Topics in random interfaces

Tadahisa Funaki

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Abstract

Randomly fluctuating interfaces, which arise in the situation of phase coexistence to separate two distinct phases, are studied in several different settings such as Ising model, effective interface models like $\nabla\varphi$-interface model or dynamic Young diagrams, sharp interface limit for stochastic Allen-Cahn equation (sometimes called time-dependent Ginzburg-Landau model or stochastic quantization, dynamic $P(\phi)$-model), Kardar-Parisi-Zhang (KPZ) equation and others.

This course focuses on three different topics and approaches, that is, (1) scaling limits for $\nabla\varphi$-interface model with pinning (static theory), (2) sharp interface limits for stochastic Allen-Cahn equation (dynamic theory) and (3) KPZ equation.

Assuming that the interface is represented as a height function measured from a fixed reference discretized hyperplane, the system is governed by the Hamiltonian of (gradient of) the height functions. This is called the $\nabla\varphi$-interface model. I will discuss the scaling limits for Gaussian (or non-Gaussian) random fields with a pinning effect under the situation that the rate functional of the corresponding large deviation principle has non-unique minimizers (joint works with E. Bolthausen and others [11], [12], [51], [46]).

Sharp interface limit for Allen-Cahn equation, that is a reaction-diffusion equation with bistable reaction term, leads to a motion of mean curvature for the interface. Its stochastic perturbation will be discussed. After mentioning some examples of stochastic partial differential equations (SPDEs), a brief introduction to the Brownian motions, martingales and stochastic integrals will be given in an infinite dimensional setting. Regularity property of solutions of SPDEs of parabolic type with additive noises will be discussed. The references for this part are papers [39], [40], [41], [57], [89]. A survey is given in [43].

KPZ equation describes a fluctuation of growing interfaces, and recently attracts a lot of attentions. This is an ill-posed SPDE and requires a renormalization. I will discuss its invariant measures (joint work with J. Quastel [52], [49]).

Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan.
e-mail: funaki@ms.u-tokyo.ac.jp

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Contents

1 Scaling limits for pinned Gaussian random fields under the presence of two possible candidates

The macroscopic shape of crystals is usually described by variational problems. We first explain those characterizing Wulff shape, Winterbottom shape and also variational problems with two phases (or two media), variational problem with pinning effect. We give some examples of minimizers. Next, we explain underlying microscopic models such as Ising model and $\nabla\varphi$-interface model. Macroscopic variational problems and microscopic models are linked by large deviation principle, or law of large numbers. We will focus on the $\nabla\varphi$-interface model with pinning. For such model, results for $d = 1, n \geq 1$ (with Bolthausen, Otobe [12], [51]) and those for $d \geq 3, n = 1$ (with Bolthausen, Chiyonobu [11]) will be presented, where d is the dimension of base space, while n is the dimension of value space. See [55], [46] for the $\nabla\varphi$-interface model.

1.1 Macroscopic variational problems

1.1.1 Wulff shape and Winterbottom shape

Let a direction-dependent (anisotropic) surface tension $\sigma : S^{d-1} \to (0, \infty)$ be given. Then, for a domain V (a droplet of water located in vapor) in $\mathbb{R}^d$ with a nice boundary $S = \partial V$, we define the total surface tension of S (total energy of the interface S) by

$$J(S) = \int_S \sigma(\vec{n}(x))dx,$$

where $\vec{n}(x)$ is the outward normal vector at $x \in S$ and dx is the surface element of S. Wulff shape (introduced in 1902) is determined as the minimizer of the variational problem:

$$\min_{\mathrm{vol}\, V=v} J(S),$$

for each given volume $v > 0$.

When the droplet is placed upon solid substrates, in addition to the water-vapor surface tension σ, one needs to consider the effect of the water-solid surface tension σ_{WS}. The equilibrium shape of droplet is called Winterbottom shape (introduced in 1967). In the context of the $\nabla\varphi$-interface model discussed later, if the solid substrates are located at the height level 0, the variational problem is formulated with pinning effect (see Section 1.1.3) and considered under the conditions $h \geq 0$ and the total volume $\int h dx = v > 0$, see [13].

1.1.2 Variational problem with two phases

We now consider the situation that the interface S can be described by a height function h measured from a fixed reference hyperplane: $S = \{(x, y); y = h(x), x \in D\}$, that is, S is represented as a graph of $y = h(x)$. Such model is called an effective model.

Let $D \Subset \mathbb{R}^d$ be given and for $h : D \to \mathbb{R}$ (or $\mathbb{R}^n$ later), consider the total energy given by

$$J(h) = \int_D \left\{\sigma(\nabla h(x)) - Q(x) 1_{\{h(x)\leq 0\}}\right\} dx, \tag{1.1}$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}$ is a convex function (e.g., $\sigma(u) = \frac{1}{2}|u|^2$) and $Q(x) \geq 0$. If h takes negative values, $J(h)$ becomes smaller and this means that the negative side has advantage. In other words, we consider a domain $D \times \mathbb{R}$, which is filled with a predominant media in the lower half $D \times (-\infty, 0]$ of this domain and an inferior media in the upper half $D \times (0, \infty)$. The analytic theory for J was studied by [1], [2], [91] and others.

It is easy to see that the minimizer h satisfies Euler equation:

$$\mathrm{div}\{\nabla\sigma(\nabla h)\} \equiv \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\frac{\partial \sigma}{\partial u_i}(\nabla h)\right) = 0 \quad \text{on } D \setminus \Gamma,$$

where $\Gamma = \{h = 0\}$, and the free boundary condition:

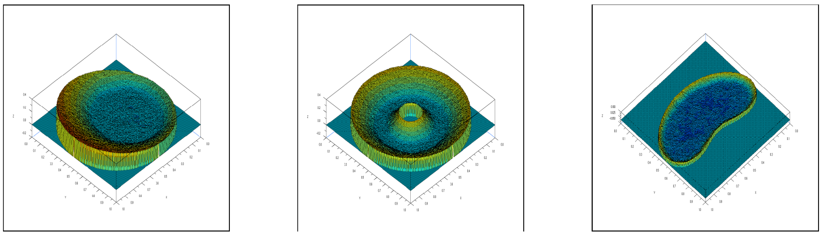
$$\Psi(\nabla h^+) - \Psi(\nabla h^-) = Q \quad \text{on } \Gamma,$$

where $\Psi(u) = u \cdot \nabla\sigma(u) - \sigma(u)$.
In particular, when $\sigma(u) = \frac{1}{2}|u|^2$, Euler equation and the free boundary condition are given by

$$\Delta h = 0 \quad \text{and} \quad |\nabla h^+|^2 - |\nabla h^-|^2 = 2Q(x),$$

respectively.

Another interpretation of this variational principle is that J describes the energy of flow of two liquids in models of jets and cavities and in this setting h is called a stream function.



$D =$ disc, $h|_{\partial D} \geq 0$ $\qquad$ $D =$ annulus

Figure 1: Shapes of minimizers

1.1.3 Variational problem with pinning effect

The last section discussed the situation that the negative side has advantage. Here, we consider the situation that a single point $\{0\}$ (or $D \times \{0\}$) has advantage. This is called a pinning effect. The problem becomes a bit singular compared with that discussed in the last section.

The energy of the height of interface $h : D \to \mathbb{R}$ (or $\mathbb{R}^n$) with pinning at $0 \in \mathbb{R}$ (or $\mathbb{R}^n$) is given by

$$J(h) = \int_D \left\{ \sigma(\nabla h(x)) - \xi 1_{\{h(x)=0\}} \right\} dx, \quad \xi > 0. \tag{1.2}$$

Euler equation and the free boundary condition for the minimizer h are similar to the two phases' problem at least if $h|_{\partial D} \geq 0$.

1.1.4 Examples of minimizers of J with pinning effect

Here, we consider the simple case: $d = 1, n \geq 1$, $\sigma = \frac{1}{2}|u|^2, D = [0,1]$. In this case, for $h : [0,1] \to \mathbb{R}^n$ satisfying $h(0) = a, h(1) = b$, the energy J with pinning at 0 is given by

$$J(h) = \frac{1}{2} \int_0^1 |\dot{h}(x)|^2 \, dx - \xi \Big| \{ x \in [0,1]; h(x) = 0 \} \Big|,$$

where $|\{\cdots\}|$ denotes the Lebesgue measure.

This energy J has two possible candidates $\bar{h}$ and $\hat{h}$, where $\bar{h}$ is a straight line connecting a and b.

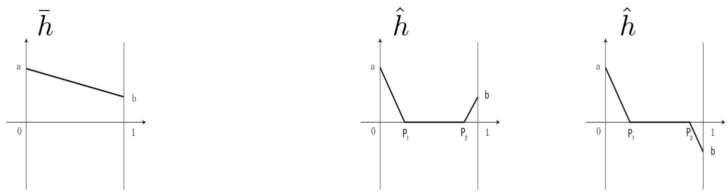


Figure 2: Possible candidates of minimizers

The Euler equation is given by $\Delta h = 0$. This means that the minimizers should be line except when they touch 0. In $\hat{h}$, let us denote $P_1(x_1, 0)$ and $P_2(1 - x_2, 0)$. Then, the free boundary condition called Young's relation is determined by

$$|a|/x_1 = |b|/x_2 = \sqrt{2\xi}.$$

We call critical (coexisting) case if $J(\bar{h}) = J(\hat{h})$ holds.

Proposition 1.1. *Assume $d = n = 1$, then the coexisting region is given by $\sqrt{|a|} + \sqrt{|b|} = (2\xi)^{1/4}, ab > 0$, see Figure 3.*

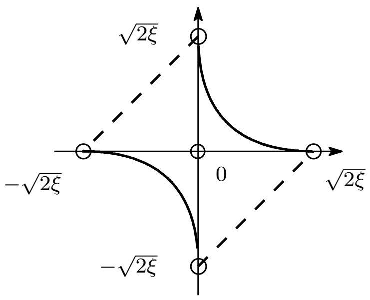


Figure 3: Coexisting region

The question we want to ask is the following: Which minimizer really appears at critical case. Macroscopically, these two minimizers have the same superiority. Our hope is that we can find out the difference if we observe the system from microscopic level.

1.2 Microscopic models

1.2.1 Wulff shape from microscopic models

Dobrushin-Kotecký-Shlosman [25], Ioffe and others [66], [67], [68] proved large deviation principle (LDP) for Ising model and derived Wulff shape under a scaling limit. Ising model is a microscopic model, while Wulff shape is a macroscopic shape of crystal. There are several results for the so-called $\nabla\varphi$-interface model taking as a microscopic model. Bolthausen-Ioffe [13] studied the $\nabla\varphi$-interface model with wall and pinning when $d = 2$ and derived Winterbottom shape as a macroscopic shape of crystal. Deuschel-Giacomin-Ioffe [23] proved the LDP for the $\nabla\varphi$-interface model and derived Wulff shape under the scaling limit in this setting. Funaki-Sakagawa [53] studied the $\nabla\varphi$-interface model with weak self potential, proved the LDP and derived the free boundary problem discussed in Section 1.1.2.

As we mentioned, macroscopic crystal shapes are characterized by variational problems and the large deviation principle (LDP) connects microscopic models with macroscopic variational problems. We quickly recall this procedure: Let $\{\mu_N\}_N$ be a sequence of probability measures describing microscopic model. Let h^N be the scaled heights (random variables) defined under μ_N. We call LDP holds with rate function $J^*(h)$ $\big(= J(h) - \inf J\big) \geq 0$ if

$$\mu_N(h^N \sim h) \sim e^{-NJ^*(h)}, \quad \text{as } N \to \infty.$$

Under the LDP, we easily have the concentration properties: Set $\mathcal{H} = \{h^*; \text{Minimizers of } J\}$, then for any $\delta > 0$, there exists $c = c_\delta > 0$ such that

$$\mu_N\big(\mathrm{dist}(h^N, \mathcal{H}) \geq \delta\big) \leq e^{-cN}$$

As we already stated, our question is to see what happens if the variational problem for J^* (so that for J) has several minimizers from microscopic model described by μ_N.

1.2.2 $\nabla\varphi$-interface model with pinning

Let us introduce our microscopic model called the $\nabla\phi$-interface model with pinning. The system is defined on a d-dim square lattice cylinder with size N, being periodic in 2nd-dth coordinates:

$$D_N = \{0, 1, 2, \dots, N\} \times \mathbb{T}_N^{d-1}, \quad \mathbb{T}_N^{d-1} = (\mathbb{Z}/N\mathbb{Z})^{d-1}.$$

We denote $D_N^\circ = D_N \setminus \partial D_N$, $\partial D_N = \partial_L D_N \cup \partial_R D_N$, $\partial_L D_N = \{0\} \times \mathbb{T}_N^{d-1}$ and $\partial_R D_N = \{N\} \times \mathbb{T}_N^{d-1}$.

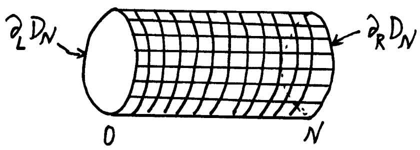


Figure 4: Lattice cylinder

The microscopic objects (height function when $n = 1$) $\phi = (\phi_i)_{i \in D_N} : D_N \to \mathbb{R}^n$ is a field defined on D_N. The energy (Hamiltonian) of ϕ is given as the sum over all bonds (i.e., pairs of nearest neighbor sites) $\langle i, j \rangle$ in D_N:

$$H_N(\phi) = \frac{1}{2} \sum_{\langle i,j \rangle \subset D_N} |\phi_i - \phi_j|^2, \tag{1.3}$$

with the boundary conditions at ∂D_N determined by given macroscopic values $a, b \in \mathbb{R}^n$:

$$\phi_i = aN, i \in \partial_L D_N \text{ and } \phi_i = bN, i \in \partial_R D_N. \tag{1.4}$$

Then, the microscopic model for ϕ is described by the Gibbs measure with pinning:

$$\mu_N^\varepsilon(d\phi) = \frac{1}{Z_N^\varepsilon} e^{-H_N(\phi)} \prod_{i \in D_N^\circ} \big(\varepsilon\delta_0(d\phi_i) + d\phi_i\big), \tag{1.5}$$

where $\varepsilon \geq 0$ is a parameter called the strength of pinning and Z_N^ε is the normalizing constant called the partition function. Note that μ_N^0 (i.e. $\varepsilon = 0$) is Gaussian. When $d = 1$, μ_N^ε defines a random walk and we will consider general non-Gaussian case in Section 1.3.

Scaling from microscopic object ϕ to macroscopic object h^N is defined as follows: Let $D = [0, 1] \times \mathbb{T}^{d-1}$ be a continuous cylinder of size 1, where $\mathbb{T}^{d-1} = (\mathbb{R}/\mathbb{Z})^{d-1}$. Then the macroscopic field $h^N = (h^N(x))_{x \in D} \in C(D, \mathbb{R}^n)$ is defined as

$$h^N\left(\frac{i}{N}\right) = \frac{1}{N}\phi_i, \quad i \in D_N, \tag{1.6}$$

and its extension to D as a step function. We sometimes take a polilinear interpolation instead of step extension, see [11], [23]. Our problem is to find the limit of h^N under μ_N^ε as $N \to \infty$. Microscopic model corresponding to two phases' case was studied by [53].

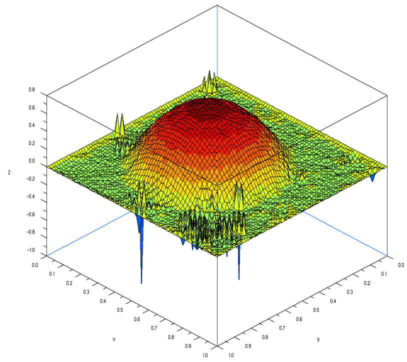


Figure 5: Microscopic droplet

The effect of microscopic pinning is macroscopically reflected in the pinning free energy defined by

$$\xi^\varepsilon = \lim_{\ell \to \infty} \frac{1}{|\Lambda_\ell|} \log \frac{Z^{0,\varepsilon}_{\Lambda_\ell}}{Z^0_{\Lambda_\ell}}, \tag{1.7}$$

where $\Lambda_\ell = \{1, 2, \dots, \ell\}^d \Subset \mathbb{Z}^d$, $|\Lambda_\ell| = \ell^d$ and $Z^{0,\varepsilon}_{\Lambda_\ell}, Z^0_{\Lambda_\ell}$ are partition functions on Λ_ℓ with 0-boundary condition with/without pinning, respectively.

It is known that there exists $\varepsilon_c \geq 0$ such that $\varepsilon > \varepsilon_c \Leftrightarrow \xi^\varepsilon > 0$ (localization) holds. Moreover, we have that

$$d = 1, n \geq 3 \Rightarrow \varepsilon_c > 0 \;\; \text{(Pinning transition occurs)},$$
$$d \geq 1, n = 1 \text{ or } d = 1, n = 2 \Rightarrow \varepsilon_c = 0 \;\; \text{(No transition occurs; always localized)}.$$

See Theorem 1.1 of [51] for $d = 1$ and Section 7 of [46] for localization when $d \geq 2, n = 1$.

1.3 Results for $d = 1, n \geq 1$

We take $D_N = \{0, 1, 2, \dots, N\}$ and consider the microscopic system $\phi = \{\phi_i\}_{i \in D_N}$ distributed under the measure $\mu_N \equiv \mu_N^\varepsilon$ on $(\mathbb{R}^n)^{D_N^\circ}$ defined by

$$\mu_N(d\phi) = \frac{1}{Z_N^\varepsilon} \prod_{i=1}^{N} p(\phi_i - \phi_{i-1}) \prod_{i=1}^{N-1} \big(\varepsilon \delta_0(d\phi_i) + d\phi_i\big),$$

satisfying the boundary condition (1.4): $\phi_0 = aN, \phi_N = bN$ with $a, b \in \mathbb{R}^n$, where p is a probability density function on $\mathbb{R}^n$: $\int_{\mathbb{R}^n} p(x)dx = 1$. We assume that p satisfies

$$\sup_{x \in \mathbb{R}^n} e^{\lambda \cdot x} p(x) < \infty$$

for all $\lambda \in \mathbb{R}^n$. Note that μ_N is a generalization of the Gibbs measure with pinning defined by (1.5) when $d = 1$. In fact, we may take $p(x) = (2\pi)^{-n/2}e^{-|x|^2/2}$.

The macroscopic height h^N is defined by (1.6) and its linear interpolation, that is,

$$h^N(x) = \frac{[Nx] - Nx + 1}{N}\phi_{[Nx]} + \frac{[Nx] - Nx}{N}\phi_{[Nx]+1}, \quad x \in D = [0, 1].$$

We define $\sigma(u), u \in \mathbb{R}^n$ by the Legendre transform

$$\sigma(u) = \sup_{\lambda\in\mathbb{R}^n} \{\lambda \cdot u - \Lambda(\lambda)\}$$

of

$$\Lambda(\lambda) = \log \int_{\mathbb{R}^n} e^{\lambda\cdot x}p(x)dx.$$

Consider J defined by (1.2) with this σ, $\xi = \xi^\varepsilon$ and $D = [0, 1]$; note that ξ^ε can be defined also in this setting. Then, one can show the sample path LDP [53], [12], [51], which is roughly formulated as

$$\mu_N(h^N \underset{L^\infty}{\sim} h) \sim e^{-NJ^*(h)}, \quad \text{as } N \to \infty,$$

for $h \in C([0, 1], \mathbb{R}^n)$. The rate function is given by $J^*(h) = J(h) - \inf J$. Here, "$h^N \underset{L^\infty}{\sim} h$" means h^N and h are close in $L^\infty([0, 1])$-norm. Note that this result is well-known when $\varepsilon = 0$ and ϕ_N is free (instead of $\phi_N = bN$), and $\sigma(u) = \frac{1}{2}|u|^2$ in the Gaussian case. The concentration property implies that

$$h^N \underset{N\to\infty}{\longrightarrow} \{\text{Minimizers of } J\},$$

in probability in L^∞-norm.

