Gaussian Free Fields with Boundary Points, Multiple SLEs, and Log-Gases

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Stochastic Analysis, Random Fields and Integrable Probability

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Plan

1. Introduction
1.1 Stochastic log-gases in R

- **For $N \in \mathbb{N} := \{1, 2, \dots\}$, consider a system of interacting Brownian motions on $\mathbb{R}$, $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in S^N$, $S \subset \mathbb{R}$, $t \geq 0$, following the SDEs,**

$$dX_i(t) = \sqrt{\kappa} dB_i(t) + F_i(\boldsymbol{X}(t))dt, \quad t \geq 0, \quad 1 \leq i \leq N,$$

where $\{B_i(t) : t \geq 0\}_{i=1}^N$ are mutually independent one-dimensional standard Brownian motions, and $\kappa > 0$. (Note that $\sqrt{\kappa} B(t) \overset{\text{(law)}}{=} B(\kappa t), t \geq 0$.)

- **Example 1: Dyson model with parameter $\beta > 0$. Set $S = \mathbb{R}$.**

Consider the case that

$$F_i(\boldsymbol{x}) = \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{4}{x_i - x_j}, \quad 1 \le i \le N.$$

A time change of the obtained SDEs ($\kappa t \to t$, $\boldsymbol{X}(t/\kappa) \to \boldsymbol{X}(t)$) gives

$$dX_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{dt}{X_i(t) - X_j(t)}, \quad t \ge 0, \quad 1 \le i \le N \quad \textbf{with} \quad \beta = \frac{8}{\kappa}.$$

The Dyson model generates the dynamical extensions of the eigenvalue statistics in Gaussian random-matrix ensembles.

- **Example 2:** **Bru–Wishart process** **with parameters** (β, ν). **Set** $S = \mathbb{R}_{\geq 0}$.

Consider the case that

$$F_i(\boldsymbol{x}) = \sum_{\substack{1 \leq j \leq N, \\ j \neq i}} \left(\frac{4}{x_i - x_j} + \frac{4}{x_i + x_j} \right) + \frac{4\alpha}{x_i}, \quad 1 \leq i \leq N, \quad \alpha \in \mathbb{R}.$$

A time change of the obtained SDEs ($\kappa t \to t$, $\boldsymbol{X}(t/\kappa) \to \boldsymbol{X}(t)$) **gives**

$$dX_i(t) = dB_i(t) + \frac{1}{2} \left[\frac{\beta(\nu+1) - 1}{X_i(t)} + \beta \sum_{\substack{1 \leq j \leq N, \\ j \neq i}} \left(\frac{1}{X_i(t) - X_j(t)} + \frac{1}{X_i(t) + X_j(t)} \right) \right] dt,$$

$$t \geq 0, \quad 1 \leq i \leq N \quad \textbf{with} \quad \beta = \frac{8}{\kappa}, \quad \nu = \alpha - 1 + \frac{\kappa}{8}.$$

The Bru–Wishart process generates the dynamical extensions of the singular-value/eigenvalue statistics in chiral-Gaussian/Laguerre random-matrix ensembles.

Stochastic Log-Gases in $\mathbb{R}$
(=Stochastic 2D Coulomb Gases Confined in $\mathbb{R}$)

Stochastic processes in $\mathbb{R}$

$$dX_i(t) = dB_i(t) - \frac{1}{2}\frac{\partial \Phi(\mathbf{x})}{\partial x_i}\bigg|_{\mathbf{x}=\mathbf{X}(t)} dt, \quad t \geq 0, \quad 1 \leq i \leq N,$$

driven by logarithmic potentials,

$$\Phi(\mathbf{x}) := \begin{cases} -\beta \displaystyle\sum_{1\leq i<j\leq N} \log(x_j - x_i), & \textbf{for the Dyson model in } \mathbb{R}, \\ -\beta \displaystyle\sum_{1\leq i<j\leq N} \Big[\log(x_j - x_i) + \log(x_j + x_i)\Big] - \{\beta(\nu+1) - 1\} \displaystyle\sum_{i=1}^{N} \log x_i & \\ \qquad \textbf{for the Bru–Wishart process in } \mathbb{R}_{\geq 0}. & \end{cases}$$

1.2 Loewner equation for multi-slit

- Denote the upper half of complex plane by $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

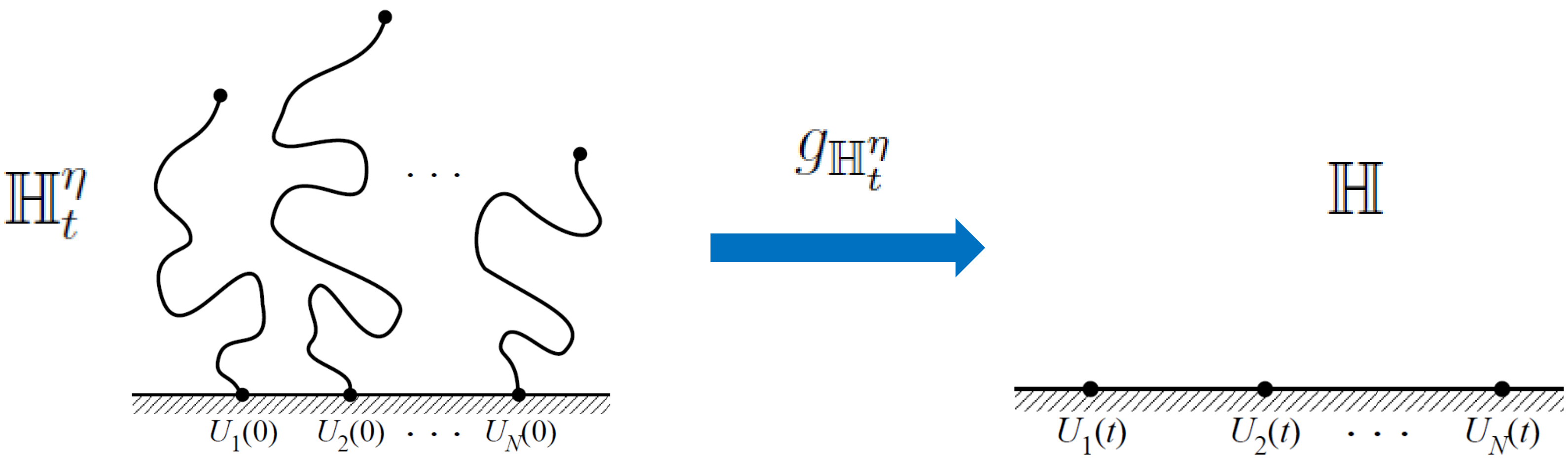
- A multi-slit $\bigcup_{i=1}^{N} \eta_i$ is defined as a union of non-colliding and non-self-intersecting curves in $\mathbb{H}$ anchored at N distinct and ordered points on $\mathbb{R}$.

