Loop Measures and Loop-Erased Random Walk (LERW)

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Models from equilibrium statistical mechanics

- Relatively simple definition on discrete lattice. Interest in behavior as lattice size gets large (or lattice spacing shrinks to zero)
- Fractal nonMarkovian random curves or surfaces at criticality.
- Can describe the distribution of curves directly or in terms of a surrounding field
  - (Discrete or continuous) Gaussian free field, Liouville quantum gravity
  - Measures and soups of Brownian (random walk) loops.
  - Isomorphism theorems relate these.
- Discrete models can be analyzed using combinatorial techniques.

- Hope to define and describe continuous object that is scaling limit. Try to use analytic and continous probability tools to analyze.
- Behavior strongly dependent on spatial dimension. (Upper) critical dimension above which behavior is relatively easy to describe.
- Nontrivial below critical dimension.
- If $d = 2$, limit is conformally invariant.
- Considering negative and complex measures can be very useful.
- We will consider one model loop-erased random walk (LERW) and the closely related uniform spanning tree as well as the "field" given by the (random walk and Brownian motion) loop measures and soups

Outline of mini-course

1. Loop measures and soups and relations to LERW, spanning trees, Gaussian field
2. General facts about LERW in $\mathbb{Z}^d$
3. Four-dimensional case (slowly recurrent sets)
4. Two dimensions and exact Green's function
5. Continuum limit in two dimensions, Schramm-Loewner evolution (SLE) and natural parametrization
6. Two-sided LERW
7. The transition probability for two-sided LERW in $d = 2$ and potential theory of "random walk with zipper".
8. For three dimensions see talks of X. Li and D. Shiraishi.

Part 1
(Discrete Time) Loop Measure and Soup

- Discrete analog of Brownian loop measure (work with J. Trujillo Ferreras and V. Limic)
- Le Jan independently developed a continuous time version. Developed further by Lupu with cable systems.
- There are advantages in each approach.
- Discrete time is more closely related to loop-erased walk and is easier to generalize to non-positive weights.
  - Discrete time Markov processes reduce to multiplication of nonnegative matrices.
  - For many purposes, no need to require nonnegative entries (and there are good reasons not to!)

General set-up

- Finite set of vertices A and a function p or q on $A \times A$.
- When we use p the function will be nonnegative. When we use q negative and complex values are possible.
- Symmetric: $p(x, y) = p(y, x)$;
  Hermitian: $q(x, y) = \overline{q(y, x)}$
- Examples
  - irreducible Markov chain on $\overline{A} = A \cup \partial A$ with transition probabilities p, viewed as a subMarkov chain on A.
  - (Simple) random walk in $A \subset \mathbb{Z}^d$:

$$p(x, y) = \frac{1}{2d}, \quad |x - y| = 1.$$

- Measure on paths $\omega = [\omega_0, \ldots, \omega_k]$,

$$q(\omega) = \prod_{j=1}^{k} q(\omega_{j-1}, \omega_j).$$

$q(\omega) = 1$ for trivial paths (single point).

- Green's function

$$G(x, y) = G^q(x, y) = \sum_{\omega: x \to y} q(\omega).$$

The weight q is integrable if for all x, y,

$$\sum_{\omega: x \to y} |q(\omega)| < \infty.$$

- Δ denotes Laplacian: $P - I$ or $Q - I$

$$\Delta f(x) = \Delta^p f(x) = \left[\sum_y p(x, y)\, f(y)\right] - f(x).$$

Usually using $-\Delta = I - P = I - Q = G^{-1}$.

- Rooted loop : path $l = [l_0, \ldots, l_k]$ with $l_0 = l_k$. Nontrivial if $|l| := k \geq 1$.
- The rooted loop measure $\tilde{m} = \tilde{m}^q$ gives each nontrivial loop l measure

$$\tilde{m}(l) = \frac{1}{|l|}\, q(l).$$

- $F(A)$ defined by

$$F(A) = F^q(A) := \exp\left\{ \sum_l \tilde{m}^q(l) \right\} = \frac{1}{\det(I - Q)}.$$

- One way to see the last equality,

$$-\log\det(I - Q) = \sum_{j=1}^{\infty} \frac{1}{j} \mathrm{tr}(Q^j).$$

(Unrooted) loop measure

- An (oriented) unrooted loop ℓ is a rooted loop that forgets the root.
- More precisely, it is an equivalence class of rooted loops under the equivalence relation

$$[l_0, \ldots, l_k] \sim [l_1, \ldots, l_k, l_1] \sim [l_2, \ldots, l_k, l_1, l_2] \sim \cdots .$$

- (Unrooted) loop measure

$$m(\ell) = m^q(\ell) = \sum_{l \in \ell} \tilde{m}(l) = \frac{K(\ell)}{|\ell|} q(\ell),$$

where $K(\ell)$ is the number of rooted representatives of ℓ. (Note that $K(\ell)$ divides $|\ell|$.)
- For example, if $[x, y, x, y, x] \in \ell$, then $|\ell| = 4$ and $K(\ell) = 2$.

$$F(A) = \exp\left\{\sum_{\ell} m(\ell)\right\} = \frac{1}{\det(I-Q)}.$$

Another way to compute $F(A)$

- Let $A = \{x_1, \ldots, x_n\}$ be an ordering of A. Let $A_j = A \setminus \{x_1, \ldots, x_{j-1}\}$. Then

$$F(A) = \prod_{j=1}^{n} G_{A_j}(x_j, x_j).$$

- In particular, the right-hand side is independent of the ordering of the vertices.

- More generally, if $V \subset A$, define

$$F_V(A) = \exp\left\{\sum_{\ell \cap V \neq \emptyset} m(\ell)\right\}.$$

- If $V = \{x_1, \ldots, x_k\}$ and $A_j = A \setminus \{x_1, \ldots, x_{j-1}\}$,

$$F_V(A) = \prod_{j=1}^{k} G_{A_j}(x_j, x_j).$$

  Again, the right-hand side is independent of the ordering of V.

- Note that

$$F_{V_1 \cup V_2}(A) = F_{V_1}(A)\, F_{V_2}(A \setminus V_1).$$

(Chronological) Loop-erasure

- Start with path $\omega = [\omega_0, \ldots, \omega_n]$
- Let
$$s_0 = \max\{t : \omega_t = \omega_0\}.$$
- Recursively, if $s_j < n$, let
$$s_{j+1} = \max\{t : \omega_t = \omega_{s_j+1}\}.$$
- When $s_j = n$, we stop and $LE(\omega) = \eta$ where
$$\eta = LE(\omega) = [\omega_{s_0}, \omega_{s_1}, \ldots, \omega_{s_j}].$$
- η is a self-avoiding walk (SAW) contained in ω with the same initial and terminal points.

Poisson and boundary Poisson kernels

- Assume q is defined on $\overline{A} \times \overline{A}$ where $\overline{A} = A \cup \partial A$.
- If $z \in A, w \in \partial A$,

$$H_A(z,w) = H_A^q(z,w) = \sum_{\omega: z \to w} q(\omega),$$

  where the sum is over all paths ω starting at z, ending at w, and otherwise staying in A.
- If $z \in \partial A, w \in \partial A$,

$$H_{\partial A}(z,w) = H_{\partial A}^q(z,w) = \sum_{\omega: z \to w} q(\omega),$$

  where the sum is over all paths (of length at least 2) starting at z, ending at w, and otherwise staying in A.

LERW from A to ∂A

- For each SAW η starting at $x \in A$, ending at ∂A, and otherwise in A define

$$\hat{q}(\eta) = \sum_{\omega : x \to \partial A, LE(\omega) = \eta} q(\omega).$$

- Here the sum is over all paths starting at x, ending at ∂A, and otherwise in A.
- Note that

$$\sum_{\eta} \hat{q}(\eta) = \sum_{\omega} q(\omega) = \sum_{y \in \partial A} H_A^q(x, y).$$

- In particular, if $q = p$ is a Markov chain, then $\hat{p}$ is a probability measure.

