Divergence of the non-radom fluctuation in First-passage percolation

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Introduction

- In this talk, we discuss the behavior of fluctuations in First-passage percolation.
- There are many results on the upper bound of fluctuations.
- However, there are few results on the lower bound.
- My motivation is to give a method to get the lower bound.

Setting (FPP)

- $\mathrm{E}^d = \{\{x,y\} |\ x,y \in \mathbb{Z}^d,\ |x-y|_1 = 1|\}$.
- $\tau = \{\tau_e\}_{e \in \mathrm{E}^d}$: I.I.D non-negative random variables.
- $\Gamma(x,y)$: the set of all paths from x to y.

First Passage time $(x, y \in \mathbb{Z}^d)$

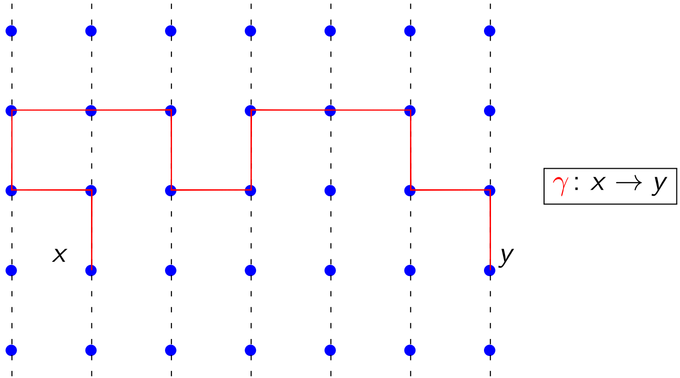
$$\mathrm{T}(x,y) := \inf \left\{ \sum_{e \in \gamma} \tau_e \,|\, \gamma \in \Gamma(x,y) \right\}$$
$$=: \inf_{\gamma \in \Gamma(x,y)} \mathrm{T}(\gamma).$$

optimal paths

$$\mathbb{O}(x,y) := \{\gamma \in \Gamma(x,y) |\ \mathrm{T}(\gamma) = \mathrm{T}(x,y)\}.$$

First Passage Time

$$\mathrm{T}(x, y) = \inf \left\{ \sum_{e \in \gamma} \tau_e \mid \gamma \in \Gamma(x, y) \right\}$$



"Law of large number" for $\mathrm{T}(x, y)$

For $x, y \in \mathbb{R}^d$, $\mathrm{T}(x, y) := \mathrm{T}([x], [y])$, where $[\cdot]$ is a floor function.

Theorem 1 (Kingman '68)

Suppose that $\mathbb{E}[\tau_e] < \infty$. *For any* $x \in \mathbb{R}^d$,

$$\lim_{n \to \infty} \frac{1}{n} \mathrm{T}(0, nx) = \mathrm{g}(x) \quad \text{a.s.},$$

where $\mathrm{g}(x) := \inf_{n \in \mathbb{N}} \frac{1}{n} \mathbb{E}[\mathrm{T}(0, nx)]$ (time constant).

Proof.

Apply Kingman's sub-additive ergodic theorem. □

Q. How fast does it converge? (rate of conergence)

Fluctuation exponent

Conjectures

- There exists $\chi(d) \geq 0$ such that for any $x \in \mathbb{Z}^d \backslash \{0\}$,

$$\mathrm{T}(0, nx) - \mathrm{g}(nx) \text{ grows like } n^{\chi(d)} \text{ as } n \to \infty.$$

  This $\chi(d)$ is called a fluctuation exponent.

- $\chi(2) = 1/3$.

- $\lim_{d \to \infty} \chi(d) = 0$.

Controversial Issue

- For sufficiently large d, $\chi(d) = 0$?

Random and non-random fluctuation

- Kesten considered the following decompositon to estimate the rate of convergence:

$$\mathrm{T}(0,x) - \mathrm{g}(x) = \underbrace{\mathrm{T}(0,x) - \mathbb{E}\mathrm{T}(0,x)}_{random} + \underbrace{\mathbb{E}\mathrm{T}(0,x) - \mathrm{g}(x)}_{non-random}.$$

- The key point is that we can estimate the non-random fluctuation (from above) by using the estimate of the random fluctuation (Kesten, Alexander, etc.).