The results on the scaling limits at criticality ($d = 1$) are summarized in the following:

Theorem 1.2. *([51], [12]) Let $\bar{h}$ and $\hat{h}$ be two possible minimizers given in Section 1.1.4 and assume the condition $J(\bar{h}) = J(\hat{h})$.*
(1) When $n = 1$, the limit of h^N under μ_N is $\hat{h}$: For every $\delta > 0$,

$$\lim_{N\to\infty} \mu_N(\|h^N - \hat{h}\|_\infty \le \delta) = 1,$$

where $\|\cdot\|_\infty$ is the supremum norm on $[0, 1]$.
(2) When $n = 2$, a coexistence occurs as $N \to \infty$, that is, the limit of h^N under μ_N is a mixture of $\bar{h}$ and $\hat{h}$: For every $\delta > 0$ small enough,

$$\lim_{N\to\infty} \mu_N(\|h^N - \hat{h}\|_\infty \le \delta) = \hat{\lambda},$$
$$\lim_{N\to\infty} \mu_N(\|h^N - \bar{h}\|_\infty \le \delta) = \bar{\lambda},$$

with some $0 < \hat{\lambda} < 1$, $\hat{\lambda} + \bar{\lambda} = 1$.
(3) When $n \ge 3$, the limit of h^N under μ_N is $\bar{h}$: For every $\delta > 0$,

$$\lim_{N\to\infty} \mu_N(\|h^N - \bar{h}\|_\infty \le \delta) = 1.$$

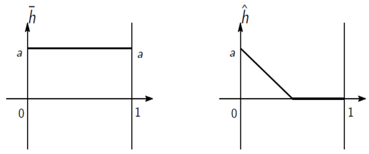


Figure 6: Possible minimizers (Free boundary case)

This can be extended to the case of free boundary condition at $\partial_R D = \{N\}$ for the microscopic system. In fact, the pinning free energy $\xi^{\varepsilon,F}$ is similar but a bit modified and LDP is essentially the same with $\xi = \xi^{\varepsilon,F}$. Possible minimizers $\bar h$ and $\hat h$ in this setting are as in Figure 6.

Critical (coexisting) case ($n = 1$) is:

$$J(\bar h) = J(\hat h) \Leftrightarrow a = \pm\sqrt{\xi/2}$$

Theorem 1.3. *([51], [12]) Assume $J(\bar h) = J(\hat h)$.*
(1) When $n = 1$, a coexistence occurs as $N \to \infty$.
(2) When $n \geq 2$, the limit of h^N is $\bar h$.

Another extension is the case with wall effect, that is, we replace $d\phi_i$ in the definition of the Gibbs measure with the Lebesgue measure $d\phi_i^+$ on $\mathbb{R}^+$. Then, the free energy is replaced by $\xi^{\varepsilon,+}$ or $\xi^{\varepsilon,F,+}$ and it is known that there exists $\varepsilon_c^+ \geq 0$ such that $\varepsilon > \varepsilon_c^+ \Leftrightarrow \xi^{\varepsilon,+} > 0$. Moreover,

$$\begin{aligned}
&d = 1, n \geq 1 \Rightarrow \varepsilon_c^+ > \varepsilon_c\, (\geq 0) \ \ \text{(in particular } \varepsilon_c^+ > 0),\\
&d = 2, n = 1 \Rightarrow \varepsilon_c^+ > 0,\\
&d \geq 3, n = 1 \Rightarrow \varepsilon_c^+ = 0 \text{ (No transition)}.
\end{aligned}$$

See Theorem 1.1 of [12] for $d = 1$ and Section 7.3 of [46] for $d \geq 2, n = 1$. We assume $a, b \in \mathbb{R}_+^n$. Then, under the balance condition $J^+(\bar h) = J^+(\hat h)$, theorems analogous to Theorems 1.2 and 1.3 hold. When $d = 1, n \geq 1$, critical exponents of $\xi^\varepsilon, \xi^{\varepsilon,+}$ at $\varepsilon = \varepsilon_c, \varepsilon_c^+$ can be computed.

1.4 Outline of the proof

The proofs of Theorems 1.2 and 1.3 go as follows. The balance condition $J(\bar h) = J(\hat h)$ implies that the (leading) exponential order of R_N vanishes, where

$$R_N = \frac{\mu_N(h^N \in \text{nbd of } \hat h)}{\mu_N(h^N \in \text{nbd of } \bar h)}, \tag{1.8}$$

and the neighborhoods are in L^∞-sense. We then compute its prefactor of the form N^α and finally obtain that

$$R_N \sim \begin{cases} N^{1-\frac{n}{2}}, & \text{in the Dirichlet case,} \\ N^{\frac12-\frac{n}{2}}, & \text{in the free case.} \end{cases}$$

This leads to the conclusion of theorems. We show this in the Dirichlet case.

We denote the partition function Z_N^ε by $Z_N^{aN,bN,\varepsilon}$ to indicate the boundary conditions and $Z_N^{aN,bN,0}$ (i.e. $\varepsilon = 0$) simply by $Z_N^{aN,bN}$, respectively. Similarly, we denote μ_N^ε by $\mu_N^{aN,bN,\varepsilon}$ and $Z_N^{aN,bN,0}$ simply by $Z_N^{aN,bN}$, respectively. Then, since the condition: "$h^N \in L^\infty$-nbd of $\bar{h}$" implies that ϕ never touch 0 (if the line connecting a and b does not touch 0 and if the L^∞-neighborhood is chosen small enough), we see that

$$Z_N^{aN,bN,\varepsilon}\mu_N(h^N \in \text{nbd of } \bar{h}) \sim Z_N^{aN,bN}.$$

Therefore, we have

$$\begin{aligned} R_N \sim & \frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}}\mu_N(h^N \in \text{nbd of } \hat{h}) \\ \sim & \sum_{j<k} \Xi_{N,j,k}^\varepsilon \, \mu_j^{aN,0}(h^N \in \text{nbd of } \hat{h}) \\ & \qquad \times \mu_{k-j}^{0,0,\varepsilon}(h^N \in \text{nbd of } \hat{h})\, \mu_{N-k}^{0,bN}(h^N \in \text{nbd of } \hat{h}), \end{aligned} \tag{1.9}$$

where

$$\begin{aligned} \Xi_{N,j,k}^\varepsilon := & \frac{Z_j^{aN,0} Z_{k-j}^{0,0,\varepsilon} Z_{N-k}^{0,bN}}{Z_N^{aN,bN}} \\ = & \frac{Z_j^{aN,0} Z_{k-j}^{0,0} Z_{N-k}^{0,bN}}{Z_N^{aN,bN}} \times \frac{Z_{k-j}^{0,0,\varepsilon}}{Z_{k-j}^{0,0}} =: \mathrm{A} \times \mathrm{B}. \end{aligned}$$

To derive the second line for R_N, we expand the product measure $\prod_{i=1}^{N-1}(\varepsilon\delta_0(d\phi_i) + d\phi_i)$ and show that the probability that ϕ touches 0 at most once is negligible. Thus, we may only consider the probability that ϕ touches 0 at least twice. In this event, j and k mean the first and last hitting times of ϕ at 0, respectively.

By the local central limit theorem, as $k \to \infty$ such that $k/N \to r \in (0,1]$, we have

$$Z_k^{a,b} \sim \frac{1}{(2\pi k)^{n/2}\sqrt{\det Q((b-a)/r)}} \exp\left\{-k\sigma\left(\frac{N(b-a)}{k}\right)\right\},$$

where $Q(v)$ is the covariance matrix of the Cramér transform $p_{\lambda(v)}$ of p, which has mean $v \in \mathbb{R}^n$. Therefore, we obtain the asymptotic behavior of A:

$$\mathrm{A} \sim \left(\frac{N}{j(k-j)(N-k)}\right)^{n/2} \exp\left\{-N\widetilde{f}(s_1, s_2)\right\},$$

where $s_1 = j/N, s_2 = 1 - k/N$ and

$$\tilde{f}(s_1, s_2) = s_1\sigma\left(-\frac{a}{s_1}\right) + s_2\sigma\left(\frac{b}{s_2}\right) + (1 - s_1 - s_2)\sigma(0) - \sigma(b-a).$$

On the other hand, applying the renewal theory based on the renewal equation for $Z_N^{0,0,\varepsilon}$:

$$Z_N^{0,0,\varepsilon} = Z_N^{0,0} + \varepsilon \sum_{i=1}^{N-1} Z_i^{0,0} Z_{N-i}^{0,0,\varepsilon}, \quad N \geq 2,$$

with $Z_1^{0,0,\varepsilon} = Z_1^{0,0} = 1$, we have

$$\mathrm{B} \sim (k-j)^{n/2} \exp\big\{N\xi^\varepsilon(1-s_1-s_2)\big\}.$$

These asymptotics of A and B are summarized into

$$\begin{aligned} R_N &\sim \left(\frac{N}{N^3}\right)^{n/2} \times N^{n/2} \times \sum_{j\sim t_1 N, k\sim(1-t_2)N} e^{-Nf(s_1,s_2)} \\ &\sim N^{-n/2} \times (\sqrt{N})^2 = N^{-n/2+1}, \end{aligned}$$

where $f(s_1,s_2) \sim c_1(t_1-s_1)^2 + c_2(t_2-s_2)^2$ with some $c_1, c_2 > 0$. Note that

$$\begin{aligned} \sum_{j\sim t_1 N} e^{-Nf(s_1,s_2)} &\sim \sum_j e^{-c_1 N (j/N)^2} \\ &= \sum_j e^{-c_1 (j/\sqrt{N})^2} \sim \sqrt{N} \int_{\mathbb{R}} e^{-c_1 x^2}\, dx. \end{aligned}$$

1.5 Results for $d \geq 3, n = 1$

We now consider a random field ϕ defined on a higher dimensional lattice cylinder D_N. We assume $d \geq 3$ from a technical reason and discuss the case of $n = 1$ only. The microscopic model is determined by the Gibbs measure μ_N^ε with pinning on $\mathbb{R}^{D_N^\circ}$ introduced in (1.5) under the boundary condition $\phi = aN$ on $\partial_L D_N$ and $\phi = bN$ on $\partial_R D_N$ with given $a, b \in \mathbb{R}$. Actually we take $a, b > 0$. Recall that μ_N^0 (i.e. pinning $\varepsilon = 0$) is Gaussian.

The corresponding macroscopic variational problem with pinning is determined as follows: Let $D = [0,1] \times \mathbb{T}^{d-1}$ be the continuous cylinder. We write its coordinate as $x = (x_1, \underline{x}) \in D$. Let $\xi > 0$ be given and consider

$$J(h) = \frac{1}{2}\int_D |\nabla h(x)|^2\, dx - \xi \Big|\{x \in D; h(x) = 0\}\Big|$$

for $h : D \to \mathbb{R}$ satisfying $h(0,\underline{x}) = a, h(1,\underline{x}) = b$, where $|\{\cdots\}|$ stands for the Lebesgue measure.

The microscopic model and the macroscopic energy functional J are linked by the (expected) LDP:

$$\mu_N^\varepsilon(h^N \underset{L^p}{\sim} h) \sim e^{-N^d J^*(h)}, \quad \text{as } N \to \infty,$$

where $2 \leq p < \frac{2d}{d-2}$ and the rate function should be given by $J^*(h) = J(h) - \inf J$ with $\xi = \xi^\varepsilon$ in (1.7).

Because of the special choice of the domain D, we see that there are only two possible candidates $\bar{h}$ and $\hat{h}$ of minimizers of J determined by

$$\bar{h}(x) = \bar{h}^{(1)}(x_1), \quad \hat{h}(x) = \hat{h}^{(1)}(x_1),$$

where $\bar{h}^{(1)}$ and $\hat{h}^{(1)}$ are possible minimizers in one-dimensional problem; see Figure 7.

The result on the scaling limits at criticality ($d \geq 3$) is formulated as follows:

Figure 7: Possible minimizers ($d \geq 2$)

Theorem 1.4. *([11]) We assume $J(\bar{h}) = J(\hat{h})$. If $d \geq 3, n = 1$ and if $\varepsilon > 0$ is sufficiently large, h^N converges to $\hat{h}$ in $L^1(D)$ in probability: For every $\delta > 0$,*

$$\lim_{N\to\infty} \mu_N^\varepsilon(\|h^N - \hat{h}\|_{L^1(D)} \leq \delta) = 1.$$

In higher dimensions, differently from the one-dimensional case, it is known that sharp spikes appear at microscopic level as Figure 5 shows. Therefore, the norm in $L^1(D)$ cannot be replaced by that of $L^\infty(D)$.

1.6 Outline of the proof

If we consider the ratio R_N of two probabilities as in (1.8), the leading order (exponential volume order) vanishes under the balance condition. The proof is reduced to show the following three assertions. We consider the quantity appearing in (1.9) instead of R_N.

- Lower bound (Surface order): For every $0 < \alpha < 1$ and $1 \leq p \leq 2$,

$$p_N := \frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \mu_N^\varepsilon(\|h^N - \hat{h}\|_{L^p(D)} \leq N^{-\alpha}) \geq e^{cN^{d-1}}$$

  with $c = c_\varepsilon (= c_{\varepsilon,\alpha}) > 0$ for $N \geq N_0$ if $\varepsilon > 0$ is large enough. The upper suffixes aN, bN in the partition functions indicate the boundary conditions and $Z_N^{aN,bN} = Z_N^{aN,bN,0}$ (i.e. $\varepsilon = 0$) as before.

- Upper bound (Capacity order): There exists $\alpha_0 > 0$ such that

$$q_N := \frac{Z_N^{aN,bN,\varepsilon}}{Z_N^{aN,bN}} \mu_N^\varepsilon(\|h^N - \bar{h}\|_{L^p(D)} \leq (\log N)^{-\alpha_0}) \leq 2$$

  for $N \geq N_0$. (Note that if $\|\cdot\|_{L^p}$ could be replaced by $\|\cdot\|_{L^\infty}$, then "$q_N \leq 1$" is trivial. But this is not the case when $d \geq 2$ because of spikes as we pointed out.)

- LD type estimate: There exists $\alpha_1 > 0$ such that

$$\lim_{N\to\infty} \mu_N^\varepsilon\big(\operatorname{dist}_{L^1}(h^N, \{\hat{h}, \bar{h}\}) \geq N^{-\alpha_1}\big) = 0.$$

1.6.1 Proof of lower bound

This part is a rather rude estimate and we need to assume that the strength of pinning $\varepsilon > 0$ is large enough. We divide D_N° as $D_N^\circ = A_L \cup \gamma_L \cup B \cup \gamma_R \cup A_R$:

$$\begin{aligned} A_L &= \big([1, Ns_1^L - K - 1] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\ \gamma_L &= \big([Ns_1^L - K, Ns_1^L] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\ B &= \big([Ns_1^L + 1, Ns_1^R] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\ \gamma_R &= \big([Ns_1^R + 1, Ns_1^R + K] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \\ A_R &= \big([Ns_1^R + K + 1, N - 1] \cap \mathbb{Z}\big) \times \mathbb{T}_N^{d-1}, \end{aligned}$$

with some $K \in \mathbb{N}$, where s_1^L and $s_1^R \in (0,1)$ are the first and last s's such that $\hat{h}^{(1)}(s) = 0$. Then, restricting the probability appearing in p_N on the event that $\phi_i = 0$ for all $i \in \gamma_L \cup \gamma_R$, the Markov property of $\mu_N^{aN,bN,\varepsilon}$ proves that

$$\begin{aligned} p_N \geq & \frac{Z_{A_L}^{aN,0} Z_B^{0,0,\varepsilon} Z_{A_R}^{0,bN}}{Z_N^{aN,bN}} \varepsilon^{|\gamma_L|+|\gamma_R|} \times \mu_{A_L}^{aN,0}(\|h^N - \hat{h}\|_{L^p(D_L)} \leq \delta) \\ & \times \mu_B^{0,0,\varepsilon}(\|h^N - \hat{h}\|_{L^p(D_M)} \leq \delta) \mu_{A_R}^{0,bN}(\|h^N - \hat{h}\|_{L^p(D_R)} \leq \delta), \end{aligned}$$

where the definitions of the measures $\mu_{A_L}^{aN,0}, \mu_B^{0,0,\varepsilon}, \mu_{A_R}^{0,bN}$ are similar as above and clear, and D_L, D_M, D_R are the macroscopic regions corresponding to A_L, B, A_R, respectively. However, LDP for $\mu_{A_L}^{aN,0}, \mu_B^{0,0}, \mu_{A_R}^{0,bN}$ and FKG type argument applied for $\mu_B^{0,0,\varepsilon}$ show that the three probabilities in the right hand side are all close to 1 for N sufficiently large. Thus, for every $c > 0$, we obtain

$$p_N \geq \frac{Z_{A_L}^{aN,0} Z_B^{0,0,\varepsilon} Z_{A_R}^{0,bN}}{Z_N^{aN,bN}} \varepsilon^{|\gamma_L|+|\gamma_R|} \times (1 - c) =: \Xi_N \times (1 - c),$$

for $N \geq N_0$.

Next, we need estimates on the partition functions. By the Gaussian property, we have

$$\begin{aligned} Z_N^{aN,bN} &= \exp\left\{-\frac{N^d}{2}(a-b)^2\right\} Z_N^{0,0}, \\ Z_{A_L}^{aN,0} &= \exp\left\{-\frac{a^2 N^d}{2(s_1^L - K/N)}\right\} Z_{A_L}^{0,0}, \\ Z_{A_R}^{0,bN} &= \exp\left\{-\frac{b^2 N^d}{2(1 - s_1^R - K/N)}\right\} Z_{A_R}^{0,0}. \end{aligned}$$

However, we have the expansion $1/(s_1^L - K/N) = 1/s_1^L + KN^{-1}/(s_1^L)^2 + O(N^{-2})$ and a similar one for $1/(1 - s_1^R - K/N)$ as $N \to \infty$. Thus, we have

$$\Xi_N^1 = \exp\left\{f(a,b)N^d - K\tilde{f}(a,b)N^{d-1} - O(N^{d-2})\right\} \Xi_N^0,$$

where Ξ_N^0 is Ξ with $a = b = 0$, and

$$f(a,b) = \frac{1}{2}(a-b)^2 - \frac{a^2}{2s_1^L} - \frac{b^2}{2(1 - s_1^R)}$$

$$= J(\bar{h}) - J(\hat{h}) - \xi^\varepsilon (s_1^R - s_1^L),$$

$$\tilde{f}(a,b) = \frac{a^2}{2(s_1^L)^2} + \frac{b^2}{2(1-s_1^R)^2}.$$

Note that $\tilde{f}(a,b) = 2\xi^\varepsilon$ holds from Young's relation for the angles of $\hat{h}$ at $s = s_1^L$ and s_1^R: $a/s_1^L = b/(1-s_1^R) = \sqrt{2\xi^\varepsilon}$.

On the other hand, by the random walk representation of partition functions, we have

$$\frac{Z_{A_L}^{0,0} Z_{A_R}^{0,0}}{Z_N^{0,0}} \geq \exp\left\{ \hat{q}^0 (|A_L| + |A_R| - |D_N^\circ|) - C_1 N^{d-1} \right\},$$

where

$$\hat{q}^0 = \frac{1}{2}(\log \frac{\pi}{d} + q), \quad q = \sum_{k=1}^\infty \frac{1}{2k} P_0^{RW^d}(\eta_{2k} = 0)$$

Moreover, by the decoupling estimate [13], we have

$$Z_B^{0,0,\varepsilon} \geq \exp\left\{ \hat{q}^\varepsilon |B| - \frac{3}{2} c N^{d-1} \right\},$$

where $\hat{q}^\varepsilon = \hat{q}^0 + \xi^\varepsilon$, $c = G(0,0)$ and G is the Green function of the d-dimensional random walk on $\mathbb{Z}^d$; recall $d \geq 3$.

Summarizing these estimates, the leading term of $e^{O(N^d)}$ cancels by the balance condition, and then by noting $\xi^\varepsilon \leq \log 2\varepsilon$ (for $\varepsilon \geq 1$), we have

$$\begin{aligned}
\log p_N &\geq \big((2K+1)(\log \varepsilon - \hat{q}^0) - 2K\xi^\varepsilon - C_1 - \frac{3}{2} c \big) N^{d-1} - O(N^{d-2}) \\
&\geq \big(\log \varepsilon - (2K+1)\hat{q}^0 - 2K \log 2 - C_1 - \frac{3}{2} c \big) N^{d-1} - O(N^{d-2}).
\end{aligned}$$

The coefficient of N^{d-1} is positive if $\varepsilon > 0$ is large enough. This completes the proof of the lower bound.