Fact: $\hat{q}(\eta) = q(\eta)\, F^q_\eta(A).$

- Write $\eta = [\eta_0, \ldots, \eta_k]$.
- Decompose any ω with $LE(\omega) = \eta$ uniquely as

$$l_0 \oplus [\eta_0, \eta_1] \oplus l_1 \oplus [\eta_1, \eta_2] \oplus l_2 \oplus \cdots \oplus l_{k-1} \oplus [\eta_{k-1}, \eta_k],$$

  where l_j is a loop rooted at η_j avoiding $[\eta_0, \ldots, \eta_{j-1}]$.
- Measure of possible l_j is $G^q_{A_j}(\eta_j, \eta_j)$ where $A_j = A \setminus \{\eta_0, \ldots, \eta_{j-1}\}$.
- Each $[\eta_{j-1}, \eta_j]$ gives a factor of $q(\eta_{j-1}, \eta_j)$.
- Multiplying we get

$$\prod_{j=1}^{k} q(\eta_{j-1}, \eta_j) \prod_{j=0}^{k-1} G^q_{A_j}(\eta_j, \eta_j) = q(\eta)\, F^q_\eta(A).$$

Wilson's Algorithm

- $\overline{A} = A \cup \partial A$ and p a Markov chain on $\overline{A}$.
- $V = A \cup \{\partial A\}$ (wired boundary)
- Choose a spanning tree of V as follows
  - Choose $z \in A$, run MC until reaches ∂A; erase loops, and add those edges to the tree.
  - If there is a vertex that is not in the tree yet, run MC from there until it reaches a vertex in the tree. Erase loops, and add those edges to the tree.
  - Continue until a spanning tree $\mathcal{T}$ is produced.
- **Fact:** The probability that $\mathcal{T}$ is chosen is $p(\mathcal{T})\, F^p(A)$.

$$p(\mathcal{T}) = \prod_{\vec{xy} \in \mathcal{T}} p(x,y),$$

where $\vec{xy}$ is oriented towards the root ∂A.

Uniform Spanning Trees (UST)

- If G is an undirected graph with vertices $A \cup \partial A$ and p is simple random walk on the graph, then each $\mathcal{T}$ has the same probability of being chosen in Wilson's algorithm

$$p(\mathcal{T})\, F(A) = \left[\prod_{x \in A} \deg(x)\right]^{-1} \frac{1}{\det(I-P)},$$

- The number of spanning trees is given by

$$\left[\prod_{x \in A} \deg(x)\right] \det(I-P) = \det(Deg - Adj)$$

where Deg, Adj are the degree and adjacency matrices of G restricted to rows, columns in A. (Kirchhoff).

Random Walk Loop Soup

- If p is a positive weight, the random walk loop soup with intensity λ is a Possonian realization from $\lambda\tilde{m}$ or λm.
- For the unrooted loop soup can use m or can use $\tilde{m}$ and then forget the root.
- Can be considered as an independent collection of Poisson processes $\{N_\lambda^\ell\}$ with rate $m(\ell)$ where N_λ^ℓ denotes the number of times that unrooted loop ℓ has appeared by time λ.

Loop soup with nonpositive weights?

- Sometimes one wants a Poissonian realization from a negative weight.
- The soup at intensity λ gives a distribution μ_λ on the set of $\mathbb{N}$-valued functions $\mathbf{k} = (k_\ell)$ that equal zero except for a finite number of loops.

$$\mu_\lambda(\mathbf{k}) = \prod_\ell \left[e^{-\lambda m(\ell)} \frac{m(\ell)^{k_\ell}}{k_\ell!} \right] = F(A)^{-\lambda} \prod_\ell \frac{m(\ell)^{k_\ell}}{k_\ell!}.$$

  Here

$$k_\ell = \# \text{ of times } \ell \text{ appears.}$$

- This definition can be extended to nonpositive weights q.

Putting loops back on

- A be a set, $z \in A$. p Markov chain on $\overline{A}$
- Take independently:
  - A loop-erased walk from z to ∂A outputting η
  - A realization of the loop soup with intensity 1 outputting a collection of unrooted loops $\ell_1, \ell_2, \ldots$ ordered by the time that they occurred.
- For each loop ℓ that intersects η choose the first point on η, say η_j that ℓ hits.
- Choose a rooted representative of ℓ that is rooted at η_j and add it to the curve. (If more than one choice, choose randomly.)
- The curve one gets has the distribution of the MC from z to ∂A.

Brownian Loop Measure/Soup (L-Werner)

- Scaling limit of random walk loop
- Rooted (Brownian) loop measure in $\mathbb{R}^d$: choose $(z, t, \tilde{\gamma})$ according to

$$\text{(Lebesgue)} \times \frac{1}{t} \, \frac{dt}{(2\pi t)^{d/2}} \times \text{(Brownian bridge of time 1)}.$$

and output

$$\gamma(s) = z + \sqrt{t}\, \tilde{\gamma}(s/t), \quad 0 \leq s \leq t.$$

- (Unrooted) Brownian loop measure: rooted loop measure "forgetting the root".
- Poissonian realizations are called Brownian loop soup.

- The measure of loops restricted to a bounded domain is infinite because of small loops.
- Measure of loops of diameter $\geq \epsilon$ in a bounded domain is finite.
- If $d = 2$, then the Brownian loop measure (on unrooted loops) is conformally invariant: if $f : D \to f(D)$ is a conformal transformation and $f \circ \gamma$ is defined with change of parametrization, then for every set of curves V,

$$\mu_{f(D)}(V) = \mu_D\{\gamma : f \circ \gamma \in V\}.$$

- True for unrooted loops but not true for rooted loops.

Convergence of Random Walk Soup

- Consider (simple) random walk measure on $\mathbb{Z}^2$ scaled to $N^{-1}\mathbb{Z}^2$.
- Scale the paths using Brownian scaling but do not scale the measure.
- The limit is Brownian loop measure in a strong sense. (L-Trujillo Ferreras).
- Given a bounded, simply connected domain D, we can couple the Brownian soup and the random walk soup with scaling N^{-1} such that, except for an event of probability $O(N^{-\alpha})$, the loops of time duration at least $N^{-\beta}$ are very close.
- A version for all loops, viewing the soup as a field, in preparation (L-Panov).

Loop soups and Gaussian Free Field

- Let A be a finite set with real-valued, symmetric, integrable weight q. Let $G = (I - Q)^{-1}$ be the Green's function which is positive definite.
- If q is a positive weight, G has all nonnegative entries. However, negative q allow for G to have some negative entries.
- The corresponding (discrete) Gaussian free field (with Dirichlet boundary conditions) is a centered multivariate normal $Z_x, x \in A$ with covariance matrix G.
- (Le Jan) Use the random walk loop soup to sample from $Z_x^2/2$.
- (Lupu) If Q is positive, find way to add signs to get Z_x.

Discrete time version of isomorphism theorem

- Consider the loop soup at intensity $1/2$. For each configuration of loops, let N_x denote the number of times that vertex x is visited.
- The random walk loop measure gives a measure on possible values $\{N_x : x \in A\}$.
- Take independent Gamma processes $\Gamma_x(t)$ of rate 1 at each $x \in A$ and let $T_x = \Gamma_x(\frac{1}{2} + N_x)$.
- Theorem: $\{T_x : x \in A\}$ has the same distribution as $\{Z_x^2/2 : x \in A\}$.
- As an example, if $q \equiv 0$, so that there are no loops then $N \equiv 0$, and $\{T_x : x \in A\}$ are independent $\Gamma(\frac{1}{2})$, that is, have the distribution of $Y^2/2$ where Y is a standard normal.

Proof of Isomorphism Theorem

- Just check it.
- (L-Perlman) Using Laplace transform adapting proof of Le Jan. Does not need positive weights.
- Can give a direct proof at intensity $1/2$ using a combinatorial graph identity and get the joint distribution of T_x and the current (local time on undirected edges).
- (L-Panov) Direct proof with intensity 1 for the sum of two indpendent copies (or for $|Z|^2$ for a complex field $Z = X + iY$). Uses an easier combinatorial identity.
- Intensity λ is related to central charge $\mathbf{c}$ of conformal field theory, $\lambda = \pm\frac{\mathbf{c}}{2}$.

Part 2
(One-sided) LERW in $\mathbb{Z}^d$, $d \geq 2$

- $(d \geq 3)$ Take simple random walk (SRW) and erase loops chronologically. This gives an infinite self-avoiding path.
- We get the same measure by starting with SRW conditioned to never return to the origin.
- The latter definition extends to $d = 2$ by using SRW "conditioned to never return to 0", more precisely, tilted by the potential kernel (Green's function).
- This is equivalent to other natural definitions such as take SRW stopped when it reaches distance R, erase loops, and take the (local) limit of measure as $R \to \infty$.

LERW as the Laplacian Random Walk

- Start with $\hat{S}_0 = 0$.
- Given $[\hat{S}_0, \ldots, \hat{S}_n] = \eta = [x_0, \ldots, x_n]$ choose x_{n+1} among nearest neighbors of x_n using distribution $c\,\phi$ where
  - $\phi = \phi_\eta$ is the unique function that vanishes on η; is (discrete) harmonic on $\mathbb{Z}^d \setminus \eta$ and has asymptotics

$$\phi(z) \to 1, \quad d \geq 3,$$

$$\phi(z) \sim \frac{2}{\pi} \log|z|, \quad d = 2.$$

- Could also consider Laplacian-b walk where we use $c\,\phi^b$ with $b \neq 1$ but this is much more difficult and very little is known about.