- For each time $t \in (0, \infty)$, $\mathbb{H}_t^{\eta} := \mathbb{H} \setminus \bigcup_{i=1}^{N} \eta_i(0, t]$ is a simply connected domain in $\mathbb{C}$ and then by the Riemann mapping theorem there exists a unique analytic function $g_{\mathbb{H}_t^{\eta}}$ such that

$$g_{\mathbb{H}_t^{\eta}} : \text{conformal map } \mathbb{H}_t^{\eta} \to \mathbb{H},$$

satisfying the hydrodynamic normalization condition

$$g_{\mathbb{H}_t^{\eta}}(z) = z + \frac{\operatorname{hcap}(\bigcup_{i=1}^{N} \eta_i(0, t])}{z} + \mathrm{O}(|z|^{-2}) \quad \text{as } z \to \infty.$$



Theorem 1.1 (RS17) For $N \in \mathbb{N}$, let $\bigcup_{i=1}^{N} \eta_i$ be a multi-slit in $\mathbb{H}$ such that $\mathrm{hcap}(\bigcup_{i=1}^{N} \eta(0,t]) = 2t, t \in (0,\infty)$. Then there exists a set of weight functions $\lambda_i(t), t \geq 0, 1 \leq i \leq N$ satisfying $\sum_{i=1}^{N} \lambda_i(t) = 1, t \geq 0$ and an N-variate continuous driving function $\boldsymbol{U}(t) = (U_1(t), \ldots, U_N(t)) \in \mathbb{R}^N, t \in (0,\infty)$ such that the solution g_t of the differential equation

$$\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \frac{2\lambda_i(t)}{g_t(z) - U_i(t)}, \qquad t \geq 0, \qquad g_0(z) = z,$$

gives $g_t = g_{\mathbb{H}_t^\eta}, t \in (0,\infty)$.

- $U_i(t) = \lim_{z\to 0} g_t(\eta(t) + z) \iff \eta_i(t) = \lim_{z \to 0} g_t^{-1}(U_i(t) + z),\ 1 \leq i \leq N,\ t \in (0,\infty)$.
- Roth and Schleissinger called this the Loewner equation for multi-slit.

[RS17] D. Roth, S. Schleissinger : The Schramm-Loewner equation for multiple slits, J. Anal. Math. 131, 73–99 (2017).

- The Loewner equation for the multi-slit given for $D = \mathbb{H}$ can be mapped to other simply connected domains $D \subsetneq \mathbb{C}$ by conformal transformations.

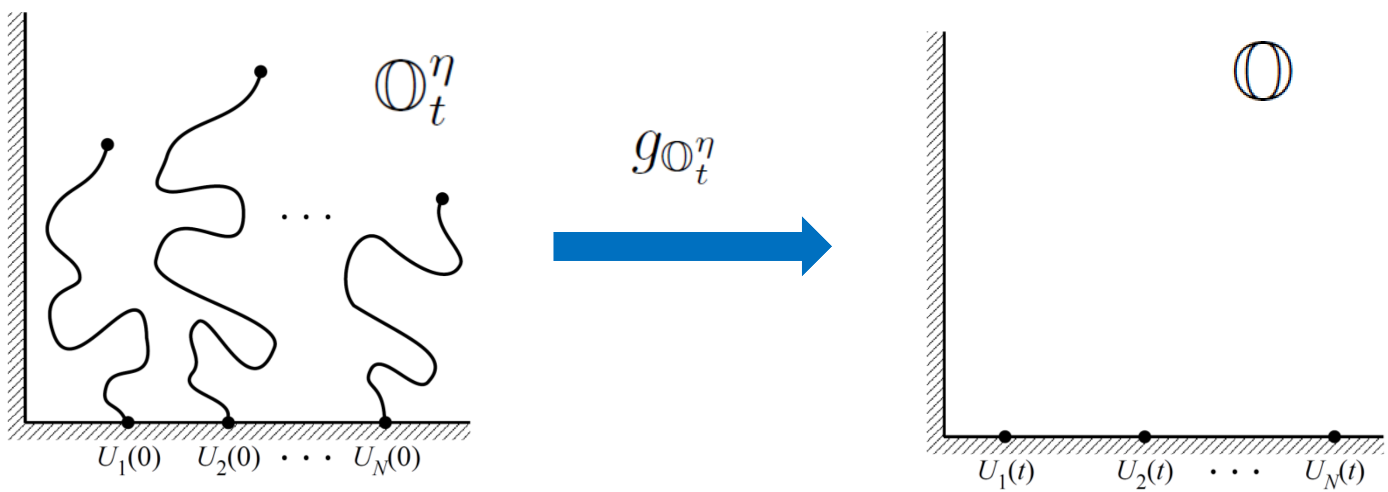
- Here we consider a conformal transformation $\varphi(z) = \sqrt{z} : \mathbb{H} \to \mathbb{O}$, where $\mathbb{O}$ denotes the first orthant in $\mathbb{C}$; $\mathbb{O} := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \mathrm{Im}\, z > 0\}$.