1.6.2 Proof of upper bound

We expand the measure $\prod_{i \in D_N^\circ} \big(\varepsilon \delta_0(d\phi_i) + d\phi_i\big)$ to have

$$q_N = \sum_{A \subset D_N^\circ} \varepsilon^{|A^c|} \frac{Z_A^{aN,bN,0}}{Z_N^{aN,bN}} \mu_A^{aN,bN,0}(\|h^N - \bar{h}\|_{L^p(D)} \leq \delta),$$

with $\delta = (\log N)^{-\alpha_0}, \alpha_0 > d/p$. Here, $\mu_A^{aN,bN,0}$ denotes the Gibbs measure without pinning and with boundary conditions 0 on $A^c = D_N^\circ \setminus A$ and (1.4) on ∂D_N. If $|A^c| \geq (N/\log N)^d$, then $h^N = 0$ on $\frac{1}{N} A^c$ so that $\|h^N - \bar{h}\|_{L^p} \geq c(\log N)^{-d/p}$ and $\mu_A^{aN,bN,0}(\|h^N - \bar{h}\|_{L^p(D)} \leq \delta) = 0$. Therefore, we may prove

$$r_N := \sum_{A \subset D_N^\circ : |A^c| \leq (N/\log N)^d} \varepsilon^{|A^c|} \frac{Z_A^{aN,bN,0}}{Z_N^{aN,bN}} \leq 2.$$

To this end, we first introduce several notation:

$$\begin{aligned}
&P_{A^c}(i,j) = P_i(\text{RW enters } A^c \text{ at } j \text{ before reaching } \partial D_N),\\
&p_L(i) = P_i(\text{RW does not return to } A^c \text{ and leave } D_N \text{ on } \partial_L D_N),\\
&p_R(i) = P_i(\text{RW does not return to } A^c \text{ and leave } D_N \text{ on } \partial_R D_N),\\
&e_{A^c}(i) = 1 - \sum_{j\in A^c} P_{A^c}(i,j)\big(= p_L(i) + p_R(i)\big),\\
&\mathrm{cap}_{D_N}(A^c) = \sum_{i\in A^c} e_{A^c}(i),
\end{aligned}$$

where RW means the random walk on $\mathbb{Z}\times\mathbb{T}_N^{d-1}$ starting at i. $e_{A^c}(i)$ and $\mathrm{cap}_{D_N}(A^c)$ are called the escape probability and the capacity. Then, by the Gaussian property, we get

$$\frac{Z_A^{aN,bN,0}}{Z_N^{aN,bN}} = \frac{1}{\sqrt{(2\pi)^{|A^c|}\det\Gamma_{A^c}}}\exp\left\{-d\left\langle m,(I-P_{A^c})\,m\right\rangle_{A^c}\right\}, \tag{1.10}$$

where $P_{A^c} = (P_{A^c}(i,j))_{i,j\in A^c}, \Gamma_{A^c} = (I-P_{A^c})^{-1}$ and m is the linear interpolation on D° satisfying the boundary condition (1.4). The denominator in the right hand side of (1.10) can be estimated by $e^{-d|A^c|}$ from below.

On the other hand, for the numerator, we see that

$$m\,(i) = \sum_{j\in A^c} P_{A^c}\,(i,j)\,m\,(j) + p_L\,(i)\,aN + p_R\,(i)\,bN.$$

Therefore, we obtain

$$(I-P_{A^c})m(i) \geq \min\,(a,b)\,Ne_{A^c}\,(i)\,,\quad i\in A^c.$$

Since we also have $m\,(i)\geq \min\,(a,b)\,N$, this combined with the estimate on the denominator leads to

$$\frac{Z_A^{aN,bN,0}}{Z_N^{aN,bN}} \leq e^{d|A^c|-cN^2\mathrm{cap}_{D_N}(A^c)}, \tag{1.11}$$

with some $c>0$.

However, we have the capacity bound:

$$\mathrm{cap}_{D_N}(A^c) \geq c|A^c|^{(d-2)/d}. \tag{1.12}$$

In fact, this can be shown as follows. First we chop the cylinder D_N into several boxes and show that (1.12) follows once we can show this for A which is not connected in the vertical direction (i.e. the direction of $\mathbb{T}_N^{d-1}$). This reduces the problem on $\mathbb{Z}^d$. But (1.12) on $\mathbb{Z}^d$ in place of D_N follows from the isoperimetric inequality:

$$|\partial A| \geq c|A|^{(d-1)/d}$$

combined with the result of [69].

Now, setting $\chi(k) = \sharp\{A; |A^c| = k\}$, we have from (1.11) and (1.12)

$$r_N \le \sum_{k=0}^{(N/\log N)^d} \chi(k)\varepsilon^k e^{dk-\bar{c}N^2k^{(d-2)/d}}.$$

The term with $k=0$ is 1, while the sum of other terms ≤ 1 if N is large. This concludes the proof of the upper bound.

1.6.3 Proof of LD type estimate

We first show the stability of J at macroscopic level:

$$J^*(h) \le \delta \Longrightarrow \mathrm{dist}_{L^1}(h, \{\bar{h}, \hat{h}\}) \le C\sqrt{\delta}. \tag{1.13}$$

Next, we introduce mesoscopic regions as follows: Given $0 < \beta < 1$, we divide D_N into $N^{d(1-\beta)}$ subboxes of sidelength N^β. We write $\mathcal{B}_N$ for the set of these subboxes, and $\hat{\mathcal{B}}_N$ for the set of unions of boxes in $\mathcal{B}_N$. For $B \in \hat{\mathcal{B}}_N$, which is called a mesoscopic region, set

$$\begin{aligned} E_{N,0}(B) &= \inf_{\phi\in\mathbb{R}^{D_N}:(1.14)} H_N(\phi), \\ E_N(B) &= E_{N,0}(B) - \xi^\varepsilon|B^c|, \\ E_N^*(B) &= E_N(B) - \min_{B\in\hat{\mathcal{B}}_N} E_N(B), \end{aligned}$$

where $H_N(\phi)$ is defined in (1.3) and the infimum is taken over all $\phi \in \mathbb{R}^{D_N}$ satisfying the condition:

$$\phi_i = \begin{cases} aN & \text{if } i \in \partial_L D_N \\ bN & \text{if } i \in \partial_R D_N \\ 0 & \text{if } i \in D_N^\circ \backslash B. \end{cases} \tag{1.14}$$

Then, the stability (1.13) of J at macroscopic level can be extended to the mesoscopic level: For $\alpha > 0$, there exists $\delta = \delta(\alpha) > 0$ such that, if N is large enough,

$$E_N^*(B) \le N^{d-\delta} \Longrightarrow \mathrm{dist}_{L^1}(h_B^N, \{\bar{h}, \hat{h}\}) \le N^{-\alpha},$$

holds for $B \in \hat{\mathcal{B}}_N$, where h_B^N is defined as follows: Let $\bar{\phi}^B$ be the harmonic function on B subject to the condition (1.14). Then, the macroscopic profile h_B^N is defined from the microscopic profile $\bar{\phi}^B$ by polilinearly interpolating $\frac{1}{N}\bar{\phi}^B_{[Nx]}, x \in D$.

We further introduce mesoscopic wetted region: Fix $\gamma > 0$ and

$$\mathcal{M}_N \equiv \mathcal{M}_N(\phi) \stackrel{\mathrm{def}}{=} \bigcup \left\{ C \in \mathcal{B}_N : \phi_C^{\mathrm{cg},\beta,N} \ge N^\gamma \right\},$$

where, for $C \in \mathcal{B}_N$,

$$\phi_C^{\mathrm{cg},\beta,N} \stackrel{\mathrm{def}}{=} N^{-d\beta} \sum_{j\in C} \phi_j.$$

We now decompose and estimate the probability of target as follows:

$$\mu_N^\varepsilon\left(\mathrm{dist}_{L^1}(h^N,\{\bar h,\hat h\})\geq N^{-\alpha}\right)=\sum_{B\in\hat{\mathcal{B}}_N}\mu_N^\varepsilon\left(\mathrm{dist}_{L^1}(h^N,\{\bar h,\hat h\})\geq N^{-\alpha},\mathcal{M}_N=B\right)$$
$$\leq \sharp\{\hat{\mathcal{B}}_N\}\max_{B\in\hat{\mathcal{B}}_N}\mu_N^\varepsilon\left(\mathrm{dist}_{L^1}(h^N,\{\bar h,\hat h\})\geq N^{-\alpha},\mathcal{M}_N=B\right).$$

Since the number of mesoscopic regions $\sharp\{\hat{\mathcal{B}}_N\}=e^{N^{d(1-\beta)}\log 2}$ is sub-exponential in N^d, if one can show that the above probability for each B is bounded by $e^{-N^{d-\delta}}$ with some $\delta<d\beta$, we obtain the conclusion.

If $E_N^*(B)\leq N^{d-\delta'}$, by mesoscopic stability, this event is hard to happen so that the probability can be bounded by $e^{-N^{d-\delta}}$ with a suitable choice of δ'. On the other hand, if $E_N^*(B)\geq N^{d-\delta'}$, this probability is bounded by

$$\leq \mu_N^\varepsilon\left(\mathcal{M}_N=B\right)\asymp e^{-E_N^*(B)}\leq e^{-N^{d-\delta'}}$$

by the mesoscopic averaging effect, under which the free energy ξ^ε arises.

The last part of the above explanation is rough. In the course of the proof, we use super-exponential estimate, analysis of super harmonic functions on D_N and volume filling lemma. To avoid technical difficulty caused by the (aN,bN)-boundary condition, we actually consider on an extended region and replace it by the 0-boundary condition.

1.7 open problems

There are several unsolved questions in higher dimensional setting. The case $d=2$ is unsolved because the Green function diverges. But we think the limit should be $\hat h$, since this is the case for both $d=1$ and $d\geq 3$. When $d\geq 3$, we assume that $\varepsilon>0$ is sufficiently large. But, the conjecture is that this assumption can be removed. We need to refine the proof of the lower bound.

Other questions: What happens for general domain D or D_N? What happens when $d\geq 3$ and $n\geq 2$? How about the non-Gaussian case? What happens for different boundary conditions as we discussed when $d=1$? What happens when the set of minimizers is continuous (as the example in p.1010 of [51])?

2 Preliminaries for stochastic partial differential equations

Stochastic partial differential equations (SPDEs) are partial differential equations containing random terms or space-time noises. They appear in a wide variety of fields such as physics, biology, engineering (e.g., in control theory, filtering), economy (e.g., in finance) and others.

Before giving precise formulation, we state motivations for the problems discussed in Sections 3 and 4. SPDEs used in physics are sometimes ill-posed.

2.1 Examples

2.1.1 TDGL equation

The first example is an equation for stochastic quantization or dynamic $P(\phi)_d$-model:

$$\partial_t \phi = \Delta\phi - \phi^3 + \dot{W}(t,x), \quad x \in \mathbb{R}^d, \tag{2.1}$$

where $\dot{W}(t,x)$ is a space-time (Gaussian) white noise, that is, a Gaussian system with mean 0 and (formal) covariance structure:

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y), \quad t,s \geq 0,\ x,y \in \mathbb{R}^d. \tag{2.2}$$

It is known that the noise is very irregular: $\dot{W} \in C^{-\frac{d+1}{2}-} := \cap_{\delta>0} C^{-\frac{d+1}{2}-\delta}$ a.s. The solution at least in the linear case (without ϕ^3) has a better regularity: $\phi \in C_{t,x}^{\frac{2-d}{4}-,\frac{2-d}{2}-}$ (a.s.) because of the regularization property of the Laplacian Δ. This implies ϕ can be a usual function only when $d=1$ so that the nonlinear equation (2.1) is well-posed only when $d=1$.

Hairer [63] introduced the theory of regularity structures and systematic way of renormalization for these ill-posed SPDEs. For (2.1) with $d=2$ or 3, we need to replace $\dot{W}$ by a smeared noise $\dot{W}^\varepsilon$ and introduce a renormalization factor $-C_\varepsilon\phi$. Then, the limit of $\phi=\phi^\varepsilon$ as $\varepsilon \downarrow 0$ exists (in time locally). The solution obtained in the limit is not continuous in ξ (in place of $\dot{W}^\varepsilon$), but continuous in ξ and their (finitely many) polynomials.

The equation (2.1) is sometimes called the time-dependent Ginzburg-Landau (TDGL) equation. TDGL equation plays a vital role as a model equation in physics. It has a form for $\mathbb{R}$-valued $u=u(t,x)$:

$$\partial_t u = -\frac{1}{2}(-\Delta)^\alpha \frac{\delta H}{\delta u(x)}(u) + (-\Delta)^{\alpha/2}\dot{W}(t,x), \quad x \in \mathbb{R}^d, \tag{2.3}$$

where $\dot{W}(t,x)$ is the space-time Gaussian white noise and $H(u)$ is a (formal) Hamiltonian, called Ginzburg-Landau-Wilson free energy, given by

$$H(u) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2}|\nabla u(x)|^2 + V(u(x)) \right\} dx, \tag{2.4}$$

with a self-potential $V:\mathbb{R}\to\mathbb{R}$; see Hohenberg-Halperin [65]. This is a kind of Langevin equation. The formula (2.2) is formal and more precise definition due to the white noise process will be explained in Section 2.2. When $\alpha=0$, the equation is called Model A (non-conservative system), whereas when $\alpha=1$, it is called Model B (system with conservation law). In the latter case, the integral $\int_{\mathbb{R}^d} u(t,x)dx$ is preserved under the time evolution at least at heuristic level. This integral represents total mass, total volume or other quantities.

Since the functional derivative of H is given by

$$\frac{\delta H}{\delta u(x)} = -\Delta u + V'(u(x)),$$

the SPDE (2.3) has the forms

$$\partial_t u = \frac{1}{2}\Delta u - \frac{1}{2}V'(u) + \dot{W}(t,x), \tag{2.5}$$

in the case $\alpha = 0$ and

$$\partial_t u = -\frac{1}{2}\Delta^2 u + \frac{1}{2}\Delta\{V'(u)\} + \sqrt{-\Delta}\dot{W}(t,x), \tag{2.6}$$

in the case $\alpha = 1$. The SPDEs (2.5) and (2.6) are called the TDGL equation of non-conservative type and the TDGL equation of conservative type, respectively. The noise $\sqrt{-\Delta}\dot{W}$ can be interpreted as the time derivative of a Q-cylindrical Brownian motion on $L^2(\mathbb{R}^d, dx)$ with a covariance operator $Q = -\Delta$; see Section 2.2. If V is a double-well potential: $V(u) = \frac{1}{4}(u^2-1)^2$ with two bottoms at $u = \pm 1$ of same depth, then $-V'(u) = u - u^3$ gives a bistable reaction term ($u = \pm 1$ are stable and $u = 0$ is unstable). When $\dot{W} = 0$ (i.e., noise is not added), (2.5) is known as Allen-Cahn equation or the reaction-diffusion equation of bistable type, while (2.6) is known as Cahn-Hilliard equation.

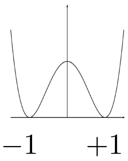


Figure 8: Double-well potential

The SPDEs (2.3) are studied related to the sharp interface limit (for Model A: Kawasaki-Ohta [72], Funaki [39], [40], [41], Weber [88], [89], for Model B: Antonopoulou-Karali-Kossioris [4], Antonopoulou-Blömker-Karali [3]) and the hydrodynamic limit (for Model B: Funaki [34], [35]).

The sharp interface limit or the problem of the dynamic phase transition is to study the limit as $\varepsilon \downarrow 0$ of TDGL equation (=stochastic Allen-Cahn equation):

$$\partial_t u = \Delta u + \frac{1}{\varepsilon}f(u) + \dot{W}(t,x), \quad x \in \mathbb{R}^d, \tag{2.7}$$

where $f = -V'$ with a potential V of double-well type, e.g., $f(u) = u - u^3$. The limit is expected to satisfy:

$$u(t,x) \xrightarrow[\varepsilon\downarrow 0]{} \begin{cases} +1 \\ -1 \end{cases}$$

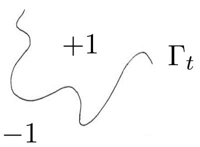


Figure 9: Phase separation

In other words, a random phase separating hyperplane Γ_t appears and the problem to be discussed is to determine its dynamics under a proper time scaling.

The SPDEs (2.3) with reflection (i.e., u is confined to be $u \geq 0$ or stay between two walls) are studied by Nualart-Pardoux [80], Funaki-Olla [50] for Model A and Debussche-Zambotti [22] for Model B. If the self-potential V has a discontinuity such as $V(u) =$

$\beta 1_{[0,\infty)}(u)$, then $V'(u) = \beta\delta_0(u)$ and the SPDE admits a singular drift. This potential physically means that the space is filled by two different media separated by an interface located at the level $u = 0$. Note that J in (1.1) essentially has the same form as H in (2.4) in this case. If $\beta > 0$ and $\alpha = 0$, the solution u of (2.5) is pushed toward the negative side with strength β when it touches the interface located at 0. Bounebache-Zambotti [15] studied such SPDE with singular drift.

The SPDE (2.3) has mathematically meaningful solution, only when the spatial dimension $d = 1$; see Section 2.3. For higher dimensions, SPDEs do not have solutions caused by the roughness of the noise, cf. [63]. Instead, one can consider spatially discretized equations. The $\nabla\varphi$-interface model is a typical such example with the self-potential $\int V(u(x))dx$ replaced by the nearest neighbor pair-potential $\sum_{x,y\in\mathbb{Z}^d:|x-y|=1} V(u(x) - u(y))$; see Funaki-Spohn [55], Funaki [46].

2.1.2 KPZ equation

The second example is the Kardar-Parisi-Zhang (KPZ) equation which describes a motion of a growing interface with fluctuation:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \dot{W}(t,x), \quad x \in \mathbb{R}, \tag{2.8}$$

where $h(t,x)$ denotes the height of interface. As we saw, in the linear case (without $\frac{1}{2}(\partial_x h)^2$), $h \in C_{t,x}^{\frac{1}{4}-,\frac{1}{2}-}$ a.s. Therefore, the KPZ equation (2.8) itself is ill-posed. The right form is the equation with renormalization:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left\{(\partial_x h)^2 - \delta_x(x)\right\} + \dot{W}(t,x), \quad x \in \mathbb{R}.$$

See Section 4.

KPZ equation and its Cole-Hopf solution are one of the recent trends in the study of probability theory. See Sasamoto-Spohn [84], [85], Hairer [62], [63], Quastel [82], Ferrari, Corwin, O'Connell, Takeuchi-Sano, Funaki-Quastel [52] and others.

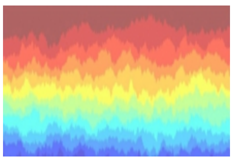


Figure 10: Color changes in time.

2.1.3 Other examples

Stochastic Navier-Stokes or Euler equations are typical examples of SPDEs, but we don't touch. Random traveling waves are studied in the case when $\alpha = 0, \dot{W} = 0$ and $V(x,u,\omega) = g(x,\omega)V_0(u)$ by Nolen-Ryzhik [78], Nolen [79]. Continuous version of parabolic Anderson model, that is, the heat equation with random potential term is given by

$$\partial_t u = \Delta u + V(t,x,\omega)u, \tag{2.9}$$

and studied with motivation to the random environment, homogenization (results depend on dimension and structure of randomness, Bal [6], Pardoux-Piatnitski [81]), localization of wave in random media. The equation (2.9) is in the class of the equations (2.3) with $\alpha = 0$, $\dot{W} = 0$ and $V(u)$ replaced by a random quadratic potential $\frac{1}{2}V(t, x, \omega)u^2$.

Fluctuation limit in the hydrodynamic limit for microscopic interacting (particle) systems leads to SPDEs. Related fluctuation-dissipation theorem or Green-Kubo formula are studied. See Funaki-Olla [50], Funaki-Sasada-Sauer-Xie [54].

Strings in $\mathbb{R}^d$ (Funaki [33]), strings on a manifold M (Funaki [38]) and strings in a convex domain D with reflection (Bounebache [14]) are studied, and provide $\mathbb{R}^d$ or M or D-valued SPDEs.

