,y) = \frac{1}{2\pi i}\oint_{C(\xi)} d\zeta\, p(s,x|\zeta)\int_S dw\, p(t,w|c^{-1}y)\frac{1}{cw-\zeta}\prod_{r\in\operatorname{supp}\xi}\left(\frac{cw-r}{\zeta-r}\right)^{\xi(\{r\})}$$
$$-\mathbf{1}(s>t)p(s-t,x|y), \qquad (s,x),(t,y)\in[0,\infty)\times S, \tag{5.39}$$

where $C(\xi)$ denotes a counterclockwise contour on $\mathbb{C}$ encircling the points in $\operatorname{supp}\xi$ on S but not point $cw, w \in S$; $C(\xi) = \sum_{v\in\operatorname{supp}\xi} C(\delta_v)$.

Corollary 5.9 *The noncolliding BM and the noncolliding BESQ$^{(\nu)}$ are determinantal for any initial configuration with fine number of particles; $\xi \in \mathfrak{M}$, $\xi(\mathbb{R}) < \infty$, or $\xi \in \mathfrak{M}^+$, $\xi(\mathbb{R}_+) < \infty$ with correlation kernels (5.39).*

In the papers [40, 41], we proved this statement by deriving the double integral representations (5.39) for the spatio-temporal correlation kernels. There we used the multiple orthogonal polynomials [39, 40, 41] in order to obtain the expression (5.39). As shown in this lectures, however, it is not necessary to use multiple orthogonal polynomials to obtain the results.

As an example, we consider the extreme case such that all N points are concentrated on an origin,

$$\xi = N\delta_0 \iff \xi_{\rm s} = \delta_0 \quad \text{with} \quad \xi(\{0\}) = N. \tag{5.40}$$

In this case (5.34) and (5.35) become

$$\begin{aligned}\phi^0_{N\delta_0}((s,x);z,\zeta) &= \frac{p(s,x|\zeta)}{p(s,x|0)}\frac{1}{z-\zeta}\left(\frac{z}{\zeta}\right)^N \\ &= \frac{p(s,x|\zeta)}{p(s,x|0)}\sum_{\ell=0}^{\infty}\frac{z^{N-\ell-1}}{\zeta^{N-\ell}},\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}\Phi^0_{N\delta_0}((s,x);z) &= \frac{1}{p(s,x|0)}\sum_{\ell=0}^{\infty} z^{N-\ell-1}\frac{1}{2\pi i}\oint_{C(\delta_0)} d\zeta\,\frac{p(s,x|\zeta)}{\zeta^{N-\ell}} \\ &= \frac{1}{p(s,x|0)}\sum_{\ell=0}^{N-1} z^{N-\ell-1}\frac{1}{2\pi i}\oint_{C(\delta_0)} d\zeta\,\frac{p(s,x|\zeta)}{\zeta^{N-\ell}},\end{aligned} \tag{5.42}$$

since the integrands are holomorphic when $\ell \geq N$, where we have assumed $\nu > -1$ for BESW$^{(\nu)}$.

For BM with the transition probability density (3.3), (5.42) gives

$$\begin{aligned}
\Phi^0_{N\delta_0}((s,x);z) &= \sum_{\ell=0}^{N-1} z^{N-\ell-1}\frac{1}{2\pi i}\oint_{C(\delta_0)} d\zeta\, \frac{e^{x\zeta/s-\zeta^2/2s}}{\zeta^{N-\ell}} \\
&= \sum_{\ell=0}^{N-1}\left(\frac{z}{\sqrt{2s}}\right)^{N-\ell-1}\frac{1}{2\pi i}\oint_{C(\delta_0)} d\eta\, \frac{e^{2(x/\sqrt{2s})\eta-\eta^2}}{\eta^{N-\ell}} \\
&= \sum_{\ell=0}^{N-1}\left(\frac{z}{\sqrt{2s}}\right)^{N-\ell-1}\frac{1}{(N-\ell-1)!}H_{N-\ell-1}\left(\frac{x}{\sqrt{2s}}\right), \qquad (5.43)
\end{aligned}$$

where we have used the contour integral representation of the Hermite polynomials [68]

$$H_n(x) = \frac{n!}{2\pi i}\oint_{C(\delta_0)} d\eta \frac{e^{2x\eta-\eta^2}}{\eta^{n+1}}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}. \qquad (5.44)$$

Thus its integral transform is calculated as

$$\begin{aligned}
&\mathsf{M}\left[\Phi^0_{N\delta_0}((s,x);iW)\middle| (t,y)\right] \\
&= \sum_{\ell=0}^{N-1}\frac{1}{(N-\ell-1)!}H_{N-\ell-1}\left(\frac{x}{\sqrt{2s}}\right)\frac{1}{(2s)^{(N-\ell-1)/2}}\mathsf{M}[(iW)^{N-\ell-1}|(t,y)] \\
&= \sum_{\ell=0}^{N-1}\frac{1}{(N-\ell-1)!}H_{N-\ell-1}\left(\frac{x}{\sqrt{2s}}\right)\frac{1}{(2s)^{(N-\ell-1)/2}}m_{N-\ell-1}(t,y) \\
&= \sum_{\ell=0}^{N-1}\frac{1}{(N-\ell-1)!2^{N-\ell-1}}\left(\frac{t}{s}\right)^{(N-\ell-1)/2} H_{N-\ell-1}\left(\frac{x}{\sqrt{2s}}\right)H_{N-\ell-1}\left(\frac{y}{\sqrt{2t}}\right),
\end{aligned}$$

where we have used Lemma 3.1. Then we obtain the following,

$$\begin{aligned}
\mathcal{M}^0_{N\delta_0}((s,x)|(t,B(t))) &= \sum_{n=0}^{N-1}\frac{1}{n!2^n}m_n(s,x)m_n(t,B(t)) \\
&= \sum_{n=0}^{N-1}\frac{1}{n!2^n}\left(\frac{t}{s}\right)^{n/2} H_n\left(\frac{x}{\sqrt{2s}}\right)H_n\left(\frac{B(t)}{\sqrt{2t}}\right) \\
&= \sqrt{\pi}e^{x^2/4s+B(t)^2/4t}\sum_{n=0}^{N-1}\left(\frac{t}{s}\right)^{n/2}\varphi_n\left(\frac{x}{\sqrt{2s}}\right)\varphi_n\left(\frac{B(t)}{\sqrt{2t}}\right), \qquad (5.45)
\end{aligned}$$

where

$$\varphi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}H_n(x)e^{-x^2/2}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}.$$

Similarly, for $\mathrm{BESQ}^{(\nu)}, \nu > -1$ with the transition probability density (3.12), we obtain

$$\begin{aligned}
\Phi^{(\nu),0}_{N\delta_0}((s,x);z) &= \frac{(2s)^\nu \Gamma(\nu+1)}{x^{\nu/2}} \sum_{\ell=0}^{N-1} z^{N-\ell-1} \frac{1}{2\pi i} \oint_{C(\delta_0)} d\zeta \, \frac{e^{-\zeta/2s}}{\zeta^{N-\ell+\nu/2}} I_\nu \left(\frac{\sqrt{x\zeta}}{s} \right) \\
&= \Gamma(\nu+1) \sum_{\ell=0}^{N-1} \left(-\frac{z}{2s} \right)^{N-\ell-1} \frac{1}{\Gamma(N-\ell+\nu)} L^{(\nu)}_{N-\ell-1} \left(\frac{x}{2s} \right), \qquad (5.46)
\end{aligned}$$

where we used the contour integral representation of the Laguerre polynomials

$$L^{(\nu)}_n(x) = \frac{\Gamma(n+\nu+1)}{x^{\nu/2}} \frac{1}{2\pi i} \oint_{C(\delta_0)} d\eta \, \frac{e^\eta}{\eta^{n+1+\nu/2}} J_\nu(2\sqrt{\eta x}) \qquad (5.47)$$

with the relation $I_\nu(iz) = (-1)^{\nu/2} J_\nu(z), -\pi < \arg(z) \le \pi/2$. By using Lemma 3.2, we have

$$\begin{aligned}
&\mathcal{M}^{(\nu),0}_{N\delta_0}((s,x)|(t, R^{(\nu)}(t))) \\
&= \mathsf{M}^{(\nu)} \left[\Phi^{(\nu),0}_{N\delta_0}((s,x);-W) \Big| (t, R^{(\nu)}(t)) \right] \\
&= \Gamma(\nu+1) \sum_{n=0}^{N-1} \frac{1}{\Gamma(n+1)\Gamma(n+\nu+1)(2s)^{2n}} m^{(\nu)}_n(s,x) m^{(\nu)}_n \left(t, R^{(\nu)}(t)\right) \\
&= \Gamma(\nu+1) \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \left(\frac{t}{s} \right)^n L^{(\nu)}_n \left(\frac{x}{2s} \right) L^{(\nu)}_n \left(\frac{R^{(\nu)}(t)}{2t} \right) \\
&= \Gamma(\nu+1) \left(\frac{x}{2s} \right)^{-\nu/2} \left(\frac{R^{(\nu)}(s)}{2s} \right)^{-\nu/2} e^{x/4s + R^{(\nu)}(t)/4t} \\
&\qquad \times \sum_{n=0}^{N-1} \frac{\Gamma(n+1)}{\Gamma(n+\nu+1)} \left(\frac{t}{s} \right)^n \varphi^{(\nu)}_n \left(\frac{x}{2s} \right) \varphi^{(\nu)}_n \left(\frac{R^{(\nu)}(t)}{2t} \right), \qquad (5.48)
\end{aligned}$$

where

$$\varphi^{(\nu)}_n(x) = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}} x^{\nu/2} L^{(\nu)}_n(x) e^{-x/2}, \quad n \in \mathbb{N}_0, \quad x \in \mathbb{R}_+.$$

The processes (5.45) and (5.48) are local martingales and

$$\mathrm{E}_0 \left[\mathcal{M}^0_{N\delta_0}((s,x)|(t, \boldsymbol{V}(t))) \right] = \mathrm{E}_0 \left[\mathcal{M}^0_{N\delta_0}((s,x)|(0, \boldsymbol{V}(0))) \right] = 1 \qquad (5.49)$$

for $0 < t \le T < \infty, (s,x) \in [0,T] \times S$.

By the formula (1.11), we obtain the correlation kernels as

$$\begin{aligned}
\mathbb{K}_{N\delta_0}(s,x;t,y) &= p(s,x|0) \mathcal{M}^0_{N\delta_0}((s,x)|(t,y)) \\
&= \frac{e^{-x^2/4s}}{e^{-y^2/4t}} \mathbf{K}_H(s,x;t,y) \qquad (5.50)
\end{aligned}$$

with

$$\mathbf{K}_H(s,x;t,y) = \frac{1}{\sqrt{2s}} \sum_{n=0}^{N-1} \left(\frac{t}{s}\right)^{n/2} \varphi\left(\frac{x}{\sqrt{2s}}\right) \varphi\left(\frac{y}{\sqrt{2t}}\right) - \mathbf{1}(s>t) p(s-t,x|y), \qquad (5.51)$$

and

$$\begin{aligned} \mathbb{K}^{(\nu)}_{N\delta_0}(s,x;t,y) &= p^{(\nu)}(s,x|0)\mathcal{M}^{(\nu),0}_{N\delta_0}((s,x)|(t,y)) \\ &= \frac{(x/2s)^{\nu/2}e^{-x/4s}}{(y/2t)^{\nu/2}e^{-y/4t}} \mathbf{K}_{L^{(\nu)}}(s,x;t,y) \end{aligned} \qquad (5.52)$$

with

$$\mathbf{K}_{L^{(\nu)}}(s,x;t,y) = \frac{1}{2s} \sum_{n=0}^{N-1} \left(\frac{t}{s}\right)^{n} \varphi^{(\nu)}\left(\frac{x}{\sqrt{2s}}\right) \varphi^{(\nu)}\left(\frac{y}{\sqrt{2t}}\right) - \mathbf{1}(s>t) p^{(\nu)}(s-t,x|y), \quad (5.53)$$

where $\mathbf{K}_H(\cdot;\cdot)$ and $\mathbf{K}_{L^{(\nu)}}(\cdot;\cdot)$ are known as the *extended Hermite and Laguerre kernels*, respectively (see, for instance, [18]). Here we would like to emphasize the fact that these kernels have been derived here by not following any 'orthogonal-polynomial arguments' but by only using proper martingales associated with the chosen initial configuration (5.40). In this special case, they are expressed by the Hermite and Laguerre polynomials in the forms (5.45) and (5.48), respectively. In the present new approach, the martingale properties (5.49) and the reducibility (1.10) coming from the independence of diffusion processes play essential roles instead of orthogonality in the theory of orthogonal ensembles in random matrix theory [53, 18].