- We set $\widehat{g}_t(z) = \sqrt{g_t(z^2) + c(t)}$, $t \geq 0$, $z \in \mathbb{O}$ with a function of time $c(t), t \geq 0$. Then we can see that the Loewner equation for the multi-slit is transfered to the following form, $\widehat{g}_0(z) = z \in \mathbb{O}$,

$$\frac{d\widehat{g}_t(z)}{dt} = \sum_{i=1}^{N} \left(\frac{2\widehat{\lambda}_i(t)}{\widehat{g}_t(z) - \widehat{U}_i(t)} + \frac{2\widehat{\lambda}_i(t)}{\widehat{g}_t(z) + \widehat{U}_i(t)} \right) + \frac{2\widehat{\lambda}_0(t)}{\widehat{g}_t(z)}, \qquad t \geq 0,$$

- The solution of this equation gives the uniformization map to $\mathbb{O}$;

$$\widehat{g}_t = g_{\mathbb{O}_t^\eta} : \text{conformal map } \mathbb{O}_t^\eta := \mathbb{O} \Big\backslash \sum_{i=1}^{N} \eta_i(0, t] \to \mathbb{O}.$$



1.3 Multiple Schramm-Leowner evolution (multiple SLE)

- **For simplicity, we assume the uniform weight** $\lambda_i(t) \equiv 1/N, t \geq 0, 1 \leq i \leq N$ **in Theorem 1.1. Then by a simple time change** $t/N \to t$ **associated with a change of notation,** $g_{Nt} \to g_t$**, the Loewner equation for the multi-slit in** $\mathbb{H}$ **is written as**

$$\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \frac{2}{g_t(z) - X_i(t)}, \quad g_0(z) = z, \quad t \geq 0.$$

Then we ask what is the suitable family of driving stochastic processes of N **particles on** $\mathbb{R}$**,** $\mathbf{X}(t) = (X_1(t), \ldots, X_N(t)), t \geq 0$ **?**

The Loewner equation for the multi-slit in $\mathbb{H}$ is written as

$$\frac{dg_t(z)}{dt} = \sum_{i=1}^{N} \frac{2}{g_t(z) - X_i(t)}, \quad g_0(z) = z, \quad t \geq 0.$$

What is the suitable family of driving stochastic processes of N particles on $\mathbb{R}$, $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)), t \geq 0$?

- As Schramm (2000) argued for his original SLE, conformal invariance implies that the driving process $X(t), t \geq 0$ should be a continuous Markov process which has in a particular parameterization independent increments.

- Moreover, Bauer, Bernard, and Kytölä (2005) and Graham (2007) showed that $X_i(t), t \geq 0, 1 \leq i \leq N$ are semi-martingales and the quadratic variations should be given by $\langle dX_i, dX_j \rangle_t = \kappa \delta_{ij} dt, t \geq 0$, $1 \leq i, j \leq N$ with $\kappa > 0$.

- Then we can assume that the system of SDEs for $\boldsymbol{X}(t), t \geq 0$ is in the form,

$$\boxed{dX_i(t) = \sqrt{\kappa} dB_i(t) + F_i(\boldsymbol{X}(t))dt, \quad t \geq 0, \quad 1 \leq i \leq N,}$$

  where $B_i(t), t \geq 0, 1 \leq i \leq N$ are independent one-dimensional standard Brownian motions, $\kappa > 0$, and $\{F_i(\boldsymbol{x})\}_{i=1}^{N}$ are suitable functions of $\boldsymbol{x} = (x_1, \dots, x_N)$ which do not explicitly depend on t.

- **In the orthant system, we put** $\widehat{\lambda}_i(t) \equiv \tau/(2N)$**,** $t \geq 0, \tau \in [0,1], 1 \leq i \leq N$ **and** $dc(t)/dt = 4, t \geq 0$**, and perform a time change** $\alpha t/(2N) \to t$ **associated with a change of notation** $\widehat{g}_{2Nt/\tau} \to \widehat{g}_t$**. Then the Loewner equation for the multi-slit is written in this system in the form,**

$$\frac{d\widehat{g}_t(z)}{dt} = \sum_{i=1}^{N} \left(\frac{2}{\widehat{g}_t(z) - \widehat{X}_i(t)} + \frac{2}{\widehat{g}_t(z) + \widehat{X}_i(t)} \right) + \frac{4\delta}{\widehat{g}_t}, \quad \widehat{g}_0(z) = z, \quad t \geq 0,$$

where $\delta := N(1-\tau)/\tau \geq 0$**. We assume that the system of SDEs for** $\widehat{\boldsymbol{X}}(t) = (\widehat{X}_1(t), \dots, \widehat{X}_N(t)), t \geq 0$ **is in the same form,**

$$\boxed{d\widehat{X}_i(t) = \sqrt{\kappa} dB_i(t) + \widehat{F}_i(\widehat{\boldsymbol{X}}(t))dt, \quad t \geq 0, \quad 1 \leq i \leq N,}$$

where the range of $\widehat{\boldsymbol{X}}(t), t \geq 0$ **shall be in** $(\mathbb{R}_{\geq 0})^N$**.**

1.4 Gaussian free field (GFF)

- First we define a functional of positive type.

Definition 1.2 Let $\mathcal{V}$ be a finite or infinite dimensional vector space. A function $\psi : \mathcal{V} \to \mathbb{C}$ is said to be a functional of positive type if for arbitrary $N \in \mathbb{N}$, $\xi_1, \ldots, \xi_N \in \mathcal{V}$, and $z_1, \ldots, z_N \in \mathbb{C}$, we have $\displaystyle\sum_{n=1}^{N}\sum_{m=1}^{N} \psi(\xi_n - \xi_m) z_n \overline{z_m} \geq 0.$

- For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$, the standard inner product is denoted by $(\boldsymbol{x}, \boldsymbol{y})$ and we write $||\boldsymbol{x}|| := \sqrt{(\boldsymbol{x}, \boldsymbol{x})}$. Let $\mathcal{B}^N$ be the family of Borel sets in $\mathbb{R}^N$. Then the following is known as the Bochner theorem.

Theorem 1.3 (Bochner theorem) Let $\psi : \mathbb{R}^N \to \mathbb{C}$ be a continuous functional of positive type such that $\psi(0) = 1$. Then there exists a unique probability measure P on $(\mathbb{R}^N, \mathcal{B}^N)$ such that

$$\psi(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} e^{\sqrt{-1}(\boldsymbol{\xi}, \boldsymbol{x})} \mathrm{P}(d\boldsymbol{x}) \quad \text{for} \quad \boldsymbol{\xi} \in \mathbb{R}^N.$$

- If we consider the case that the functional of positive type $\psi(\boldsymbol{\xi})$ is especially given by

$$\Psi(\boldsymbol{\xi}) := e^{-||\boldsymbol{\xi}||^2/2}, \quad \boldsymbol{\xi} \in \mathbb{R}^N,$$

then the probability measure P given by the Bochner theorem is the finite-dimensional standard Gaussian measure,

$$\mathrm{P}(d\mathbf{x}) = \frac{1}{(2\pi)^{N/2}} e^{-||\mathbf{x}||^2/2} d\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^N.$$