2.2 Brownian motions, martingales and stochastic integrals in an infinite dimensional setting

We now discuss more precisely. To construct the space-time Gaussian white noise $\dot{W}(t, x)$ having the covariance structure (2.2), take a complete orthonormal system (CONS) $\{\psi_k\}_{k=1}^\infty$ of $L^2(\mathbb{R}^d, dx)$ and a system of independent 1-dimensional Brownian motions $\{B_t^k\}_{k=1}^\infty$ realized on a certain probability space $(\Omega, \mathcal{F}, P)$, and consider a formal Fourier series with random coefficients:

$$W(t, x) = \sum_{k=1}^\infty B_t^k \psi_k(x). \tag{2.10}$$

Then, one would have that

$$\begin{aligned} E[W(t,x)W(s,y)] &= \sum_{k,j=1}^\infty E[B_t^k B_s^j]\psi_k(x)\psi_j(y) \\ &= \sum_{k=1}^\infty (t \wedge s)\psi_k(x)\psi_k(y) = (t \wedge s)\delta(x - y), \end{aligned}$$

and thus its time derivative is expected to satisfy the relation (2.2). As the CONS $\{\psi_k\}_{k=1}^\infty$, for example, one can take the Hermite functions:

$$h_k(x) = \left(2^k k! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_k(x),$$

where $H_k(x)$ are the Hermite polynomials:

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

Unfortunately, the series (2.10) does not converge in the space $L^2(\mathbb{R}^d, dx)$. If we add a dumping factor $\{\lambda_k > 0\}_{k=1}^\infty$ satisfying $\sum_{k=1}^\infty \lambda_k < \infty$ and consider

$$W(t, x) = \sum_{k=1}^\infty \sqrt{\lambda_k} B_t^k \psi_k(x), \tag{2.11}$$

then it converges in $L^2(\mathbb{R}^d, dx)$ or even in smaller spaces; see (a) below. Its time derivative is called a colored noise.

To discuss the construction of noises rigorously, we extend the definitions of Brownian motions, martingales and stochastic integrals to an infinite dimensional setting in a systematic way. This part is taken from my Japanese book [47].

2.2.1 Brownian motions and martingales on a Hilbert space

(a) $\boldsymbol{H}$-valued Brownian motions

Let H be a separable real Hilbert space equipped with an inner product $(\cdot, \cdot)_H$. A typical example is $H = L^2(\mathbb{R}^d, dx)$. Let Q be a self-adjoint nonnegative linear operator on H of nuclear type. In other words, there exists $\{\lambda_k \geq 0\}_{k=1}^\infty$ and CONS $\{\psi_k\}_{k=1}^\infty$ of H such that Q is orthogonalized as $Q\psi_k = \lambda_k \psi_k$ and $\mathrm{Tr} Q := \sum_{k=1}^\infty \lambda_k < \infty$. We denote the family of all such Q by $\mathfrak{L}_{+,1}(H)$, cf. Kuo [75].

Definition 2.1. *An H-valued continuous process $W = (W_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is called a Brownian motion with covariance operator $Q \in \mathfrak{L}_{+,1}(H)$ or Q-Brownian motion in short, if $(W_t, \psi)_H / \sqrt{(Q\psi, \psi)_H}$ is a 1-dimensional Brownian motion for every $\psi \in H$ such that $Q\psi \neq 0$.*

Let $\{B_t^k\}_{k=1}^\infty$ be a system of independent 1-dimensional Brownian motions defined on $(\Omega, \mathcal{F}, P)$ and set

$$W_t = \sum_{k=1}^\infty \sqrt{\lambda_k} B_t^k \psi_k := \lim_{n \to \infty} \sum_{k=1}^n \sqrt{\lambda_k} B_t^k \psi_k. \tag{2.12}$$

Then, the right hand side converges in $L^2(\Omega, H)$ for every $t \geq 0$ and converges a.s. as random variables taking values in $C([0, T], H)$ for all $T > 0$. This can be roughly seen from the computation:

$$E[\|W_t\|_H^2] = \sum_{k=1}^\infty \lambda_k E[(B_t^k)^2] = t \sum_{k=1}^\infty \lambda_k = t \, \mathrm{Tr}\, Q < \infty.$$

If we take $Q = I$, the above computation suggests that $W_t \notin H$ and indeed this is true in an infinite dimensional setting. The process W_t constructed in this way is a Q-Brownian motion. In fact, we have that

$$\begin{aligned} E[(W_t, \psi)_H^2] &= E\left[\left\{ \sum_{k=1}^\infty \sqrt{\lambda_k} B_t^k (\psi_k, \psi)_H \right\}^2 \right] \\ &= t \sum_{k=1}^\infty \lambda_k (\psi_k, \psi)_H^2 = t(Q\psi, \psi)_H, \quad \forall \psi \in H. \end{aligned} \tag{2.13}$$

Its characteristic functional is given by

$$E[e^{i(W_t, \psi)_H}] = e^{-t(Q\psi, \psi)_H / 2}, \quad \psi \in H. \tag{2.14}$$

Even though $Q \notin \mathfrak{L}_{+,1}(H)$, if Q is a bounded operator, the computation (2.13) suggests that there might be a possibility to define $(W_t, \psi)_H$, that is, to give a meaning to W_t in a weak sense. This is called the cylindrical Brownian motion and will be discussed in the next paragraph (b).

Example 2.1. (i) *(finite-dimensional case) Let α be a $d \times N$ matrix and let B_t be an N-dimensional Brownian motion. Then, αB_t is an H-valued Brownian motion with $H = \mathbb{R}^d$ and its covariance operator is determined by $Q = \alpha\, {}^t\alpha \in \mathfrak{L}_{+,1}(H)$, where the inner product of $H = \mathbb{R}^d$ is the usual one.*
(ii) *For a given $\mu = \{\mu_x > 0\}_{x\in\mathbb{Z}^d}$, set $\ell^2(\mu) = \ell^2(\mathbb{Z}^d, \mu) := \{\psi = (\psi(x))_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}; ||\psi||_\mu^2 = \sum_{x\in\mathbb{Z}^d} \psi(x)^2\mu_x < \infty\}$. Then, $\ell^2(\mu)$ is a real Hilbert space equipped with the inner product $(\psi, \varphi)_\mu = \sum_{x\in\mathbb{Z}^d} \psi(x)\varphi(x)\mu_x$. We assume $Tr\,\mu := \sum_x \mu_x < \infty$. Let $\{B_t(x)\}_{x\in\mathbb{Z}^d}$ be a system of independent 1-dimensional Brownian motions. Then, $W_t := (B_t(x))_{x\in\mathbb{Z}^d}$ is an $\ell^2(\mu)$-valued Brownian motion with a covariance operator $Q\psi(x) = \mu_x\psi(x), x \in \mathbb{Z}^d$. In fact, from $E[||W_t||_\mu^2] = t\,Tr\mu < \infty$, we find that W_t is $\ell^2(\mu)$-valued and we can easily see that $E[(W_t, \psi)_H^2] = t(Q\psi, \psi)_H$.*
(iii) *We regard $\mathbb{Z}^d$ as a square lattice embeded in $\mathbb{R}^d$. Then, the Brownian motion $W_t = (B_t(k))_{k\in\mathbb{Z}^d}$ introduced in* (ii) *can be regarded as a process taking values in the space of generalized functions on $\mathbb{R}^d$ by means of*

$$\tilde{W}_t(x) = \sum_{k\in\mathbb{Z}^d} B_t(k)\delta_k(dx), \quad x \in \mathbb{R}^d,$$

where δ_k stands for the Dirac's δ-measure at k. We can find a Hilbert space H, which is a subclass of generalized functions, such that $\tilde{W}_t$ becomes an H-valued Brownian motion.

***H*-valued martingales.** Let a reference family $(\mathcal{F}_t)_{t\geq 0}$ be given on a probability space $(\Omega, \mathcal{F}, P)$. Let us denote the family of all H-valued continuous square integrable $(\mathcal{F}_t)$-martingales $M = (M_t)_{t\in[0,T]}$ satisfying $M_0 = 0$ by $\mathcal{M}_T^2(H)$. Namely, for every $t \geq 0$, M_t is an H-valued $\mathcal{F}_t$-measurable random variable such that $E[||M_t||_H^2] < \infty$, and $E[M_t|\mathcal{F}_s] = M_s$ (i.e., $E[(M_t, \psi)_H|\mathcal{F}_s] = (M_s, \psi)_H, \forall \psi \in H$) is satisfied if $0 \leq s \leq t \leq T$. We simply write $\mathcal{M}_T^2$ for $\mathcal{M}_T^2(\mathbb{R})$. The space $\mathcal{M}_T^2(H)$ is a Hilbert space equipped with an inner product

$$(M, N)_{\mathcal{M}_T^2(H)} = E[(M_T, N_T)_H],$$

in particular, $\mathcal{M}_T^2(H)$ is complete. As usual, we identify $M_t, N_t \in \mathcal{M}_T^2(H)$ if these two processes satisfy $P(M_t = N_t, \forall t \in [0, T]) = 1$. The quadratic variation V_t of $M \in \mathcal{M}_T^2(H)$ is defined by the following two conditions and such V_t always exists.

(1) V_t is an $\mathfrak{L}_{+,1}(H)$-valued $(\mathcal{F}_t)$-adapted process which is increasing as operators: $0 \leq s \leq t$ implies $0 = V_0 \leq V_s \leq V_t$, where $V \leq V'$ means that $V' - V$ is a nonnegative operator.

(2) For every $\varphi, \psi \in H$, $\langle (M_t, \varphi)_H, (M_t, \psi)_H\rangle = (V_t\varphi, \psi)_H$ holds, where the left hand side is the quadratic variation (cross variation) of $\mathbb{R}$-valued martingales.

Lévy's theorem, which characterizes the Brownian motion by means of martingales, can be extended to the infinite-dimensional setting:

Theorem 2.1. *Let $Q \in \mathfrak{L}_{+,1}(H)$. Then, M is a Q-Brownian motion if and only if $M \in \mathcal{M}_T^2(H)$ has a quadratic variation $V_t = t\,Q$.*

(b) Cylindrical Brownian motions

Let Q be more general self-adjoint nonnegative bounded linear operator on H. Here we assume $Q > 0$ for simplicity, that is $Q\psi \neq 0$ for every $\psi \neq 0$. We denote the family of all such Q by $\mathfrak{L}_+(H)$. Then, the Brownian motion with covariance operator Q cannot be realized as an H-valued process in general, however we can define it in the following weak sense.

Definition 2.2. *A cylindrical Brownian motion on H with covariance operator $Q \in \mathfrak{L}_+(H)$, or Q-cylindrical Brownian motion in short, is a family of $\mathbb{R}$-valued processes $W = \{W_t(\psi); t \geq 0, \psi \in H\}$ parametrized by H satisfying the following two conditions:*

(1) *For every $\psi \in H, \psi \neq 0$, $W_t(\psi)/\sqrt{(Q\psi,\psi)_H}$ is a 1-dimensional Brownian motion.*

(2) *For every $\alpha, \beta \in \mathbb{R}$, $\psi, \varphi \in H$, $W_t(\alpha\psi + \beta\varphi) = \alpha W_t(\psi) + \beta W_t(\varphi)$, $\forall t \geq 0$ holds a.s.*

If $Q \in \mathfrak{L}_{+,1}(H)$ and W_t is a Q-Brownian motion, then $W_t(\psi) := (W_t, \psi)_H, \psi \in H$ becomes a Q-cylindrical Brownian motion. In this way, Q-cylindrical Brownian motion is realized as an H-valued process if and only if $Q \in \mathfrak{L}_{+,1}(H)$. In particular, when $H = L^2(\mathbb{R}^d)$ and $Q = I_d$ (=identity operator), $W = \{W_t(\psi); t \geq 0, \psi \in L^2(\mathbb{R}^d)\}$ is called a d-dimensional white noise process.

Let $\{\psi_k\}_{k=1}^\infty$ be a CONS of H and let $\{B_t^k\}_{k=1}^\infty$ be a system of independent 1-dimensional Brownian motions. Then, for every $Q \in \mathfrak{L}_+(H)$,

$$W_t(\psi) := \sum_{k=1}^\infty B_t^k(\sqrt{Q}\psi_k, \psi)_H, \quad \psi \in H, \tag{2.15}$$

converges in $\mathcal{M}_T^2$ and defines a Q-cylindrical Brownian motion, where $\sqrt{Q}$ is defined by means of the spectral decomposition of Q and the reference family $(\mathcal{F}_t)_{t\geq 0}$ for $\mathcal{M}_T^2$ is that naturally defined from $\{B_t^k\}_{k=1}^\infty$.

The computation (2.13) holds in this generalized setting and we have that

$$E[W_t(\psi)^2] = t(Q\psi,\psi)_H, \quad \psi \in H,$$

or more generally,

$$\begin{aligned} E[W_t(\psi)W_s(\varphi)] &= t \wedge s \sum_{i=1}^\infty (\sqrt{Q}\psi_i, \psi)_H(\sqrt{Q}\psi_i, \varphi)_H \\ &= t \wedge s(Q\psi, \varphi)_H, \quad \psi, \varphi \in H, \end{aligned}$$

which is finite. This computation suggests that the derivative $\dot{W}_t(x)$ of the white noise process in t would have the covariance structure (2.2).

Remark 2.1. *For the white noise in time $\dot{B}_t = \frac{dB_t}{dt}$, we formally have*

$$E[\dot{B}_t\dot{B}_s] = \delta(t-s),$$

since we would have for $\varphi, \psi \in C_0^\infty((0,\infty))$

$$\begin{aligned}\int_0^\infty dt \int_0^\infty ds\, \varphi(t)\psi(s)E[\dot{B}_t\dot{B}_s] &= E[\langle \varphi, \dot{B}\rangle\langle \psi, \dot{B}\rangle] \\ = E[\langle \dot{\varphi}, B\rangle\langle \dot{\psi}, B\rangle] &= \int_0^\infty dt \int_0^\infty ds\, \dot{\varphi}(t)\dot{\psi}(s)E[B_tB_s] \\ = \int_0^\infty \dot{\varphi}(t)dt \int_0^\infty \dot{\psi}(s)(t\wedge s)ds &= \int_0^\infty \varphi(t)\psi(t)dt.\end{aligned}$$

Example 2.2. (i) *$\mu = \mathbf{1}$ denotes $\mu_x = 1$ for all $x \in \mathbb{Z}^d$ and write $\ell^2(\mathbb{Z}^d, \mathbf{1})$ simply ℓ^2. Then, independent system of Brownian motions $W_t = (B_t(x))_{x\in\mathbb{Z}^d}$ considered in Example 2.1-*(ii) *is $W_t \notin \ell^2$ a.s. However, the series*

$$W_t(\psi) = \sum_{x\in\mathbb{Z}^d} B_t(x)\psi(x) := \lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d:|x|\le n} B_t(x)\psi(x), \quad \psi \in \ell^2,$$

converges in $\mathcal{M}_T^2$ for every $T > 0$ and the limit is a cylindrical Brownian motion with covariance operator $Q = I_d \in \mathfrak{L}_+(\ell^2)$. As is stated in Example 2.1, $W_t \in \ell^2(\mu)$ if $Tr\,\mu < \infty$. This means that I_d-cylindrical Brownian motion on ℓ^2 can be realized as $\ell^2(\mu)$-valued Brownian motion by extending the Hilbert space ℓ^2 to $\ell^2(\mu)$.
(ii) *The space-time scaling applied to $\tilde{W}_t(x), x \in \mathbb{R}^d$ considered in Example 2.1-*(iii) *determines*

$$\tilde{W}_t^{(n)}(x) := \sum_{k\in\mathbb{Z}^d} B_{t/n}(k)\delta_{k/n}(dx), \quad x \in \mathbb{R}^d,\, n \in \mathbb{N}.$$

Then, $\tilde{W}_t^{(n)}(x)$ as generalized functions valued process satisfies $\langle \tilde{W}_t^{(n)}, \psi\rangle = \sum_{k\in\mathbb{Z}^d} B_{t/n}(k)\psi(k/n)$, and therefore we have

$$E[\langle \tilde{W}_t^{(n)}, \psi\rangle^2] = \frac{t}{n}\sum_{k\in\mathbb{Z}^d} \psi\left(\frac{k}{n}\right)^2 \underset{n\to\infty}{\longrightarrow} t||\psi||^2_{L^2(\mathbb{R}^d)}.$$

This suggests that the white noise process appears as the limit of independent Brownian motions under a suitable scaling in lattice width and time length.

2.2.2 Stochastic integrals

Let $W = (W_t)_{t\ge 0}$ be the $(\mathcal{F}_t)$-cylindrical Brownian motion on H with covariance operator $Q \in \mathfrak{L}_+(H)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with $(\mathcal{F}_t)_{t\ge 0}$. In particular, for every $\psi \in H$, $W_t(\psi)$ is $(\mathcal{F}_t)$-adapted and $W_t(\psi) - W_s(\psi)$ and $\mathcal{F}_s$ are independent if $0 \le s < t$. The goal is to define stochastic integrals with respect to W_t.

(a) *H*-valued integrands

Let $\mathcal{L}^2_{T,Q}(H)$ be the family of all $(\mathcal{F}_t)$-adapted H-valued stochastic processes $f = (f_t(\omega))_{t\in[0,T]}$ satisfying

$$\|f\|^2_{\mathcal{L}^2_{T,Q}(H)} = E[\int_0^T ||\sqrt{Q}f_t||^2_H\, dt] < \infty.$$

We define the stochastic integral $M_t(f) \equiv \int_0^t (f_s, dW_s)_H \in \mathcal{M}_T^2$ of $f \in \mathcal{L}_{T,Q}^2(H)$ by

$$M_t(f) := \sum_{k=1}^{\infty} \int_0^t (f_s, \psi_k)_H \, dW_s(\psi_k), \tag{2.16}$$

where $\{\psi_k\}_{k=1}^{\infty}$ is an arbitrary CONS of H. Each term in the right hand side is defined as a usual stochastic integral since $W_t(\psi_k)$ is a 1-dimensional Brownian motion except a constant multiplier. The series (2.16) is defined by expanding W_t like $W_t = \sum_k W_t(\psi_k)\psi_k$ though it is not necessarily H-valued process. To observe that the right hand side of (2.16) converges in the space $\mathcal{M}_T^2$, we set the sum in (2.16) up to nth term as $M_t^{(n)} \in \mathcal{M}_T^2$. Then, noting that $\langle W(\psi_k), W(\psi_{k'})\rangle_t = t(Q\psi_k, \psi_{k'})_H$, we see that

$$\begin{aligned} E[|M_T^{(n)} - M_T^{(m)}|^2] &= E\left[\sum_{k,k'=n+1}^{m} \int_0^T (f_t, \psi_k)_H (f_t, \psi_{k'})_H (Q\psi_k, \psi_{k'})_H \, dt\right] \\ &= E\left[\int_0^T (Qf_t^{(n,m)}, f_t^{(n,m)})_H \, dt\right], \end{aligned}$$

where $f_t^{(n,m)} := \sum_{k=n+1}^{m} (f_t, \psi_k)_H \psi_k$. Since $f \in \mathcal{L}_{T,Q}^2(H)$, by means of Lebesgue's convergence theorem, this tends to 0 as $m > n \to \infty$. Thus, $\{M_t^{(n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{M}_T^2$ and by its completeness we can find the limit $M_t(f)$. It can be shown that the limit $M_t(f)$ is uniquely determined independently of the choice of the CONS $\{\psi_k\}_{k=1}^{\infty}$.

Proposition 2.2. *The quadratic variation (cross variation) of the stochastic integral $M_t(f)$ is given by*

$$\langle M(f), M(g)\rangle_t = \int_0^t (Qf_s, g_s)_H \, ds, \quad f, g \in \mathcal{L}_{T,Q}^2(H).$$

In particular, Itô isometry: $\|M(f)\|_{\mathcal{M}_T^2} = \|f\|_{\mathcal{L}_{T,Q}^2(H)}$ holds.