- In this talk, we only discuss the lower bound of the non-random flucuation.

Previous researches

In this slide, we suppose

- the distribution is non-degenerate, $(\mathbb{P}(\tau_e = a) < 1 \ \forall a \in \mathbb{R})$
- $\mathbb{P}(\tau_e = 0) < p_c(d)$, (subcritical regime)
- $\exists \alpha > 0$ such that $\mathbb{E} e^{\alpha \tau_e} < \infty$. (finite exponential moment)

Theorem 2 (Kesten '93)

For any $x \in \mathbb{Z}^d \setminus \{0\}$ and $\epsilon > 0$, there exists $c > 0$ s.t.

$$\mathbb{E}\mathrm{T}(0, nx) - \mathrm{g}(nx) \geq c n^{-1-\epsilon}, \quad \forall n \in \mathbb{N}.$$

Theorem 3 (Auffinger-Damron-Hanson '15)

For any $x \in \mathbb{Z}^d \setminus \{0\}$ and $\epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ s.t.

$$\mathbb{E}\mathrm{T}(0, nx) - \mathrm{g}(nx) \geq n^{-\frac{1}{2}-\epsilon}.$$

Main result I

Theorem 4 (N)

Suppose the distribution is non-degenerate and $\mathbb{E}\tau_e < \infty$. *Then,*

$$\inf_{x \in \mathbb{Z}^d \setminus \{0\}} (\mathbb{E}\mathrm{T}(0,x) - \mathrm{g}(x)) > 0.$$

As before, we expect that there exists $\chi'(d)$ such that

$$\mathbb{E}\mathrm{T}(0,nx) - \mathrm{g}(nx) \text{ grows like } n^{\chi'(d)}.$$

The above result shows that $\chi'(d) \geq 0$ if exists.

Useful distributions

Let τ^- be the infimum of the support of the distribution of τ_e.

Definition 1

τ is useful $\overset{def}{\Leftrightarrow}$ the following hold:

- there exists $\alpha > 0$ such that $\mathbb{E}\tau_e^{2+\alpha} < \infty$,
- $\mathbb{P}(\tau_e = \tau^-) < \begin{cases} p_c(d) & \text{if } \tau^- = 0, \\ \vec{p}_c(d) & \text{otherwise,} \end{cases}$

  where $p_c(d)$ and $\vec{p}_c(d)$ are the critical probabilities of d-dim percolation, oriented percolation model, resp.

Conjecture

Useful $\Leftrightarrow \mathbb{B}_d = \{x \in \mathbb{R}^d |\ \mathrm{g}(x) \leq 1\}$ is compact & strictly convex.

Main result II

Theorem 5 (N)

Suppose τ is useful. There exist $c > 0$ and a sequence (x_n) of $\mathbb{Z}^d$ such that $|x_n|_1 = n$,

$$\mathbb{E}\mathrm{T}(0, x_n) - \mathrm{g}(x_n) \geq c(\log\log n)^{1/d}.$$

Note that by Jensen's inequality,

$$\begin{aligned}\mathbb{E}|\mathrm{T}(0, x_n) - \mathrm{g}(x_n)| &\geq |\mathbb{E}\mathrm{T}(0, x_n) - \mathrm{g}(x_n)| \\ &\geq c(\log\log n)^{1/d}.\end{aligned}$$

$\Rightarrow$ Divergence of the fluctuation around the time constant.

Some open problems

We collect some open problems:

- Divergence of the random fluctuation for a fixed direction: for any $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\lim_{n \to \infty} \mathbb{E}\mathrm{T}(0, nx) - \mathrm{g}(nx) = \infty.$$

- Divergence of the random fluctuation: for $d \geq 3$,

$$\sup_{x \in \mathbb{Z}^d} \mathbb{E}|\mathrm{T}(0, x) - \mathbb{E}[\mathrm{T}(0, x)]| = \infty.$$

- The existence of $\chi'(d)$: there exists $\chi'(d)$ such that

$$\mathbb{E}\mathrm{T}(0, nx) - \mathrm{g}(nx) \asymp n^{\chi'(d)}.$$