Basic idea for understanding LERW

- If the number of points in the first n steps of the walk remaining after loop-erasure is $f(n)$ then

$$|\hat{S}_{f(n)}|^2 = |S_n|^2 \asymp n, \quad |\hat{S}_m|^2 \asymp f^{-1}(m).$$

- The point S_n is not erased if and only if

$$LE(S[0,n]) \cap S[n+1,\infty) = \emptyset.$$

Hence,

$$f(n) \asymp n \, \mathbb{P}\{LE(S[0,n]) \cap S[n+1,\infty) = \emptyset\}.$$

Critical Exponent

- Let $S^1, S^2, \ldots$ be independent SRWs and

$$T_n^j = \min\{t : |S_t^j| \geq e^n\}.$$

$$\omega_n^j = S^j[1, T_n^j], \quad \eta_n^j = LE(S^j[0, T_n^j]).$$

- Interested in

$$\hat{p}_{1,1}(n) = \mathbb{P}\{\eta_n^1 \cap \omega_n^2 = \emptyset\} \approx e^{-\xi n}.$$

  This should be comparable to $e^{-2n} f(e^{2n})$ of previous slide.

- Fractal dimension of LERW should be $2 - \xi$.

Similar problem — SRW intersection exponent

$$p_{1,k}(n) = \mathbb{P}\{\omega_n^1 \cap [\omega_n^2 \cup \cdots \cup \omega_n^{k+1}] = \emptyset\}.$$

- $d = 4$ is critical dimension for intersections of two-dimensional sets.
- If $d \geq 5$, $p_{1,k}(\infty) > 0$.
- Using relation with harmonic measure, we can show

$$p_{1,2}(n) \asymp \begin{cases} e^{n(d-4)} & d < 4 \\ n^{-1} & d = 4. \end{cases}$$

- Cauchy-Schwarz gives

$$\left.\begin{matrix} e^{n(d-4)} \\ n^{-1} \end{matrix}\right\} \lesssim p_{1,1}(n) \lesssim \begin{cases} e^{n(d-4)/2} & d < 4 \\ n^{-1/2} & d = 4. \end{cases}$$

- For $d = 4$, "mean-field behavior" holds, that is

$$p_{1,1}(n) \asymp [p_{1,2}(n)]^{1/2} \asymp n^{-1/2}.$$

- For $d < 4$, mean-field behavior does not hold. In fact,

$$p_{1,1}(n) \sim c\, e^{-\xi n}$$

  where $\xi = \xi_d(1,1) \in (\frac{4-d}{2}, 4-d)$ is the Brownian intersection exponent.
- For $d = 2$, $\xi = 5/4$. Proved by L-Schramm-Werner using Schramm-Loewner evolution (SLE).
- For $d = 3$, ξ is not known and may never be known exactly. Numerically $\xi \approx .58$ and rigorously $1/2 < \xi < 1$.

- $\hat{S}$ infinite LERW obtained from SRW S; X, independent SRW, started distance $R = e^n$ away

$$T_n = \min\{j : |X_j| \geq e^n\}.$$

- Long range intersection

$$\mathbb{P}\{X[T_n, T_{n+1}] \cap \hat{S} \neq \emptyset\} \asymp \begin{cases} 1, & d < 4 \\ n^{-1}, & d = 4 \\ e^{(4-d)n} & d > 4. \end{cases}$$

- Two exact exponents — third moment and three-arm exponent. Both obtained by considering the event $S[T_n, T_{n+1}] \cap \hat{S} \neq \emptyset$ and considering the "first" intersection.
- The difference comes from whether one takes the first on S or the first on X.

Let $S^1, S^2, \ldots$ be independent simple random walk starting at the origin and

$$\eta_n^j = LE(S^j[0, T_n^j]), \quad \omega_n^j = S^j[1, T_n^j].$$

Third moment estimate

$$\mathbb{P}\{\eta_n^1 \cap (\omega_n^2 \cup \omega_n^3 \cup \omega_n^4) = \emptyset\} \asymp \begin{cases} n^{-1}, & d = 4 \\ e^{(d-4)n}, & d < 4. \end{cases}$$

Three-arm estimate

$$\mathbb{P}\{\eta_n^1 \cap (\omega_n^2 \cup \omega_n^3) = \emptyset, \eta_n^2 \cap \omega_n^3 = \emptyset\} \asymp \begin{cases} n^{-1}, & d = 4 \\ e^{(d-4)n}, & d < 4. \end{cases}$$

- Let $Z_n = \mathbb{P}\{\eta_n^1 \cap \omega_n^2 = \emptyset \mid \eta_n^1\}$. We are interested in

$$\mathbb{P}\{\eta_n^1 \cap \omega_n^2 = \emptyset\} = \mathbb{E}\left[Z_n\right].$$

- The third moment estimate tells us

$$\mathbb{E}[Z_n^3] \asymp \begin{cases} n^{-1}, & d = 4 \\ e^{(d-4)n}, & d < 4. \end{cases}$$

$$\left.\begin{matrix} n^{-1} \\ e^{(d-4)n} \end{matrix}\right\} \lesssim \mathbb{E}[Z_n] \lesssim \begin{cases} n^{-1/3}, & d = 4 \\ e^{(d-4)n/3}, & d < 4. \end{cases}$$

- Mean-field or non-multifractal behavior would be $\mathbb{E}[Z_n^\lambda] \asymp \mathbb{E}[Z_n]^\lambda$.
- Basic principle: Mean-field behavior holds at the critical dimension $d = 4$ but not below the critical dimension.

Part 3
Slowly recurrent set in $\mathbb{Z}^d$

- Let $A \subset \mathbb{Z}^d, d \geq 2$ and let X be a simple random walk starting at the origin with stopping times $T_n = \min\{j : |X_j| \geq e^n\}$. Let E_n be the event

$$E_n = \{X[T_{n-1}, T_n] \cap A \neq \emptyset\}.$$

- A is recurrent if X visits A infintely often, that is, if $\mathbb{P}\{E_n \text{ i.o.}\} = 1$. This is equivalent to (Wiener's test)

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) = \infty.$$

It is slowly recurrent if also

$$\mathbb{P}(E_n) \to 0.$$

Mostly interested in sets with $\mathbb{P}(E_n) \asymp 1/n$.

Examples of slowly recurrent sets

- A single point in $\mathbb{Z}^2$.
- Line or a half-line in $\mathbb{Z}^3$

$$A = \{(j, 0, 0) : j \in \mathbb{Z}\},$$

$$A_+ = \{(j, 0, 0) : j \in \mathbb{Z}_+\}.$$

- A simple random walk path $A = S[0, \infty)$ in $\mathbb{Z}^4$.
- A loop-erased walk $A = \hat{S}[0, \infty)$ in $\mathbb{Z}^4$.
- The intersection of two simple random walk paths in $\mathbb{Z}^3$.

Basic Idea for Slowly Recurrent Sets

$$E_n = \{X[T_{n-1}, T_n] \cap A \neq \emptyset\}.$$

$$V_n = \mathbb{P}\{X[1, T_n] \cap A = \emptyset\} = \mathbb{P}(E_1^c \cap \cdots \cap E_n^c).$$

$$\mathbb{P}(V_n) = \prod_{j=1}^{n} \mathbb{P}(E_j^c \mid V_{j-1}).$$

- Although $\mathbb{P}(V_{n-1})$ is small it is asymptotic to $\mathbb{P}(V_{n-\log n})$. Hence

$$\mathbb{P}(E_n \mid V_{n-1}) \sim \mathbb{P}(E_n \mid V_{n-\log n}).$$

- The distribution of $X(T_{n-1})$ given $V_{n-\log n}$ is almost the same as the unconditional distribution. Hence,

$$\mathbb{P}(E_n \mid V_{n-\log n}) \sim \mathbb{P}(E_n).$$

- More precisely, find summable δ_n such that

$$\mathbb{P}(E_n \mid V_{n-1}) = \mathbb{P}(E_n) + O(\delta_n).$$

Suppose that

$$\mathbb{P}(E_j) = \frac{\alpha_j}{j}.$$

Then,

$$\begin{aligned}
\mathbb{P}(V_n) &= \prod_{j=1}^{n} \mathbb{P}(E_j^c \mid V_{j-1}) \\
&= \prod_{j=1}^{n} \left[1 - \frac{\alpha_j}{j} + O(\delta_j)\right] \\
&\sim c \exp\left\{-\sum_{j=1}^{n} \frac{\alpha_j}{j}\right\}.
\end{aligned}$$

If $X^1, \ldots, X^k$ are independent simple random walks and

$$V_n^j = \{X^j[1, T_n^j] \cap A = \emptyset\},$$

then

$$\mathbb{P}(V_n^1 \cap \cdots \cap V_n^k) = \mathbb{P}(V_n^1)^k \sim c' \exp\left\{-\sum_{j=1}^{n} \frac{k\alpha_j}{j}\right\}.$$

Example: line A and half-line A_+ in $\mathbb{Z}^3$

$$\mathbb{P}(E_n) = \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

$$\mathbb{P}(E_n^+) = \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

$$\mathbb{P}(V_n) \sim \frac{c}{n},$$

$$\mathbb{P}(V_n^+) \sim \frac{c'}{\sqrt{n}} \asymp \sqrt{\mathbb{P}(V_n)}.$$

(Not quite precise) description of LERW in $\mathbb{Z}^4$

- $\hat{S}[0,\infty)$ infinite LERW in $\mathbb{Z}^4$.
- Let Γ_n be $\hat{S}[0,\infty)$ from the first visit to $\{|z| > e^{n-1}\}$ to the first visit to $\{|z| > e^n\}$ (almost the same as $LE(S[T_{n-1}, T_n))$.
- Let X_t be an independent simple random walk and let K_n be the event that X intersects Γ_n.

$$\mathbb{P}(K_n) = H(\Gamma_n) = \frac{Y_n}{n},$$

  where Y_n has a limit distribution.
- Let

$$Z_n = \mathbb{P}\left[(K_1 \cup \cdots \cup K_n)^c \mid \hat{S}\right].$$

- If the events K_n were independent we would have

$$Z_n = \prod_{j=1}^{n} \left[1 - \frac{Y_j}{j} \right].$$

- 4-d LERW has the same behavior as the toy problem where $Y_1, Y_2 . . . .$ are independent, nonnegative random variables (with an exponential moment).