Proof. Similar computations for $M_t^{(n)}$ presented above show that

$$\langle M^{(n)}\rangle_t = \int_0^t (Qf_s^{(n)}, f_s^{(n)})_H \, ds, \quad f_s^{(n)} := \sum_{k=1}^{n} (f_s, \psi_k)_H \psi_k.$$

Letting $n \to \infty$, this implies $\langle M(f)\rangle_t = \int_0^t ||\sqrt{Q} f_s||_H^2 \, ds$. The cross variation $\langle M(f), M(g)\rangle_t$ can be computed easily from this. □

For the white noise process $\{W_t(x)\}$ on $\mathbb{R}^d$, we denote the stochastic integral as

$$\int_0^t \big(f_s, dW_s\big)_{L^2(\mathbb{R}^d)} = \int_0^t \int_{\mathbb{R}^d} f_s(x) W(dsdx).$$

(b) Operator-valued integrands

We extend the result obtained in (a) to stochastic integrals with operator-valued integrands. Let us introduce another separable real Hilbert space $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$. We denote the family of all linear operators $\Phi : H \to \mathcal{H}$ such that $\Phi\sqrt{Q}$ are Hilbert-Schmidt operators by $\mathfrak{L}_{2,Q}(H, \mathcal{H})$. Namely, Φ satisfies that

$$||\Phi||_{2,Q}^2 := \sum_{k=1}^{\infty} ||\Phi\sqrt{Q}\psi_k||_{\mathcal{H}}^2 < \infty, \tag{2.17}$$

for every CONS $\{\psi_k\}_{k=1}^{\infty}$ of H, cf. Kuo [75]. Note that $||\Phi||_{2,Q}$ defines a norm of the space $\mathfrak{L}_{2,Q}(H, \mathcal{H})$, which does not depend on the choice of $\{\psi_k\}_{k=1}^{\infty}$, and

$$||\Phi||_{2,Q}^2 = \sum_{j=1}^{\infty} ||(\Phi\sqrt{Q})^* \varphi_j||_H^2 = \mathrm{Tr}(\Phi Q \Phi^*) \tag{2.18}$$

holds for a CONS $\{\varphi_j\}_{j=1}^{\infty}$ of $\mathcal{H}$, where Φ^* is the adjoint operator of Φ. In fact, this follows from $||\Phi||_{2,Q}^2 = \sum_{k,j}(\Phi\sqrt{Q}\psi_k, \varphi_j)_{\mathcal{H}}^2 = \sum_{k,j}(\psi_k, (\Phi\sqrt{Q})^*\varphi_j)_H^2 = \sum_j ||(\Phi\sqrt{Q})^*\varphi_j||_H^2$.

We denote the family of all $(\mathcal{F}_t)$-adapted and $\mathfrak{L}_{2,Q}(H, \mathcal{H})$-valued processes $\Phi = (\Phi_t(\omega))_{t\in[0,T]}$ satisfying

$$\|\Phi\|_{\mathcal{L}_T^2(\mathfrak{L}_{2,Q}(H,\mathcal{H}))}^2 = E[\int_0^T ||\Phi_t||_{2,Q}^2 \, dt] < \infty$$

by $\mathcal{L}_T^2(\mathfrak{L}_{2,Q}(H, \mathcal{H}))$. Then, for each $\Phi \in \mathcal{L}_T^2(\mathfrak{L}_{2,Q}(H, \mathcal{H}))$, we define the stochastic integral $M_t(\Phi) \equiv \int_0^t \Phi_s \, dW_s \in \mathcal{M}_T^2(\mathcal{H})$ by M_t, which satisfies

$$(M_t, \varphi)_{\mathcal{H}} = \int_0^t (\Phi_s^* \varphi, dW_s)_H, \quad \forall t \geq 0 \quad \text{a.s.} \tag{2.19}$$

for every $\varphi \in \mathcal{H}$. The right hand side of (2.19) is defined as a stochastic integral introduced in (a) by noting that $(\Phi_t^*\varphi)_{t\in[0,T]} \in \mathcal{L}_{T,Q}^2(H)$. Note that the Brownian motion W_s is not necessarily H-valued, but the stochastic integral obtained by acting $\Phi_s : H \to \mathcal{H}$ becomes $\mathcal{H}$-valued process.

Proposition 2.3. (i) *$M_t(\Phi) \in \mathcal{M}_T^2(\mathcal{H})$ exists uniquely.*
(ii) *The quadratic variation V_t of $M_t(\Phi)$ is determined by*

$$V_t = \int_0^t \Phi_s Q \Phi_s^* \, ds \in \mathfrak{L}_{+,1}(\mathcal{H}).$$

In particular, $|| \int_0^t \Phi_s \, dW_s||_{\mathcal{H}}^2 - \int_0^t ||\Phi_s||_{2,Q}^2 \, ds$ is a martingale, and this implies Itô isometry:

$$\|M(\Phi)\|_{\mathcal{M}_T(\mathcal{H})} = \|\Phi\|_{\mathcal{L}_T^2(\mathfrak{L}_{2,Q}(H,\mathcal{H}))}.$$

(iii) *(Burkholder's inequality) For every $p \geq 1$, there exists $C_p > 0$ such that*

$$E\left[\sup_{0\leq t\leq T} \left|\left|\int_0^t \Phi_s \, dW_s\right|\right|_{\mathcal{H}}^{2p}\right] \leq C_p E\left[\left(\int_0^T ||\Phi_t||_{2,Q}^2 \, dt\right)^p\right]. \tag{2.20}$$

Proof. (i) The uniqueness is immediate from the definition (2.19). For the existence, consider an approximation by

$$M_t^{(n)} := \sum_{j=1}^{n} \int_0^t (\Phi_s^* \varphi_j, dW_s)_H \varphi_j \in \mathcal{M}_T^2(\mathcal{H})$$

for a CONS $\{\varphi_j\}_{j=1}^\infty$ of $\mathcal{H}$. Then, by Proposition 2.2 and (2.18), $\{M_t^{(n)}\}_n$ is a Cauchy sequence in $\mathcal{M}_T^2(\mathcal{H})$ and the limit satisfies the property (2.19).
(ii) For every $\varphi, \psi \in \mathcal{H}$, from (2.19) and Proposition 2.2, we have that

$$\langle (M_t, \varphi)_{\mathcal{H}}, (M_t, \psi)_{\mathcal{H}} \rangle = \int_0^t (Q\Phi_s^* \varphi, \Phi_s^* \psi)_H \, ds = (V_t \varphi, \psi)_{\mathcal{H}}.$$

This implies that the quadratic variation of M_t is given by V_t. In particular, $(\int_0^t \Phi_s \, dW_s, \varphi_j)_{\mathcal{H}}^2 - (V_t \varphi_j, \varphi_j)_{\mathcal{H}}$ is a martingale and thus $|| \int_0^t \Phi_s \, dW_s||_{\mathcal{H}}^2 - \sum_{j=1}^\infty (V_t \varphi_j, \varphi_j)_{\mathcal{H}}$ is also a martingale. Therefore, noting (2.18), we obtain that

$$\begin{aligned} \sum_{j=1}^\infty (V_t \varphi_j, \varphi_j)_{\mathcal{H}} &= \sum_{j=1}^\infty \int_0^t (\Phi_s Q \Phi_s^* \varphi_j, \varphi_j)_{\mathcal{H}} \, ds \\ &= \sum_{j=1}^\infty \int_0^t ||(\Phi_s \sqrt{Q})^* \varphi_j||_H^2 \, ds = \int_0^t ||\Phi_s||_{2,Q}^2 \, ds. \end{aligned}$$

(iii) We omit the proof. The details are in [47]. □

Remark 2.2. *With $f \in H$, associate an operator Φ_f determined by $\Phi_f \psi := (f, \psi)_H, \psi \in H$. Then, since $||\Phi_f||_{2,Q} = ||\sqrt{Q} f||_H < \infty$, we see that $\Phi_f \in \mathfrak{L}_{2,Q}(H, \mathbb{R})$. In this way, the stochastic integral introduced in* (a) *can be regarded as a special case introduced in* (b) *by taking $\mathcal{H} = \mathbb{R}$.*

2.3 Stochastic partial differential equations of parabolic type with additive noises

We consider the TDGL equations (2.5) of non-conservative type and (2.6) of conservative type in extended forms. This section is mostly taken from [37].

Let us consider the following SPDEs of parabolic type with additive noises for $u = u(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$:

$$\partial_t u = Au + B(u) + C\dot{W}(t,x), \tag{2.21}$$

where

(1) $A = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$ with $a_\alpha \in C_b^\infty(\mathbb{R}^d)$, $m \in \mathbb{N}$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$. The derivative is the usual $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$. The coefficients satisfy the uniform ellipticity condition:

$$\inf_{x, \sigma \in \mathbb{R}^d, |\sigma|=1} (-1)^{m+1} \sum_{|\alpha|=2m} a_\alpha(x) \sigma^\alpha > 0,$$

where $\sigma^\alpha = \sigma_1^{\alpha_1} \cdots \sigma_d^{\alpha_d}$ for $\sigma = (\sigma_1, \ldots, \sigma_d)$.

(2) $B(u)$ is a nonlinear term. To be precise, we assume that it has the form: $B(u)(x) = B\{b(x,u)\}$, where $B = \sum_{|\alpha| \leq n} b_\alpha(x) D^\alpha$ with $b_\alpha \in C_b^\infty(\mathbb{R}^d)$, $n \in \mathbb{Z}$, and $b(u) = b(x,u)$ is a nonlinear functional of u defined on a suitable function space and taking values in another function space.

(3) $C = \sum_{|\alpha|\le \ell} c_\alpha(x) D^\alpha$ with $c_\alpha \in C_b^\infty(\mathbb{R}^d)$, $\ell \in \mathbb{Z}$. The integers n and ℓ may be negative, then B and C are regarded as integral operators. Here we assume $n, \ell \ge 0$.

(4) $\dot{W}(t,x)$ is the space-time Gaussian white noise, i. e. a Gaussian system with mean 0 and covariance structure (2.2).

Note that $m = 1, n = 0, \ell = 0$ for the TDGL equation (2.5) of non-conservative type, while $m = 2, n = 2, \ell = 1$ for the TDGL equation (2.6) of conservative type.

2.3.1 Concepts of Solutions

We take the weighted L^2-spaces

$$L_r^2 = L^2(\mathbb{R}^d, e^{-2r\chi(x)}dx), \quad r > 0,$$

over $\mathbb{R}^d$ as the state spaces for solutions of (2.21), where $\chi \in C^\infty(\mathbb{R}^d)$ such that $\chi(x) = |x|$ for $|x| \ge 1$.

Definition 2.3. *$u(t,x)$ is called a solution of* (2.21) *with an initial value u_0 in the sense of generalized functions, if it satisfies*

$$\langle u(t), \varphi\rangle = \langle u_0, \varphi\rangle + \int_0^t \{\langle u(s), A^*\varphi\rangle + \langle b(\cdot, u(s)), B^*\varphi\rangle\} ds + W_t(C^*\varphi), \tag{2.22}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, where $\langle u, \varphi\rangle = \int_{\mathbb{R}^d} u(x)\varphi(x)dx$.

We have multiplied (2.21) by the test function φ and integrate it in t and x. Then, we formally obtain (2.22) by regarding

$$\int_0^t ds \int_{\mathbb{R}^d} C\dot{W}(s,x)\varphi(x)dx = \int_0^t \int_{\mathbb{R}^d} C^*\varphi(x) W(ds,dx) = W_t(C^*\varphi).$$

Another way to give a mathematical meaning to (2.21) is due to Duhamel's principle:

Definition 2.4. *$u(t,x)$ is called a mild solution of* (2.21), *if it satisfies*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)B(u(s))ds + \int_0^t T(t-s)C dW_s,$$

where $T(t)$ is the semigroup generated by the operator A in the space L_r^2; see also (2.24) *below. The last term is defined as a stochastic integral introduced in Section 2.2-*(b) *with non-random operator as its integrand.*

In some typical cases the two notions of solutions are equivalent, see [47], [56]. If $b(x,u)$ is Lipschitz continuous as a map from $u \in L_r^2$ to $b(\cdot, u) \in L_r^2$, the (mild) solution of the SPDE (2.21) exists uniquely under the conditions on m, n, ℓ of Proposition 2.4 stated below. One can apply the standard method of successive approximations. The pathwise uniqueness holds.

2.3.2 Regularity of Solutions

Since the noise $W_t(x)$ lives only in a bad space, we need the regularizing properties of the operator A.

Proposition 2.4. *Assume $2m > 2\ell + d$ and in addition, for simplicity, $n < 2\ell + \frac{d}{2}$. Then, for the solution $u(t,x)$ of (2.21), we have that*

$$u(t,x) \in \bigcap_{\delta>0} C^{\alpha-\delta,\beta-\delta}((0,\infty)\times\mathbb{R}^d), \quad a.s., \tag{2.23}$$

with

$$\alpha = \frac{2m-2\ell-d}{4m} \quad \text{and} \quad \beta = \frac{2m-2\ell-d}{2}.$$

In particular, for the TDGL equation (2.5) of non-conservative type, we have that

$$u(t,x) \in \bigcap_{\delta>0} C^{\frac{2-d}{4}-\delta,\frac{2-d}{2}-\delta}((0,\infty)\times\mathbb{R}^d),$$

and for the TDGL equation (2.6) of conservative type, we have that

$$u(t,x) \in \bigcap_{\delta>0} C^{\frac{2-d}{8}-\delta,\frac{2-d}{2}-\delta}((0,\infty)\times\mathbb{R}^d).$$

Therefore the solutions live in the usual function spaces, only when $d = 1$. If $d = 2$, the solutions are already generalized functions. Of course, this can be improved, if we take ℓ to be negative, which results in a more regular noise in the space variable, that is, a colored noise. Otherwise, the solution cannot be defined in a classical sense.

Proof of Proposition 2.4. Let $q(t,x,y)$ be the fundamental solution of the parabolic operator $\partial_t - A$. Then, the following estimates are known:

$$\left|\partial_t^j D_x^\alpha D_y^\beta q(t,x,y)\right| \le t^{-\frac{|\alpha|+|\beta|}{2m}-j}\bar{q}(t,x,y), \quad t\in(0,T], x,y\in\mathbb{R}^d,$$

where

$$\bar{q}(t,x,y) = K_1 t^{-\frac{d}{2m}} \exp\left\{-K_2\left(\frac{|x-y|^{2m}}{t}\right)^{\frac{1}{2m-1}}\right\}, \quad t\in(0,T], x,y\in\mathbb{R}^d.$$

We consider the mild solution and set

$$u(t,x) = u_1(t,x) + u_2(t,x) + u_3(t,x), \tag{2.24}$$

where

$$\begin{aligned} u_1(t,x) &= \int_{\mathbb{R}^d} q(t,x,y)u_0(y)dy, \\ u_2(t,x) &= \int_0^t \int_{\mathbb{R}^d} C_y^* q(t-s,x,y)dW_s(y)dy, \end{aligned}$$

$$u_3(t,x) = \int_0^t \int_{\mathbb{R}^d} B_y^* q(t-s,x,y) b(y,u(s)) ds dy.$$

Then, for the term $u_2(t,x)$ involving the stochastic integrals, using Itô isometry, one can show that

$$E\left[\left| D^\alpha u_2(t,x) - D^\alpha u_2(t',x') \right|^2 \right] \\ \le C \left\{ |t-t'|^{\frac{2m-2l-d-2|\alpha|}{2m}} + |x-x'|^{(2m-2l-d-2|\alpha|-\delta)\wedge 2} \right\}, \\ t,t' \in (0,T], x,x' \in \mathbb{R}^d, \ \delta > 0,$$

as long as both exponents are positive. Noting that $u_2(t,x)$ is Gaussian and applying Kolmogorov-Čentsov's theorem (see, for example, Kunita [74]), we obtain (2.23) for $u_2(t,x)$. Other terms u_1 and u_3 have better regularity at least if $n < 2\ell + \frac{d}{2}$; see [37] for details. □

If we consider the stochastic Navier-Stokes equations as an example of the equation (2.21), the nonlinear term $u \cdot \nabla u$ appears, so that the noise has to be colored, or some special way to interpret the nonlinearity is required. See Section 6 for the KPZ equation.

2.3.3 Invariant measures

Once the existence, uniqueness and regularity of solutions are established, the next interest might be in studying their asymptotic behavior as time t becomes large, in particular, the existence and uniqueness of invariant measures and ergodicity. For finite-dimensional Markov processes, one can apply the well-known lower bound technique due to Doeblin. However, the problem becomes delicate for processes taking values in infinite-dimensional spaces. The solutions of SPDEs take values in certain Polish spaces. In order to investigate the above mentioned properties for such processes, the following general methods are known: (1) strong-Feller property (Da Prato-Zabczyk [21]), (2) asymptotic strong-Feller property (Hairer-Mattingly [64], Hairer [61]), and (3) e-property (Komorowski-Peszat-Szarek [73]). These methods are applied to various types of SPDEs.

Funaki [36] studied the invariant or reversible measures of the TDGL equation. It is shown that the (grandcanonical) Gibbs measure associated with the Hamiltonian $H(u)$ is reversible under the TDGL equation (2.5) of non-conservative type and the uniqueness of (tempered) invariant measure is shown under the convexity of the potential V. Contrarily, the reversible measures of the TDGL equation (2.6) of conservative type are not unique. This equation has a family of canonical Gibbs measures (Gibbs measures associated with the Hamiltonian $H_{\lambda(\cdot)} = H(u) - \int \lambda(x)u(x)dx$ with external fields λ which satisfy $\Delta\lambda = 0$) as its reversible measures. This fact is essential to discuss the hydrodynamic limit.

3 Sharp interface limit for stochastic Allen-Cahn equations

Let us consider the stochastic Allen-Cahn equation (stochastic reaction-diffusion equation or modified TDGL equation) with a small parameter $\varepsilon > 0$:

$$\partial_t u = \Delta u + \frac{1}{\varepsilon} f(u) + \dot{W}^\varepsilon(t,x), \quad t > 0, \ x \in D, \tag{3.1}$$

where $\dot{W}^\varepsilon(t,x)$ is a space-time noise depending on ε, which will be specified later, and D is a domain in $\mathbb{R}^d$. In the TDGL equation (2.5) of non-conservative type, the noise was the space-time Gaussian white noise and the nonlinear term was given by $-\frac{1}{2}V'(u)$. In this section, we will treat different noises, as well. We assume the reaction term $f \in C^\infty(\mathbb{R})$ is bistable, that is,

$$\exists u_* \in (-1,1) \quad \text{s.t.} \quad f(\pm 1) = f(u_*) = 0, \;\; f'(\pm 1) < 0, \, f'(u_*) > 0.$$

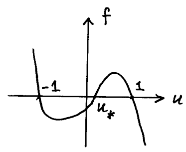


Figure 11: Bistable reaction term

A typical example is $f(u) = u - u^3$. We would expect that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = +1$ or -1 holds for the solution $u = u^\varepsilon(t,x)$ of (3.1), since ± 1 are stable points of f (or of the reaction dynamical system $\dot{u} = f(u)$). Note that u_* is an unstable point of f. The goal is to find the evolutional law of the random interface separating two stable phases ± 1.

Before precisely stating results on the sharp interface limit for stochastic Allen-Cahn equations, we give a quick overview:

(1) $d = 1$ (Funaki [39], [40]) Replace $\dot{W}(t,x)$ by a small space-time Gaussian white noise $\varepsilon^\gamma \dot{W}(t,x)$ with $\gamma > 0$. Recall that the SPDE is well-posed only in one-dimension. Then, under a proper initial condition, we can show that

$$u^\varepsilon(t,x) := u(\varepsilon^{-2\gamma-\frac{1}{2}}t, x) \longrightarrow \chi_{\eta_t}(x) = 1_{(-\infty,\eta_t)} - 1_{(\eta_t,\infty)},$$

where η_t behaves as a Brownian motion (BM) multiplied by a constant called inverse surface tension. Numerical simulation due to Y. Otobe will be shown:
(a) $\varepsilon = 0.01, \gamma = 0.25$: strong force to ± 1 compared to (b), BM is observed.
(b) $\varepsilon = 0.1, \gamma = 0.25$: BM is observed.
(c) $\varepsilon = 0.1, \gamma = 0.375$: a bit small fluctuation compared to (b), BM is observed.
(d) $\varepsilon = 0.1, \gamma = 2.0$: very small but some fluctuation is observed. Time is too short to see BM.

(2) $d \geq 2$ (Funaki [41], Weber [89]) Take $\dot{W}^\varepsilon(t,x) \equiv \dot{W}^\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}}\xi^\varepsilon(t)$ with $\xi^\varepsilon(t) \sim \dot{W}_t$ (time-dependent white noise) as $\varepsilon \downarrow 0$. Then, the dynamics of the phase separating hyperplane Γ_t appearing in the limit is given by

$$V = \kappa + c_0 \dot{w}_t,$$

where V is an inward normal velocity of Γ_t, κ is the mean curvature of Γ_t and $c_0 = \frac{\sqrt{2}}{\int_{-1}^1 \sqrt{V(u)}du}$.

(3) Mass conserving stochastic Allen-Cahn equation (Funaki-Yokoyama [57]):

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon}\left(f(u^\varepsilon) - \fint_D f(u^\varepsilon) \right) + \alpha \dot{W}^\varepsilon(t), \quad t > 0, \; x \in D.$$

The limit is governed by

$$v = \kappa - \fint_{\Gamma_t} \kappa + \frac{\alpha|D|}{2|\Gamma_t|} \circ \dot{W}(t),$$

where $\fint_U$ means the average over U.

