Sharp interface limit of stochastic Cahn-Hilliard equation

Rongchan Zhu

Beijing Institute of Technology/Bielefeld University

Joint work with Lubomir Banas and Huanyu Yang

[Banas/Yang/Z.: arXiv:1905.09182]
[Yang/Z.:arXiv:1905.07216]

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The minimizers of the energy (2) are the constant functions $u \equiv 1$ and $u \equiv -1$, which represent the “pure phases” of the system.

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- As $\varepsilon \to 0$, $F(u^\varepsilon) \to 0$, which implies $u^\varepsilon \to -1 + 2\mathbf{1}_E$ for some $E \subset [0, T] \times \mathcal{D}$. $\Gamma_t := \partial E_t$ is the interface.

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- $\Delta v = 0$ in $\mathcal{D} \setminus \Gamma_t$;
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- Formally,

$$\begin{aligned} \int_0^t \int_{\mathcal{D}} \partial_t \mathbf{1}_{E_t} \psi &= -\frac{1}{2} \int_0^t \int_{\mathcal{D}^+} \nabla v \nabla \psi - \frac{1}{2} \int_0^t \int_{\mathcal{D}^-} \nabla v \nabla \psi \\ &= \frac{1}{2} \int_0^t \int_{\mathcal{D}^+} \mathrm{div}(\nabla v \psi) + \frac{1}{2} \int_0^t \int_{\mathcal{D}^-} \mathrm{div}(\nabla v \psi) \\ &= \frac{1}{2} \int_0^t \int_{\Gamma_t} (\partial_n v^+ - \partial_n v^-) \psi. \end{aligned}$$

Hele-Shaw model

Formally derived by [Pego: 1989] and rigorous proved by [Alikakos, Bates,Chen: 1994]: $v^\varepsilon \to v$, (v, Γ) solves the following free boundary problem:

$$\begin{cases} \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial \mathcal{D}, \\ \ v = \dfrac{2}{3} H \ \text{ on } \ \Gamma_t, \\ \ \mathcal{V} = \dfrac{1}{2}(\partial_n v^+ - \partial_n v^-) \ \text{ on } \ \Gamma_t, \end{cases} \tag{6}$$

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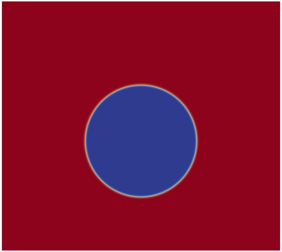

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[Alikakos, Bates, Chen 1994]: construct a pair $(u_A^\varepsilon, v_A^\varepsilon)$ such that

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- $\|u^\varepsilon - u_A^\varepsilon\|_{C(\mathcal{D}_T)} \to 0$.

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$$\begin{cases} du^\varepsilon = \Delta v^\varepsilon dt + \varepsilon^\sigma dW \text{ in } \mathcal{D}_T, \\ v^\varepsilon = \frac{1}{\varepsilon} F'(u^\varepsilon) - \varepsilon \Delta u^\varepsilon \text{ in } \mathcal{D}_T, \\ \frac{\partial u^\varepsilon}{\partial n} = \frac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D} \end{cases}$$

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Problems: Singular noise or Small $\sigma \Rightarrow^?$ Stochastic Hele shaw

Singular noise: space-time white noise

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[Banas, Yang, Z. 19] For $\sigma > \frac{107}{12}$, $\|R^\varepsilon\|_{L^3_t L^3_x}$ converges to 0 in probability.

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This method can be applied to

$$du^{\varepsilon,h} = \Delta\left(-\varepsilon\Delta u^{\varepsilon,h} + \frac{1}{\varepsilon}\left(F'(u^{\varepsilon,h}) - 3c^{\varepsilon}_{h,t}u^{\varepsilon,h}\right)\right)dt + \varepsilon^{\sigma}\nabla\cdot dW^h_t,$$

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$$du^{\varepsilon,h} = \Delta\left(-\varepsilon\Delta u^{\varepsilon,h} + \frac{1}{\varepsilon}\left(F'(u^{\varepsilon,h}) - 3c^{\varepsilon}_{h,t}u^{\varepsilon,h}\right)\right)dt + \varepsilon^{\sigma}\nabla\cdot dW^h_t,$$

- $W^h = W * \rho_h$
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Main results for $\sigma \geq 1/2$

We consider the stochastic Cahn-Hilliard equation on a bounded smooth open domain $\mathcal{D} \subset \mathbb{R}^d$ ($d = 2, 3$):

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We conjecture in general case for $\sigma \geq 1/2$, the limit is the deterministic Hele Shaw model.