- $$Z_n = c_n \prod_{j=1}^{n} \left[1 - \frac{Y_j - \mu}{j} \right].$$

  where $\mu = \mathbb{E}[Y_j]$ and

$$c_n = \prod_{j=1}^{n} \left[1 - \frac{\mu}{j} \right] \sim C_\mu \, n^{-\mu}.$$

- There exists a random variable Z such that with probability one
$$Z = \lim_{n\to\infty} n^{\mu} Z_n.$$
- The convergence is in every L^p. Indeed,
$$\begin{aligned} Z_n^p &= c_n^p \prod_{j=1}^{n} \left[1 - \frac{Y_j - \mu}{j}\right]^p \\ &= c_n^p \prod_{j=1}^{n} \left[1 - \frac{p(Y_j - \mu)}{j} + O(j^{-2})\right] \\ &\sim c\, n^{-p\mu} \prod_{j=1}^{n} \left[1 - \frac{p(Y_j - \mu)}{j}\right]. \end{aligned}$$

Theorem (L-Sun-Wu)

Let S, X be independent simple random walks starting at the origin in $\mathbb{Z}^4$ and let $\hat{S}$ denote the loop-erasure of S. Let T_n be the first time that X reaches $\{|z| \geq e^n\}$, and let

$$Z_n = \mathbb{P}\{X[1, T_n] \cap \hat{S}[0, \infty) = \emptyset \mid S[0, \infty)\}.$$

Then the limit

$$Z = \lim_{n \to \infty} n^{1/3} Z_n$$

exists with probability one and in L^p for all p. In particular,

$$\mathbb{E}[Z_n^p] \sim c_p \, n^{-p/3}.$$

- The third-moment estimate tells us that $\mathbb{E}[Z_n^3] \asymp n^{-1}$ which allows us to determine the exponent $1/3$.

Combining with earlier results:

- S simple random walk in $\mathbb{Z}^4$ with loop-erasure $\hat{S}$.
- Define $\sigma(k) = \max\{n : S(n) = \hat{S}(k)\}$. That is, $\hat{S}(k) = S(\sigma(k))$.
- There exists c such that

$$\sigma(k) \sim c\, k\, (\log k)^{1/3}.$$

- Let

$$W_t^{(n)} = \frac{\hat{S}(tn)}{\sqrt{n\,(\log n)^{1/3}}}, \quad 0 \leq t \leq 1.$$

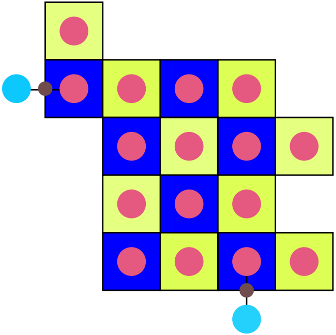
  Then $W^{(n)}$ converges to a Brownian motion.
- For $d \geq 5$, holds without log correction.

Part 4

Two dimensions and conformal invariance

- Associate to each finite $z = x + iy \in \mathbb{Z} + i\mathbb{Z}$, $\mathcal{S}_z$, the closed square of side length 1 centered at z.
- If $A \subset \mathbb{Z}^2$, there is the associated domain

$$\text{int}\left[\bigcup_{z \in A} \mathcal{S}_z\right].$$



- Take $D \subset \mathbb{C}$ a bounded (simply) connected domain containing the origin.
- For each N, let A_N be the connected component of

$$\{z \in \mathbb{Z}^2 : \mathcal{S}_z \subset ND\}.$$

  containing the origin. If D is simply connected, then so is A. We write $D_N \subset D$ for the domain associated to $N^{-1} A_N$.
- If $z, w \in \partial D$ are distinct, we write z_N, w_N for appropraite boundary points (edges) in ∂A_N so that $N^{-1} z_N \sim z, N^{-1} w_N \sim w$.
- Take simple random walk from z_N to w_N in A_N and erase loops.

Main questions

- Let $\eta = [\eta_0, \ldots, \eta_n]$ denote a loop-erased random walk from z_N to w_N in A_N.
- Find fractal dimension d such that typically $n \asymp N^d$.
- Consider the scaled path

$$\gamma_N(t) = N^{-1}\,\eta(tN^d), \quad 0 \leq t \leq n/N^d.$$

  What measure on paths on D does this converge to?
- Reasonable to expect the limit to be conformally invariant: the limit of simple random walk is c.i. and "loop-erasing" seems conformally invariant since it depends only on the ordering of the points.

Possible approaches

- Start by trying to find d directly.
- Assume that the limit is conformally invariant and see what possible limits there are. Determine which one has to be LERW limit. Then try to justify it.
- Both techniques work and both use conformal invariance.
- We will first consider the direct method looking at the discrete process.

- If A is a finite, simply connected subset of $\mathbb{Z} + i\mathbb{Z}$ containing the origin with corresponding domain D_A, let $f = f_A$ be a conformal transformation from D_A to the unit disk with $f(0) = 0$. (Riemann mapping theorem)
- Associate to each boundary edge of $\partial_e A$, the corresponding point z on ∂D_A which is the midpoint of the edge.
- Define $\theta_z \in [0, \pi)$ by $f(z) = e^{2i\theta_z}$
- The conformal radius of A (with respect to the origin) is defined to be

$$r_A(0) = |f'(0)|^{-1}.$$

It is comparable to $\mathrm{dist}(0, \partial A)$ (Koebe $1/4$-theorem)

Theorem (Beneš-L-Viklund)

There exists $\hat{c}, u > 0$ *such that if* A *is a finite simply connected subset of* $\mathbb{Z}^2$ *and* $z, w \in \partial_e A$, *then the probability that loop-erased random walk from* z *to* w *in* A *goes through the origin is*

$$\hat{c}\, r_A^{-3/4} \left[\sin^3 |\theta_z - \theta_w| + O(r_A^{-u})\right].$$

- The constant $\hat{c}$ is lattice dependent and the proof does not determine it. We could give a value of u that works but we do not know the optimum value.
- The exponents $3/4$ and 3 are universal.
- The estimate is uniform over all A with no smoothness assumptions on ∂A (this is important for application).
- A weaker version was proved by Kenyon (2000) and the proof uses an important idea from his paper.

- Let $H_A(0,z)$ be the Poisson kernel.
- $H_{\partial A}(z,w)$ the boundary Poisson kernel. This is also the total mass of the loop-erased measure.
- (Kozdron-L):

$$H_{\partial A}(z,w) = \frac{c' \, H_A(0,z) \, H_A(0,w)}{\sin^2(\theta_z - \theta_w)} [1 + O(r_A^{-u})].$$

- We prove that the $\hat{p}_A$ measure of paths from z to w that go through the origin is asymptotic to

$$\sum_{\eta : z \to w, \, 0 \in \eta} \hat{p}_A(\eta) \sim$$

$$c_* \, H_A(0,z) \, H_A(0,w) \, \sin|\theta_z - \theta_w| \, r_A^{-3/4}.$$

Fomin's identity (two path case)

- Let A be a bounded set and z_1, w_1, z_2, w_2 distinct points on ∂A. Let

$$\hat{H}^q_A(z_1 \leftrightarrow w_1, z_2 \leftrightarrow w_2) = \sum_{\omega^1, \omega^2} q(\omega^1)\, q(\omega^2),$$

  where the sum is over all paths $\omega^j : z_j \to w_j$ in A such that

$$\omega^2 \cap LE(\omega^1) = \emptyset.$$

-

$$\hat{H}^q_A(z_1 \leftrightarrow w_1, z_2 \leftrightarrow w_2) = \sum_{\boldsymbol{\eta} = (\eta^1, \eta^2)} q(\eta^1)\, q(\eta^2)\ F^q_{\boldsymbol{\eta}}(A).$$

  where the sum is over all nonintersecting pairs of SAWs $\boldsymbol{\eta} = (\eta^1, \eta^2)$ with $\eta^j : z_j \to w_j$.