- Hence we can say that the finite-dimensional standard Gaussian measure P is determined by the characteristic function $\Psi(\boldsymbol{\xi})$ as

$$\Psi(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} e^{\sqrt{-1}(\boldsymbol{\xi},\mathbf{x})} \mathrm{P}(d\mathbf{x})$$
$$= e^{-||\boldsymbol{\xi}||^2/2} \quad \text{for} \quad \boldsymbol{\xi} \in \mathbb{R}^N.$$

- **Let $D \subsetneq \mathbb{C}$ be a simply connected domain, which is bounded. We consider the L^2 space on D with the inner product, $(f, g) := \int_D f(z)g(z)d\mu(z)$, $f, g \in L^2(D)$, where $d\mu(z) = dzd\overline{z}$.**

- **Let Δ be the Dirichlet Laplacian acting on $L^2(D)$. Then $-\Delta$ has positive discrete eigenvalues so that**

$$-\Delta e_n = \lambda_n e_n, \quad e_n \in L^2(D), \quad n \in \mathbb{N}.$$

We assume that the eigenvalues are labeled in a non-decreasing order;

$$0 < \lambda_1 \le \lambda_2 \le \cdots .$$

The system of eigenvalue functions $\{e_n\}_{n \in \mathbb{N}}$ forms a CONS of $L^2(D)$.

- **We write $\mathcal{C}_0^\infty(D)$ for the space of real smooth functions on D with a compact support. For two functions $f, g \in \mathcal{C}_0^\infty(D)$, their Dirichlet inner product is defined as**

$$(f, g)_\nabla := \frac{1}{2\pi} \int_D (\nabla f)(z) \cdot (\nabla g)(z) d\mu(z).$$

The Hilbert space completion of $\mathcal{C}_0^\infty(D)$ with respect to the Dirichlet inner product will be denoted by $W(D)$. We write $||f||_\nabla = \sqrt{(f, f)_\nabla}, f \in W(D)$.

- **If we set $u_n = \sqrt{\frac{2\pi}{\lambda_n}} e_n, n \in \mathbb{N}$, then by integration by parts, we have**

$$(u_n, u_n)_\nabla = \frac{1}{2\pi}(u_n, (-\Delta)u_m) = \delta_{nm}, \quad n, m \in \mathbb{N}.$$

Therefore $\{u_n\}_{n\in\mathbb{N}}$ forms a CONS of $W(D)$.

- **Let $\widehat{\mathcal{H}}(D)$ be the space of formal infinite series in $\{u_n\}_{n\in\mathbb{N}}$.**

- **For two formal series $f = \sum_{n\in\mathbb{N}} f_n u_n$, $g = \sum_{n\in\mathbb{N}} g_n u_n \in \widehat{\mathcal{H}}(D)$ such that $\sum_{n\in\mathbb{N}} |f_n g_n| < \infty$, we define their pairing as $(f, g)_\nabla := \sum_{n\in\mathbb{N}} f_n g_n$.**

  (In case when $f, g \in W(D)$, their pairing of course coincides with the Dirichlet inner product.)

- **For any $a \in \mathbb{R}$, the operator $(-\Delta)^a$ acts on $\widehat{\mathcal{H}}(D)$ as**

$$(-\Delta)^a \sum_{n\in\mathbb{N}} f_n u_n := \sum_{n\in\mathbb{N}} \lambda_n^a f_n u_n, \quad (f_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$

- Using this fact, we define $\mathcal{H}_a := (-\Delta)^a W(D)$, $a \in \mathbb{R}$, each of which is a Hilbert space with inner product

$$\langle f, g \rangle_a := ((-\Delta)^{-a} f, (-\Delta)^{-a} g)_\nabla, \quad f, g \in \mathcal{H}_a(D).$$

  We write $||\cdot||_a := \sqrt{\langle \cdot, \cdot \rangle_a}$, $a \in \mathbb{R}$.

- We can prove that
  - $\boxed{\mathcal{H}_a(D) \subset \mathcal{H}_b(D) \text{ for } a < b,}$
  - the dual Hilbert space of $\mathcal{H}_a(D)$ is given by $\mathcal{H}_{-a}(D)$.

Example When $a = 1/2$, we have

$$\langle f, g \rangle_{1/2} = \left((-\Delta)^{-1/2} f, (-\Delta)^{-1/2} g \right)_\nabla = \frac{1}{2\pi}(f, g), \quad f, g \in \mathcal{H}_{1/2}(D).$$

Therefore $\mathcal{H}_{1/2}(D) \simeq L^2(D)$.

Remark Since $\mathcal{H}_{1/2}(D) = L^2(D)$ as shown in above, the members of $\mathcal{H}_a(D)$ with $a > 1/2$ cannot be functions, but are distributions.

- Define $\mathcal{E}(D) := \bigcup_{a>1/2} \mathcal{H}_a(D)$. Then its dual Hilbert space is identified with $\mathcal{E}(D)^* := \bigcap_{a<-1/2} \mathcal{H}_a(D)$ and

$$\mathcal{E}(D)^* \subset W(D) \subset \mathcal{E}(D)$$

is established. Here $(\mathcal{E}(D)^*, W(D), \mathcal{E}(D))$ is called a **Gel'fand triple**. We set $\Sigma_{\mathcal{E}(D)} = \sigma(\{(\cdot, f)_\nabla : f \in \mathcal{E}(D)^*\})$.

- On such a setting, the following is obtained. This theorem is the extension of the Bochner theorem (Theorem 1.3) and is called the **Bochner–Minlos theorem**.

Theorem 1.4 [Bochner–Minlos theorem] Let ψ be a continuous function of positive type on $W(D)$ such that $\psi(0) = 1$. Then there exists a unique probability measure P on $(\mathcal{E}(D), \Sigma_{\mathcal{E}(D)})$ such that

$$\psi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1}(h,f)_\nabla} \mathbf{P}(dh) \quad \text{for } f \in \mathcal{E}(D)^*.$$

See, for instance,

[Hida80] T. Hida : Brownian Motion, Application of Mathematics, vol.11, Springer, (1980), Heidelberg.

[Asai10] A. Asai: Functional Integral Methods in Quantum Mathematical Physics, (in Japanese), Kyoritu-Shuppan, (2010), Tokyo.