5.9 Martingales associated with infinite particle systems

In [40] we gave useful sufficient conditions of ξ so that the noncolliding BM, $(\Xi(t), \mathbb{P}_\xi)$ is well defined as a determinantal process even if $\xi(\mathbb{R}) = \infty$. For $L > 0, \alpha > 0$ and $\xi \in \mathfrak{M}$ we put

$$M(\xi, L) = \int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{x}, \qquad M_\alpha(\xi, L) = \left(\int_{[-L,L]\setminus\{0\}} \frac{\xi(dx)}{|x|^\alpha} \right)^{1/\alpha}, \qquad (5.54)$$

and

$$M(\xi) = \lim_{L\to\infty} M(\xi, L), \quad M_\alpha(\xi) = \lim_{L\to\infty} M_\alpha(\xi, L), \qquad (5.55)$$

if the limits finitely exist. Then

(**C.1**) there exists $C_0 > 0$ such that $|M(\xi, L)| < C_0$, $L > 0$,

(**C.2**) (i) there exist $\alpha \in (1,2)$ and $C_1 > 0$ such that $M_\alpha(\xi) \leq C_1$,
(ii) there exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a^2}\xi^{\langle 2\rangle}) \leq C_2(\max\{|a|, 1\})^{-\beta} \quad \forall a \in \operatorname{supp}\xi.$$

It was shown that, if $\xi \in \mathfrak{M}_0$ satisfies the conditions (**C.1**) and (**C.2**), then for $a \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$\Phi_\xi^a(z) \equiv \lim_{L\to\infty} \Phi_{\xi\cap[a-L,a+L]}^a(z) \quad \text{finitely exists}, \tag{5.56}$$

and

$$|\Phi_\xi^a(z)| \le C \exp\left\{c(|a|^\theta + |z|^\theta)\right\} \left|\frac{z}{a}\right|^{\xi(\{0\})} \left|\frac{a}{a-z}\right|, \quad a \in \operatorname{supp}\xi,\ z \in \mathbb{C}, \tag{5.57}$$

for some $c, C > 0$ and $\theta \in (\max\{\alpha, (2-\beta)\}, 2)$, which are determined by the constants C_0, C_1, C_2 and the indices α, β in the conditions (Lemma 4.4 in [40]). We have noted that in the case that $\xi \in \mathfrak{M}_0$ satisfies the conditions (**C.1**) and (**C.2**) with constants C_0, C_1, C_2 and indices α and β, then $\xi \cap [-L, L], \forall L > 0$ does as well. Hence we can obtain the convergence of moment generating functions

$$\Psi_{\xi\cap[-L,L]}^{\boldsymbol{t}}[\boldsymbol{f}] \to \Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] \quad \text{as} \quad L \to \infty, \tag{5.58}$$

which implies the convergence of the probability measures

$$\mathbb{P}_{\xi\cap[-L,L]} \to \mathbb{P}_\xi \quad \text{in } L \to \infty \tag{5.59}$$

in the sense of finite dimensional distributions. Moreover, even if $\xi(\mathbb{R}) = \infty$, $\mathbb{K}_\xi$ is well-defined as a correlation kernel and dynamics of the noncolliding BM with an infinite number of particles $(\Xi(t), \mathbb{P}_\xi)$ exists as a determinantal process [40].

Similarly, in [41], the following sufficient conditions for initial configurations $\xi \in \mathfrak{M}^+$ were given so that the noncolliding BESQ, $(\Xi^{(\nu)}(t), \mathbb{P}_\xi^{(\nu)}), \nu > -1$ is well-defined as determinantal processes even if $N = \xi(\mathbb{R}_+) = \infty$.

(C.A) (i) There exists $\alpha \in (1/2, 1)$ and $C_1 > 0$ such that $M_\alpha(\xi) \le C_1$.
(ii) There exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a}\xi) \le C_2(|a| \vee 1)^{-\beta}, \quad \forall a \in \operatorname{supp}\xi.$$

The families of ξ satisfying the conditions are denoted by $\hat{\mathfrak{X}} = \mathfrak{X}$ for the noncolliding BM and $\hat{\mathfrak{X}} = \mathfrak{X}^+$ for the noncolliding BESQ$^{(\nu)}$, respectively.

Proposition 5.10 *Suppose that $0 < t \le T < \infty$. Then the noncolliding BM, $(\Xi(t), \mathbb{P}_\xi)$, started at $\xi \in \mathfrak{X}_0$ and the noncolliding BESQ$^{(\nu)}$, $(\Xi^{(\nu)}(t), \mathbb{P}_\xi^{(\nu)}), \nu > -1$, started at $\xi \in \mathfrak{X}_0^+$ have DMR for any $\mathcal{F}(t)$-measurable polynomial function also in the case with $N = \xi(S) = \infty$.*

For $(\Xi(t), \mathbb{P}_\xi)$, the similar statement was proved for the complex BM representation in [43] (Corollary 1.3). Here by the reducibility of the determinantal martingale given by Lemma 1.1, this proposition is readily concluded.

There are two interesting examples of local martingales for infinite particle systems. First we consider the configuration

$$\xi_{\mathbb{Z}}(\cdot) = \sum_{j\in\mathbb{Z}} \delta_j(\cdot), \tag{5.60}$$

that is, the configuration in which every integer point $\mathbb{Z}$ is occupied by one particle. It is easy to confirm that $\xi_{\mathbb{Z}} \in \mathfrak{X}_0$ and the noncolliding BM started at $\xi_{\mathbb{Z}}$, $(\Xi(t), \mathbb{P}_{\xi_{\mathbb{Z}}})$, is a determinantal process with an infinite number of particles [40]. Since $\prod_{n\in\mathbb{N}}(1-x^2/n^2) = \sin(\pi x)/(\pi x)$,

$$\begin{aligned} \Phi^u_{\xi_{\mathbb{Z}}}(z) &= \prod_{r\in\mathbb{Z}, r\neq u} \frac{z-r}{u-r} \\ &= \frac{\sin\{\pi(z-u)\}}{\pi(z-u)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\lambda\, e^{i\lambda(z-u)}, \quad z, u \in \mathbb{C}. \end{aligned} \tag{5.61}$$

Its integral transform is calculated as

$$\begin{aligned} \mathsf{M}\left[\Phi^u_{\xi_{\mathbb{Z}}}(iw)\middle| (t,x)\right] &= \int_{\mathbb{R}} dw\, q(t,w|x)\Phi^u_{\xi_{\mathbb{Z}}}(iw) \\ &= \int_{-\infty}^{\infty} dw\, \frac{1}{\sqrt{2\pi t}} e^{-(ix+w)^2/2t}\Phi^u_{\xi_{\mathbb{Z}}}(iw) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\lambda\, e^{t\lambda^2/2+i\lambda(x-u)}. \end{aligned} \tag{5.62}$$

Then we have local martingales

$$\mathcal{M}^k_{\xi_{\mathbb{Z}}}(t, B_j(t)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\lambda\, \exp\left\{\frac{\lambda^2}{2}t + i\lambda(B_j(t)-k)\right\}, \quad j,k\in\mathbb{Z}, \quad 0 < t \le T < \infty. \tag{5.63}$$

We see that

$$\begin{aligned} \mathrm{E}_{\xi_{\mathbb{Z}}}\left[\mathcal{M}^k_{\xi_{\mathbb{Z}}}(t, B_j(t))\right] &= \mathrm{E}_{\xi_{\mathbb{Z}}}\left[\mathcal{M}^k_{\xi_{\mathbb{Z}}}(0, B_j(0))\right] \\ &= \delta_{jk}, \qquad 0 < t \le T < \infty. \end{aligned} \tag{5.64}$$

If $\nu > -1$, the Bessel function $J_\nu(z)$ given by (4.16) has an infinite number of pairs of positive and negative zeros with the same absolute value, which are all simple. We write the positive zeros of $J_\nu(z)$ arranged in ascending order of the absolute values as

$$0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \cdots. \tag{5.65}$$

Then, $J_\nu(z)$ has the following infinite product expression [73],

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{j=1}^{\infty}\left(1 - \frac{z^2}{j_{\nu,j}^2}\right). \tag{5.66}$$

For the noncolliding BESQ$^{(\nu)}$, we consider the initial configuration in which every point of the squares of positive zeros of $J_\nu(z)$ is occupied by one particle, which is denoted as

$$\xi^{\langle 2\rangle}_{J_\nu}(\cdot) = \sum_{j=1}^{\infty} \delta_{j^2_{\nu,j}}(\cdot). \tag{5.67}$$

We can see that $\xi^{\langle 2 \rangle}_{J_\nu} \in \mathfrak{X}_0^+$ and thus $(\Xi^{(\nu)}(t), \mathbb{P}^{(\nu)}_{\xi^{\langle 2 \rangle}_{J_\nu}})$ is a determinantal process with an infinite number of particles [41]. For $k \in \mathbb{N}$ we find that

$$\Phi^{(j_{\nu,k})^2}_{\xi^{\langle 2 \rangle}_{J_\nu}}(z) = \left(\frac{(j_{\nu,k})^2}{z} \right)^{\nu/2} \frac{1}{(J_{\nu+1}(j_{\nu,k}))^2} \int_0^1 d\lambda \, J_\nu(\sqrt{\lambda z}) J_\nu(\sqrt{\lambda} j_{\nu,k}), \tag{5.68}$$

and their integral transforms gives the martingales,

$$\begin{aligned}
\mathcal{M}^{(j_{\nu,k})^2}_{\xi^{\langle 2 \rangle}_{J_\nu}}(t, R^{(\nu)}_j(t)) &= \mathsf{M}^{(\nu)} \left[\Phi^{(j_{\nu,k})^2}_{\xi^{\langle 2 \rangle}_{J_\nu}}(-W) \middle| (t, R^{(\nu)}_j) \right] \\
&= \left(\frac{(j_{\nu,k})^2}{R^{(\nu)}_j(t)} \right)^{\nu/2} \frac{1}{(J_{\nu+1}(j_{\nu,k}))^2} \int_0^1 d\lambda \, e^{\lambda t/2} J_\nu \left(\sqrt{\lambda R^{(\nu)}_j(t)} \right) J_\nu(\sqrt{\lambda} j_{\nu,k}), \\
&\qquad j, k \in \mathbb{N}, \quad 0 < t \leq T \leq \infty.
\end{aligned} \tag{5.69}$$

We see that for $0 < t \leq T < \infty$,

$$\begin{aligned}
\mathrm{E}^{(\nu)}_{\xi^{\langle 2 \rangle}_{J_\nu}} \left[\mathcal{M}^{(j_{\nu,k})^2}_{\xi^{\langle 2 \rangle}_{J_\nu}}(t, R^{(\nu)}_j(t)) \right] &= \mathrm{E}^{(\nu)}_{\xi^{\langle 2 \rangle}_{J_\nu}} \left[\mathcal{M}^{(j_{\nu,k})^2}_{\xi^{\langle 2 \rangle}_{J_\nu}}(0, R^{(\nu)}_j(0)) \right] \\
&= \delta_{jk}.
\end{aligned} \tag{5.70}$$