3.1 Known results without noises

First, assume $\dot{W}^\varepsilon = 0$ and set $A(f) := \int_{-1}^1 f(u)\,du = V(-1) - V(1)$, where V is the potential corresponding to f such that $f = -V'$. A **traveling wave solution** $m = m(y), y \in \mathbb{R}$ with speed $c = c(f) \in \mathbb{R}$ is determined as an increasing solution of the ordinary differential equation:

$$\begin{cases} m'' + c\,m' + f(m) = 0, & y \in \mathbb{R}, \\ \qquad\qquad m(\pm\infty) = \pm 1, \end{cases}$$

where $m' = dm/dy$ and $m'' = d^2m/dy^2$. In particular, $v(t,y) = m(y - ct)$ is a solution of the 1-dimensional reaction-diffusion equation:

$$\partial_t v = \partial_y^2 v + f(v), \quad t > 0,\ y \in \mathbb{R}, \tag{3.2}$$

see [60], [30], [5]. We normalize the function m as $m(0) = 0$. It is known that $A(f)$ and $-c(f)$ have the same signs; especially, $A(f) = 0$ is equivalent to $c(f) = 0$. If $A(f) > 0$, then, since $V(-1) > V(1)$, the solution v moves from the metastable state -1 to the lower bottom $+1$ of V and this yields a wave moving to the left so that $c(f) < 0$.

(a) The case $A(f) \neq 0$: The proper time scale is $O(\varepsilon^{1/2})$, i.e., for the solution u^ε of (3.1) with $\dot{W}^\varepsilon = 0$, we have that

$$\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{1/2}t, x) \longrightarrow \chi_{\Gamma_t}(x) \qquad (\varepsilon \downarrow 0),$$

where Γ_t is a hyperplane in D and $\chi_{\Gamma_t}(x) = 1(x \in \text{outside of } \Gamma_t)$, $\chi_{\Gamma_t}(x) = -1(x \in \text{inside of } \Gamma_t)$. Γ_t evolves according to the **Huygens' principle**: waves with speed $c(f)$ are created from each point of Γ_t to all outward directions, and Γ_t is determined as the envelope of the wave fronts. See [31], [32], [26] for f of KPP type, Gärtner [58] for f of bistable type in higher dimensional spaces and [29] for f of bistable type in 1-dimensional space.

(b) The case $A(f) = 0, d \geq 2$: Under the condition $A(f) = 0$, we have that $c(f) = 0$ and therefore the wave front is immobile. Such wave is called the **standing wave**. The result in (a) shows that Γ_t does not move under the time scale $O(\varepsilon^{1/2})$ so that one should consider longer time scale. In fact, the proper time scale is $O(1)$, i.e.,

$$u^\varepsilon(t,x) \longrightarrow \chi_{\Gamma_t}(x) \qquad (\varepsilon \downarrow 0)$$

and Γ_t moves according to the **motion by mean curvature**; see [77], [27] and many other references.

(c) The case $A(f) = 0, d = 1$: This is a plane wave so that the curvature is 0. The proper time scale is therefore much longer than $O(1)$. In fact, Carr-Pego [18] showed that the proper scale is $O(\exp C\varepsilon^{-1/2})$, which is extremely long.

3.2 Results with noises

We only consider the cases (b) and (c) with noises, so that our assumptions in this subsection are: f is bistable, $A(f) = 0$ together with a technical condition

$$\exists C, p > 0 \text{ s.t. } |f(u)| \le C(1 + |u|^p), \quad \sup_{u \in \mathbb{R}} f'(u) < \infty. \tag{3.3}$$

3.2.1 The case $d = 1, D = \mathbb{R}, \dot{W}^\varepsilon(t, x) = \varepsilon^\gamma a(x)\dot{W}(t, x)$

Let $\gamma > 0$ and $a \in C_0^2(\mathbb{R})$ be an intensity of the noise (we assume that it has a compact support to localize the problem and to kill the fluctuation near $x = \pm\infty$), and $\dot{W}(t, x)$ be the space-time Gaussian white noise with the covariance structure (2.2). Under the condition (3.3) on f, the SPDE (3.1) with $\dot{W}^\varepsilon(t, x) = \varepsilon^\gamma a(x)\dot{W}(t, x)$ has a unique solution (in generalized functions' sense or in mild sense) which is Hölder continuous:

$$u^\varepsilon(t, x) \in \cap_{\delta > 0} C^{\frac{1}{4}-\delta, \frac{1}{2}-\delta}((0, \infty) \times \mathbb{R}), \quad \text{a.s.}$$

Theorem 3.1. (Funaki [39]) *If the initial value has the form $u^\varepsilon(0, x) = m((x - \xi)/\sqrt{\varepsilon})$ with some $\xi \in \mathbb{R}$ and the reaction term has the symmetry $f(u) = -f(-u)$, then for all sufficiently large $\gamma > 0$, we have the convergence in law:*

$$\bar{u}^\varepsilon(t, x) := u^\varepsilon(\varepsilon^{-2\gamma - 1/2} t, x) \Longrightarrow \chi_{\xi_t}(x) \quad (\varepsilon \downarrow 0), \tag{3.4}$$

where $\chi_\xi(x) = 1(x > \xi)$, $\chi_\xi(x) = -1(x < \xi)$. The phase separation point ξ_t moves according to the following stochastic differential equation (SDE):

$$d\xi_t = \alpha_1 a(\xi_t) dB_t + \alpha_2 a(\xi_t) a'(\xi_t) dt, \quad \xi_0 = \xi, \tag{3.5}$$

where B_t is a 1-dimensional Brownian motion, $\alpha_1 = ||m'||_{L^2(\mathbb{R})}^{-1}$,

$$\alpha_2 = -||m'||_{L^2(\mathbb{R})}^{-2} \int_0^\infty dt \int_{\mathbb{R}^2} x p(t, x, y)^2 f''(m(y)) m'(y) dx dy,$$

and $p(t, x, y)$ is a fundamental solution of the linearized operator $\partial_t - \{\partial_y^2 + f'(m(y))\}$.

This theorem shows that the diffusion coefficient (mobility) α_1^2 is given by the inverse of the surface tension $||m'||_{L^2(\mathbb{R})}^2$ and this coincides with the conjecture made by Kawasaki-Ohta [72] and Spohn [86]. The condition on γ being sufficiently large guarantees that the effect of the reaction term $\frac{1}{\varepsilon} f$ is dominant over the random fluctuation term $\varepsilon^\gamma a(x)\dot{W}(t, x)$. If the random fluctuation is much stronger, it may happen that the shape of the wave front formed in the solution u is totally destroyed by the fluctuation.

We can also study the self-similar space-time Gaussian (colored) noise $\{\dot{W}_h(t, x), 1/2 \le h \le 1\}$ with mean 0 and the covariance structure:

$$E[\dot{W}_h(t, x)\dot{W}_h(s, y)] = \delta_0(t - s) Q_h(x - y),$$

where Q_h is the Riesz potential kernel of $(2h - 1)$th order:

$$Q_h(x) = \begin{cases} h(2h-1)|x|^{2h-2}, & 1/2 < h \le 1, \\ \delta_0(x), & h = 1/2. \end{cases}$$

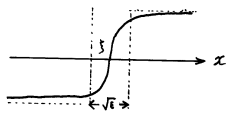
Note that $\dot{W}_{\frac{1}{2}}(t,x)$ is the space-time Gaussian white noise and $\dot{W}_1(t,x) = \dot{w}(t)$ is the white noise in t. One can show that $\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-h}t, x)$ converges to $\chi_{\xi_t}(x)$ as $\varepsilon \downarrow 0$ and the same SDE (3.5) is obtained in the limit with the constants α_1, α_2 modified according to the kernel Q_h. It is reasonable to choose the self-similar noises from the viewpoint of the scalings.

The time change $\varepsilon^{-2\gamma-1/2}$ in (3.4) is very different from the case without noise, recall Section 3.1-(c). The intuitive reason for the properness of this time scale is explained as follows: $\bar{u} = \bar{u}^\varepsilon$ satisfies (in law) the SPDE:

$$\partial_t \bar{u} = \varepsilon^{-2\gamma-1/2}\left\{\Delta\bar{u} + \frac{1}{\varepsilon}f(\bar{u})\right\} + \left(\varepsilon^{-2\gamma-1/2}\right)^{1/2} \cdot \varepsilon^\gamma a(x)\dot{W}(t,x). \tag{3.6}$$

Note that the noise term becomes $a(x)\varepsilon^{-1/4}\dot{W}(t,x)$. The strong drift $\varepsilon^{-2\gamma-1/2}$ pushes $\bar{u}$ to the neighborhood of

$$\begin{aligned} M^\varepsilon := &\left\{\bar{u};\ \Delta\bar{u} + \frac{1}{\varepsilon}f(\bar{u}) = 0,\ \bar{u}(\pm\infty) = \pm 1\right\} \\ =&\{m\left((x-\xi)/\sqrt{\varepsilon}\right);\ \xi \in \mathbb{R}\}, \end{aligned}$$



so that $\bar{u}^\varepsilon(t,x)$ is close to $m\left((x-\xi_t)/\sqrt{\varepsilon}\right)$ with some ξ_t. In particular, the width of the interface is $O(\varepsilon^{1/2})$. The contribution of the noise $\dot{W}(t,x)$ comes only from this region, therefore its order is $O(\{\varepsilon^{1/2}\}^{1/2}) = O(\varepsilon^{1/4})$ by self-similarity. This balances with the factor $\varepsilon^{-1/4}$ in front of the noise. On the other hand, since $\varepsilon^{-2\gamma-1/2} \ll \exp(C\varepsilon^{-1/2})$, the time scale is too short to observe the deterministic movement found by Carr-Pego [18].

If the centering condition on f (i.e., the oddness of f) is violated ($A(f) = 0$ is always assumed), we can show the law of large numbers:

$$u^\varepsilon(\varepsilon^{-2\gamma}t, x) \Longrightarrow \chi_{\xi_t}(x), \qquad \dot{\xi}_t = \alpha_3 a^2(\xi_t),$$

with the constant

$$\alpha_3 = -\frac{1}{2\|m'\|^2_{L^2(\mathbb{R})}}\int_0^\infty dt \int_{\mathbb{R}^2} p(t,x,y)^2 f''(m(y))m'(y)dxdy.$$

The centering condition implies $\alpha_3 = 0$, so that we get the central limit theorem and obtain the random motion of the interface in the limit under longer time scale as in Theorem 3.1. Brassesco-De Masi-Presutti [17] discussed a similar problem at microscopic level under the centering condition. Brassesco-Buttà [16] studied the existence of non-odd function f for which $\alpha_3 \neq 0$ holds. See Bertini-Brassesco-Buttà [8] for recent progress.

The proof of Theorem 3.1 consists of the following two steps:

(1) To show that $\bar{u}^\varepsilon$ stays near M^ε, we take Ginzburg-Landau-Wilson free energy

$$H^\varepsilon(u) := \int_{\mathbb{R}}\left\{\frac{1}{2}|\nabla u|^2(x) + \frac{1}{\varepsilon}V(u(x))\right\}dx$$

as a Lyapunov function, where V is the potential corresponding to f (i.e., $f = -V'$). However, since u^ε is not differentiable in x, we cannot insert u^ε into H^ε and require some extra trick. The convergence speed of u^ε to M^ε is controlled by the spectral gap property of the Hesse operator (Schrödinger operator) $-\partial_y^2 + V''(m)$ of $H^1(u)$ (i.e., $\varepsilon = 1$) at $u = m$.

(2) We introduce a nice coordinate in the tubular neighborhood of M^ε (or on M^1 under the spatial scaling $x = \varepsilon^{1/2}y$). Consider the PDE (3.2). If its initial data v_0 is in an L^2-tubular neighborhood of M^1, the solution $v = v(t, y)$ converges to a certain $m_\zeta(y) := m(y - \zeta)$ in M^1 as $t \to \infty$. The limit ζ depends on the initial value v_0 so that we denote it by $\zeta = \zeta(v_0) \in \mathbb{R}$. This defines a nice coordinate in an L^2-tubular neighborhood of M^1. In fact, if we compute the time derivative of $\zeta(u^\varepsilon(t))$, the diverging factor cancels.

We outline the derivation of the SDE (3.5): We first introduce $v^\varepsilon(t, y) := \bar{u}^\varepsilon(t, \varepsilon^{1/2}y)$ by observing $\bar{u}^\varepsilon(t, x)$ under the microscopic spatial variable y. Then, from (3.6), v^ε satisfies the SPDE:

$$\partial_t v = \varepsilon^{-2\gamma-3/2} \{\Delta v + f(v)\} + \varepsilon^{-1/2} a(\varepsilon^{1/2}y)\dot{W}(t, y), \tag{3.7}$$

in law sense. The coordinate $\zeta(v) \in \mathbb{R}$ defined in the L^2-tubular neighborhood of M^1 introduced above enjoys the following properties. We denote its first and second Fréchet derivatives by $D\zeta(y, v)$ and $D^2\zeta(y_1, y_2, v)$, respectively. The shifted standing wave m is defined by $m_\eta(y) = m(y - \eta), y \in \mathbb{R}$ for $\eta \in \mathbb{R}$.

Lemma 3.2. (1) *For every v in the neighborhood of M^1, we have that*

$$\langle D\zeta(\cdot, v), \Delta v + f(v)\rangle_{L^2} = 0.$$

(2) *For every $\eta \in \mathbb{R}$, we have that*

$$D\zeta(y, m_\eta) = -\frac{m'_\eta(y)}{||m'||^2_{L^2}}.$$

(3) *For every $\eta \in \mathbb{R}$, we have that*

$$D^2\zeta(y, y, m_\eta) = -\frac{1}{||m'||^2_{L^2}} \int_0^\infty dt \int_{\mathbb{R}} p(t, y, z; m_\eta)^2 f''(m_\eta(z)) m'_\eta(z)\, dz,$$

where $p(t, y, z; m_\eta)$ denotes the fundamental solution of $\partial_t - \{\partial_y^2 + f'(m_\eta(y))\}$.

Indeed, (1) follows from the observation that $\zeta(v(t)) = \text{const}$ in t along the solution $v(t)$ of the PDE (3.2). We may let $t \downarrow 0$ in the identity

$$0 = \frac{d}{dt}\zeta(v(t)) = \langle D\zeta(\cdot, v(t)), \Delta v(t) + f(v(t))\rangle,\ t > 0.$$

Another coordinate $\eta(v) \in \mathbb{R}$ in an L^2-tubular neighborhood of M^1 is definable as the minimizer of $\text{dist}\,(v, M^1) = \min_{\eta\in\mathbb{R}} \|v - m_\eta\|_{L^2}$ and this is called the Fermi coordinate. The first Fréchet derivatives of two coordinates $\zeta(v)$ and $\eta(v)$ actually coincide at $v = m_\eta$, however the second derivatives are different. (2) and (3) are some concrete computations and we omit them.

Define the macroscopic phase separation point of $v^\varepsilon(t)$ by $\xi^\varepsilon_t := \varepsilon^{1/2}\zeta(v^\varepsilon(t))$. Then, applying Itô's formula and from the SPDE (3.7), we have that

$$d\xi^\varepsilon_t = \int_{\mathbb{R}} D\zeta(y, v^\varepsilon(t)) a(\varepsilon^{1/2}y) W(dtdy) + \frac{1}{2}\varepsilon^{-1/2} \int_{\mathbb{R}} D^2\zeta(y, y, v^\varepsilon(t)) a^2(\varepsilon^{1/2}y) dy\, dt. \tag{3.8}$$

Note that the diverging factor (the first term in (3.7)) vanishes due to Lemma 3.2-(1). The quadratic variation of the first term in (3.8) is given by

$$\int_{\mathbb{R}} D\zeta(y, v^\varepsilon(t))^2 a^2(\varepsilon^{1/2}y)dy\, dt.$$

We can assume that $v^\varepsilon(t)$ is close to $m_{\varepsilon^{-1/2}\xi_t}$ for some ξ_t, and thus, from Lemma 3.2-(2), this integral is close to

$$a^2(\xi_t)\int_{\mathbb{R}} \frac{(m_{\varepsilon^{-1/2}\xi_t}(y))^2}{||m'||_{L^2}^4} dy\, dt = a^2(\xi_t)\alpha_1^2\, dt,$$

which leads to the first term in the SDE (3.5). Note that $a^2(\varepsilon^{1/2}y)$ is close to $a^2(\xi_t)$ under the integration in y as we explain below.

On the other hand, in the second term in (3.8), the contribution of $D^2\zeta(y, y, m_{\varepsilon^{-1/2}\xi_t})$ comes only from the vicinity of $y = \varepsilon^{-1/2}\xi_t$. Therefore, we may expand $a^2(\varepsilon^{1/2}y)$ as

$$a^2(\varepsilon^{1/2}y) = a^2(\xi_t) + \frac{1}{2}(a^2)'(\xi_t)\cdot \varepsilon^{1/2}(y - \varepsilon^{-1/2}\xi_t) + \cdots .$$

However, the contribution of the first $a^2(\xi_t)$ vanishes under the integration in y, since

$$\int_{\mathbb{R}} D^2\zeta(y, y, m_\eta)\, dy = 0,$$

which follows from Lemma 3.2-(3) noting the symmetry of f. The contribution of the second term, after cancellation of $\varepsilon^{-1/2}$ and $\varepsilon^{1/2}$, gives

$$\frac{1}{2}(a^2)'(\xi_t)\alpha_2\, dt,$$

from Lemma 3.2-(3), and this is just the second term in the SDE (3.5).

S. Weber [90] studied the case where several interfaces (multi-kinks) appear on $[0, 1]$ with periodic boundary conditions. The noise is $\varepsilon^\gamma \dot{W}(t, x)$ (space-time white noise). Annihilating Brownian motions are obtained in the limit. His methods are: (1) Consider approximate slow manifold $\mathcal{M}$ and coordinate system around $\mathcal{M}$ (PDE case: Carr-Pego, Xinfu Chen). (2) Use the idea of expansion in stochastic case due to Antonopoulou-Blömker-Karali [3] when u is close to $\mathcal{M}$. (3) Show annihilation when two interfaces touch.

Antonopoulou-Blömker-Karali [3] studied Cahn-Hilliard equation with smooth noise (TDGL equation of conservative type with Q-BM in place of space-time white noise) on $[0, 1]$ with no-flux boundary conditions of Neumann type:

$$\partial_x u = \partial_x^3 u = 0 \quad \text{at } x = 0, 1.$$

SDEs are obtained in the sharp interface limit for multi-kinks before collisions (result is local in time). The arguments are rather heuristic.

Bertini-Brassesco-Buttà [9] considered Cahn-Hilliard equation with the noise $\sqrt{-\Delta}\dot{W}(t, x)$ as in (2.6) multiplied by $\varepsilon^{1/2}$. The motion of the phase separation point is governed by a fractional Brownian motion with self-similarity parameter $\frac{1}{4}$.

3.2.2 The case $d = 2$ on a bounded domain with smooth boundary

Consider the SPDE (3.1) in higher dimensions with Neumann boundary condition: $\partial u/\partial n = 0$ $(x \in \partial D)$. We assume $A(f) = 0$, but don't require the oddness assumption on f. The noise $\dot{W}^\varepsilon(t,x) = \xi_t^\varepsilon/\sqrt{\varepsilon}$ depends only on t, and ξ_t^ε has the form $\xi_t^\varepsilon = \varepsilon^{-\gamma}\xi(\varepsilon^{-2\gamma}t), 0 < \gamma < 1/3$, where $\xi(t) \in C^1(\mathbb{R}_+)$ (a.s.) is a stationary process with mean 0 satisfying the strong mixing property. We have the convergence in law: $\xi_t^\varepsilon \Rightarrow \alpha\dot{W}(t)$ $(\varepsilon \downarrow 0)$. Unfortunately, we cannot treat the white noise $\alpha\dot{W}(t)$ directly. Instead, we consider a mild noise ξ_t^ε converging to $\alpha\dot{W}(t)$. Here, $W(t)$ is a 1-dimensional Brownian motion and α is a constant given by

$$\alpha = \sqrt{2\int_0^\infty E[\xi(0)\xi(t)]\,dt}.$$

Then, under some condition on the initial value, which will be stated below in the outline of the proof, we have the following theorem for the solution $u^\varepsilon(t,x)$ of (3.1) with $\dot{W}^\varepsilon(t,x) = \xi_t^\varepsilon/\sqrt{\varepsilon}$ when $D \subset \mathbb{R}^2$.