- **Under certain conditions for ψ, the domain of test functions f can be extended from $\mathcal{E}(D)^*$ to $W(D)$.**

- **We can verify that the functional $\Psi(f) := e^{-\|f\|_{\nabla}^2/2}$ satisfies the conditions. Then the following is established with a probability measure $\mathbb{P}$ on $(\mathcal{E}(D), \Sigma_{\mathcal{E}(D)})$,**

$$\boxed{\Psi(f) = \int_{\mathcal{E}(D)} e^{\sqrt{-1}(h,f)_{\nabla}} \mathbb{P}(dh) = e^{-\|f\|_{\nabla}^2/2} \quad \textbf{for } f \in W(D).}$$

- **We define the Gaussian free field (GFF) with the Dirichlet boundary condition $H \in \mathcal{E}(D)$ by an isotopy**

$$\boxed{H : W(D) \to L^2(\mathcal{E}(D), \mathbb{P}), \quad \textbf{such that} \quad W(D) \ni f \mapsto (H, f)_{\nabla} \in L^2(\mathcal{E}(D), \mathbb{P}).}$$

- **The following linearity holds,**

$$(H, af + bg)_{\nabla} = a(H, f)_{\nabla} + b(H, g)_{\nabla} \quad \textbf{for} \quad a, b \in \mathbb{R}, \quad f, g \in W(D).$$

- Assume that $D, D' \subsetneq \mathbb{C}$ are simply connected domains and let

$$\phi : \text{conformal transformation } D' \to D.$$

Lemma 1.5 The Dirichlet inner product is conformal invariant, that is,

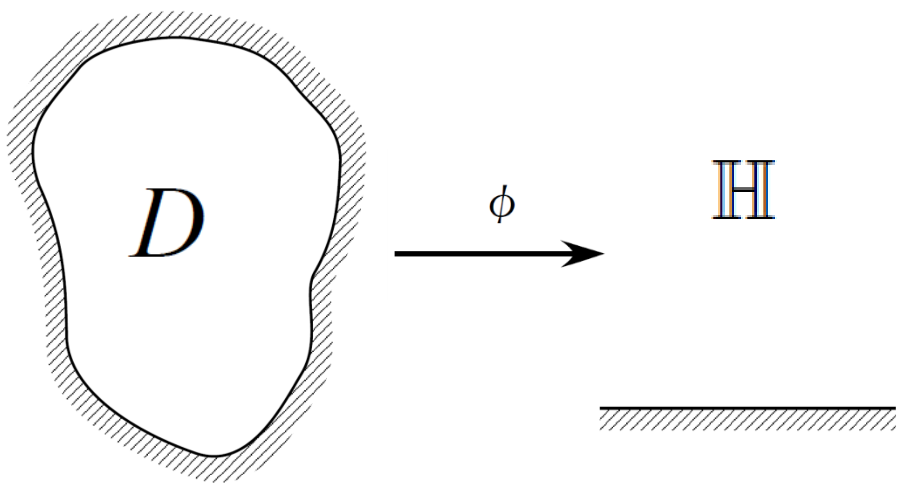
$$\int_D (\nabla f)(z) \cdot (\nabla g)(z) d\mu(z) = \int_{D'} (\nabla(f \circ \phi))(z) \cdot (\nabla(g \circ \phi))(z) d\mu(z) \quad \text{for } f, g \in \mathcal{C}_0^\infty(D).$$

- From the above lemma, we see that

$$\phi^* : W(D) \to W(D') \text{ such that } W(D) \ni f \mapsto f \circ \phi \in W(D')$$

is an isomorphism.

- This allows one to consider GFF on an unbounded domain.



- **Namely, if D' is bounded, we already have a family $\{(H, f)_\nabla : f \in W(D')\}$ of random variables. Then, even if D is unbounded, we can define a family $\{(\phi_* H, f)_\nabla : f \in W(D)\}$ by**

$$(\phi_* H, f)_\nabla := (H, \phi^* f)_\nabla, \quad f \in W(D)$$

so as to have the following covariance structure,

$$\mathbb{E}\Big[(\phi_* H, f)_\nabla (\phi_* H, g)_\nabla\Big] = (\phi^* f, \phi^* g)_\nabla = (f, g)_\nabla \quad \textbf{for } f, g \in W(D).$$

- **Relying on the following formal computation**

$$\begin{aligned}(\phi_* H, f)_\nabla = (H, \phi^* f)_\nabla &= \frac{1}{2\pi} \int_{D'} (\nabla H)(z) \cdot (\nabla f \circ \phi)(z) d\mu(z) \\ &= \frac{1}{2\pi} \int_D (\nabla H \circ \phi^{-1})(z) \cdot (\nabla f)(z) d\mu(z)\end{aligned}$$

we understand the equality $\phi_* H = H \circ \phi^{-1}$.

On the Green's funciton

- We have constructed a family $\{(H,f)_\nabla : f \in W(D)\}$ of random variables whose covariance structure is given by

$$\mathbb{E}\Big[(H,f)_\nabla (H,g)_\nabla\Big] = (f,g)_\nabla \quad \text{for } f,g \in W(D).$$

- By a formal integration by parts, we see that

$$(H,f)_\nabla = \frac{1}{2\pi}\int_D (\nabla H)(z)\cdot(\nabla f)(z)d\mu(z) = \frac{1}{2\pi}\int_D H(z)(-\Delta f)(z)d\mu(z) = \frac{1}{2\pi}(H,(-\Delta)f).$$

Motivated by this observation, we define

$$(H,f) := 2\pi(H,(-\Delta)^{-1}f)_\nabla \quad \text{for } \in \mathsf{D}((-\Delta)^{-1}),$$

where $\mathsf{D}((-\Delta)^{-1})$ denotes the domain of $(-\Delta)^{-1}$ in $W(D)$.