Theorem (Fomin)

$$\hat{H}^q_A(z_1 \leftrightarrow w_1, z_2 \leftrightarrow w_2) - \hat{H}^q_A(z_1 \leftrightarrow w_2, z_2 \leftrightarrow w_1)$$
$$= H^q_A(z_1, w_1)\, H^q_A(z_2, w_2) - H^q_A(z_1, w_2)\, H^q_A(z_2, w_1).$$

- Gives LERW quantities in terms of random walk quantities
- Generalization of Karlin-MacGregor formula for Markov chains.
- There is an n-path version giving a determinantal identity.
- If A is simply connected then at most one term on the left-hand side is nonzero.

- Consider a slightly different quantity

$$\Lambda_A(z,w) = \Lambda_{A,+}(z,w) + \Lambda_{A,-}(z,w) = \sum_{\eta: z\to w,\ \overline{01}\in\eta} \hat{p}_A(\eta)$$

where the sum is over all paths whose loop-erasure uses the edge $\vec{01}$ or its reversal $\vec{10}$.

- $$\Lambda_{A,+}(z,w) = \frac{1}{4}\, F_{01}(A)\, \hat{H}_{A'}(z\leftrightarrow 0, w\leftrightarrow 1),$$
  $$\Lambda_{A,-}(z,w) = \frac{1}{4}\, F_{01}(A)\, \hat{H}_{A'}(z\leftrightarrow 1, w\leftrightarrow 0),$$

  where $A' = A \setminus \{0,1\}$.
- Fomin's identity gives an expression for the difference of the right-hand side in terms of Poisson kernels.

Negative weights (zipper)

- Take a path (zipper) on the dual lattice starting at $\frac{1}{2} - \frac{i}{2}$ going to the right.
- Let q be the measure that equals p except if an edge crosses the zipper

$$q(n, n-i) = -p(n, n-i) = -\frac{1}{4}, \qquad n > 0.$$

$$\Lambda_A^{q}(z,w) = \Lambda_{A,+}^{q}(z,w) + \Lambda_{A,-}^{q}(z,w) = \sum_{\eta : z \to w,\ \overline{01} \in \eta} \hat{q}_A(\eta)$$

$$\Lambda_{A,+}^{q}(z,w) = \frac{1}{4}\, F_{01}^{q}(A)\, \hat{H}_{A'}^{q}(z \leftrightarrow 0, w \leftrightarrow 1),$$

$$\Lambda_{A,-}^{q}(z,w) = \frac{1}{4}\, F_{01}^{q}(A)\, \hat{H}_{A'}^{q}(z \leftrightarrow 1, w \leftrightarrow 0),$$

where $A' = A \setminus \{0,1\}$.

- Fomin's identity gives

$$\sum_{\eta: z \to w,\, \vec{01} \in \eta} \hat{q}_A(\eta) - \sum_{\eta: z \to w,\, \vec{10} \in \eta} \hat{q}_A(\eta) =$$

$$\frac{1}{4} F^{q}_{01} \left[H^{q}_{A'}(z,0)\, H^{q}_{A'}(w,1) - H^{q}_{A'}(z,1) H^{q}_{A'}(w,0) \right].$$

- $$\hat{q}_A(\eta) = q(\eta)\, F^{q}_{\eta}(A).$$

- Two topological facts: first, (with appropriate order of z, w):

$$q_A(\eta) = \begin{cases} p_A(\eta), & \vec{01} \in \eta \\ -p_A(\eta), & \vec{10} \in \eta. \end{cases} .$$

- Second, if ℓ is a loop then $q(\ell) = \pm\, p(\ell)$ where the sign is negative iff ℓ has odd winding number about $\frac{1}{2} - \frac{i}{2}$. Any loop with odd winding number intersects every SAW from z to w in A using $\overline{01}$.
- Therefore,

$$F_\eta(A) = F_\eta^q(A)\, \exp\left\{2\, m(O_A)\right\},$$

where O_A is the set of loops in A with odd winding number about $\frac{1}{2} - \frac{i}{2}$.

Main combinatorial Identity

$$\Lambda_A(z,w) = \sum_{\eta: z\to w,\ \vec{01}\in\eta} \hat{p}_A(\eta) + \sum_{\eta: z\to w,\ \vec{10}\in\eta} \hat{p}_A(\eta)$$

$$= \exp\{2\, m(O_A)\} \left[\sum_{\eta: z\to w,\ \vec{01}\in\eta} \hat{q}_A(\eta) - \sum_{\eta: z\to w,\ \vec{10}\in\eta} \hat{q}_A(\eta) \right]$$

$$= \frac{F^q_{01}(A)}{4}\, e^{2m(O_A)} \times$$

$$[H^q_{A'}(z,0)\, H^q_{A'}(w,1) - H^q_{A'}(z,1) H^q_{A'}(w,0)].$$

- Here, $A' = A \setminus \{0,1\}$ and O_A is the set of loops in A with odd winding number about $\frac{1-i}{2}$.
- $m = m^p$ is the usual random walk loop measure.

The proof then boils down to three estimates:

- $$F^q_{0,1}(A) = c_1 + O(r_A^{-u}).$$
- $$m(O_A) = \frac{\log r_A}{8} + c_2 + O(r_A^{-u}).$$
  $$e^{2m(O_A)} = c_3\, r_A^{1/4}\left[1 + O(r_A^{-u})\right].$$
- $$H^q_{A'}(z,0)\, H^q_{A'}(w,1) - H^q_{A'}(z,1) H^q_{A'}(w,0) =$$
  $$c_4\, r_A^{-1}\, H_A(0,z)\, H_A(0,w)\left[|\sin(\theta_z - \theta_w)| + O(r_A^{-u})\right].$$
- The first one is easiest (although takes some argument).
- The others strongly use conformal invariance of Brownian motion.

Loops with odd winding number

- First consider $A_n = C_n = \{|z| < e^n\}$. Let $O_n = O_{A_n}$.
- $O_n \setminus O_{n-1}$ is the set of loops in C_n of odd winding number that are not contained in C_{n-1}. Macroscopic loops.
- Consider Brownian loops in C_n of odd winding number about the origin that do not lie in C_{n-1}. The measure is independent of n (conformal invariance) and a calculation shows the value is $1/8$.
- Using coupling with random walk measure, show

$$m(O_n) - m(O_{n-1}) = m(O_n \setminus O_{n-1}) = \frac{1}{8} + O(e^{-un}).$$

- $$m(O_n) = \frac{n}{8} + c_2 + O(e^{-un}).$$
- For more general A with $e^n \le r_A \le e^{n+1}$ first approximate by C_{n-4} and then attach the last piece. Uses strongly conformal invariance of Brownian measure.

- $$H^q_{A'}(0,z) = H_{A'}(0,z)\,\mathbb{E}[(-1)^J],$$
  where the expectation is with respect to an h-process from 0 to z in A' and J is the number of times the process crosses the zipper.
- Example: $A = \{x+iy : |x|,|y| < n\}$, $z=-n, w=n$. $H^q_{A'}(0,z)$ is the measure of paths starting at 0, leaving A at z, and not returning to the positive axis.
- Paths that return to the postive axis "from above" cancel with those that return "from below".
- $H^q_{A'}(0,z) \sim c\,n^{-1/2}$.
- Combine this discrete cancellation with macroscropic comparisons to Brownian motion.

Part 5

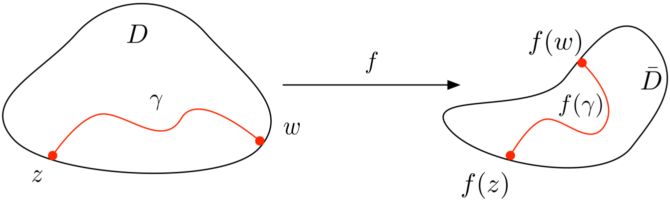
Continuous limit: Schramm-Loewner evolution (SLE)

- Family of probability measures $\{\mu_D(z,w)\}$ on simple curves $\gamma : (0, t_\gamma) \to D$ from z to w in D.
- Supported on curves of fractal dimension $\frac{5}{4} = 2 - \frac{3}{4}$. .
- Suppose $f : D \to f(D)$ is a conformal transformation. Define $f \circ \gamma$ to be the image of γ parametrized so that the time to traverse $f(\gamma[r,s])$ is

$$\int_r^s |f'(\gamma(t))|^{5/4}\, dt.$$

- Conformal invariance:

$$f \circ \mu_D(z,w) = \mu_{f(D)}(f(z), f(w)).$$



- Here $f \circ \mu$ is the pull-back

$$f \circ \mu\,(V) = \mu\{\gamma : f \circ \gamma \in V\}.$$

- Domain Markov property: in the probability measure $\mu_D(z,w)$, suppose that an initial segment $\gamma[0,t]$ is observed. Then the distribution of the remainder of the path is

$$\mu_{D\setminus\gamma[0,t]}(\gamma(t),w).$$

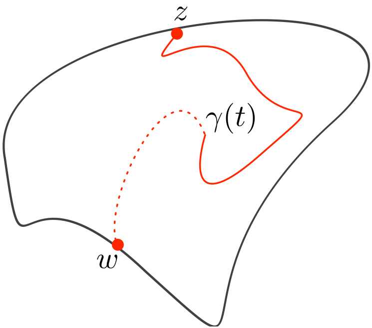


Figure: Domain Markov property (M. Jahangoshahi)

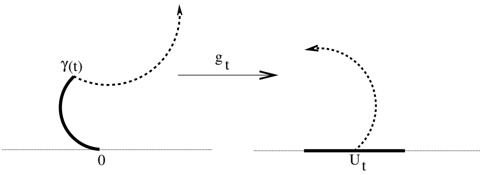
Theorem (Schramm, ...)

