Scaling limit of uniform spanning tree in three dimensions

Daisuke Shiraishi, Kyoto University

ongoing work with Omer Angel (UBC), David Croydon (Kyoto University) and Sarai Hernandez Torres (UBC)

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- UST has important connections to several areas:
  - Loop-erased random walk (LERW)
  - Loop soup
  - Conformally invariant scaling limits
  - The Abelian sandpile model
  - Gaussian free field
  - Domino tiling
  - Random cluster model
  - Random interlacements
  - Potential theory
  - Amenability $\cdots$

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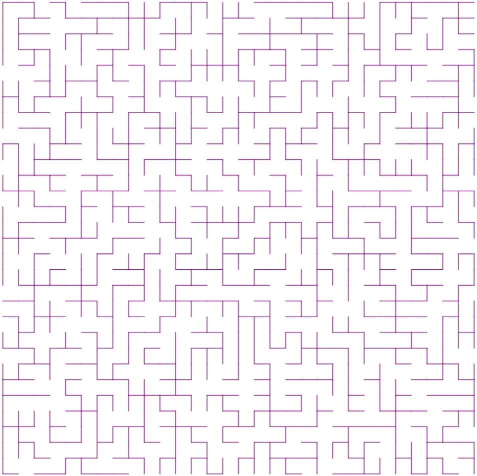

2D UST in a fine grid.
Picture credit: Adrien Kassel.

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  In particular, we want to define a random metric χ in $\mathbb{R}^3$ which is the limit of the rescaled graph distance in UST.

  Namely, χ satisfies that for all $x, y \in \mathbb{R}^3$, the rescaled graph distance between x and y in UST in $\delta\mathbb{Z}^3$ converges weakly to $\chi(x, y)$.

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- The distortion of the correspondence $\mathcal{R}$ is defined by $\mathrm{dis}(\mathcal{R}) = \sup\left\{\left|d_1(x_1, y_1) - d_2(x_2, y_2)\right| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\right\}$.

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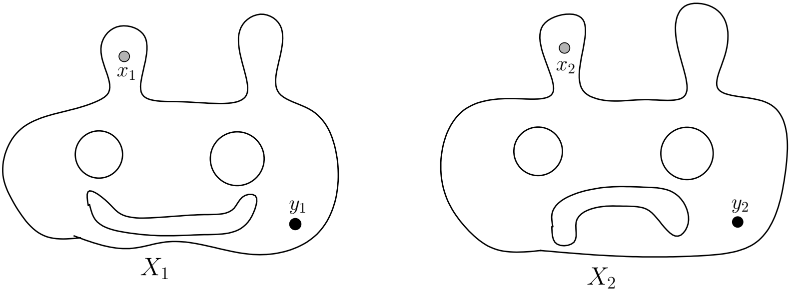

Correspondence between X_1 and X_2.
Picture credit: Daisuke Shiraishi.

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- For two pointed compact metric spaces (X_1, ρ_1) and (X_2, ρ_2), define the distance $d_{\mathrm{GH}}(X_1, X_2)$ by

$$d_{\mathrm{GH}}(X_1, X_2) = \inf \mathrm{dis}(\mathcal{R}),$$

  where the infimum is over all correspondences $\mathcal{R}$ between X_1 and X_2 with $(\rho_1, \rho_2) \in \mathcal{R}$.

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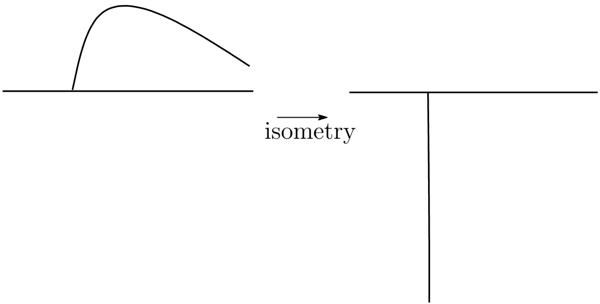

Two equivalent trees in the Gromov-Hausdorff topology.