- The action of $(-\Delta)^{-1}$ is expressed as an integral operator and the integral kernel is known as the Green's function $G_D(z,w)$.

$$((-\Delta)^{-1}f)(z) = \frac{1}{2\pi}\int_D G_D(z,w)f(w)d\mu(w), \quad \text{a.e. } z \in D, \quad f \in \mathsf{D}((-\Delta)^{-1}).$$

- **Hence the covariance of (H, f) and (H, g) with $f, g \in \mathsf{D}((-\Delta)^{-1})$ is written as**

$$\mathbb{E}[(H, f)(H, g)] = \int_{D \times D} f(z) G_D(z, w) g(w) d\mu(z) d\mu(w).$$

- **When we symbolically write**

$$(H, f) = \int_D H(z) f(z) d\mu(z), \quad f \in \mathsf{D}((-\Delta)^{-1}),$$

the covariance structure can be understood as

$$\mathbb{E}[H(z), H(w)] = G_D(z, w), \quad z, w \in D, \quad n \neq w.$$

Example **When D is the upper half plane $\mathbb{H}$,**

$$G_{\mathbb{H}}(z, w) = \log \left| \frac{z - \overline{w}}{z - w} \right| = -\log |z - w| + \log |z - \overline{w}|, \quad z, w \in \mathbb{H}, \quad z \neq w.$$

1.5 Imaginary surface (IS)

- Now we define an equivalent class of pairs (D, H) of simply connected domains $D \subsetneq \mathbb{C}$ and distribution-valued random field H_D on D (e.g., GFF) induced by the conformal equivalence.

Definition 1.6 (Imaginary surface (IS)) Let $\gamma \in (0, 2]$ and put $\chi = \dfrac{2}{\gamma} - \dfrac{\gamma}{2}$.

A χ-imaginary surface is a collection of pairs (D, H_D) subject to the condition that, for all simply connected domains $D_1, D_2 \subsetneq \mathbb{C}$ and conformal map $\psi : D_1 \to D_2$, the following equality holds,

$$H_{D_1} = H_{D_2} \circ \psi - \chi \arg \psi' \quad \text{in } \mathbb{P}.$$

where $\psi'(z) := d\psi(z)/dz$.

- See for more details,

[MS16] J. Miller, S. Sheffield : Imaginary geometry I : Interacting SLEs, Probab.Theory Relat.Fields 164, 553–705 (2016).

[She16] S. Sheffield : Conformal weldings of random surfaces: SLE and the quantum gravity zipper, Ann. Probab. 44, 3474–3545 (2016).

2. Imaginary Surface with Boundary Points (IS-BPs)

- **Here we consider the two cases** $D = \mathbb{H}$ **or** $D = \mathbb{O}$**.**
- **Consider the Weyl chambers.**

$$\mathbb{W}_N(S) = \{\boldsymbol{x} = (x_1, \dots, x_N) \in S^N : x_1 < \cdots < x_N\}, \quad S = \mathbb{R} \textbf{ or } S = \mathbb{R}_{\geq 0}.$$

- **Let** $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$**.**
- **We consider the following complex-valued logarithmic potentials,**

$$\Phi_{\mathbb{H}}(z; \{\boldsymbol{x}, \infty\}, \boldsymbol{\alpha}) = \sum_{i=1}^{N} \alpha_i \log(z - x_i), \quad z \in \mathbb{H}, \ \boldsymbol{x} \in \mathbb{W}_N(\mathbb{R}),$$

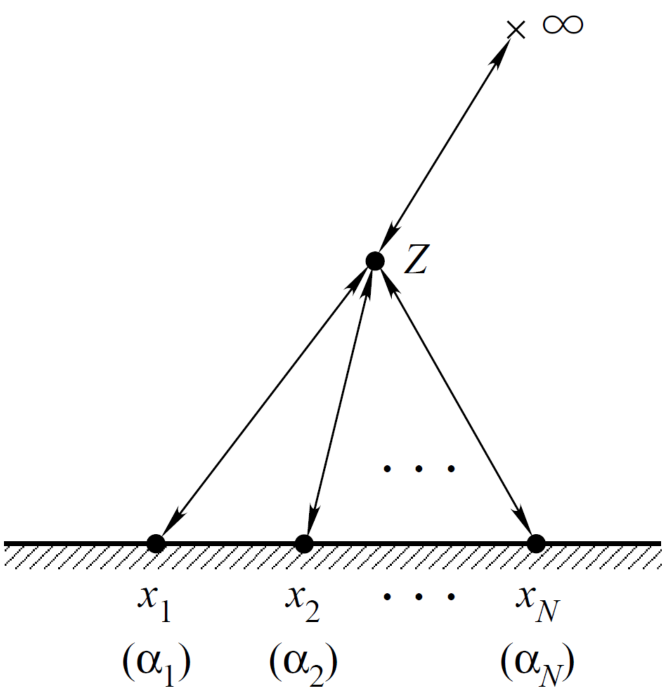
$$\Phi_{\mathbb{O}}(z; \{\boldsymbol{x}, \infty\}, \boldsymbol{\alpha}) = \sum_{i=1}^{N} \alpha_i \Big[\log(z - x_i) + \log(z + x_i) \Big] + \alpha_0 \log z, \ \ z \in \mathbb{O}, \ \boldsymbol{x} \in \mathbb{W}_N(\mathbb{R}_{\geq 0}).$$

Here $\{\boldsymbol{x}, \infty\} := \{x_1, \dots, x_N, \infty\}$**. We see that** ∞ **is also a singular point of** $\Phi_D(\,\cdot\,; \{\boldsymbol{x}, \infty\}, \boldsymbol{\alpha})$**.**

- **Note that each** α_i **seems to be a 2D Coulomb charge of the particle at the boundary points** $x_i \in \partial D$**,** $1 \leq i \leq N$**.**

- Now we take the imaginary part of $\overline{\Phi_D}$ to define real-valued fields ϕ_D on D;

$$\begin{aligned}
\phi_{\mathbb{H}}(z;\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}) &:= \operatorname{Im}\overline{\Phi_{\mathbb{H}}(z;\boldsymbol{x},\boldsymbol{\alpha})} \\
&= -\sum_{i=1}^{N}\alpha_i \arg(z-x_i), \quad z\in\mathbb{H}, \quad \boldsymbol{x}\in\mathbb{W}_N(\mathbb{R}), \\
\phi_{\mathbb{O}}(z;\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}) &:= \operatorname{Im}\overline{\Phi_{\mathbb{O}}(z;\boldsymbol{x},\boldsymbol{\alpha})} \\
&= -\left\{\sum_{i=1}^{N}\alpha_i\Big[\arg(z-x_i)+\arg(z+x_i)\Big]+\alpha_0 \arg z\right\}, \\
&\qquad z\in\mathbb{O}, \quad \boldsymbol{x}\in\mathbb{W}_N(\mathbb{R}_{\geq 0}).
\end{aligned}$$



- **We introduce new distribution-valued random fields by**

$$H_{\mathbb{H}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} := H_{\mathbb{H}} + \phi_{\mathbb{H}}(\,\cdot\,;\boldsymbol{x},\boldsymbol{\alpha}),$$
$$H_{\mathbb{O}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} := H_{\mathbb{O}} + \phi_{\mathbb{O}}(\,\cdot\,;\boldsymbol{x},\boldsymbol{\alpha}),$$

  where $H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ denotes the GFFs with the Dirichlet boundary conditions in $D = \mathbb{H}$ and $D = \mathbb{O}$, respectively.