By the formula (1.14), these martingales determine the correlation kernels, which are denoted as $\mathbb{K}_{\xi_\mathbb{Z}}$ and $\mathbb{K}^{(\nu)}_{\xi^{\langle 2 \rangle}_{J_\nu}}$. In the previous papers [40, 41], we showed

$$\begin{aligned}
&\lim_{\tau \to \infty} \mathbb{K}_{\xi_\mathbb{Z}}(s + \tau, x; t + \tau, y) = \mathbf{K}_{\sin}(t - s, y - x), \\
&\lim_{\tau \to \infty} \mathbb{K}^{(\nu)}_{\xi^{\langle 2 \rangle}_{J_\nu}}(s + \tau, x; t + \tau, y) = \left(\frac{x}{y} \right)^{\nu/2} \mathbf{K}_{\mathrm{J}_\nu}(t - s, y|x),
\end{aligned} \tag{5.71}$$

and proved that the noncolliding BM started at (5.60) and the noncolliding BESQ$^{(\nu)}$ started at (5.67) converge in the long-term limit to the equilibrium determinantal processes governed by the extended sine kernel

$$\mathbf{K}_{\sin}(t, x) = \begin{cases} \displaystyle\int_0^1 d\lambda \, e^{\pi^2 \lambda^2 t/2} \cos(\pi \lambda x), & \text{if } t > 0 \\[2ex] \displaystyle\frac{\sin(\pi x)}{\pi x} & \text{if } t = 0 \\[2ex] \displaystyle -\int_1^\infty d\lambda \, e^{\pi^2 \lambda^2 t/2} \cos(\pi \lambda x), & \text{if } t < 0, \end{cases} \tag{5.72}$$

and the extended Bessel kernel [18]

$$\mathbf{K}_{J_\nu}(t,y|x) = \begin{cases} \dfrac{1}{4}\displaystyle\int_0^1 d\lambda\, e^{\lambda t/2} J_\nu(\sqrt{\lambda x}) J_\nu(\sqrt{\lambda y}), & \text{if} \quad t>0 \\[2ex] \dfrac{J_\nu(\sqrt{x})\sqrt{y}J'_\nu(\sqrt{y}) - \sqrt{x}J'_\nu(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)}, & \text{if} \quad t=0 \\[2ex] -\dfrac{1}{4}\displaystyle\int_1^\infty d\lambda\, e^{\lambda t/2} J_\nu(\sqrt{\lambda x}) J_\nu(\sqrt{\lambda y}), & \text{if} \quad t<0, \end{cases} \tag{5.73}$$

respectively. These *relaxation phenomena* of infinite particle systems are caused by the following properties of the present martingales,

$$\begin{aligned} &\lim_{\tau\to\infty} \sum_{k\in\mathbb{Z}} p(\tau,x|k) \mathcal{M}^k_{\xi_\mathbb{Z}}(t+\tau, B(t)) = \mathcal{M}^x_{\xi_\mathbb{Z}}(t, B(t)), \quad x\in\mathbb{R}, \\ &\lim_{\tau\to\infty} \sum_{k\in\mathbb{N}} \frac{4p^{(\nu)}(\tau,x|(j_{\nu,k})^2)}{(J_{\nu+1}(x))^2} \mathcal{M}^{(j_{\nu,k})^2}_{\xi^{\langle 2\rangle}_{J_\nu}}(t+\tau, R^{(\nu)}(t)) = \mathcal{M}^x_{\xi^{\langle 2\rangle}_{J_\nu}}(t, R^{(\nu)}(t)) \\ &= \frac{x^{\nu/2}}{(J_{\nu+1}(x))^2} \frac{1}{(R^{(\nu)}(t))^{\nu/2}} \int_0^1 d\lambda\, e^{\lambda t/2} J_\nu(\sqrt{\lambda x}) J_\nu\left(\sqrt{\lambda R^{(\nu)}(t)}\right), \; x\in\mathbb{R}_+, \end{aligned} \tag{5.74}$$

for $0 < t \leq T < \infty$.

6 Noncolliding Random Walk

6.1 Construction

Consider a random walk (RW), $\boldsymbol{V}(t) = (V_1(t), \dots, V_N(t)), t \in \mathbb{N}_0$ on $\mathbb{Z}^N$, such that the components $V_j(t), j = 1, 2, \dots, N$ are independent simple and symmetric RWs;

$$\begin{aligned} V_j(0) &= u_j \in \mathbb{Z}, \\ V_j(t) &= u_j + \zeta_j(1) + \zeta_j(2) + \cdots + \zeta_j(t), \qquad t \in \mathbb{N}, \quad 1 \le j \le N, \end{aligned} \tag{6.1}$$

where $\{\zeta_j(t) : 1 \le j \le N, t \in \mathbb{N}\}$ is a family of i.i.d. random variables binomially distributed as

$$\mathrm{P}[\zeta_j(1) = 1] = \frac{1}{2}, \quad \mathrm{P}[\zeta_j(1) = -1] = \frac{1}{2}, \quad 1 \le j \le N. \tag{6.2}$$

For each component, $V_j(\cdot), 1 \le j \le N$, the transition probability is given by

$$\begin{aligned} p(t-s, y|x) &= \mathrm{P}[V_j(t) = y | V_j(s) = x] \\ &= \begin{cases} \dfrac{1}{2^{t-s}} \dbinom{t-s}{[(t-s)+(y-x)]/2}, \\ \qquad \text{if } t \ge s,\ -(t-s) \le y-x \le t-s,\ [(t-s)+(y-x)]/2 \in \mathbb{Z}, \\ 0, \\ \qquad \text{otherwise}. \end{cases} \end{aligned} \tag{6.3}$$

Put $\mathbb{Z}^N = \mathbb{Z}^N_{\mathrm{e}} \sqcup \mathbb{Z}^N_{\mathrm{o}}$ with

$$\begin{aligned} \mathbb{Z}^N_{\mathrm{e}} &= \{\boldsymbol{x} = (x_1, \dots, x_N) : x_j \in 2\mathbb{Z}, 1 \le j \le N\}, \\ \mathbb{Z}^N_{\mathrm{o}} &= \{\boldsymbol{x} = (x_1, \dots, x_N) : x_j \in 1 + 2\mathbb{Z}, 1 \le j \le N\}. \end{aligned}$$

We always take the initial point $\boldsymbol{u} = (u_1, \dots, u_N) = \boldsymbol{V}(0)$ from $\mathbb{Z}^N_{\mathrm{e}}$, then $\boldsymbol{V}(t) \in \mathbb{Z}^N_{\mathrm{e}}$, if t is even, and $\boldsymbol{V}(t) \in \mathbb{Z}^N_{\mathrm{o}}$, if t is odd. The probability space is denoted as $(\Omega, \mathcal{F}, \mathrm{P}_{\boldsymbol{u}})$. The expectation is written as $\mathrm{E}_{\boldsymbol{u}}$, which is given by the summation over all walks $\{\boldsymbol{V}(t) : t \in \mathbb{N}_0\}$ started at $\boldsymbol{u}$ with the transition probability (6.3) for each component.

Let

$$\mathbb{W}_N = \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < \cdots < x_N\}$$

be the Weyl chamber of type A_{N-1}. Define $\tau_{\boldsymbol{u}}$ be the exit time from the Weyl chamber of the RW started at $\boldsymbol{u} \in \mathbb{Z}^N_{\mathrm{e}} \cap \mathbb{W}_N$,

$$\tau_{\boldsymbol{u}} = \inf\{t \ge 1 : \boldsymbol{V}(t) \notin \mathbb{W}_N\}. \tag{6.4}$$

In the present paper, we study the RW *conditioned to stay in* $\mathbb{W}_N$ *forever*, that is, $\tau_{\boldsymbol{u}} = \infty$ is conditioned. We call such a conditional RW the *(simple and symmetric) noncolliding RW*, since when we regard the j-th component $V_j(\cdot)$ as the position of j-th particle on $\mathbb{Z}, 1 \le j \le N$, if $\tau_{\boldsymbol{u}} < \infty$, then at $t = \tau_{\boldsymbol{u}}$ there is at least one pair of particles $(j, j+1)$, which collide with each other; $V_j(\tau_{\boldsymbol{u}}) = V_{j+1}(\tau_{\boldsymbol{u}}), 1 \le j \le N-1$. Such a conditional

RW is also called *vicious walkers* in statistical physics [17, 11], *non-intersecting paths*, *non-intersecting walks*, and *ordered random walks* (see [14]).

Let $\mathfrak{M}$ be the space of nonnegative integer-valued Radon measure on $\mathbb{Z}$. We consider the noncolliding RW as a process in $\mathfrak{M}$ and represent it by

$$\Xi(t,\cdot)=\sum_{j=1}^{N}\delta_{X_j(t)}(\cdot), \quad t\in\mathbb{N}_0, \tag{6.5}$$

where

$$\boldsymbol{X}(t)=(X_1(t),\dots,X_N(t))\in\mathbb{Z}^N\cap\mathbb{W}_N, \quad t\in\mathbb{N}_0. \tag{6.6}$$

The configuration $\Xi(t,\cdot)\in\mathfrak{M}, t\in\mathbb{N}_0$ is unlabeled, while $\boldsymbol{X}(t)\in\mathbb{Z}^N\cap\mathbb{W}_N, t\in\mathbb{N}_0$ is labeled. We write the probability measure for $\Xi(t,\cdot), t\in\mathbb{N}_0$ started at $\xi\in\mathfrak{M}$ as $\mathbb{P}_\xi$ with expectation $\mathbb{E}_\xi$, and introduce a filtration $\{\mathcal{F}(t):t\in\mathbb{N}_0\}$ defined by $\mathcal{F}(t)=\sigma(\Xi(s):0\le s\le t, s\in\mathbb{N}_0)$. Then the above definition of the noncolliding RW gives the follows: Let $\xi=\sum_{j=1}^N\delta_{u_j}$ with $\boldsymbol{u}\in\mathbb{Z}_{\rm e}^N\cap\mathbb{W}_N$, and $t\in\mathbb{N}$, $t\le T\in\mathbb{N}$. For any $\mathcal{F}(t)$-measurable bounded function F,

$$\mathbb{E}_\xi\Big[F(\Xi(\cdot))\Big]=\lim_{n\to\infty}\mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^N\delta_{V_j(\cdot)}\right)\middle|\tau_{\boldsymbol{u}}>n\right]. \tag{6.7}$$

The important fact is that, if we write the Vandermonde determinant as

$$h(\boldsymbol{x})=\det_{1\le j,k\le N}[x_j^{k-1}]=\prod_{1\le j<k\le N}(x_k-x_j), \tag{6.8}$$

the expectation (6.7) is obtained by an h-transform in the sense of Doob of the form

$$\mathbb{E}_\xi\Big[F(\Xi(\cdot))\Big]=\mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^N\delta_{V_j(\cdot)}\right)\mathbf{1}(\tau_{\boldsymbol{u}}>T)\frac{h(\boldsymbol{V}(T))}{h(\boldsymbol{u})}\right]. \tag{6.9}$$

See, for instance, Lemma 4.4 in [45].

The formula (6.9) is a discrete analogue of the construction of noncolliding Brownian motion (BM) by Grabiner [24] as an h-transform of absorbing BM in $\mathbb{W}_N$. The noncolliding BM is equivalent to Dyson's BM model (with parameter $\beta=2$) and the latter is known as an eigenvalue process of Hermitian matrix-valued BM [13, 53, 67, 24, 26, 37, 42, 69, 58, 59]. Then the noncolliding RW has been attracted much attention as a discretization of models associated with the Gaussian random matrix ensembles [2, 27, 55, 36, 28, 3, 18, 16].