The spatial Gromov-Hausdorff convergence

- A quadruplet $\underline{X} = (X, d_X, \rho_X, \phi_X)$ is called a pointed spatial compact metric space if (X, d_X, ρ_X) is a pointed compact metric space and ϕ_X is a continuous map from (X, d_X) to $\mathbb{R}^3$.

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- For two pointed spatial compact metric spaces $\underline{X_i} = (X_i, d_i, \rho_i, \phi_i)$ $(i = 1, 2)$, define $d_{\mathrm{GH}}^{\mathrm{sp}}(\underline{X_1}, \underline{X_2})$ by

$$d_{\mathrm{GH}}^{\mathrm{sp}}(\underline{X_1}, \underline{X_2}) = \inf \left(\mathrm{dis}(\mathcal{R}) \vee \sup_{(x_1, x_2) \in \mathcal{R}} d_{\mathrm{Euclid}}\big(\phi_1(x_1), \phi_2(x_2)\big) \right),$$

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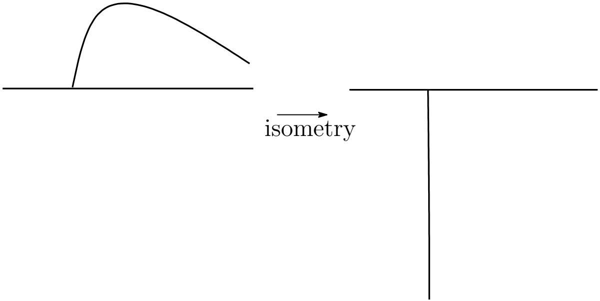

These two trees are distinguished in the **spatial** Gromov-Hausdorff topology.

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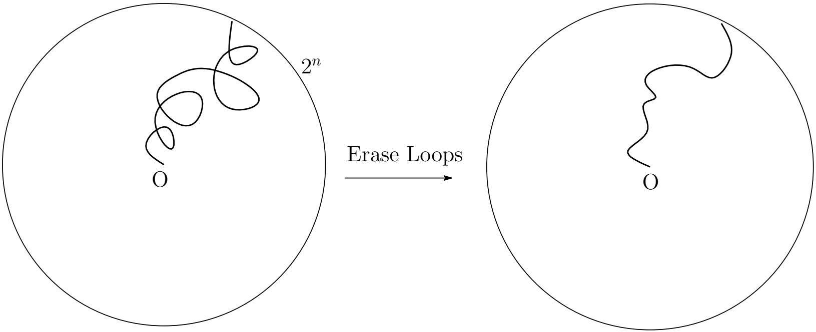
- Let LERW_n be the loop-erased random walk from 0 to $\partial B(2^n)$ in $\mathbb{Z}^3$. Denote the number of steps of LERW_n by $\left|\mathrm{LERW}_n\right|$.

SRW and LERW



SRW (left) and Loop-erased random walk (right) in $\mathbb{Z}^3$.

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Theorem (Angel-Croydon-S.-Hernandez Torres. '19+)

As $n \to \infty$, the pointed spatial tree $(\mathcal{U}, 2^{-\beta n} d_{\mathcal{U}}, 0, 2^{-n}\phi_{\mathcal{U}})$ converges weakly w.r.t. the metric d_{GH}^{sp}.

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- **Remark 2**: One of the key ingredient is the convergence of 3D LERW in the natural parametrization established by Li-S. ('18).
- **Remark 3**: Kozma ('07) proved the existence of weak convergence limit of 3D LERW w.r.t. the Hausdorff metric. But the topology he used is weaker than we want.

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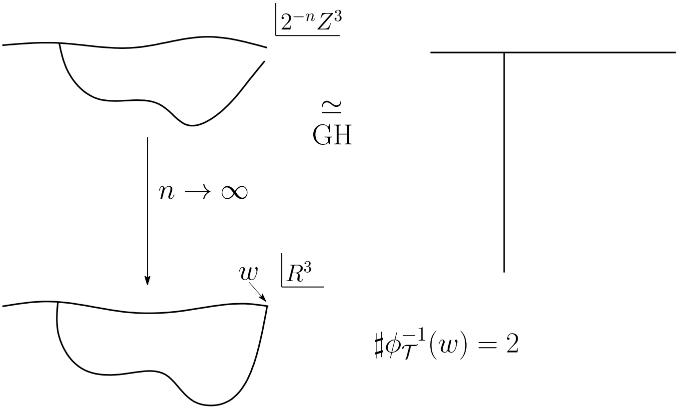

A subtree in the UST (top left) and its limit in the Euclidean topology (bottom left). The right tree is equivalent to the UST subtree in the Gromov-Hausdorff topology.