Idea of Proof: Lyapunov property

Recall

$$\mathcal{E}^\varepsilon(u^\varepsilon) := \frac{\varepsilon}{2}\int_{\mathcal{D}} |\nabla u^\varepsilon(x)|^2 dx + \frac{1}{\varepsilon}\int_{\mathcal{D}} F(u^\varepsilon(x))dx.$$

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$$d\mathcal{E}^\varepsilon(u^\varepsilon) = \langle D\mathcal{E}^\varepsilon(u^\varepsilon), du^\varepsilon\rangle + \frac{\varepsilon^{2\sigma}}{2}\mathrm{Tr}(QD^2\mathcal{E}^\varepsilon(u^\varepsilon))dt$$

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Lemma 3

(Lyapunov property) Assume $Tr(-\Delta Q) < \infty$ and $\sup_{0<\varepsilon<1}\mathcal{E}^\varepsilon(u_0^\varepsilon) < \mathcal{E}_0$ then there exists $\varepsilon_0 \in (0,1)$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and any $p \geq 1$,

$$\mathbb{E}\sup_{t\in[0,T]}\mathcal{E}^\varepsilon(t)^p \lesssim (\varepsilon^{2\sigma-1} + \mathcal{E}_0)^p,$$

$$\mathbb{E}\left(\int_0^T \|\nabla v^\varepsilon\|_{L^2}^2 dt\right)^p \lesssim (\varepsilon^{2\sigma-1} + \mathcal{E}_0)^p.$$

Idea of Proof: Tightness for $\sigma \geq \frac{1}{2}$

Lemma 4

Assume $\sigma \geq \frac{1}{2}$. For any $\beta \in (0, \frac{1}{12})$

$$\mathbb{E}\left(\|u^\varepsilon\|_{C^\beta([0,T];L^2)}\right) \lesssim 1$$

For any $\delta > 0$, there exists a constant $C \equiv C(\delta, T) > 0$, such that

$$\mathbb{P}\left(\int_0^T \|v^\varepsilon(t)\|_{H^1}^2 dt \leq C\right) \geq 1 - \delta.$$

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Tightness + Skorohod theorem $\Rightarrow$

Theorem 5

[Y, Zhu: 19] For any $\sigma \geq \frac{1}{2}$, $\mathbb{P} - a.s.\ \omega$,

- $u^\varepsilon \to -1 + 2\mathbf{1}_E$ *in* $C([0,T], L^2_w)$,
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$$\begin{cases} \Delta v = 0 \text{ in } \mathcal{D} \setminus \Gamma_t, \ t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\mathcal{D}, \\ v = \dfrac{2}{3} H \ \text{ on } \ \Gamma_t, \\ \mathcal{V} = \dfrac{1}{2}(\partial_n v^+ - \partial_n v^-) \ \text{ on } \ \Gamma_t. \end{cases} \tag{9}$$

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$\Rightarrow$ The only possibility to converge to "stochastic Hele-Shaw model" is $\sigma = 0$.

Equation driven by "smeared" noise

We consider the following random PDE:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = \Delta v^\varepsilon + \varepsilon^\sigma \xi_t^\varepsilon, \quad (t,x) \in [0,T] \times \mathcal{D}, \\ v^\varepsilon = -\varepsilon \Delta u^\varepsilon(t) + \frac{1}{\varepsilon} F'(u^\varepsilon(t)), \quad (t,x) \in [0,T] \times \mathcal{D}, \\ \frac{\partial u^\varepsilon}{\partial n} = \frac{\partial v^\varepsilon}{\partial n} = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{D}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad x \in \mathcal{D}, \end{cases} \tag{10}$$

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$\sigma > 0 \Rightarrow$ deterministic model (9).

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- $-\Delta v = \xi := \frac{dW_t}{dt}$ in $\mathcal{D} \setminus \Gamma_t$;
- Let $\hat{v} = v + \Delta^{-1}\xi$

$$\begin{aligned}\int_0^t \int_{\mathcal{D}} \partial_t \mathbf{1}_{E_t} \psi &= -\frac{1}{2}\int_0^t \int_{\mathcal{D}^+} \nabla \hat{v} \nabla \psi - \frac{1}{2}\int_0^t \int_{\mathcal{D}^-} \nabla \hat{v} \nabla \psi \\ &= \frac{1}{2}\int_0^t \int_{\Gamma_t} (\partial_n \hat{v}^+ - \partial_n \hat{v}^-)\psi.\end{aligned}$$

Stochastic Hele-Shaw model

For $\sigma = 0$, we proved in radial symmetric case, that the sharp interface limit of (10) is the weak formula of the following "stochastic Hele-Shaw" model:

$$\begin{cases} \Delta v dt = -dW_t \text{ in } \mathcal{D} \setminus \Gamma_t, \; t > 0, \\ \dfrac{\partial v}{\partial n} = 0 \text{ on } \partial \mathcal{D}, \\ v = \dfrac{2}{3} H \text{ on } \Gamma_t, \\ \mathcal{V} dt = \dfrac{1}{2} \left[\dfrac{\partial}{\partial n} \right]_{\Gamma_t} (v dt + \Delta^{-1} dW_t), \end{cases} \tag{11}$$

where H: mean curvature

$$\left[\frac{\partial}{\partial n} \right]_{\Gamma_t} f = \frac{\partial f^+}{\partial n} - \frac{\partial f^-}{\partial n}.$$

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Thank you!