There is a unique family of measures satisfying the above properties, the (chordal) Schramm-Loewner evolution with parameter 2 (SLE_2) with natural parametrization.

- SLE_κ exists for other values of κ but the curves have different fractal dimension.
- Schramm only considered simply connected domains. In general, extending to multiply connected is difficult but $\kappa = 2$ is special where it is more straightforward.

Definition of SLE_2

- $g_t : \mathbb{H} \setminus \gamma(0,t] \to \mathbb{H}$



- Reparametrize (by capacity) and then g_t satisfies

$$\partial_t g_t(z) = \frac{1}{g_t(z) - U_t}, \quad g_0(z) = z.$$

where U_t is a standard Brownian motion.

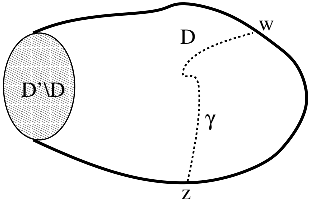
- Extend to simply connected domains by conformal invariance. For other domains use the (generalized) restriction property.

(Generalized) restriction property

- If $D \subset D'$, the Radon-Nikodym derivative

$$\frac{d\mu_D(z,w)}{d\mu_{D'}(z,w)}\,(\gamma)$$

is proportional to e^{-L} where L is the measure of loops in D' that intersect both γ and $D' \setminus D$. (Conformally invariant)



SLE Green's function

- Suppose D is a simply connected domain containing the origin and $\gamma : z \to w$ is an SLE_2 path.
- There exists c_* such that

$$\mathbb{P}\{\mathrm{dist}(0,\gamma) \leq r\} \sim c_* \, r^{3/4} \, \sin^3 |\theta_z - \theta_w|, \quad r \downarrow 0.$$

- More generally for SLE_κ with $\kappa < 8$,

$$\mathbb{P}\{\mathrm{dist}(0,\gamma) \leq r\} \sim c_*(\kappa) \, r^{1-\frac{\kappa}{8}} \, \sin^{\frac{8}{\kappa}-1} |\theta_z - \theta_w|, \quad r \downarrow 0.$$

Parametrization

- The SLE path is parametrized by (half-plane) capacity so that

$$g_t(z) = z + \frac{1}{z} + O(|z|^{-2}), \quad z \to \infty.$$

  This is singular with respect to the "natural parametrization".
- How does one parametrize a $(5/4)$-dimensional fractal curve?
- Hausdorff $(5/4)$-measure is zero.
- Hausdorff measure with "gauge function" might be possible but too difficult for SLE paths.

Minkowski content

- Let $\gamma_t = \gamma[0,t]$.
- (L-Rezaei) With probability one,

$$\text{Cont}_{5/4}(\gamma_t) = \lim_{r\downarrow 0} r^{-3/4} \text{Area}(\{z : \text{dist}(z,\gamma_t) \le r\})$$

  exists, is continuous and strictly increasing in t.
- Natural parametrization: $\text{Cont}_{5/4}(\gamma_t) = t$.
- Chordal SLE with the natural parametrization is the measure on curves with properties described before.

Convergence result (L - Viklund)

- D a bounded, analytic domain containing the origin with distinct boundary points a, b.
- For each N, let A be the connected component containing the origin of all $z \in \mathbb{Z}^2$ such that $\mathcal{S}_z \subset N \cdot D$. where $\mathcal{S}_z$ is the closed square centered at z of side length 1.
- Let $a_N, b_N \in \partial_e A_N$ with $a_N/N \to a, b_N/N \to b$.
- Let μ_N be the probability measure on paths obtained as follows:
- Take LERW from a_N to b_N in A_N. Write such a path as

$$\eta = [a_-, a_+, \eta_2, \dots, \eta_{k-1}, b_+, b_-].$$

- Scale the path η by scaling space by N^{-1} and time by $c\,N^{-5/4}$. Use linear interpolation to make this a continuous path. This defines the probability measure μ_N.
- Define a metric $\rho(\gamma^1, \gamma^2)$ on paths $\gamma^j : [s_j, t_j] \to \mathbb{C}$,

$$\inf \left\{ \sup_{s_1 \le t \le t_1} |\alpha(t) - t| + \sup_{s_1 \le t \le t_1} |\gamma^2(\alpha(t)) - \gamma^1(t)| \right\}.$$

  where the infimum is over all increasing homeomorphisms $\alpha : [s_1, t_1] \to [s_2, t_2]$.
- Let $\mathfrak{p}$ denote the corresponding Prokhorov metric.

Theorem (L-Viklund)

As $n \to \infty$,

$$\mu_N \to \mu$$

in the Prokhorov metric.

- Convergence for curves modulo parametrization (and in capacity parametrization) was proved by L-Schramm-Werner.
- The new part is the convergence in the natural parametrization.

Part 6
Two-sided loop-erased random walk

- The infinite two-sided loop-erased random walk (two-sided LERW) is the limit measure of the "middle" of a LERW.
- Probability measure on pairs of nonintersecting infinite self-avoiding starting at the origin.
- Straightforward to construct if $d \geq 5$.
- This construction can be adapted for $d = 4$ using results of L-Sun-Wu. It will not work for $d = 2, 3$.
- New result constructs the process for $d = 2$ and $d = 3$.

Constructing two-sided LERW for $d \geq 4$

- Start with independent simple random walks starting at the origin S, X.
- Erase loops from S giving the (one-sided) LERW $\hat{S}[0,\infty)$. Reverse time so that it goes from time $-\infty$ to 0.
- Tilt the measure on $\hat{S}$ by $\tilde{Z} := Z/\mathbb{E}[Z]$, where

$$Z = \mathbb{P}\{X[1,\infty) \cap \hat{S}[0,\infty) = \emptyset \mid \hat{S}\}, \quad d \geq 5,$$

$$Z = \lim_{n\to\infty} n^{1/3}\, \mathbb{P}\{X[1,T_n] \cap \hat{S}[0,\infty) = \emptyset \mid \hat{S}\}, \quad d = 4.$$

- If $d \geq 5$, $\tilde{Z}$ is bounded. If $d = 4$, it is not bounded but has all moments.

- Given $\hat{S}$, choose X as random walk conditioned to avoid $\hat{S}[0,\infty)$. For $d=4$, one does an h-process with harmonic function

$$Z_x = \lim_{n\to\infty} n^{1/3}\, \mathbb{P}^x\{X[1,T_n] \cap \hat{S}[0,\infty) = \emptyset\}.$$

- Erase loops from X to give the "future" of the two-sided LERW.
- Uses reversibility of (the distribution of) LERW.
- If $d<4$, the marginal distribution of one path is not absolutely continuous with respect to one-sided measure so this does not work.

Notation

- $C_n = \{z \in \mathbb{Z}^d : |z| < e^n\}$.
- $\mathcal{W}_n$ is the set of SAWs η starting at the origin, ending in ∂C_n and otherwise in C_n.
- $\mathcal{A}_n$ is the set of ordered pairs $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{W}_n^2$ such that
$$\eta^1 \cap \eta^2 = \{0\}.$$
- $\mathcal{A}_n(a, b)$ is the set of such $\boldsymbol{\eta}$ such that η^1 ends at a and η^2 ends at b.
- By considering $(\eta^1)^R \oplus \eta^2$, we see there is a natural bijection between $\mathcal{A}_n(a, b)$ and the set of SAWs from a to b in C_n that go through the origin.