- **Moreover, we consider the situation such that the boundary points are random variables $\boldsymbol{X} = (X_1, \dots, X_N) \in \mathbb{W}_N(S)$.**

- The probability law of GFF H_D (resp. $\boldsymbol{X} \in \partial D$) is denoted by $\mathbb{P}$ (resp. P).
- We consider an equivalent class of the triplets, $(D, H_D^{\boldsymbol{X},\boldsymbol{\alpha}}, \boldsymbol{X})$, induced by the conformal equivalence.

Definition 2.1 (**Imaginary surface with boundary points (IS-BPs)**) **Let** $\gamma \in (0,2]$, $\boldsymbol{\alpha} \in \mathbb{R}^N$, **and put** $\chi = \dfrac{2}{\gamma} - \dfrac{\gamma}{2}$.
A χ**-imaginary surface with** $(N+1)$**-boundary points is a collection of triplet** $(D, H_D^{\boldsymbol{X},\boldsymbol{\alpha}}, \boldsymbol{X})$ **subject to the condition that,**
for all simply connected domains $D_1, D_2 \subsetneq \mathbb{C}$ **and conformal map** $\psi : D_1 \to D_2$,
the following equalities holds in probability law $\mathbb{P} \otimes \mathrm{P}$,

$$H_{D_1} \overset{\text{(law)}}{=} H_{D_2} \circ \psi - \chi \arg \psi',$$

$$\boldsymbol{X}_{D_1} = (X_{D_1,1}, \dots, X_{D_1,N}) \overset{\text{(law)}}{=} \psi^{-1}(\boldsymbol{X}_{D_2}) := (\psi^{-1}(X_{D_2,1}), \dots, \psi^{-1}(X_{D_2,N})).$$

where $\psi'(z) := d\psi(z)/dz$.

Remark

- We can construct a GFF with the free boundary condition on a simply connected domain $D \subsetneq \mathbb{C}$, which is denoted by $\widetilde{H}_D$.
- For $\widetilde{H}_D$, we consider the real part of $\overline{\Phi_D}$, and define

$$\widetilde{H}_D^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} := \widetilde{H}_D + \widetilde{\phi}_D(\,\cdot\,;\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}),$$

where, for $D = \mathbb{H}$ and $\mathbb{O}$,

$$\widetilde{\phi}_{\mathbb{H}}(z;\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}) = \sum_{i=1}^{N} \alpha_i \log|z - x_i|,\ z \in \mathbb{H},\ \boldsymbol{x} \in \mathbb{W}_N(\mathbb{R}),$$

$$\widetilde{\phi}_{\mathbb{O}}(z;\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}) = \sum_{i=1}^{N} \alpha_i \Big[\log|z - x_i| + \log|z + x_i|\Big] + \alpha_0 \log|z|,\ z \in \mathbb{O},\ \boldsymbol{x} \in \mathbb{W}_N(\mathbb{R}_{\geq 0}).$$

- The equivalence class of triplets $(D, \widetilde{H}_D^{\boldsymbol{X},\boldsymbol{\alpha}}, \boldsymbol{X})$, induced by the conformal equivalence is called quantum surface with boundary points.

  [KK19+] M. K., S. Koshida : Conformal welding problem, flow line problem, and multiple Schramm–Loewner evolution, arXiv:math/PR:1903.09925

3. Two Ways of Sampling IS-BPs

Setting

- Let $0 < T < \infty$ and consider a time duration $t \in [0, T]$.
- Give an initial configuration of BPs, $\boldsymbol{X}(0) = \boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{W}_N(S)$, $S = \mathbb{R}$ or $\mathbb{R}_{\geq 0}$.
- We consider the situation such that BPs evolve in time as a system of interacting Brownian motions

$$\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in W_N(S), \quad t \geq 0 \quad \text{with } S = \mathbb{R} \text{ or } \mathbb{R}_{\geq 0},$$

which solves the SDEs in the form,

$$dX_i(t) = \sqrt{\kappa} dB_i(t) + F_i(\boldsymbol{X}(t))dt, \quad t \geq 0, \quad 1 \leq i \leq N.$$

Here $B_i(t), t \geq 0, 1 \leq i \leq N$ are independent one-dimensional standard Brownian motions.

Sampling A

- **Sample a GFF :** H_D.
- **Then obtain an instance of IS-BPs,**

$$H_D^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} := H_D + \phi_D(\cdot;\boldsymbol{x},\boldsymbol{\alpha}).$$

Sampling B

- **Sample a GFF :** H_D.
- **Sample a time-evolution of BPs on** $S(=\mathbb{R}$ **or** $\mathbb{R}_{\geq 0})$ **starting from given** $\boldsymbol{x}$ **:**

$$\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{W}_N(S), \quad t \in [0, T], \quad \boldsymbol{X}(0) = \boldsymbol{x}.$$

- **Generate multiple slits** $\bigcup_{i=1}^{N} \eta^{(i)}(0, T]$ **by the multiple SLE** $g_{D_t^\eta}, t \in [0, T]$, **which is driven by** $\boldsymbol{X}(t), t \in [0, T]$.
- **Erase the multiple slits by the conformal map** $g_{D_T^\eta}$.
- **Then obtain an instance**

$$\begin{aligned} g_{D_T^\eta} * H_D^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}} &:= H_D^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}} \circ g_{D_T^\eta} - \chi \arg g'_{D_T^\eta} \\ &= H_D \circ g_{D_T^\eta} + \phi_D(g_{D_T^\eta}(\cdot); \boldsymbol{X}(T), \boldsymbol{\alpha}) - \chi \arg g'_{D_T^\eta}. \end{aligned}$$

Sampling B

- **Sample a GFF :** H_D.
- **Sample a time-evolution of BPs on** $S(=\mathbb{R}$ **or** $\mathbb{R}_{\geq 0})$ **starting from given** $\boldsymbol{x}$ **:**