Nagao and Forrester [55] studied a 'bridge' of noncolliding RW started from $\boldsymbol{u}_0=(2j)_{j=0}^{N-1}$ at $t=0$ and returned to the same configuration $\boldsymbol{u}_0$ at time $t=2M, M\in\mathbb{N}_0$. They showed that at time $t=M$ the spatial configuration provides a determinantal point process and the correlation kernel is expressed by using the symmetric Hahn polynomials. Johansson [28] generalized the process to a bridge from $\boldsymbol{u}_0$ at $t=0$ to $M_2-M_1+\boldsymbol{u}_0$ at $t=M_1+M_2$, $M_1,M_2\in\mathbb{N}_0, M_2>M_1$, and proved that the process is determinantal. The dynamical correlation kernel is of the Eynard-Mehta type and called the extended Hahn kernel. For the noncolliding RW defined for infinite time-period $t\in\mathbb{N}_0$ by (6.7) or (6.9) [45, 14], however, determinantal structure of spatio-temporal correlations has not been clarified so far.

6.2 Determinantal martingales for noncolliding RW

Let $\widetilde{W}_j(t), t \in \mathbb{N}_0, 1 \leq j \leq N$ be independent copies of (4.47). Set $\widetilde{\boldsymbol{W}}(t) = (\widetilde{W}_1(t), \dots, \widetilde{W}_N(t)) \in \mathbb{R}^N, t \in \mathbb{N}_0$ in the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$. We consider a complex process $\boldsymbol{Z}(t) = (Z_1(t), \dots, Z_N(t)), t \in \mathbb{N}_0$ with (1.22). The probability space for (1.22) is a product of the probability space $(\Omega, \mathcal{F}, \mathrm{P}_{\boldsymbol{u}})$ for the RW, $\boldsymbol{V}(t), t \in \mathbb{N}_0$, and $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathrm{P}})$ for $\widetilde{\boldsymbol{W}}(t), t \in \mathbb{N}_0$. Let $\mathbf{E}_{\boldsymbol{u}}$ be the expectation for the process $\boldsymbol{Z}(t), t \in \mathbb{N}_0$ with the initial condition $\boldsymbol{Z}(0) = \boldsymbol{u} \in \mathbb{Z}_{\mathrm{e}} \cap \mathbb{W}_N$.

By multilinearity of determinant, the Vandermonde determinant (6.8) does not change in replacing x_j^{k-1} by any monic polynomial of x_j of degree $k-1$, $1 \leq j, k \leq N$. Note that $m_{k-1}(t, x_j)$ is a monic polynomial of x_j of degree $k-1$. Then

$$\begin{aligned}
\frac{h(\boldsymbol{V}(t))}{h(\boldsymbol{u})} &= \frac{1}{h(\boldsymbol{u})} \det_{1 \leq j,k \leq N} [m_{k-1}(t, V_j(t))] \\
&= \frac{1}{h(\boldsymbol{u})} \det_{1 \leq j,k \leq N} \left[\widetilde{\mathrm{E}}[Z_j(t)^{k-1}] \right] \\
&= \widetilde{\mathrm{E}} \left[\frac{1}{h(\boldsymbol{u})} \det_{1 \leq j,k \leq N} [Z_j(t)^{k-1}] \right],
\end{aligned}$$

where we have used the multilinearity of determinant and independence of $Z_j(t)$'s. Therefore, we have obtained the equality,

$$\frac{h(\boldsymbol{V}(t))}{h(\boldsymbol{u})} = \widetilde{\mathrm{E}} \left[\frac{h(\boldsymbol{Z}(t))}{h(\boldsymbol{u})} \right], \quad t \in \mathbb{N}_0. \tag{6.10}$$

Now we consider the determinant identity [43],

$$\frac{h(\boldsymbol{z})}{h(\boldsymbol{u})} = \det_{1 \leq j,k \leq N} \left[\Phi_\xi^{u_k}(z_j) \right], \tag{6.11}$$

where $\xi = \sum_{j=1}^N \delta_{u_j}, \boldsymbol{u} = (u_1, \dots, u_N) \in \mathbb{W}_N$, and $\Phi_\xi^{u_k}(z)$ is given by

$$\Phi_\xi^{u_k}(z) = \prod_{\substack{1 \leq j \leq N, \\ j \neq k}} \frac{z - u_j}{u_k - u_j}, \quad 1 \leq k \leq N. \tag{6.12}$$

Let

$$\mathcal{M}_\xi^{u_k}(t, V_j(t)) \equiv \widetilde{\mathrm{E}} \left[\Phi_\xi^{u_k}(Z_j(t)) \right], \quad t \in \mathbb{N}_0, \quad 1 \leq j \leq N. \tag{6.13}$$

Since $\Phi_\xi^{u_k}(z)$ is a polynomial of z of degree $N-1$, $\mathcal{M}_\xi^{u_k}(t, V_j(t))$ is expressed by a linear combination of the polynomial martingales $\{m_n(t, V_j(t)) \ : 0 \leq n \leq N-1\}$. Then $\mathcal{M}_\xi^{u_k}(t, V_j(t)), 1 \leq j \leq N, t \in \mathbb{N}_0$ are independent martingales and

$$\begin{aligned}
\mathrm{E}_{\boldsymbol{u}}[\mathcal{M}_\xi^{u_k}(t, V_j(t))] &= \mathrm{E}_{\boldsymbol{u}}[\mathcal{M}_\xi^{u_k}(0, V_j(0))] \\
&= \mathcal{M}_\xi^{u_k}(0, u_j) \\
&= \Phi_\xi^{u_k}(u_j) = \delta_{jk}, \quad 1 \leq j, k \leq N.
\end{aligned} \tag{6.14}$$

Using the identity (6.11) for $h(\boldsymbol{Z}(t))/h(\boldsymbol{u})$ in (6.10), we have

$$\begin{aligned}\frac{h(\boldsymbol{V}(t))}{h(\boldsymbol{u})} &= \widetilde{\mathrm{E}}\left[\det_{1\le j,k\le N}[\Phi_\xi^{u_k}(Z_j(t))]\right] \\ &= \det_{1\le j,k\le N}\left[\widetilde{\mathrm{E}}[\Phi_\xi^{u_k}(Z_j(t))]\right],\end{aligned}$$

where independence of $Z_j(t)$'s is used. Let

$$\mathcal{D}_\xi(t,\boldsymbol{V}(t)) = \det_{1\le j,k\le N}[\mathcal{M}_\xi^{u_k}(t,V_j(t))], \quad t\in\mathbb{N}_0. \tag{6.15}$$

We obtain the equality

$$\frac{h(\boldsymbol{V}(t))}{h(\boldsymbol{u})} = \mathcal{D}_\xi(t,\boldsymbol{V}(t)), \quad t\in\mathbb{N}_0. \tag{6.16}$$

In other words, for the complex processes (1.22), $\Phi_\xi^{u_k}(Z_j(\cdot)), 1\le j,k\le N$ are discrete-time complex martingales (see Section V.2 of [61]) such that, for any $t\in\mathbb{N}_0$,

$$\mathbf{E}_{\boldsymbol{u}}[\Phi_\xi^{u_k}(Z_j(t))] = \mathbf{E}_{\boldsymbol{u}}[\Phi_\xi^{u_k}(Z_j(0))] = \delta_{jk}, \quad 1\le j,k\le N. \tag{6.17}$$

6.3 DMR and CPR for noncolliding RW

Since we consider the noncolliding RW as a process represented by an unlabeled configuration (1.15), measurable functions of $\Xi(\cdot)$ are only symmetric functions of N variables, $X_j(\cdot), 1\le j\le N$. Then by the equality (6.16), we obtain the following DMR and CPR for the present noncolliding RW [35].

Theorem 6.1 *Suppose that $N\in\mathbb{N}$ and $\xi=\sum_{j=1}^N \delta_{u_j}$ with $\boldsymbol{u}=(u_1,\dots,u_N)\in\mathbb{Z}_{\mathrm{e}}^N\cap\mathbb{W}_N$. Let $t\in\mathbb{N}, t\le T\in\mathbb{N}$. For any $\mathcal{F}(t)$-measurable bounded function F we have*

$$\begin{aligned}\mathbb{E}_\xi\left[F(\Xi(\cdot))\right] &= \mathrm{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^N \delta_{V_j(\cdot)}\right)\mathcal{D}_\xi(T,\boldsymbol{V}(T))\right] \\ &= \mathbf{E}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^N \delta_{\Re Z_j(\cdot)}\right)\det_{1\le j,k\le N}[\Phi_\xi^{u_k}(Z_j(T))]\right].\end{aligned} \tag{6.18}$$

Proof. It is sufficient to consider the case that F is given as $F(\Xi(\cdot))=\prod_{m=1}^M g_m(\boldsymbol{X}(t_m))$ for $M\in\mathbb{N}$, $t_m\in\mathbb{N}, 1\le m\le M$, $t_1<\dots<t_M\le T\in\mathbb{N}$, with symmetric bounded measurable functions g_m on $\mathbb{Z}^N$, $1\le m\le M$. Here we prove the equalities

$$\begin{aligned}\mathbb{E}_\xi\left[\prod_{m=1}^M g_m(\boldsymbol{X}(t_m))\right] &= \mathrm{E}_{\boldsymbol{u}}\left[\prod_{m=1}^M g_m(\boldsymbol{V}(t_m))\mathcal{D}_\xi(T,\boldsymbol{V}(T))\right] \\ &= \mathbf{E}_{\boldsymbol{u}}\left[\prod_{m=1}^M g_m(\boldsymbol{V}(t_m))\det_{1\le j,k\le N}[\Phi_\xi^{u_k}(Z_j(T))\right].\end{aligned} \tag{6.19}$$

By (6.9), the LHS of (6.19) is given by

$$\mathrm{E}_{\boldsymbol{u}}\left[\prod_{m=1}^{M} g_m(\boldsymbol{V}(t_m))\mathbf{1}(\tau_{\boldsymbol{u}} > t_M)\frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right], \tag{6.20}$$

where we used the fact that $h(\boldsymbol{V}(\cdot))/h(\boldsymbol{u})$ is martingale. At time $t = \tau_{\boldsymbol{u}}$, there are at least one pair $(j, j+1)$ such that $V_j(\tau_{\boldsymbol{u}}) = V_{j+1}(\tau_{\boldsymbol{u}}), 1 \leq j \leq N-1$. We choose the minimal j. Let $\sigma_{j,j+1}$ be the permutation of the indices j and $j+1$ and for $\boldsymbol{v} = (v_1, \dots, v_N) \in \mathbb{Z}^N$ we put $\sigma_{j,j+1}(\boldsymbol{v}) = (v_{\sigma_{j,j+1}(k)})_{k=1}^{N} = (v_1, \dots, v_{j+1}, v_j, \dots, v_N)$. Let $\boldsymbol{u}'$ be the labeled configuration of the process at time $t = \tau_{\boldsymbol{u}}$. Since $u'_j = u'_{j+1}$ by the above setting, under the probability law $\mathrm{P}_{\boldsymbol{u}'}$ the processes $\boldsymbol{V}(t), t > \tau_{\boldsymbol{u}}$ and $\sigma_{j,j+1}(\boldsymbol{V}(t)), t > \tau_{\boldsymbol{u}}$ are identical in distribution. Since $g_m, 1 \leq m \leq M$ are symmetric, but h is antisymmetric, the Markov property of the process $\boldsymbol{V}(\cdot)$ gives

$$\mathrm{E}_{\boldsymbol{u}}\left[\prod_{m=1}^{M} g_m(\boldsymbol{V}(t_m))\mathbf{1}(\tau_{\boldsymbol{u}} \leq t_M)\frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right] = 0.$$

Therefore, (6.20) is equal to

$$\mathrm{E}_{\boldsymbol{u}}\left[\prod_{m=1}^{M} g_m(\boldsymbol{V}(t_m))\frac{h(\boldsymbol{V}(t_M))}{h(\boldsymbol{u})}\right].$$

By the equality (6.16) and the martingale property of $\mathcal{D}_\xi(\cdot, \boldsymbol{V}(\cdot))$, we obtain the first line of (6.19). By definitions of $\mathbf{E}_{\boldsymbol{u}}$ and $\mathcal{D}_\xi$, the second line is valid. Then the proof is completed. ∎

Note that the CPR in the second line of (6.18) may correspond to the complex BM representation reported in [43] for the noncolliding BM.