Theorem 3.3. (Funaki [41]) *As long as the limit phase separation curve Γ_t is strictly convex and stays inside D, we have the convergence in law:*

$$u^\varepsilon(t,x) \Longrightarrow \chi_{\Gamma_t}(x) \quad (\varepsilon \downarrow 0),$$

where the curve Γ_t moves according to the **randomly perturbed curvature flow***:*

$$V = \kappa + (c_0\alpha)\dot{W}(t), \tag{3.9}$$

where V denotes the inward normal velocity of Γ_t, κ is the curvature of Γ_t and

$$c_0 = \sqrt{2}\Big/\int_{-1}^1 du\sqrt{\int_u^1 f(v)\,dv}.$$

A heuristic derivation of (3.9) is given as follows: Since ξ_t^ε is close to $\alpha\dot{W}(t)$, (3.1) is almost

$$\partial_t u = \Delta u + \frac{1}{\varepsilon}\left\{f(u) + \sqrt{\varepsilon}\alpha\dot{W}(t)\right\}.$$

In other words, the potential V is randomly perturbed to $V(u) - (\sqrt{\varepsilon}\alpha\dot{W}(t))u$ and this yields a small traveling wave toward the minimizer of the perturbed fluctuating potential. This gives $c_0\alpha\dot{W}_t$ in (3.9).

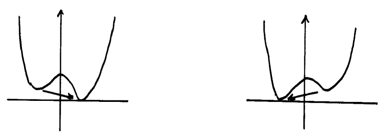


Figure 12: Fluctuating potential

More precisely, for $a \in \mathbb{R}$ (with small $|a|$), define the traveling wave solution $m = m(y;a)$ and its speed $c = c(a)$ by

$$\begin{cases} m'' + cm' + \{f(m) + a\} = 0, \quad y \in \mathbb{R}, \\ m(\pm\infty) = m_\pm^*, \end{cases} \tag{3.10}$$

where $m_\pm^* \equiv m_\pm^*(a) = \pm 1 + O(a)\ (a \to 0)$ are solutions of $f(m_\pm^*) + a = 0$. Then, since the solution of (3.1) behaves as

$$u^\varepsilon(t,x) \sim m(d(x,\Gamma_t)/\sqrt{\varepsilon}; \sqrt{\varepsilon}\alpha\dot W(t)),$$
$$d(x,\Gamma_t) = \text{signed distance between } x \text{ and } \Gamma_t,$$

by putting this in (3.1), we obtain

$$\begin{aligned} 0 &= \partial_t u^\varepsilon - \Delta u^\varepsilon - \frac{1}{\varepsilon} f(u^\varepsilon) - \frac{\alpha}{\sqrt{\varepsilon}}\dot W(t) \\ &\sim \frac{1}{\sqrt{\varepsilon}} m'\left(\frac{d}{\sqrt{\varepsilon}}\right)\partial_t d - \left\{\frac{1}{\sqrt{\varepsilon}} m'\left(\frac{d}{\sqrt{\varepsilon}}\right)\Delta d + \frac{1}{\varepsilon} m''\left(\frac{d}{\sqrt{\varepsilon}}\right)|\nabla d|^2\right\} - \frac{1}{\varepsilon} f(m) - \frac{\alpha}{\sqrt{\varepsilon}}\dot W(t) \\ &\sim \frac{1}{\sqrt{\varepsilon}} m'\left(\frac{d}{\sqrt{\varepsilon}}\right)\left\{\partial_t d - \Delta d - c_0\alpha \dot W(t)\right\}. \end{aligned}$$

The last line follows from (3.10), $|\nabla d| = 1$ near Γ_t and $c(a) = c(0) + c'(0)a + O(a^2) = -c_0 a + O(a^2)$. The ODE (3.10) was used to cancel the terms of order $O(1/\varepsilon)$. Thus the condition to cancel the terms of the order $O(1/\sqrt{\varepsilon})$ becomes

$$\partial_t d = \Delta d + c_0\alpha\dot W(t).$$

Since Δd describes the curvature on Γ_t, we obtain the limit equation (3.9).

The actual proof of Theorem 3.3 is given as follows: Since we assume the noise is mild, we can directly apply the PDE methods, in particular, we can construct super/sub solutions of (3.1) due to the comparison theorem. Those are given as functions close to $\tilde u^\varepsilon(t,x) := m(d(x,\Gamma_t^\varepsilon)/\sqrt{\varepsilon}; \sqrt{\varepsilon}\xi_t^\varepsilon)$ (assume this for $t = 0$), where the curve Γ_t^ε in D is determined by

$$V = \kappa - \frac{1}{\sqrt{\varepsilon}} c(\sqrt{\varepsilon}\xi_t^\varepsilon). \tag{3.11}$$

However, if Γ_t^ε is convex, in terms of the Gauss map ($\theta \in S^1 \mapsto x(\theta) \in \Gamma_t^\varepsilon$), (3.11) can be rewritten into a PDE for the curvature function $\kappa = \kappa^\varepsilon(t,\theta)$:

$$\partial_t\kappa = \kappa^2\left\{\partial_\theta^2\kappa + \kappa - \frac{1}{\sqrt{\varepsilon}} c(\sqrt{\varepsilon}\xi_t^\varepsilon)\right\}.$$

Then, one can study its limit as $\varepsilon \downarrow 0$ and obtain the following SPDE in the limit:

$$\partial_t\kappa = \kappa^2\left\{\partial_\theta^2\kappa + \kappa + c_0\alpha\circ\dot W(t)\right\}, \tag{3.12}$$

where $\circ$ denotes the Stratonovich's stochastic integral. The equation (3.12) gives the precise mathematical meaning to the random perturbation of the curvature flow (3.9).

H. Weber [89] extended Theorem 3.3 to arbitrary dimensions $d \geq 2$ and established short time sharp interface limit under non-convex setting of the interfaces. Convergence was shown in a.s.-sense due to the result by Dirr-Luckhaus-Novaga [24], who gave pathwise solution to $V = \kappa + \dot W(t)$.

Antonopoulou-Karali-Kossioris [4] considered Cahn-Hilliard equation with deterministic noise (under white noise scaling) and gave formal expansion of the solutions. For the deterministic Cahn-Hilliard equation, it is known that Hele-Shaw free boundary problem appears in the sharp interface limit (instead of mean curvature motion in Allen-Cahn case).

3.2.3 Mass conserving stochastic Allen-Cahn equation

Cahn-Hilliard equation has a mass conservation law. Another one which has such property is the mass conserving Allen-Cahn equation, that is (3.13) with $\alpha = 0$. We add a stochastic term similar to the one introduced in Section 3.2.2.

Let $u = u^\varepsilon(t,x)$ be a solution of the following SPDE in a smooth bounded domain D in $\mathbb{R}^n$:

$$(3.13) \qquad \begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon + \dfrac{1}{\varepsilon}\left(f(u^\varepsilon) - \displaystyle\fint_D f(u^\varepsilon)\right) + \alpha \dot{W}^\varepsilon(t), & x \in D \\ \partial_\nu u^\varepsilon = 0, & x \in \partial D \\ u^\varepsilon(\cdot,0) = g^\varepsilon(\cdot), \end{cases}$$

where $\alpha > 0$, ν is the inward normal vector on ∂D and

$$\fint_D f(u^\varepsilon) = \frac{1}{|D|}\int_D f(u^\varepsilon(t,x))dx.$$

Here, $\dot{W}^\varepsilon(t)$ is a time derivative of $W^\varepsilon(t) \in C([0,\infty))$ (a.s.) such that $W^\varepsilon(t)$ converges to one-dimensional Brownian motion $W(t)$ in a suitable sense. The reaction term $f \in C^\infty(\mathbb{R})$ is bistable satisfying $A(f) = 0$. Mass conservation law is destroyed by the noise, but the following identity holds:

$$\fint_D u^\varepsilon(t) = \fint_D u^\varepsilon(0) + \alpha W^\varepsilon(t).$$

Our goal is to study the limit of $u^\varepsilon(t,x)$ as $\varepsilon \downarrow 0$.

To state our result, let us introduce an evolution of limit hypersurfaces $\Gamma_t \subset D$ governed by

$$(3.14) \qquad V = \kappa - \fint_{\Gamma_t} \kappa + \frac{\alpha|D|}{2|\Gamma_t|} \circ \dot{W}(t), \quad t \in [0,\sigma],$$

up to a certain stopping time $\sigma > 0$ (a.s.), where V is the inward normal velocity of Γ_t, κ is the mean curvature of Γ_t (multiplied by $n-1$), $\dot{W}(t)$ is the white noise process and $\circ$ means Stratonovich's stochastic integral.

Evolution of approximating herpersurfaces $\Gamma_t^\varepsilon \subset D$ is defined by

$$(3.15) \qquad V^\varepsilon = \kappa - \fint_{\Gamma_t^\varepsilon} \kappa + \frac{\alpha|D|}{2|\Gamma_t^\varepsilon|}\dot{W}^\varepsilon(t), \quad t \in [0,\sigma^\varepsilon],$$

We assume $\Gamma_t^\varepsilon \to \Gamma_t$ in a proper sense. This can be shown at least in 2-dimensional case for convex curves.

Theorem 3.4. *([57]) Assume that Γ_0 has the form $\Gamma_0 = \partial D_0$ with some $D_0 \Subset D$ and satisfies the same condition as in [19]. Suppose that a smooth local solution $\Gamma = \cup_{0\le t\le\sigma}(\Gamma_t \times \{t\})$ of (3.14) such that $\Gamma_t \Subset D$ for all $t \in [0,\sigma]$ uniquely exists. Then, there exist a family of continuous functions $\{g^\varepsilon(\cdot)\}_{\varepsilon\in(0,1)}$ satisfying*

$$(3.16) \qquad \lim_{\varepsilon\to 0} g^\varepsilon(x) = \begin{cases} 1, & x \in D\backslash\bar{D}_0 \\ -1, & x \in D_0, \end{cases}$$

and stopping times σ^ε such that $(u^\varepsilon(t\wedge\sigma^\varepsilon,\cdot),\sigma^\varepsilon)$ converges weakly to $(\chi_{\Gamma_{t\wedge\sigma}}(\cdot),\sigma)$ on $C([0,T],L^2(D)) \times [0,\infty)$ and $\sigma > 0$ a.s.

The method of the proof is the asymptotic expansion employed by Chen-Hilhorst-Logak [19] in the case without noise. Under the expansion, diverging terms like $(\dot{W}^\varepsilon(t))^2$, $(\dot{W}^\varepsilon(t))^3$ etc. appear. Usually, we cannot control such terms, but fortunately they appear only in the higher order terms in the expansion. Therefore, if the diverging speed of derivatives of $W^\varepsilon(t)$ is sufficiently slow, we can control them.

3.3 Motion of kinks – Random Hamilton-Jacobi equation

This part is taken from unpublished notes presented at Saitama University, 1997. Motivated by the above mentioned results (cf. [90]), let us consider a simple equation like

$$V = \dot{W}(t) \quad (= \text{white noise in } t)$$

for the interface Γ_t. For instance in Theorem 3.1, only a single phase separation point arises in the limit. Here we study the case where phase separation points are multiple. We employ the level set approach and then the normal velocity V can be expressed as

$$V = \frac{u_t}{|\nabla u|} \quad \text{if} \quad \Gamma_t = \{x \in \mathbb{R}^d \,|\, u(t,x) = 0\}.$$

Therefore, our basic equation becomes

$$u_t - |\nabla u| \dot{W}(t) = 0. \tag{3.17}$$

Question: For every continuous function $W(t)$, can one give a mathematical meaning to the equation (3.17)?

The answer is “No”. We need some additional condition on $W(t)$. Our basic idea to construct a solution to (3.17) is the following: Since one can not directly give meaning to (3.17) in “viscosity sense”, we first approximate $W(t)$ by smooth functions $W^\delta(t)$ and then, take the limit $\delta \downarrow 0$.

Assuming that $W(t)$ is smooth, the viscosity solution of (3.17) is given as follows: If $\dot{W}(t) \geq 0$ on $[0, t_1]$, $\dot{W}(t) \leq 0$ on $[t_1, t_2]$ and so on, then

$$\begin{aligned}
&u(t,x) = \sup\{u_0(y)\,;\, y \in [x - W(t), x + W(t)]\}, \qquad t \in [0, t_1] \\
&\qquad\qquad \text{(i.e. } + \text{ region expands in time)}, \\
&u(t,x) = \inf\{u(t_1, y)\,;\, y \in [x - (W(t_1) - W(t)), x + (W(t_1) - W(t))]\}, t \in [t_1, t_2] \\
&\qquad\qquad \text{(i.e. } - \text{ region expands in time)}, \\
&\dots
\end{aligned}$$

Since the equation (3.17) is geometric, we may assume that the initial data is a kink:

$$u_0(x) = (-1)^i, \quad x \in (x_i, x_{i+1}), \quad 0 \leq i \leq n, \qquad x_0 = -\infty, x_{n+1} = \infty,$$

namely, taking this type of functions for initial data is enough for studying the motion of interfaces. For such initial data, we have the following picture for solutions:

If $\dot{W}(t) \geq 0$,

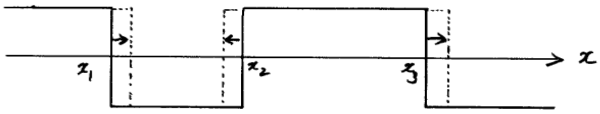


Figure 13: Motion of kinks

Here

$$x_i(t) = \begin{cases} x_i + W(t), & i : \text{odd}, \\ x_i - W(t), & i : \text{even}. \end{cases}$$

When two phase separation points intersect, both points immediately disappear. If we let $W^\delta(t) \to W(t)$, the limit has the same structure (with probability one, since the sample path of the Brownian motion is "regular"). In this sense, one can give a meaning to the equation (3.17) for $W(t)$ which is continuous in t and "regular".

Dirichlet boundary value problem: Consider the equation (3.17) on $\mathbb{R}_+ = [0, \infty)$ with Dirichlet boundary condition:

$$u(t, 0) = +1,$$

and initial value:

$$u_0(x) = 1_{[0,x_0)}(x) - 1_{(x_0,\infty)}(x), \qquad x_0 > 0.$$

Then, if $W(t)$ is smooth, $+$ region expands for t such that $\dot{W}(t) \geq 0$, while $-$ region expands for t such that $\dot{W}(t) \leq 0$, and the phase separation point moves according to

$$x(t) = x_0 + W(t)$$

as before. Once $x(t)$ reaches the boundary $x = 0$ (in this case $\dot{W}(t) < 0$), it stays at 0 until the time that $\dot{W}(t) > 0$. Therefore, the motion is given by

$$x(t) = x_0 + W(t) - \inf_{s \leq t} \left((x_0 + W(s)) \wedge 0 \right).$$

This formula is meaningful also for non-smooth curve $W(t)$. In particular, if $W(t)$ is a Brownian motion, $x(t)$ is the so-called reflecting Brownian motion.

Spatially dependent noise: For the noise $\dot{W}(t, x)$ depending on the spatial variable x, let us consider an equation:

$$V = \dot{W}(t, x).$$

Simplest such noise is

$$\dot{W}(t, x) = a(x)\dot{W}(t), \qquad a(x) > 0.$$

Then the Hamilton-Jacobi equation becomes

$$u_t - |\nabla u| a(x) \dot{W}(t) = 0.$$

If $W(t)$ is smooth, similarly as above the phase separation points move after the ODE:

$$dx^i(t) = a(x^i(t))\dot{W}(t)\, dt, \qquad i = 1, 2, \ldots, n.$$

Now, for the Brownian motion $W(t)$, we approximate it by smooth one. For instance, if we take mollified function

$$W^\delta(t) = \delta^{-1} \left(W * \rho(\cdot/\delta) \right)(t)$$

or polygonal approximation, then the limit is described by the SDE of Stratonovich type:

$$dx^i(t) = a(x^i(t)) \circ dW(t).$$

However, one can take other approximations to derive different types of SDEs in the limit. As a conclusion, it is not clear how to define the notion of "viscosity solutions" to the stochastic Hamilton-Jacobi equation when the noise is spatially dependent.

Related topics "Kink Stochastics" are discussed by Grant Lythe (University of Leeds):

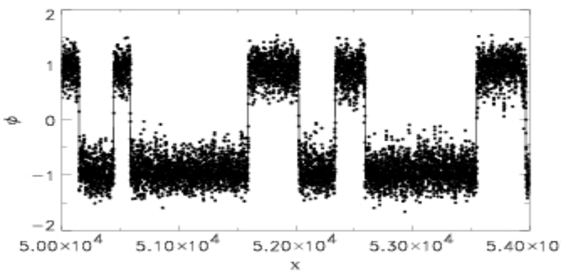


Figure 14: http://www1.maths.leeds.ac.uk/˜grant/

Zero temperature limit for interacting Brownian particles: Motion of kinks can be discussed from microscopic particle systems; see Funaki [44], [45]. The coagulation in one dimension is studied by [45], while the motion of a single crystal in higher dimensions is discussed by [44].

4 KPZ equation

4.1 KPZ equation, its ill-posedness and renormalization

Kardar-Parisi-Zhang [71] introduced the following SPDE for a height function $h(t,x)$ of a growing interface with random fluctuation:

(4.1) $$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left(\partial_x h\right)^2 + \dot{W}(t,x), \quad t>0,\ x \in \mathbb{R}\ (\text{or } S = \mathbb{R}/\mathbb{Z}),$$

where $\dot{W}(t,x)$ is the space-time Gaussian white noise. The coefficients $\frac{1}{2}$ are not important, since we can change them under space-time scaling.

If a curve $\mathcal{C}_t = \{(x,y); y = h(t,x), x \in \mathbb{R}\}$ located in a plane $\mathbb{R}^2$ evolves upward with normal velocity $V = \kappa + A$, where κ is the (signed) curvature of $\mathcal{C}_t$ and $A > 0$ is a constant, then its height function $h(t,x)$ satisfies the nonlinear PDE:

$$\partial_t h = \frac{\partial_x^2 h}{1 + (\partial_x h)^2} + A(1 + (\partial_x h)^2)^{1/2},$$

see, e.g., Matano-Nakamura-Lou [76]. The KPZ equation (4.1) is obtained by taking the leading terms in this equation (more precisely saying, in the equation for $h(t,x)-At$ rather than $h(t,x)$ itself) under the situation that the tilt $\partial_x h$ of the interface is small and taking the fluctuations caused by a space-time noise into account. Note that $\frac{1}{2}$ is put in front of the second derivative and A is chosen as $A=1$. This simplification is essential in view of the scaling property or universality of the KPZ equation.

The KPZ equation (4.1) is actually ill-posed. Indeed, in (4.1), dropping the nonlinear term, we have seen that $h \in C^{\frac{1}{4}-,\frac{1}{2}-}((0,\infty)\times\mathbb{R})$ a.s. and for such h the spatial derivative $\partial_x h$ is defined only in generalized functions' sense. Therefore, the nonlinear term $(\partial_x h)^2$ is not definable in a usual sense. The roughness caused by the noise and the nonlinearity does not match. Instead, the following renormalized KPZ equation with compensator $\delta_x(x)$ $(=+\infty)$ has a meaning:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\left\{\left(\partial_x h\right)^2 - \delta_x(x)\right\} + \dot{W}(t,x), \quad x \in \mathbb{R}, \tag{4.2}$$

as we will see later. This is discussed by Hairer [62], [63].

$\frac{1}{3}$**-power law**: The research of KPZ equation has recently attracted a lot of attention because of its special feature called $\frac{1}{3}$-power law. Balász-Quastel-Seppäläinen [7] showed that if we choose $Z(0,x)=e^{B(x)}$ with B being a two-sided Brownian motion independent of the noise $\dot{W}(t,x)$ (i.e., we consider a stationary solution) then

$$ct^{\frac{2}{3}} \le \mathrm{Var}(h(t,0)) \le Ct^{\frac{2}{3}},$$

i.e., the fluctuations of $h(t,0)$ are of order $t^{\frac{1}{3}}$. This is a different behavior from the usual central limit theorem. Sasamoto-Spohn [84], [85] showed that the limit distribution of $h(t,0)$ under the scaling is given by the so-called Tracy-Widom distribution. See Quastel [82].