- Similarly, we can define $\mathcal{A}_n(a, b; A)$ for SAWs from a to b in A.
- The loop-erased measure on $\mathcal{A}_n(a, b; A)$ is the measure

$$\hat{p}_A(\eta) = p(\eta)\, F_\eta(A) = (2d)^{-|\eta|}\, F_\eta(A).$$

  Can normalize to make it a probability measure. Same probability measure if we use

$$\hat{p}_A(\eta) = p(\eta)\, F_\eta(\hat{A}) = (2d)^{-|\eta|}\, F_\eta(\hat{A}), \quad \hat{A} = A \setminus \{0\}.$$

- If $C_k \subset A$, then this measure induces a probability measure $\mathbb{P}_{A,a \to b,k}$ on $\mathcal{A}_k$.

Theorem

For each k, there exists a probability measure $\hat{p}_k$ on $\mathcal{A}_k$ such that if $k < n$, $C_n \subset A$, $a, b \in \partial_e A$ with $\mathcal{A}(a, b; A)$ nonempty, then for all $\boldsymbol{\eta} \in \mathcal{A}_k$,

$$\mathbb{P}_{A,a\to b,k}(\boldsymbol{\eta}) = \hat{p}_k(\boldsymbol{\eta}) \left[1 + O(e^{u(k-n)})\right].$$

More precisely, there exist c, u such that for all such k, A, a, b and all $\boldsymbol{\eta} \in \mathcal{A}_k$,

$$\left| \log \left[\frac{\mathbb{P}_{A,a\to b,k}(\boldsymbol{\eta})}{\hat{p}_k(\boldsymbol{\eta})} \right] \right| \leq c\, e^{u(k-n)}.$$

- The measures $\hat{p}_k$ are easily seen to be consistent and this gives the two-sided LERW.

Slightly different setup

- Let η^1, η^2 be independent infinite LERW stopped when then reach ∂C_n. This gives a measure $\mu_n \times \mu_n$ on $\mathcal{W}_n^2$.

$$\mu_n(\eta) = (2d)^{-|\eta|} \, F_\eta(\hat{\mathbb{Z}}^d) \, \mathrm{Es}_\eta(z),$$

  where z is the endpoint of η.
- Note This is not the same as "stop a simple random walk when it reaches ∂C_n and then erase loops" which would give measure

$$(2d)^{-|\eta|} \, F_\eta(C_n).$$

- Given η^j, the remainder of the infinite LERW walk is obtained by:
  - Take simple random walk starting at the end of η^j conditioned to never return to η^j
  - Erase loops.

Tilted measure ν_n

- Obtain ν_n by tilting $\mu_n \times \mu_n$ by

$$1\{\eta \in \mathcal{A}_n\} \exp\{-L_n(\boldsymbol{\eta})\},$$

  where $L_n = L_n(\boldsymbol{\eta})$ is the loop measure of loops in $\hat{C}_n$ that intersect both η^1 and η^2.
- This is to compensate for "double counting" of loop terms.
- If $d = 2$, restrict to loops that do not disconnect 0 from ∂C_n (any disconnecting loop intersects all η^1, η^2 and hence does not contribute to the probability measure).
- If $C_{n+1} \subset A$, then

$$\mathbb{P}_{A,a \to b,n} \ll \nu_n^{\#}.$$

SLE analogue

(L- Kozdron, Lind, Werness, Jahangoshahi, Healey,...)

- Natural measure on multiple SLE_κ paths $\kappa \leq 4$ can be obtained from starting with k independent SLE_κ paths $\boldsymbol{\gamma} = (\gamma^1, \ldots, \gamma^k)$ and tilting by

$$Y(\boldsymbol{\gamma}) = I \exp\left\{ \frac{\mathbf{c}}{2} \sum_{j=2}^{k} L_j \right\}, \qquad \mathbf{c} = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa},$$

  where L_j is the Brownian loop measure of loops that hit at least j of the paths and I is the indicator that the paths are disjoint.
- The case $k = 2$ is sometimes called two-sided radial SLE_κ. The scaling limit of two-sided LERW in $\mathbb{Z}^2$ is two-sided SLE_2.

Coupling

- Let $\gamma \in \mathcal{A}_k$ and $\nu_n^{\#}(\cdot \mid \gamma)$ the conditional distribution given that the initial configuration is γ.
- Challenge: Couple $\nu_n^{\#}$ and $\nu_n^{\#}(\cdot \mid \gamma)$ so that, except for an event of probability $O(e^{-u(n-k)})$, the paths agree from their first visit to $C_{k+(n-k)/2}$ onward.
- Given this,

$$\frac{\nu_{n+1}(\mathcal{A}_{n+1})}{\nu_n(\mathcal{A}_n)} = \frac{\nu_{n+1}(\mathcal{A}_{n+1};\gamma)}{\nu_n(\mathcal{A}_n;\gamma)} \left[1 + O(e^{-u(n-k)})\right]$$

$$\frac{\nu_{n+1}(\mathcal{A}_{n+1};\gamma)}{\nu_{n+1}(\mathcal{A}_{n+1})} = \frac{\nu_n(\mathcal{A}_n;\gamma)}{\nu_n(\mathcal{A}_n)} \left[1 + O(e^{-u(n-k)})\right]$$

- Fix (large) n and $\gamma_k, \tilde{\gamma}_k \in \mathcal{A}_k$ with $k < n$.
- Couple Markov chains $\gamma_k, \gamma_{k+1}, \ldots, \gamma_n$ and $\tilde{\gamma}_k, \tilde{\gamma}_{k+1}, \ldots, \tilde{\gamma}_n$ so they have the distribution of the beginning of the paths under $\nu_n^{\#}$.
- Write $\gamma_j =_r \tilde{\gamma}_j$ if the paths agree from their first visit to ∂C_{j-r} to ∂C_j.
- Suppose we can show the following:
  - For every $j < \infty$ can find $\rho_j > 0$ such that given any $(\gamma_k, \tilde{\gamma}_k)$ we can couple so that with probability at least ρ_j, $\gamma_{k+j} =_{j-2} \tilde{\gamma}_{k+j}$.
  - If $\gamma_k =_j \tilde{\gamma}_k$, then we can couple the next step such that, except perhaps on an event of probability $O(e^{-\beta j})$,

$$\gamma_{k+1} =_{j+1} \tilde{\gamma}_{k+1}.$$

- Then there exists c, u such that for any $\boldsymbol{\gamma}_k, \tilde{\boldsymbol{\gamma}}_k$,

$$\mathbb{P}\{\boldsymbol{\gamma}_n =_{(n-k)/2} \tilde{\boldsymbol{\gamma}}_n\} \geq 1 - ce^{-u(n-k)}.$$

- Does not give a good estimate on u.
- Same basic strategy used for other problems, e..g, the measure of Brownian motion "at a random cut point".
- The hard work is showing that the conditions on previous slide hold.
- We discuss some of the ingredients of the proof.

"Obvious" fact about simple random walk

- Let $\eta \in \mathcal{W}_n$ and S a simple random walk starting at z, the endpoint of η.
- Let $\tau = \tau_r = \min\{j : |S_j - z] \geq r\}$
- Lemma: there exists uniform $\rho > 0$ such that

$$\mathbb{P}\{|S_\tau| \geq e^n + \frac{r}{3} \mid S[1,\tau] \cap \eta = \emptyset\} \geq \rho.$$

- If there were no conditioning this would follow from central limit theorem. Conditioning should only increase the probability so it is"obvious".
- Important to know that there exists ρ that works for all n, η, r.
- Various versions have been proved by L, Masson, Shiraishi
- Brownian motion version is easier — then careful approximation of BM by random walk.

- Corollary: the probability that simple random walk starting at z conditioned to avoid η enters C_{n-k} is less than $c\,e^{-k}$.
- This obviously holds for the loop-erasure as well.
- For $d \geq 3$ we use transience of the simple walk: the probability that a RW starting outside C_n reaches C_{n-k} is $O(e^{(d-2)(k-n)})$.
- For $d = 2$ we use the Beurling estimate (Kesten). The probability a random walk starting at C_n reaches C_{n-k} and then returns to ∂C_n without intersecting η is $O(e^{k-n})$.
- One of the reasons to use "infinite LERW when it reaches ∂C_n" rather than "loop erasure of RW when it reaches ∂C_n" is to use this fact.

Estimating loop measure

- If $d \geq 3$, the loop measure of loops that intersect both ∂C_n and C_{n-k} is $O(e^{(d-2)(k-n)})$.
- If $d = 2$, the loop measure of loops that intersect both ∂C_n and C_{n-k} and do not disconnect the origin from ∂C_n is $O(e^{(k-n)/2})$.
- This uses the disconnection exponent for $d = 2$ RW: the probability that a RW starting next to the origin reaches C_n without disconnecting the origin is comparable to $O(e^{-n/4})$. (L-Puckette, L-Schramm-Werner)
- For $d = 2$ focus on nondisconnecting loops. Loops that disconnect intersect all SAWs and hence do not affect the normalized probability measure on SAWs weighted by a loop term.