Then by Proposition 1.5, we will immediately conclude the following [35]. Let

$$\mathbb{K}_\xi(s, x; t, y) = \begin{cases} \displaystyle\sum_{j=1}^{N} p(s, x|u_j)\mathcal{M}_\xi^{u_j}(t, y) - \mathbf{1}(s > t)p(s-t, x|y), \\ \qquad\qquad \text{if} \quad (s, x), (t, y) \in \mathbb{N}_0 \times \mathbb{Z}, \quad s+x, t+y \in 2\mathbb{Z}, \\ 0, \qquad\qquad \text{otherwise,} \end{cases} \tag{6.21}$$

where p is the transition probability for RW (6.3). Here

$$\mathcal{M}_\xi^{u_j}(t, y) = \widetilde{\mathrm{E}}[\Phi_\xi^{u_j}(y + i\widetilde{W}(t))] \tag{6.22}$$

is a functional of initial configuration $\xi = \sum_{j=1}^{N} \delta_{u_j}$ through (6.12).

Corollary 6.2 *For any initial configuration $\xi \in \mathfrak{M}$ with $\xi(\mathbb{Z}_{\mathrm{e}}^N) = N \in \mathbb{N}$, the noncolliding RW, $(\Xi(t), t \in \mathbb{N}_0, \mathbb{P}_\xi)$ is determinantal with the kernel (6.21) with (6.22) in the sense that the moment generating function (1.17) is given by Fredholm determinant*

$$\Psi_\xi^{\boldsymbol{t}}[\boldsymbol{f}] = \mathop{\mathrm{Det}}_{\substack{(s,t) \in \{t_1, t_2, \dots, t_M\}^2, \\ (x,y) \in \mathbb{Z}^2}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}_\xi(s, x; t, y)\chi_t(y)\Big], \tag{6.23}$$

and then all spatio-temporal correlation functions are given by determinants as

$$\rho_\xi\Big(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots ; t_M, \boldsymbol{x}^{(M)}_{N_M}\Big) = \begin{cases} \displaystyle\det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \Big[\mathbb{K}_\xi(t_m, x^{(m)}_j; t_n, x^{(n)}_k)\Big], \\ \qquad \textit{if} \quad \boldsymbol{x}^{(m)}_{N_m} \in \mathbb{Z}_{\rm e}^{N_m} \cap \mathbb{W}_{N_m}, t_m = \textit{even}, \\ \qquad \textit{or} \quad \boldsymbol{x}^{(m)}_{N_m} \in \mathbb{Z}_{\rm o}^{N_m} \cap \mathbb{W}_{N_m}, t_m = \textit{odd}, \quad 1 \le m \le M, \\ 0, \quad \textit{otherwise}, \end{cases} \tag{6.24}$$

$t_m \in \mathbb{N}, 1 \le m \le M$, $t_1 < \cdots < t_M$, *and* $0 \le N_m \le N, 1 \le m \le M$.

7 DMR in O'Connell Process

7.1 Quantum Toda lattice and Whittaker function

Let $a > 0$. The Hamiltonian of the $\mathrm{GL}(N, \mathbb{R})$-quantum Toda lattice is given by

$$\mathcal{H}_N^a = -\frac{1}{2}\Delta + \frac{1}{a^2}V_N(\boldsymbol{x}/a), \quad \boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N \tag{7.1}$$

with the Laplacian $\Delta = \sum_{j=1}^N \partial^2/\partial x_j^2$ and the potential

$$V_N(\boldsymbol{x}) = \sum_{j=1}^{N-1} e^{-(x_{j+1}-x_j)}. \tag{7.2}$$

For $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_N) \in \mathbb{W}_N^{\mathrm{A}}$, the eigenfunction problem

$$\mathcal{H}_N^a \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) = \lambda(\boldsymbol{\nu})\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) \tag{7.3}$$

for the eigenvalue

$$\lambda(\boldsymbol{\nu}) = -|\boldsymbol{\nu}|^2/2 \tag{7.4}$$

is uniquely solved under the condition that

$$e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}}\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) \quad \text{is bounded} \tag{7.5}$$

and

$$\lim_{\boldsymbol{x}\to\infty, \boldsymbol{x}\in\mathbb{W}_N} e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}}\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) = \prod_{1\le j<k\le N} \Gamma(\nu_k - \nu_j), \tag{7.6}$$

where $\boldsymbol{x} \to \infty, \boldsymbol{x} \in \mathbb{W}_N$ means $x_{j+1} - x_j \to \infty, 1 \le j \le N-1$, and $\Gamma(\cdot)$ denotes the Gamma function. The eigenfunction $\psi_{\boldsymbol{\nu}}^{(N)}(\,\cdot\,)$ is called the class-one Whittaker function [4, 57].

The class-one Whittaker function $\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x})$ has several integral representations, one of which was given by Givental [23],

$$\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}) = \int_{\mathbb{T}_N(\boldsymbol{x})} \exp\left(\mathcal{F}_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{T})\right) d\boldsymbol{T}. \tag{7.7}$$

Here the integral is performed over the space $\mathbb{T}_N(\boldsymbol{x})$ of all real lower triangular arrays with size N, $\boldsymbol{T} = (T_{j,k}, 1 \le k \le j \le N)$, with $T_{N,k} = x_k, 1 \le k \le N$, and

$$\mathcal{F}_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{T}) = \sum_{j=1}^N \nu_j \left(\sum_{k=1}^j T_{j,k} - \sum_{k=1}^{j-1} T_{j-1,k}\right) - \sum_{j=1}^{N-1}\sum_{k=1}^j \left\{e^{-(T_{j,k}-T_{j+1,k})} + e^{-(T_{j+1,k+1}-T_{j,k})}\right\}. \tag{7.8}$$

We can prove that [56, 12]

$$\lim_{a\to 0} a^{N(N-1)/2}\psi_{a\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a) = \frac{\det_{1\le j,\ell\le N}\left[e^{x_j\nu_\ell}\right]}{h(\boldsymbol{\nu})}. \tag{7.9}$$

where $h(\boldsymbol{\nu})$ is the Vandermonde determinant (5.10).

The following orthogonality relation is proved for the class-one Whittaker functions [63, 72],

$$\int_{\mathbb{R}^N} \psi^{(N)}_{-i\boldsymbol{k}}(\boldsymbol{x})\psi^{(N)}_{i\boldsymbol{k}'}(\boldsymbol{x})d\boldsymbol{x} = \frac{1}{s_N(\boldsymbol{k})N!} \sum_{\sigma \in \mathcal{S}_N} \delta(\boldsymbol{k} - \sigma(\boldsymbol{k}')), \tag{7.10}$$

for $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{R}^N$, where $s_N(\cdot)$ is the density function of the Sklyanin measure [65]

$$\begin{aligned} s_N(\boldsymbol{\mu}) &= \frac{1}{(2\pi)^N N!} \prod_{1 \le j < \ell \le N} |\Gamma(i(\mu_\ell - \mu_j))|^{-2} \\ &= \frac{1}{(2\pi)^N N!} \prod_{1 \le j < \ell \le N} \left\{ (\mu_\ell - \mu_j) \frac{\sinh \pi(\mu_\ell - \mu_j)}{\pi} \right\}, \quad \boldsymbol{\mu} \in \mathbb{R}^N. \end{aligned} \tag{7.11}$$

Borodin and Corwin proved that for a class of test functions, the orthogonality relation (7.10) can be extended for any $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{C}^N$ [9]. Moreover, the following recurrence relations with respect to $\boldsymbol{\nu}$ are established [44, 9]; for $1 \le r \le N-1, \boldsymbol{\nu} \in \mathbb{C}^N$,

$$\sum_{\substack{I \subset \{1,\dots,N\}, \\ |I|=r}} \prod_{\substack{j \in I, \\ k \in \{1,2,\dots,N\} \setminus I}} \frac{1}{i(\nu_k - \nu_j)} \psi^{(N)}_{i(\boldsymbol{\nu}+i\boldsymbol{e}_I)}(\boldsymbol{x}) = \exp\left(-\sum_{j=1}^r x_j \right) \psi^{(N)}_{i\boldsymbol{\nu}}(\boldsymbol{x}), \tag{7.12}$$

where $\boldsymbol{e}_I$ is the vector with ones in the slots of label I and zeros otherwise;

$$(\boldsymbol{e}_I)_j = \begin{cases} 1, & j \in I, \\ 0, & j \in \{1, \dots, N\} \setminus I. \end{cases}$$

In particular, for $r = 1$,

$$\sum_{j=1}^N \prod_{1 \le k \le N: k \ne j} \frac{1}{i(\nu_k - \nu_j)} \psi^{(N)}_{i(\boldsymbol{\nu}+i\boldsymbol{e}_{\{j\}})}(\boldsymbol{x}) = e^{-x_1} \psi^{(N)}_{i\boldsymbol{\nu}}(\boldsymbol{x}), \tag{7.13}$$

where the ℓ-th component of the vector $\boldsymbol{e}_{\{j\}}$ is $(\boldsymbol{e}_{\{j\}})_\ell = \delta_{j\ell}, 1 \le j, \ell \le N$. As fully discussed by Borodin and Corwin [9], the recurrence relations (7.12) are derived as the $q \to 1$ limit of the eigenfunction equations associated to the Macdonald difference operators in the theory of symmetric functions [50]. For more details on Whittaker functions, see [44, 4, 30, 56, 31, 9] and references therein.

7.2 O'Connell process

O'Connell introduced an N-component diffusion process, $N \ge 2$, which can be regarded as a stochastic version of a quantum open Toda-lattice [56]. Let $a > 0$. The infinitesimal generator of the O'Connell process is given by

$$\begin{aligned} \mathcal{L}^{\nu,a}_N &= -(\psi^{(N)}_{\boldsymbol{\nu}}(\boldsymbol{x}/a))^{-1} \left(\mathcal{H}^a_N + \frac{1}{2}|\boldsymbol{\nu}|^2 \right) \psi^{(N)}_{\boldsymbol{\nu}}(\boldsymbol{x}/a) \\ &= \frac{1}{2}\Delta + \nabla \log \psi^{(N)}_{\boldsymbol{\nu}}(\boldsymbol{x}/a) \cdot \nabla, \end{aligned} \tag{7.14}$$

where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_N)$. This multivariate diffusion process is an extension of a one-dimensional diffusion studied by Matsumoto and Yor [51, 52]. (The Matsumoto-Yor process describes time-evolution of the relative coordinate of the $N = 2$ case.)