4.2 Cole-Hopf solution and linear stochastic heat equation

Another natural approach to the renormalized equation (4.2) is due to the Cole-Hopf transformation. Consider the tilt of the interface $u(t,x)=\partial_x h(t,x)$, which satisfies the viscous stochastic Burgers equation from (4.1):

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \dot{W}(t,x). \tag{4.3}$$

Here the noise has even less regularity, but a formal application of the Cole-Hopf transformation

$$Z(t,x) := e^{-\int_x^\infty u(t,y)dy} = e^{h(t,x)-h(t,\infty)},$$

or setting

$$Z(t,x) := e^{h(t,x)}$$

leads to the linear SPDE with a multiplicative noise:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}_t(x), \tag{4.4}$$

which is called the linear stochastic heat equation (SHE). In fact, the last multiplicative term in (4.4) should be interpreted in Stratonovich's sense. Or, if we start from (4.2), since $E[\dot{W}(t,x)^2] = \delta_x(x)$, a formal application of Itô's formula leads to (4.4). This SPDE is well-posed, has a unique continuous solution and we interpret (4.1) or (4.2) by (4.4) with $h(t,x) = \log Z(t,x)$. Such h is called the Cole-Hopf solution of (4.1) and was introduced by Bertini-Giacomin [10].

The solution $Z(t)$ of (4.4) is defined in a generalized functions' sense or in a mild form due to Duhamel's principle using heat kernel $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/(2t)}$. These two notions are equivalent, and there exists a unique solution s.t. $Z \in C([0,\infty) \times \mathbb{R})$ and $\sup_{x\in\mathbb{R}} e^{-r|x|}|Z(t,x)| < \infty$ for every $r > 0$ a.s. The following strong comparison is known: $Z(0,x) \geq 0$ for all $x \in \mathbb{R}$ and $Z(0,x) > 0$ for some $x \in \mathbb{R}$, then $Z(t,x) > 0$ holds for every $t > 0, x \in \mathbb{R}$ a.s. Therefore, we can define the Cole-Hopf transformation:

$$h(t,x) := \log Z(t,x). \tag{4.5}$$

Heuristic derivation of the KPZ equation (with renormalization factor $\delta_x(x)$) from SHE (4.4) under the Cole-Hopf transformation (4.5) is given as follows: Apply Itô's formula for $h = \log z$:

$$\begin{aligned}
\partial_t h &= Z^{-1}\partial_t Z - \frac{1}{2} Z^{-2}(\partial_t Z)^2 \\
&= Z^{-1}\left(\frac{1}{2}\partial_x^2 Z + Z\dot{W}\right) - \frac{1}{2}\delta_x(x) \\
&= \frac{1}{2}\{\partial_x^2 h + (\partial_x h)^2\} + \dot{W} - \frac{1}{2}\delta_x(x),
\end{aligned}$$

where the second line follows by SHE (4.4) and $(dZ(t,x))^2 = (ZdW(t,x))^2$, $dW(t,x)dW(t,y) = \delta(x-y)dt$.

This leads to the renormalized KPZ equation (4.2). The function $h(t,x)$ defined by (4.5) is meaningful and called the Cole-Hopf solution of the KPZ equation, although the equation (4.1) does not make sense.

Hairer [62], [63] gave a meaning to (4.2) without bypassing SHE. Goncalves-Jara-Sethuraman [59] introduced a notion of *probability energy solution* to (4.3).

4.3 KPZ approximating equations

Our goal is to introduce approximations for (4.2), in particular, well adapted to finding invariant measures.

4.3.1 Simple approximation

Let us first introduce a symmetric convolution kernel: Let $\eta \in C_0^\infty(\mathbb{R})$ s.t. $\eta(x) \geq 0$, $\eta(x) = \eta(-x)$ and $\int_{\mathbb{R}} \eta(x)dx = 1$ be given, and set $\eta^\varepsilon(x) := \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. Then, we define the smeared noise by

$$W^\varepsilon(t,x) = \langle W(t), \eta^\varepsilon(x-\cdot)\rangle \ \big(= W(t) * \eta^\varepsilon(x)\big).$$

Approximating Equation-1 is given by

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\big((\partial_x h)^2 - \xi^\varepsilon\big) + \dot W^\varepsilon(t,x)$$
$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot W^\varepsilon(t,x),$$

where $\xi^\varepsilon = \eta_2^\varepsilon(0)$ $(:= \eta^\varepsilon * \eta^\varepsilon(0))$. It is easy to show that $Z = Z^\varepsilon$ converges to the solution Z of SHE, and therefore $h = h^\varepsilon$ converges to the Cole-Hopf solution of the KPZ equation.

4.3.2 Approximation adapted to finding invariant measures

We want to introduce an approximation which is suitable to study the invariant measures. Here is a general principle: Consider the SPDE

$$\partial_t h = F(h) + \dot W,$$

and let A be a certain operator. Then, the structure of the invariant measures essentially does not change for

$$\partial_t h = A^2 F(h) + A\dot W.$$

This may not be true in non-reversible situation.

We introduce the KPZ approximating equation-2:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\big((\partial_x h)^2 - \xi^\varepsilon\big) * \eta_2^\varepsilon + \dot W^\varepsilon(t,x), \tag{4.6}$$

where $\eta_2(x) = \eta * \eta(x)$, $\eta_2^\varepsilon(x) = \eta_2(x/\varepsilon)/\varepsilon$ and $\xi^\varepsilon = \eta_2^\varepsilon(0)$. Note that the solution h of (4.6) is smooth in x, so that we can consider the associated tilt process $\partial_x h$.

4.4 Invariant measures at approximating level

Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$, where B is the two-sided Brownian motion. ν^ε is independent of choice of $B(0)$.

Theorem 4.1. *ν^ε is invariant for the tilt process $\partial_x h$ determined by SPDE* (4.6).

DaPrato-Debussche-Tubaro (2007) studied a similar SPDE to (4.6) on S.

Sketch of the proof:

Step 1: Consider on a discrete torus $\mathbb{T}_N = \{1, 2, \dots, N\}$. The discretization of $(\partial_x h)^2$ should be carefully chosen (cf. Sasamoto-Spohn [83]):

$$\frac{1}{3}\Big\{(h_{i+1} - h_i)^2 + (h_i - h_{i-1})^2 + (h_{i+1} - h_i)(h_i - h_{i-1})\Big\},\ i \in \mathbb{T}_N$$

Discrete version of ν^ε defined on $\mathbb{T}_N$ is invariant.

Step 2: Continuum limit as $N \to \infty$ leads to the result on S. This can be easily extended to a torus $S_M = \mathbb{R}/M\mathbb{Z}$ of size M.

Step 3: Take an infinite-volume limit as $M \to \infty$ by usual tightness and martingale problem approach.

Remark 4.1. *Infinitesimal invariance can be directly shown based on Wiener-Itô expansion of tame functions* Φ *[49]:*

$$\int \mathcal{L}^\varepsilon \Phi(h) \nu^\varepsilon(dh) = 0, \tag{4.7}$$

where $\mathcal{L}^\varepsilon$ *is (pre) generator of the SPDE* (4.6)*.*

$$\begin{aligned}
&\mathcal{L}^\varepsilon = \mathcal{L}_0^\varepsilon + \mathcal{A}^\varepsilon, \\
&\mathcal{L}_0^\varepsilon \Phi(h) = \frac{1}{2}\int_{\mathbb{R}^2} D^2\Phi(x_1, x_2; h)\eta_2^\varepsilon(x_1 - x_2)dx_1 dx_2 + \frac{1}{2}\int_{\mathbb{R}} \partial_x^2 h(x) D\Phi(x; h)dx, \\
&\mathcal{A}^\varepsilon \Phi(h) = \frac{1}{2}\int_{\mathbb{R}} \left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon(x) D\Phi(x; h)dx.
\end{aligned}$$

Combined with the well-posedness of $\mathcal{L}^\varepsilon$*-martingale problem, which can be shown at least on* S*, it is expected that the infinitesimal invariance implies Theorem 4.1. But this is not clear in infinite-dimensional setting; cf. Echeverria (1982), Bhatt-Karandikar (1993).*

4.5 Passing to the limit $\varepsilon \downarrow 0$

The goal is to pass to the limit $\varepsilon \downarrow 0$ in the KPZ approximating equation (4.6). We consider its Cole-Hopf transform: $Z\,(\equiv Z^\varepsilon) := e^h$. Then, by Itô's formula, Z satisfies the SPDE:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + A^\varepsilon(x, Z) + Z\dot{W}^\varepsilon(t, x), \tag{4.8}$$

where

$$A^\varepsilon(x, Z) = \frac{1}{2} Z(x) \left\{ \left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon(x) - \left(\frac{\partial_x Z}{Z}\right)^2 (x) \right\}.$$

The complex term $A^\varepsilon(x, Z)$ looks vanishing as $\varepsilon \downarrow 0$. But this is not true.

Indeed, under the average in time t, $A^\varepsilon(x, Z)$ can be replaced by a linear function $\frac{1}{24}Z$, see Theorem 4.2 below. Thus, the limit as $\varepsilon \downarrow 0$ (under stationarity of tilt),

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24} Z + Z\dot{W}(t, x).$$

Or, heuristically at KPZ level,

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\{(\partial_x h)^2 - \delta_x(x)\} + \frac{1}{24} + \dot{W}(t, x).$$

We prove that $A^\varepsilon(x, Z^\varepsilon(s))$ can be replaced asymptotically by $\frac{1}{24} Z^\varepsilon(s, x)$. For the proof, a similar method to showing Boltzmann-Gibbs principle is applied. To avoid the complexity arising from the infiniteness of invariant measures, we view $h^\varepsilon(t, \rho) = \int h^\varepsilon(t, x)\rho(x)dx$ (height averaged by $\rho \in C_0^\infty(\mathbb{R}), \geq 0, \int \rho(x)dx = 1$) in modulo 1 (called wrapped process).

Theorem 4.2. *For every $\varphi \in C_0(\mathbb{R})$ satisfying supp $\varphi \cap$ supp $\rho = \emptyset$, we have that*

$$\lim_{\varepsilon \downarrow 0} E^{\pi \otimes \nu^\varepsilon} \left[\left\{ \int_0^t \tilde{A}^\varepsilon(\varphi, Z^\varepsilon(s)) ds \right\}^2 \right] = 0,$$

where π is the uniform measure for $h^\varepsilon(0, \rho) \in [0, 1)$,

$$\tilde{A}^\varepsilon(\varphi, Z) = \int_{\mathbb{R}} \tilde{A}^\varepsilon(x, Z)\varphi(x)dx$$

$$\tilde{A}^\varepsilon(x, Z) = A^\varepsilon(x, Z) - \frac{1}{24} Z(x).$$

Proof. (1) Reduction of equilibrium dynamic problem to static one: The expectation is bounded by

$$\leq 20\, t \sup_{\Phi \in L^2(\pi \otimes \nu^\varepsilon)} \left\{ 2E^{\pi \otimes \nu^\varepsilon} \left[\tilde{A}^\varepsilon(\varphi, Z)\Phi \right] - \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi \rangle_{\pi \otimes \nu^\varepsilon} \right\} \left(= 20t \| A^\varepsilon(\varphi, Z) \|_{-1,\varepsilon}^2 \right),$$

where $\mathcal{L}_0^\varepsilon$ is the symmetric part of $\mathcal{L}^\varepsilon$. This is a generic bound in a stationary situation. Here,

$$2E^{\pi \otimes \nu^\varepsilon} \left[\tilde{A}^\varepsilon(\varphi, Z)\Phi \right] = E^\pi \left[Z_\rho E^{\nu^\varepsilon} \left[B^\varepsilon(\varphi, Z)\Phi(h(\rho), \nabla h) \right] \right],$$

where $Z_\rho = \exp\{\int_{\mathbb{R}} \log Z(x)\rho(x)dx\}$, $B^\varepsilon(x, Z) = \frac{2A^\varepsilon(x,Z)}{Z_\rho}$ and $B^\varepsilon(\varphi, Z) = \int_{\mathbb{R}} B^\varepsilon(x, Z)\varphi(x)dx$.
(2) The key is the following static bound:

Proposition 4.3. *For $\Phi = \Phi(\nabla h) \in L^2(\tilde{\mathcal{C}}, \nu)$ such that $\|\Phi\|_{1,\varepsilon}^2 = \langle \Phi, (-\mathcal{L}_0^\varepsilon)\Phi \rangle_{\pi \otimes \nu^\varepsilon} < \infty$, and φ satisfying the condition of Theorem 4.2, we have that*

$$\left| E^{\nu^\varepsilon} \left[B^\varepsilon(\varphi, Z)\Phi \right] \right| \leq C(\varphi)\sqrt{\varepsilon}\|\Phi\|_{1,\varepsilon}, \tag{4.9}$$

with some positive constant $C(\varphi)$, which depends only on φ, for all ε: $0 < \varepsilon \leq \frac{\delta}{2} \wedge \frac{1}{6}$.

Once this proposition is shown, the proof of Theorem 4.2 is concluded, since the sup in the last slide is bounded by

$$\leq 20t \sup\{2eC(\varphi)\sqrt{\varepsilon}\|\Phi\|_{1,\varepsilon} - \|\Phi\|_{1,\varepsilon}^2\} = \text{const}(\sqrt{\varepsilon})^2 \to 0.$$

$\square$

Proof of Proposition 4.3. First note that

$$\begin{aligned} &E^{\nu^\varepsilon} \left[B^\varepsilon(\varphi, Z)\Phi \right] \\ &= E^{\nu^\varepsilon} \left[\frac{Z(x)}{Z_\rho} \left(\{\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)\} - \frac{1}{12} \right) \Phi \right] \end{aligned}$$

To compute this expectation, since $\{\Psi^\varepsilon * \eta_2^\varepsilon(x) - \Psi^\varepsilon(x)\}$ is 2nd order Wiener functional, we need to pick up the 2nd order and 0th order terms of the products of two Wiener functionals $\frac{Z(x)}{Z_\rho} \times \Phi$. We apply the diagram formula to compute the Winer chaos expansion of products of two functions.

Note that, under ν,

$$\begin{aligned}\frac{Z(x)}{Z_\rho} &= e^{B(x)-\int_{\mathbb{R}} B(y)\rho(y)dy} \\ &= e^{a(x)}\left\{1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\mathbb{R}^n}\phi_x^{\otimes n}(u_1,\dots,u_n)dB(u_1)\cdots dB(u_n)\right\},\end{aligned}$$

where,

$$\begin{aligned}\phi_x(u) &= 1_{(-\infty,x]}(u) - \int_u^\infty \rho(y)dy, \\ a(x) &= \frac{1}{2}\int_{\mathbb{R}}\phi_x(u)^2 du.\end{aligned}$$

Note that the kernel ϕ_x has jump.

$\frac{1}{24}$is the speed of growing interface, and already appears in some previous talks and in many KPZ related papers. For general convolution kernel η, this constant is given by $J/2$, where

$$J = P(R_1+R_3>0, R_2+R_3>0) - P(R_1>0, R_2>0),$$

and $\{R_i\}_{i=1}^3$ are i.i.d. r.v.s distributed under $\eta_2(x)dx$.

If η is symmetric,

$$\begin{aligned}P(R_1+R_3>0, R_2+R_3>0) &= P(R_1-R_3>0, R_2-R_3>0) \\ &= P(R_3 = \min R_i) = \frac{1}{3},\end{aligned}$$

so that $J = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. (If the support of $\eta \subset [0,\infty)$ (or $\subset (-\infty, 0]$), then $J = 0$.) $\square$

Wrapping can be removed by showing uniform estimate:

$$\sup_{0<\varepsilon<1} E\left[\sup_{0\le t\le T} h^\varepsilon(t,\rho)^2\right] < \infty.$$

Namely, height cannot move very fast. This is shown only on a torus (since we need Poincaré inequality).

Under the stationary situation of the tilt processes, in the limit, we obtain the SHE:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24}Z + Z\dot{W}(t,x). \tag{4.10}$$

This looks different from the original SHE (4.4), but the solution Z_t of (4.10) gives the solution $\tilde{Z}_t$ of (4.4) under the simple transformation $\tilde{Z}_t := e^{-\frac{t}{24}}Z_t$. This implies the invariance of the distribution of the geometric Brownian motion for the tilt process determined by the SHE (4.4), and therefore that of BM for Cole-Hopf solution.

We formulate the results more precisely in the next section.

4.6 Invariant measures of Cole-Hopf solution and SHE

As a byproduct, one can give a class of invariant measures for the stochastic heat equation (4.4) and for the Cole-Hopf solution of the KPZ equation. Let $\mu^c, c \in \mathbb{R}$ be the distribution of $e^{B(x)+cx}, x \in \mathbb{R}$ on $\mathcal{C}_+$, where $B(x)$ is the two-sided Brownian motion such that $\mu^c(B(0) \in dx) = dx$. Let ν^c be the distribution of $B(x) + cx$ on $\mathcal{C}$. Note that these are not probability measures.

Theorem 4.4. $\{\mu^c\}_{c\in\mathbb{R}}$ *are invariant under SHE* (4.4)*, i.e.,*
$Z(0) \stackrel{law}{=} \mu^c \Rightarrow Z(t) \stackrel{law}{=} \mu^c$ *for all* $t \geq 0$ *and* $c \in \mathbb{R}$.

Corollary 4.5. $\{\nu^c\}_{c\in\mathbb{R}}$ *are invariant under the Cole-Hopf solution of the KPZ equation.*

c means the average tilt of the interface. We have different invariant measures for different average tilts. Reversibility does not hold, but a kind of Yaglom reversibility holds.

(Scale invariance) If $Z(t,x)$ is a solution of (4.4), then

$$Z^c(t,x) := e^{cx+\frac{1}{2}c^2t} Z(t, x+ct)$$

is also a solution (with a new white noise). Therefore, once the invariance of μ^0 is shown, μ^c is also invariant for every $c \in \mathbb{R}$.

One expects μ^c, $c \in \mathbb{R}$ to be all the extremal invariant measures (except constant multipliers), but this remains open; cf. Funaki-Spohn [55] for $\nabla\varphi$-interface model.

The argument at the end of the last section combined with Theorem 4.1 at approximating level shows the invariance of μ for tilt processes.

To extend this to the height processes Z_t, we introduce the transformation $h^\varepsilon(x, Z) := \log(Z * \eta^\varepsilon(x))$. Then, the evolution of $h^\varepsilon(x, Z_t)$ is governed only by the tilt variables and the initial data $h^\varepsilon(x, Z_0)$.

4.7 Multi-component KPZ equation

Ferrari-Sasamoto-Spohn [28] studied $\mathbb{R}^d$-valued KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{R}$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \dot{W}^\alpha(t,x), \ x \in \mathbb{R}, \tag{4.11}$$

where $\dot{W}(t,x) = (\dot{W}^\alpha(t,x))_{\alpha=1}^d$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise. The constants $(\Gamma^\alpha_{\beta\gamma})_{1\leq\alpha,\beta,\gamma\leq d}$ satisfy the condition:

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\gamma_{\beta\alpha}. \tag{4.12}$$

Similar SPDE appears to discuss motion of loops on a manifold, cf. Funaki [38], Hairer [63].

We introduce the smeared noise:

$$W^\varepsilon(t,x) \equiv (\dot{W}^{\varepsilon,\alpha}(t,x))_{\alpha=1}^d = \langle W(t), \eta^\varepsilon(x - \cdot)\rangle,$$

and consider $\mathbb{R}^d$-valued KPZ approximating equation for $h = h^\varepsilon(t,x) \equiv (h^{\varepsilon,\alpha}(t,x))_{\alpha=1}^d$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - \xi^\varepsilon \delta^{\beta\gamma}) * \eta_2^\varepsilon + \dot{W}^{\varepsilon,\alpha}(t,x), \tag{4.13}$$

where $\delta^{\beta\gamma}$ denotes Kronecker's δ. Let ν^ε be the distribution of $\partial_x(B * \eta^\varepsilon(x))$ on $\mathcal{C} = C(\mathbb{R};\mathbb{R}^d)$, where B is the $\mathbb{R}^d$-valued two-sided Brownian motion satisfying $B(0) = 0$.

Theorem 4.6. *The probability measure ν^ε on $\mathcal{C}$ is infinitesimally invariant for the tilt process $\partial_x h$ of the SPDE* (4.13).

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