Separation lemma

- Let $\boldsymbol{\eta} = (\eta^1, \eta^2) \in \mathcal{A}_n$.
- Consider all $\tilde{\boldsymbol{\eta}} = (\tilde{\eta}^1, \tilde{\eta}^2) \in \mathcal{A}_{n+1}$ that extend $\boldsymbol{\eta}$.
- If we tilt by the loop term $e^{-L_{n+1}}$ there is a positive probability ρ (independent of n, $\boldsymbol{\eta}$) that the endpoints of $\tilde{\boldsymbol{\eta}}$ are separated.
- First proved for nonintersecting Brownian motions.
- An analogue of (parabolic) boundary Harnack principle — if one conditions a Brownian motion to stay in a domain for a while, then the path gets away from the boundary.
- This is a key step in coupling $\boldsymbol{\eta}_n, \boldsymbol{\gamma}_n$ with positive probability.
- There is also a version for LERW in A from x to y (in ∂A) conditioned to go through the origin.

Original theorem

- Let $A \supset C_{n+1}$ and x, y distinct boundary points.
- Consider LERW from x to y in A conditioned so that paths go through the origin.
- Let $\lambda_A = \lambda_{A,x,y}$ be the probability measure obtained by truncating to paths $\in \mathcal{A}_n$. Consider

$$Y(\boldsymbol{\eta}) = \frac{d\lambda_A}{d\lambda_n}(\boldsymbol{\eta}).$$

- The distribution of Y depends on A, x, y; however
  - Y is uniformly bounded.
  - If $\boldsymbol{\eta} =_k \boldsymbol{\gamma}$, then

$$Y(\boldsymbol{\eta}) = Y(\boldsymbol{\gamma}) + O(e^{-k/2}).$$

Part 7
The distribution in $\mathbb{Z}^2 = \mathbb{Z} + i\,\mathbb{Z}$
(with F. Viklund, C. Beneš)

- The distribution of two-sided LERW for $d = 2$ is closely related to potential theory with zipper (signed weights) or in double covering of $\mathbb{Z}^2$.

$$q(\mathbf{e}) = -p(\mathbf{e}) = -\frac{1}{4}, \quad \mathbf{e} = \{x - i, x\}, x > 0,$$

$$q(\mathbf{e}) = p(\mathbf{e}) = \frac{1}{4}, \quad \text{other } \mathbf{e}.$$

- Δ denotes the usual random walk Laplacian and Δ^q the corresponding operator for q:

$$\Delta^q f(z) = \left[\sum_w q(z,w)\, f(w)\right] - f(z).$$

Fundamental solutions

- The fundamental solution $a(z)$ of Δ is the potential kernel which is a discrete harmonic approximation of $\log|z|$.
- The fundamental solutions of Δ^q are discrete q-harmonic approximations of real and imaginary parts of $\sqrt{z}$:
- Let S be a simple random walk,

$$\sigma_R = \min\{j : |S_j| \geq R\},$$

$$\tau_+ = \min\{j \geq 0 : S_j \in \{0, 1, 2, \ldots\}\}.$$

$$u(z) = \lim_{R \to \infty} R^{1/2}\, \mathbb{P}^z\{\sigma_R < \tau_+\}.$$

$$u(z) = 0, \qquad z \in \{0, 1, 2, 3, \ldots\},$$

$$u(x + iy) = u(x - iy),$$

$$\Delta u(z) = 0, \qquad z \notin \{0, 1, 2, \ldots\},$$

$$\Delta^q u(z) = 0, \qquad z \neq 0.$$

- If $f(z) = |z|^{1/2} \sin(\theta_z/2)$, then

$$\left| u(z) - \frac{4}{\pi} f(z) \right| \leq c \, \frac{f(z)}{|z|}$$

- Define the "conjugate" function v by

$$v(-x + iy) = \pm u(x + iy),$$

where the sign is chosen to be negative on $\{\mathrm{Im}(z) < 0\}$.

$$v(z) = 0, \qquad z \in \{0, -1, -2, -3, \ldots\},$$

$$v(x + iy) = -v(x - iy), \qquad y > 0,$$

$$\Delta v(z) = 0, \qquad z \notin \{0, -1, -2, \ldots\},$$

$$\Delta^q v(z) = 0, \qquad z \neq 0.$$

- If $g(z) = |z|^{1/2} \cos(\theta_z/2)$, then

$$\left| v(z) - \frac{4}{\pi} g(z) \right| \leq c \, \frac{|g(z)|}{|z|}$$

- If η is a finite set of vertices containing the origin,

$$a_\eta(z) = a(z) - \sum_{w \in \eta} H_{\mathbb{Z}^2 \setminus \eta}(z, w)\, a(w).$$

- $H_{\mathbb{Z}^2 \setminus \eta}(z, w)$ is the Poisson kernel

$$H_{\mathbb{Z}^2 \setminus \eta}(z, w) = \sum_{\omega : z \to w} p(\omega),$$

  where the sum is over all nearest neighbor paths from z to w, otherwise in $\mathbb{Z}^2 \setminus \eta$.

- Then a_η satisfies $a_\eta \equiv 0$ on η and

$$\Delta a_\eta(z) = 0, \quad z \notin \eta,$$

$$a_\eta(z) \sim \frac{2}{\pi} \log |z|, \quad z \to \infty.$$

- $$u_\eta(z) = u(z) - \sum_{w\in\eta} H^q_{\mathbb{Z}^2\setminus\eta}(z,w)\, u(w).$$
$$v_\eta(z) = v(z) - \sum_{w\in\eta} H^q_{\mathbb{Z}^2\setminus\eta}(z,w)\, v(w).$$
- $H^q_{\mathbb{Z}^2\setminus\eta}(z,w)$ is the Poisson kernel
$$H^q_{\mathbb{Z}^2\setminus\eta}(z,w) = \sum_{\omega: z\to w} q(\omega),$$
where the sum is over all nearest neighbor paths from z to w, otherwise in $\mathbb{Z}^2 \setminus \eta$.
- Then u_η, v_η satisfies $u_\eta, v_\eta \equiv 0$ on η and
$$\Delta^q u_\eta(z) = \Delta^q v_\eta(z) = 0, \qquad z \notin \eta,$$
$$u_\eta(z) = u(z) + o(1), \qquad v_\eta(z) = v(z) + o(1), \qquad z \to \infty.$$

Let $\eta = [\eta_0 = 0, \eta_1 = 1, \ldots, \eta_k]$ be a SAW starting with $[0, 1]$.

- The probability that the one-sided LERW traverses η is

$$4^{-k} F_\eta(\hat{\mathbb{Z}}^2) \Delta a_\eta(\eta_k).$$

- There exists c such that the probability that the two-sided LERW traverses η is

$$\hat{p}(\eta) := c\, 4^{-k} F^q_\eta(\hat{\mathbb{Z}}^2) \det M_\eta,$$

$$M_\eta = \begin{bmatrix} \Delta^q v_\eta(\eta_k) & \Delta^q u_\eta(0) \\ \Delta^q u_\eta(\eta_k) & \Delta^q v_\eta(0) \end{bmatrix}.$$

- Follows from Fomin's identity using the weight q (as done in BLV) and being able to take the limit (using the recent result).

Example: $\eta = \eta^k = [0, 1, \ldots, k], \quad k \geq 2$

- $u_\eta(z) = u(z)$, $v_\eta(z) = v(z-k)$, $\Delta^q u_\eta(k) = 0$
- $\Delta^q u_\eta(0) = \Delta^q v_\eta(k) = \Delta u(0)$.
- $$\frac{\hat{p}(\eta^k)}{\hat{p}(\eta^{k-1})} = \frac{1}{4}\, F_k^q(\mathbb{Z}^2 \setminus \eta^{k-1}) = \frac{1}{4}\, G^q_{\mathbb{Z}^2 \setminus \eta^{k-1}}(k,k).$$
- In the q-measure loops that hit the negative real axis have total measure zero since "positive" loops cancel with "negative" loops. Hence,

$$G^q_{\mathbb{Z}^2 \setminus \eta^{k-1}}(k,k) = G_{\mathbb{Z}^2 \setminus \{,\ldots,k-2,k-1\}}(k,k) =$$

$$G_{\mathbb{Z}^2 \setminus \{\ldots,-1,0\}}(1,1) = 4\,(\sqrt{2}-1).$$

- Therefore, $\hat{p}(\eta^k) = 4^{-1}\,(\sqrt{2}-1)^{k-1}$ (Also derived by Kenyon-Wilson)

Part 8
Three dimensions
(Li and Shiraishi using result of Kozma)

- There exists an α such that the loop-erased walk grows like n^{α}.
- Moreover, the paths scaled by number of steps (natural parametrization) converge to a scaling limit.
- α is not known (may never be known) and the nature of the limit is not known.

Open problem: Laplacian motion in $\mathbb{R}^3$

- Can we give a description in the continuum of the scaling limit of LERW in three dimensions?
- It should be "Brownian motion tilted locally by harmonic measure", that is, Laplacian motion.

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THANK YOU