We can show that the O'Connell process is realized as the following mutually killing BMs *conditioned that all particles survive forever*, if the particle position of the j-th BM is identified with the j-th component of the O'Connell process, $1 \le j \le N$ [30, 31, 32]. Let $B_j(t), 1 \le j \le N$ be independent one-dimensional standard BMs started at $B_j(0) = x_j \in \mathbb{R}$, and for $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_N) \in \mathbb{R}^N$,

$$B_j^{\nu_j}(t) = B_j(t) + \nu_j t, \qquad 1 \le j \le N \tag{7.15}$$

be drifted BMs. We consider an N-particle system of BMs with drift vector $\boldsymbol{\nu}$, $\boldsymbol{B}^{\nu}(t) = (B_1^{\nu_1}(t), \dots, B_N^{\nu_N}(t)), t \ge 0$, such that the probability $P_N^a(t|\{\boldsymbol{B}^{\nu}(s)\}_{0\le s\le t})$ that all N particles survive up to time t conditioned on a path $\{\boldsymbol{B}^{\nu}(s)\}_{0\le s\le t}$ decays following the equation

$$\frac{d}{dt} P_N^a(t|\{\boldsymbol{B}^{\nu}(s)\}_{0\le s\le t}) = -\frac{1}{a^2} V_N(\boldsymbol{B}^{\nu}(t)/a) P_N^a(t|\{\boldsymbol{B}^{\nu}(s)\}_{0\le s\le t}), \quad t \ge 0. \tag{7.16}$$

It is a system of mutually killing BMs, in which the Toda-lattice potential (7.2) determines the decay rate of the survival probability depending on a configuration $\boldsymbol{B}^{\nu}(t)$ [31]. With the initial condition $\boldsymbol{B}^{\nu}(0) = \boldsymbol{x} \in \mathbb{W}_N$, the survival probability is obtained by averaging over all paths of BMs started at $\boldsymbol{x}$ as

$$P_N^a(t; \boldsymbol{x}, \boldsymbol{\nu}) = \mathrm{E}_{\boldsymbol{x}}\Big[P_N^a(t|\{\boldsymbol{B}^{\nu}(s)\}_{0\le s\le t})\Big], \tag{7.17}$$

and we can show that [31, 57, 32]

$$\begin{aligned} \lim_{t\to\infty} P_N^a(t; \boldsymbol{x}, \boldsymbol{\nu}) &= c_1^a(N, \boldsymbol{\nu}) e^{-\boldsymbol{\nu}\cdot\boldsymbol{x}/a} \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a), \quad \text{if } \boldsymbol{\nu} \in \mathbb{W}_N, \, \boldsymbol{\nu} \neq 0, \\ P_N^a(t; \boldsymbol{x}, 0) &\sim c_2^a(N) t^{-N(N-1)/4} \psi_0^{(N)}(\boldsymbol{x}/a) \quad \text{as } t \to \infty, \end{aligned} \tag{7.18}$$

where $c_1^a(N, \boldsymbol{\nu})$ and $c_2^a(N)$ are independent of $\boldsymbol{x}$ and t. Then, conditionally on surviving of all N particles, the equivalence of this vicious BM, which has a killing term given by the Toda-lattice potential, with the O'Connell process is proved. We note that the parameter $a > 0$ in the killing rate (7.16) with (7.2) indicates the characteristic range of interaction to kill neighboring particles as well as the characteristic length in which neighboring particles can exchange their order in $\mathbb{R}$. It implies that if we take the limit $a \to 0$, the O'Connell process is reduced to the noncolliding BM. (The original vicious Brownian motion is a system of BMs such that if pair of particles collide they are annihilated immediately. The noncolliding BM is the vicious BM conditioned never to collide with each other, and thus all particles survive forever.)

The transition probability density for the O'Connell process with $\boldsymbol{\nu}$ is given by [31]

$$P_N^{\nu, a}(t, \boldsymbol{y}|\boldsymbol{x}) = e^{-t|\boldsymbol{\nu}|^2/2a^2} \frac{\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{y}/a)}{\psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a)} Q_N^a(t, \boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N, t \ge 0, \tag{7.19}$$

with

$$Q_N^a(t,\boldsymbol{y}|\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-t|\boldsymbol{k}|^2/2} \psi_{ia\boldsymbol{k}}^{(N)}(\boldsymbol{x}/a) \psi_{-ia\boldsymbol{k}}^{(N)}(\boldsymbol{y}/a) s_N(a\boldsymbol{k}) d\boldsymbol{k}. \tag{7.20}$$

(See also Proof of Proposition 4.1.32 in [9].) As a matter of fact, we can confirm that $u(t,\boldsymbol{x}) \equiv P_N^{\nu,a}(t,\boldsymbol{y}|\boldsymbol{x})$ satisfies the Kolmogorov backward equation associated with the infinitesimal generator $\mathcal{L}_N^{\nu,a}$ given by (7.14),

$$\begin{aligned} \frac{\partial u(t,\boldsymbol{x})}{\partial t} &= \mathcal{L}_N^{\nu,a} u(t,\boldsymbol{x}) \\ &= \frac{1}{2}\sum_{j=1}^N \frac{\partial^2 u(t,\boldsymbol{x})}{\partial x_j^2} + \sum_{j=1}^N \frac{\partial \log \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a)}{\partial x_j} \frac{\partial u(t,\boldsymbol{x})}{\partial x_j}, \end{aligned} \tag{7.21}$$

$\boldsymbol{x} \in \mathbb{R}^N, t \geq 0$, under the condition $u(0,\boldsymbol{x}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \equiv \prod_{j=1}^N \delta(x_j - y_j), \boldsymbol{y} \in \mathbb{R}^N$. We denote the O'Connell process by

$$\boldsymbol{X}^a(t) = (X_1^a(t), X_2^a(t), \dots, X_N^a(t)), \quad t \geq 0. \tag{7.22}$$

It is defined as an N-particle diffusion process in $\mathbb{R}$ such that its backward Kolmogorov equation is given by (7.21). Therefore, (7.22) is a unique solution of the following stochastic differential equation for given initial configuration $\boldsymbol{X}^a(0) = \boldsymbol{x} \in \mathbb{R}^N$,

$$dX_j^a(t) = dB_j(t) + \Big[\boldsymbol{F}_N^{\nu,a}(\boldsymbol{X}^a(t))\Big]_j dt, \quad 1 \leq j \leq N, t \geq 0 \tag{7.23}$$

with

$$\boldsymbol{F}_N^{\nu,a}(\boldsymbol{x}) = \nabla \log \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a), \tag{7.24}$$

where $\{B_j(t)\}_{j=1}^N$ are independent one-dimensional standard BMs and $[\boldsymbol{V}]_j$ denotes the j-th coordinate of a vector $\boldsymbol{V}$.

7.3 Special entrance law

Let $N \in \mathbb{N}$, and define

$$\boldsymbol{\rho} = \left(-\frac{N-1}{2}, -\frac{N-1}{2}+1, \dots, \frac{N-1}{2}-1, \frac{N-1}{2}\right). \tag{7.25}$$

O'Connell considered the process starting from $\boldsymbol{x} = -M\boldsymbol{\rho}$ and let $M \to \infty$ [56]. It was claimed in [56] (see also [4]) that

$$\psi_{\boldsymbol{\nu}}^{(N)}(-M\boldsymbol{\rho}) \sim Ce^{-N(N-1)M/8} \exp\left(e^{M/2}\mathcal{F}_{\mathbf{0}}(\boldsymbol{T}^0)\right) \tag{7.26}$$

as $M \to \infty$, where the coefficient C and the critical point $\boldsymbol{T}^0$ are independent of $\boldsymbol{\nu}$. Then as a limit of (7.19) with (7.20), we have a probability density function

$$\begin{aligned} \mathcal{P}_N^{\nu,a}(t,\boldsymbol{x}) &\equiv \lim_{M\to\infty} P_N^{\nu,a}(t,\boldsymbol{x}|-M\boldsymbol{\rho}) \\ &= e^{-t|\boldsymbol{\nu}|^2/2a^2} \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a) \vartheta_N^a(t,\boldsymbol{x}) \end{aligned} \tag{7.27}$$

with

$$\vartheta_N^a(t,\boldsymbol{x}) = \int_{\mathbb{R}^N} e^{-t|\boldsymbol{k}|^2/2} \psi_{-ia\boldsymbol{k}}^{(N)}(\boldsymbol{x}/a) s_N(a\boldsymbol{k}) d\boldsymbol{k} \tag{7.28}$$

for any $t > 0$. Since we have taken the limit $M \to \infty$ for the state $-M\boldsymbol{\rho}$, we cannot speak of initial configurations any longer, but for an arbitrary series of increasing times, $0 < t_1 < t_2 < \cdots < t_M < \infty$, the probability density function of the multi-time joint distributions is given by

$$\mathcal{P}_N^{\nu,a}(t_1,\boldsymbol{x}^{(1)}; t_2, \boldsymbol{x}^{(2)}; \dots; t_M, \boldsymbol{x}^{(M)}) = \prod_{m=1}^{M-1} P_N^{\nu,a}(t_{m+1}-t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)}) \mathcal{P}_N^{\nu,a}(t_1, \boldsymbol{x}^{(1)}) \tag{7.29}$$

for $\boldsymbol{x}^{(m)} \in \mathbb{R}^N, 1 \leq m \leq M$. We can call the probability measure $\mathcal{P}_N^{\nu,a}(t,\boldsymbol{x})d\boldsymbol{x}$ with (7.27) and $d\boldsymbol{x} = \prod_{j=1}^N dx_j$ an *entrance law coming from* "$-\infty\boldsymbol{\rho}$" [56] (see, for instance, Section XII.4 of [61] for entrance laws). We note that, by (7.19), (7.29) is written as

$$\begin{aligned}
&\mathcal{P}_N^{\nu,a}(t_1,\boldsymbol{x}^{(1)}; t_2, \boldsymbol{x}^{(2)}; \dots; t_M, \boldsymbol{x}^{(M)}) \\
&\quad = e^{-t_M|\boldsymbol{\nu}|^2/2a^2} \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}^{(M)}/a) \prod_{m=1}^{M-1} Q_N^a(t_{m+1}-t_m, \boldsymbol{x}^{(m+1)}|\boldsymbol{x}^{(m)}) \vartheta_N^a(t_1, \boldsymbol{x}^{(1)}).
\end{aligned}$$

The expectation with respect to the distribution of the present process started according to the special entrance law (7.27) is denoted by $\mathbb{E}^{\nu,a}$. For measurable functions $f^{(m)}, 1 \leq m \leq M$,

$$\begin{aligned}
&\mathbb{E}^{\nu,a}\left[\prod_{m=1}^M f^{(m)}(\boldsymbol{X}^a(t_m))\right] \\
&= e^{-t_M|\boldsymbol{\nu}|^2/2a^2} \left\{\prod_{m=1}^M \int_{\mathbb{R}^N} d\boldsymbol{x}^{(m)}\right\} f^{(M)}(\boldsymbol{x}^{(M)}) \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}^{(M)}/a) Q_N^a(t_M - t_{M-1}, \boldsymbol{x}^{(M)}|\boldsymbol{x}^{(M-1)}) \\
&\quad \times \prod_{m=2}^{M-1} f^{(m)}(\boldsymbol{x}^{(m)}) Q_N^a(t_m - t_{m-1}, \boldsymbol{x}^{(m)}|\boldsymbol{x}^{(m-1)}) f^{(1)}(\boldsymbol{x}^{(1)}) \vartheta_N^a(t_1, \boldsymbol{x}^{(1)}),
\end{aligned} \tag{7.30}$$

$0 < t_1 < \cdots < t_M < \infty$, where $d\boldsymbol{x}^{(m)} = \prod_{j=1}^N dx_j^{(m)}, 1 \leq m \leq M$.

The present special entrance law (7.27) is called a Whittaker measure by Borodin and Corwin [9] and denoted by $\mathbf{WM}_{(\nu;t)}(\boldsymbol{x})$. Note that in the notation of [9], a Whittaker process is a 'triangular array extension' of the Whittaker measure and is not the same as the O'Connell process.