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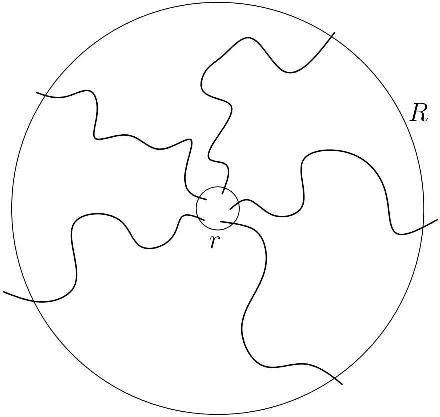

Five non-intersecting arms from $\partial B(r)$ to $\partial B(R)$ for UST in $\mathbb{Z}^3$.

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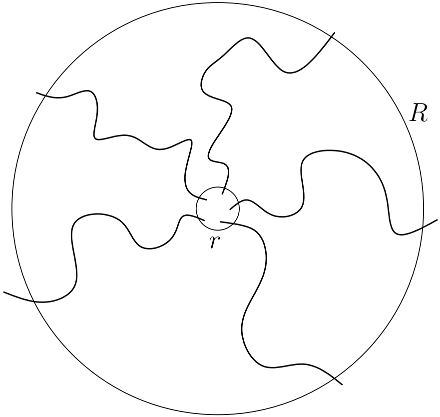

Five non-intersecting arms from $\partial B(r)$ to $\partial B(R)$ for UST in $\mathbb{Z}^3$.
It is proved that $\exists \epsilon, C > 0$ s.t.
$P\Big(\exists k \text{ arms between } \partial B(r) \text{ and } \partial B(R) \text{ in UST}\Big) \le C(r/R)^{\epsilon k}$
for all $k \ge 2$ and $r < R$ with $Cr < R$.

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\chi(x, y) &:= d_{\mathrm{Haus}}\Big(\phi_{\mathcal{T}}^{-1}(x), \phi_{\mathcal{T}}^{-1}(y)\Big) \\
&= \max\left\{ \max_{x' \in \phi_{\mathcal{T}}^{-1}(x)} d_{\mathcal{T}}\big(x', \phi_{\mathcal{T}}^{-1}(y)\big), \max_{y' \in \phi_{\mathcal{T}}^{-1}(y)} d_{\mathcal{T}}\big(y', \phi_{\mathcal{T}}^{-1}(x)\big) \right\}.
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$$\chi(x,y) := d_{\mathrm{Haus}}\Big(\phi_{\mathcal{T}}^{-1}(x), \phi_{\mathcal{T}}^{-1}(y)\Big)$$

$$= \max\left\{ \max_{x' \in \phi_{\mathcal{T}}^{-1}(x)} d_{\mathcal{T}}\big(x', \phi_{\mathcal{T}}^{-1}(y)\big), \max_{y' \in \phi_{\mathcal{T}}^{-1}(y)} d_{\mathcal{T}}\big(y', \phi_{\mathcal{T}}^{-1}(x)\big) \right\}.$$

(Note that for typical $x, y \in \mathbb{R}^3$, $\chi(x,y) = d_{\mathcal{T}}(x', y')$ a.s.)
Then χ is the limit of rescaled graph distances of UST's.

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- **Remark 6**: Our topology is stronger than the topology Oded Schramm considered in the paper ('00) introducing his SLE.
- **Remark 7**: Several properties of $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}}, \phi_{\mathcal{T}})$ as well as the SRW on $\mathcal{U}$ and its scaling limit will be studied in our forthcoming paper. (Scaling limit of the SRW on 2D UST was studied in Barlow-Croydon-Kumagai ('17).)

Big Problem

What is the scaling limit of 3D UST?

Can we give a “nice” description of it?

Thank you for your attention!