When $M = 1$, for $t > 0$, (7.30) gives

$$\begin{aligned}
&\mathbb{E}^{\nu,a}[f(\boldsymbol{X}^a(t))] \\
&\quad = e^{-t|\boldsymbol{\nu}|^2/2a^2} \int_{\mathbb{R}^N} d\boldsymbol{x} f(\boldsymbol{x}) \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a) \vartheta_N^a(t,\boldsymbol{x}) \\
&\quad = e^{-t|\boldsymbol{\nu}|^2/2a^2} \int_{\mathbb{R}^N} d\boldsymbol{x} f(\boldsymbol{x}) \psi_{\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a) \int_{\mathbb{R}^N} d\boldsymbol{k} e^{-t|\boldsymbol{k}|^2/2} \psi_{-ia\boldsymbol{k}}^{(N)}(\boldsymbol{x}/a) s_N(a\boldsymbol{k}).
\end{aligned} \tag{7.31}$$

7.4 Combinatorial limit $a \to 0$

The transition probability density of the absorbing BM in $\mathbb{W}_N^{\rm A}$ is given by the Karlin-McGregor determinant of (3.3),

$$q_N(t, \boldsymbol{y}|\boldsymbol{x}) = \det_{1 \le j,k \le N}[p(t, y_j|x_k)], \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\rm A}, t \ge 0. \tag{7.32}$$

Consider the drift transform of (7.32),

$$q_N^{\nu}(t, \boldsymbol{y}|\boldsymbol{x}) = \exp\left\{ -\frac{t}{2}|\boldsymbol{\nu}|^2 + \boldsymbol{\nu} \cdot (\boldsymbol{y} - \boldsymbol{x}) \right\} q_N(t, \boldsymbol{y}|\boldsymbol{x}).$$

Then, if $\boldsymbol{\nu} \in \overline{\mathbb{W}}_N^{\rm A} = \{\boldsymbol{x} \in \mathbb{R}^N : x_1 \le x_2 \le \cdots \le x_N\}$, the transition probability density of the noncolliding BM with drift $\boldsymbol{\nu}$ is given by [5]

$$p_N^{\nu}(t, \boldsymbol{y}|\boldsymbol{x}) = e^{-t|\boldsymbol{\nu}|^2/2} \frac{\det_{1 \le j,k \le N}[e^{\nu_j y_k}]}{\det_{1 \le j,k \le N}[e^{\nu_j x_k}]} q_N(t, \boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N, \quad t \ge 0. \tag{7.33}$$

In the limit $\nu_j \to 0, 1 \le j \le N$, (7.33) becomes

$$p_N(t, \boldsymbol{y}|\boldsymbol{x}) = \frac{h(\boldsymbol{y})}{h(\boldsymbol{x})} q_N(t, \boldsymbol{y}|\boldsymbol{x}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{W}_N^{\rm A}, t \ge 0. \tag{7.34}$$

We prove the following. (The superscript $a\nu$ is used for the processes with drift vector $a\boldsymbol{\nu} = (a\nu_1, \dots, a\nu_N)$.)

Lemma 7.1 *For* $\boldsymbol{\nu} \in \overline{\mathbb{W}}_N$,

$$\begin{aligned} \lim_{a \to 0} \mathcal{P}_N^{a\nu, a}(t, \boldsymbol{x}) d\boldsymbol{x} &= p_N(t^{-1}, \boldsymbol{x}/t|\boldsymbol{\nu}) d(\boldsymbol{x}/t) \\ &= p_N^{\nu}(t, \boldsymbol{x}|\boldsymbol{0}) d\boldsymbol{x}, \quad t > 0. \end{aligned} \tag{7.35}$$

Proof By the asymptotics (7.9) and the definition (7.32) of q_N, we have

$$\lim_{a \to 0} a^{N(N-1)/2} e^{-t|\boldsymbol{\nu}|^2/2} \psi_{a\boldsymbol{\nu}}^{(N)}(\boldsymbol{x}/a) = \left(\frac{2\pi}{t} \right)^{N/2} e^{|\boldsymbol{x}|^2/2t} \frac{q_N(t^{-1}, \boldsymbol{x}/t|\boldsymbol{\nu})}{h(\boldsymbol{\nu})}. \tag{7.36}$$

For ϑ_N^a defined by the integral (7.28), we can show that the Whittaker function with purely imaginary index multiplied by the Sklyanin density, $\psi_{-ia\boldsymbol{k}}^{(N)}(\,\cdot\,) s_N(a\boldsymbol{k})$, is uniformly integrable in $a > 0$ with respect to the Gaussian measure $e^{-t|\boldsymbol{k}|^2/2} d\boldsymbol{k}, t > 0$. Then the integral and the limit $a \to 0$ is interchangeable. Since

$$\psi_{-ia\boldsymbol{k}}^{(N)}(\boldsymbol{x}/a) \sim (-ia)^{-N(N-1)/2} \frac{\det_{1 \le j,\ell \le N}[e^{-ix_j k_\ell}]}{h(\boldsymbol{k})}, \quad \text{as } a \to 0$$

by (7.9), and (7.11) gives $s_N(a\boldsymbol{k}) \sim a^{N(N-1)}(h(\boldsymbol{k}))^2/\{(2\pi)^N N!\}$, as $a \to 0$, we have

$$\begin{aligned}
&\lim_{a\to 0} a^{-N(N-1)/2} \vartheta_N^a(t,\boldsymbol{x}) \\
&= \frac{1}{(2\pi)^N N!} \int_{\mathbb{R}^N} d\boldsymbol{k} e^{-t|\boldsymbol{k}|^2/2} \det_{1\le j,\ell\le N}[e^{-ix_j k_\ell}] h(i\boldsymbol{k}) \\
&= \frac{t^{-N(N+1)/4}}{(2\pi)^{N/2}} e^{-|\boldsymbol{x}|^2/2t} \frac{1}{N!} \int_{\mathbb{R}^N} d(\sqrt{t}\boldsymbol{k}) \det_{1\le j,\ell\le N}\left[\frac{e^{-(\sqrt{t}k_\ell + ix_j/\sqrt{t})^2/2}}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1}(i\sqrt{t}k_\ell - i\sqrt{t}k_m)\right].
\end{aligned}$$

By multi-linearity of determinant,

$$\begin{aligned}
&\frac{1}{N!}\int_{\mathbb{R}^N} d(\sqrt{t}\boldsymbol{k}) \det_{1\le j,\ell\le N}\left[\frac{e^{-(\sqrt{t}k_\ell + ix_j/\sqrt{t})^2/2}}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1}(i\sqrt{t}k_\ell - i\sqrt{t}k_m)\right] \\
&= \det_{1\le j,\ell\le N}\left[\int_{\mathbb{R}} d(\sqrt{t}k) \frac{e^{-(\sqrt{t}k + ix_j/\sqrt{t})^2/2}}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1}(i\sqrt{t}k - i\sqrt{t}k_m)\right] \\
&= \det_{1\le j,\ell\le N}\left[\int_{\mathbb{R}} du \frac{e^{-(u + ix_j/\sqrt{t})^2/2}}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1}(iu - i\sqrt{t}k_m)\right]. \qquad (7.37)
\end{aligned}$$

The integral in the determinant (7.37) can be identified with an integral representation given by Bleher and Kuijlaars [8, 40] for the multiple Hermite polynomial of type II,

$$P_{\xi_{\ell-1}}(x_j/\sqrt{t}) \quad \text{with} \quad \xi_{\ell-1}(\cdot) = \sum_{m=1}^{\ell-1} \delta_{i\sqrt{t}k_m}(\cdot).$$

(We set $\xi_0(\cdot) \equiv 0$ and $\prod_{m=1}^{0}(\cdot) \equiv 1$.) It is a monic polynomial of $x_j/\sqrt{t}$ with degree $\ell - 1$. Then (7.37) is equal to the Vandermonde determinant

$$h(\boldsymbol{x}/\sqrt{t}) = t^{N(N-1)/4} h(\boldsymbol{x}/t).$$

Therefore, we obtain

$$\lim_{a\to 0} a^{-N(N-1)/2} \vartheta_N^a(t,\boldsymbol{x}) = \frac{1}{(2\pi t)^{N/2}} e^{-|\boldsymbol{x}|^2/2t} h(\boldsymbol{x}/t). \qquad (7.38)$$

Combining (7.36) and (7.38), we obtain the equality

$$\lim_{a\to 0} \mathcal{P}_N^{a\nu,a}(t,\boldsymbol{x}) = \frac{h(\boldsymbol{x}/t)}{h(\boldsymbol{\nu})} q_N(t^{-1}, \boldsymbol{x}/t|\boldsymbol{\nu}) t^{-N}, \qquad (7.39)$$

which gives the first equality of (7.35) by the formula (7.34). The second equality is concluded by the reciprocal relation proved as Theorem 2.1 in [32] (see (7.55) below). The proof is then completed. ∎

Moreover, if we take the limit $\boldsymbol{\nu} \to 0$ in (7.35), we have the following

$$\begin{aligned} \lim_{\boldsymbol{\nu}\to 0}\lim_{a\to 0}\mathcal{P}_N^{a\nu,a}(t,\boldsymbol{x}) &= p_N(t,\boldsymbol{x}|\boldsymbol{0}) \\ &= \frac{t^{-N^2/2}}{(2\pi)^{N/2}\prod_{j=1}^N \Gamma(j)} e^{-|\boldsymbol{x}|^2/2t}(h_N(\boldsymbol{x}))^2. \end{aligned} \tag{7.40}$$

This is the probability density of the eigenvalue distribution of the Gaussian unitary ensemble (GUE) with variance $\sigma^2 = t$ of random matrix theory. It implies that *a geometric lifting* of the GUE-eigenvalue distribution is the $\boldsymbol{\nu} \to 0$ limit of the entrance law coming from "$-\infty\boldsymbol{\rho}$",

$$\begin{aligned} \mathcal{P}_N^a(t,\boldsymbol{x}) &\equiv \lim_{\boldsymbol{\nu}\to 0}\mathcal{P}_N^{\nu,a}(t,\boldsymbol{x}) \\ &= \psi_0^{(N)}(\boldsymbol{x}/a)\vartheta_N^a(t,\boldsymbol{x}) \\ &= \psi_0^{(N)}(\boldsymbol{x}/a)\int_{\mathbb{R}^N} e^{-t|\boldsymbol{k}|^2/2}\psi_{-ia\boldsymbol{k}}^{(N)}(\boldsymbol{x}/a)s_N(a\boldsymbol{k})d\boldsymbol{k}. \end{aligned} \tag{7.41}$$

7.5 Determinantal formula of Borodin and Corwin

For $x \in \mathbb{R}, a > 0$, set

$$\Theta^a(x) = \exp(-e^{-x/a}). \tag{7.42}$$

Note that $\lim_{a\to 0}\Theta^a(x) = \mathbf{1}(x > 0)$, that is, (7.42) is a softening of an indicator function $\mathbf{1}(x > 0)$.

Let $\widetilde{\delta} = \sup\{|\nu_j| : 1 \le j \le N\}$ and choose $0 < \delta < 1$ so that $\widetilde{\delta} < \delta/2$. Borodin and Corwin [9] proved that $\mathbb{E}^{\nu,a}[\Theta^a(X_1^a(t) - h)], h \in \mathbb{R}$ is given by a Fredholm determinant of a kernel $K_{e^{h/a}}$ for the contour integrals on $C((-1)\circ\nu)$, $(-1)\circ\nu(\cdot) = \sum_{j=1}^N \delta_{-\nu_j}(\cdot)$;

$$\mathbb{E}^{\nu,a}\Big[\Theta^a(X_1^a(t) - h)\Big] = \mathop{\mathrm{Det}}_{(v,v')\in C((-1)\circ\nu)^2}\Big[\delta(v - v') + K_{e^{h/a}}(v,v')\Big], \tag{7.43}$$

where

$$K_u(v,v') = \int_{-i\infty+\delta}^{i\infty+\delta}\frac{ds}{2\pi i}\Gamma(-s)\Gamma(1+s)\prod_{\ell=1}^N\frac{\Gamma(v+\nu_\ell)}{\Gamma(s+v+\nu_\ell)}\frac{u^s e^{tvs/a^2+ts^2/2a^2}}{v+s-v'}, \quad u > 0. \tag{7.44}$$

Here the Fredholm determinant is defined by the sum of infinite series of multiple contour-integrals

$$\mathop{\mathrm{Det}}_{(v,v')\in C((-1)\circ\nu)^2}\Big[\delta(v-v') + K_u(v,v')\Big] = \sum_{L=0}^\infty \frac{1}{L!}\prod_{j=1}^L \oint_{C((-1)\circ\nu)}\frac{dv_j}{2\pi i}\det_{1\le j,k\le L}[K_u(v_j,v_k)], \tag{7.45}$$

where the term for $L = 0$ is assumed to be 1. Note that (7.44) depends on ν, a and t; $K_u(\cdot,\cdot) = K_u(\cdot,\cdot;\nu,a,t)$.

The Fredholm determinant formula (7.43) discovered by Borodin and Corwin [9] is surprising, since the O'Connell process is not determinantal as mentioned above. We would like to understand the origin of such a determinantal structure surviving in the geometric lifting from the noncolliding BM to the O'Connell process.

7.6 Variations of CPR and DMR

Let

$$\widehat{\nu}(\cdot) = \sum_{j=1}^{N} \delta_{\hat{\nu}_j}(\cdot) \in \mathfrak{M}_0. \tag{7.46}$$

For such $\widehat{\nu}$, define a function of $x \in \mathbb{C}$ with parameters $\boldsymbol{u} \in \mathbb{C}, a > 0$ by

$$\Phi_{\widehat{\nu}}^{u,a}(x) = \Gamma(1 - a(u-x)) \prod_{r \in \operatorname{supp} \widehat{\nu} \cap \{u\}^{\rm c}} \frac{\Gamma(a(r-u))}{\Gamma(a(r-x))}. \tag{7.47}$$

This function has simple poles at

$$x_n = -\frac{n}{a} + u, \quad n \in \mathbb{N} \tag{7.48}$$

and satisfies

$$\Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(\widehat{\nu}_j) = \delta_{jk}, \quad 1 \leq j, k \leq N. \tag{7.49}$$

Since

$$\Gamma(ax) \sim \frac{1}{ax}, \quad x \in \mathbb{C}, \quad \text{as } a \to 0, \tag{7.50}$$

$$\lim_{a \to 0} \Phi_{\widehat{\nu}}^{u,a}(x) = \Phi_{\widehat{\nu}}^{u}(x) = \prod_{r \in \operatorname{supp} \widehat{\nu} \cap \{u\}^{\rm c}} \frac{x-r}{u-r}, \quad x, u \in \mathbb{C}. \tag{7.51}$$

We say that $\Phi_{\widehat{\nu}}^{u}(\cdot)$ is the *combinatorial limit* of $\Phi_{\widehat{\nu}}^{u,a}(\cdot)$, and that $\Phi_{\widehat{\nu}}^{u,a}(x)$ is the *geometric lifting* of $\Phi_{\widehat{\nu}}^{u}(\cdot)$. All poles (7.48) go to infinity in the limit $a \to 0$ and the function becomes entire in the combinatorial limit.

Let $Z_j(t), 1 \leq j \leq N, t \geq 0$ be a set of independent complex BM's given by (4.21). In [33], we showed that the determinant formula (7.43) with (7.45) of Borodin and Corwin is rewritten as follows.

$$\begin{aligned} &\mathbb{E}^{a \circ \widehat{\nu}, a}[\Theta^a(X_1^a(t) - h)] \\ &= \mathbf{E}_{\widehat{\boldsymbol{\nu}}}\left[\det_{1 \leq j,k \leq N} \left[\delta_{jk} - \Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(Z_j(1/t)) \mathbf{1}(\Re Z_j(1/t) < h/t) \right] \right] \end{aligned} \tag{7.52}$$

for $a > 0, h \in \mathbb{R}, t > 0$.

The functions $\Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(\cdot), 1 \leq k \leq N$ are not entire, but they are holomorphic in $\mathbb{C} \setminus \{-n/a + \widehat{\nu}_k : n \in \mathbb{N}\}$. Then $\Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(\mathbb{Z}_j(1/t)), 1 \leq j, k \leq N$ are conformal martingales. Since (7.49) holds, for each $t > 0$, the RHS of (7.52) is equal to

$$\begin{aligned} &\mathbf{E}_{\widehat{\boldsymbol{\nu}}}\left[\det_{1 \leq j,k \leq N} \left[\Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(Z_j(1/t)) - \Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(Z_j(1/t)) \mathbf{1}(\Re Z_j(1/t) < h/t) \right] \right] \\ &= \mathbf{E}_{\widehat{\boldsymbol{\nu}}}\left[\det_{1 \leq j,k \leq N} \left[(1 - \mathbf{1}(\Re Z_j(1/t) < h/t)) \Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(Z_j(1/t)) \right] \right] \\ &= \mathbf{E}_{\widehat{\boldsymbol{\nu}}}\left[\prod_{j=1}^{N} \mathbf{1}(\Re Z_j(1/t) \geq h/t) \det_{1 \leq j,k \leq N} \left[\Phi_{\widehat{\nu}}^{\widehat{\nu}_k, a}(Z_j(1/t)) \right] \right]. \end{aligned}$$

That is, we have the equality

$$
\begin{aligned}
&\mathbb{E}^{a\circ\widehat{\nu},a}[\Theta^a(X_1^a(t)-h)] \\
&= \mathbf{E}_{\widehat{\boldsymbol{\nu}}}\left[\prod_{j=1}^{N}\mathbf{1}(\Re Z_j(1/t)\geq h/t) \det_{1\leq j,k\leq N}\left[\Phi_{\widehat{\nu}}^{\widehat{\nu}_k,a}(Z_j(1/t))\right]\right], \quad a>0, h\in\mathbb{R}. \quad (7.53)
\end{aligned}
$$

The observable $\Theta^a(X_1^a(t)-h), h\in\mathbb{R}$ in the LHS is a softening of the indicator $\mathbf{1}(X_1(t)\geq h)$. Its expectation for the O'Connell process started according to the entrance law coming from "$-\infty\boldsymbol{\rho}$" has the representation as given by the RHS, in which the 'sharp' indicators $\mathbf{1}(V_j(1/t)\geq h/t), 1\leq j\leq N$ are observed, but the complex weight on paths is 'softened'(geometrically lifted). We regard (7.53) as a 'variation' of CPR.

Proposition 7.2 *The O'Connell process has a variation of CPR for $\Theta^a(X_1^a(\cdot)-h), a>0, h\in\mathbb{R}$ as given by (7.53).*

For the noncolliding BM $\boldsymbol{X}(t)=(X_1(t),\dots,X_N(t)), t\geq 0$ and $\boldsymbol{\nu}=(\nu_1,\dots,\nu_N)\in\mathbb{W}_N^{\rm A}$, we consider the noncolliding BM with drift vector $\boldsymbol{\nu}$,

$$
\begin{aligned}
&\boldsymbol{X}^{\nu}(t)=(X_1^{\nu}(t),\dots,X_N^{\nu}(t)), \\
&\text{with} \quad X_j^{\nu}(t)=X_j(t)+\nu_j t, \quad 1\leq j\leq N, \quad t\geq 0. \quad (7.54)
\end{aligned}
$$

Put $\nu(\cdot)=\sum_{j=1}^N \delta_{\nu_j}(\cdot)\in\mathfrak{M}_0$. In [32], we proved the equality

$$
\mathbb{E}_{\nu}\left[\prod_{j=1}^{N}\mathbf{1}(X_j(t)\geq h)\right]=\mathbb{E}_{N\delta_0}\left[\prod_{j=1}^{N}\mathbf{1}(X_j^{\nu}(1/t)\geq h/t)\right], \quad t>0. \quad (7.55)
$$

We call such a relation *reciprocal time relation* [32]. By applying Corollary 5.7, the LHS of (7.55) is given by CPR and we have the equality

$$
\begin{aligned}
&\mathbb{E}_{N\delta_0}\left[\prod_{j=1}^{N}\mathbf{1}(X_j^{\nu_j}(1/t)\geq h/t)\right] \\
&= \mathbf{E}_{\widehat{\boldsymbol{\nu}}}\left[\prod_{j=1}^{N}\mathbf{1}(\Re Z_j(1/t)\geq h/t) \det_{1\leq j,k\leq N}\left[\Phi_{\nu}^{\nu_k}(Z_j(1/t))\right]\right], \quad h\in\mathbb{R}. \quad (7.56)
\end{aligned}
$$

The determinant formula (7.53) can be regarded as a geometric lifting of the expression (7.56) for the noncolliding BM with drift.

The functions $\Phi_{\widehat{\nu}}^{\nu_k,a}(x), 1\leq k\leq N$ are not polynomial of x, but here we apply the equality (5.26). By multilinearity of determinant, the RHS of (7.53) becomes

$$
\begin{aligned}
&\mathrm{E}_{\widehat{\nu}}\left[\prod_{j=1}^{N}\mathbf{1}(\Re Z_j(1/t)\geq h/t) \det_{1\leq j,k\leq N}\left[\check{E}_0[\Phi_{\widehat{\nu}}^{\widehat{\nu}_k,a}(Z_j(1/t))]\right]\right] \\
&= \mathrm{E}_{\widehat{\nu}}\left[\prod_{j=1}^{N}\mathbf{1}(\Re Z_j(1/t)\geq h/t) \det_{1\leq j,k\leq N}\left[\mathsf{M}[\Phi_{\widehat{\nu}}^{\widehat{\nu}_k,a}(iW)|(1/t,B_j(1/t))]\right]\right].
\end{aligned}
$$

Then we put

$$\mathcal{M}_{\widehat{\nu}}^{u,a}(\cdot,\cdot) = \mathsf{M}[\Phi_{\widehat{\nu}}^{u,a}(iW)|(\cdot,\cdot)], \quad a > 0, u \in \mathbb{C}, \tag{7.57}$$

and

$$\mathcal{D}_{\widehat{\nu}}^{a}(1/t, \boldsymbol{B}(1/t)) = \det_{1 \le j,k \le N} \Big[\mathsf{M}[\Phi_{\widehat{\nu}}^{u,a}(iW)|(1/t, B_j(1/t))]\Big], \quad a > 0, \quad t > 0. \tag{7.58}$$

Then (7.53) is written as follows.

$$\begin{aligned} &\mathbb{E}^{a\circ\widehat{\nu},a}[\Theta^a(X_1^a(t) - h)] \\ &\quad = \mathrm{E}_{\widehat{\nu}}\left[\prod_{j=1}^{N} \mathbf{1}(B_j(1/t) \ge h/t)\mathcal{D}_{\widehat{\nu}}^{a}(1/T, \boldsymbol{B}(1/T))\right], \\ &\qquad a > 0, \quad h \in \mathbb{R}, \quad 0 < t \le T \le \infty. \end{aligned} \tag{7.59}$$

Proposition 7.3 *The O'Connell process has a variation of DMP for $\Theta^a(X_1^a(\cdot) - h), a > 0, h \in \mathbb{R}$ as given by (7.59). Then the expectation of $\Theta^a(X_1^a(\cdot) - h), a > 0, h \in \mathbb{R}$ is F-determinantal.*

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