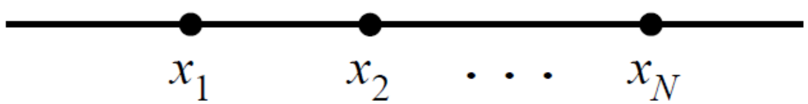
$$\boldsymbol{X}(t) = (X_1(t), \ldots, X_N(t)) \in \mathbb{W}_N(S), \quad t \in [0, T], \quad \boldsymbol{X}(0) = \boldsymbol{x}.$$

- **Generate multiple slits** $\bigcup_{i=1}^{N} \eta^{(i)}(0, T]$ **by the multiple SLE** $g_{D_t^\eta}, t \in [0, T]$, **which is driven by** $\boldsymbol{X}(t), t \in [0, T]$.
- **Erase the multiple slits by the conformal map** $g_{D_T^\eta}$.
- **Then obtain an instance**

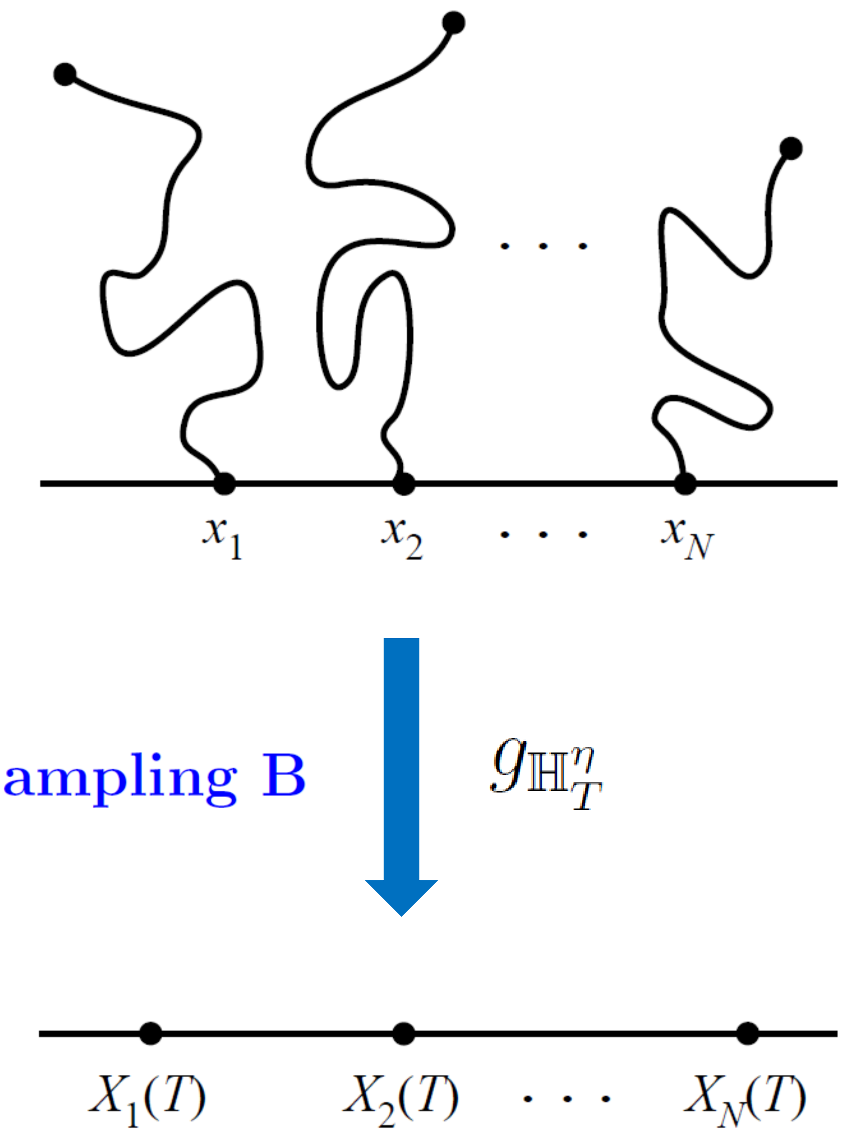
Coupling GFF and multiple SLE

$$\begin{aligned} g_{D_T^\eta} * H_D^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}} &:= H_D^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}} \circ g_{D_T^\eta} - \chi \arg g'_{D_T^\eta} \\ &= H_D \circ g_{D_T^\eta} + \phi_D(g_{D_T^\eta}(\cdot); \boldsymbol{X}(T), \boldsymbol{\alpha}) - \chi \arg g'_{D_T^\eta}. \end{aligned}$$

Sampling A



Sampling B



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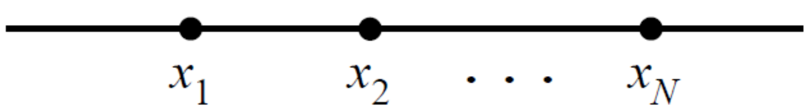
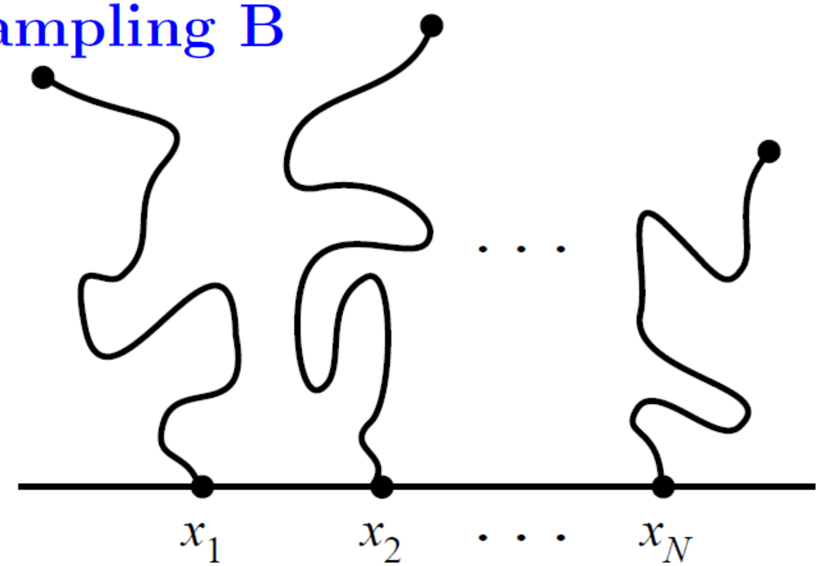
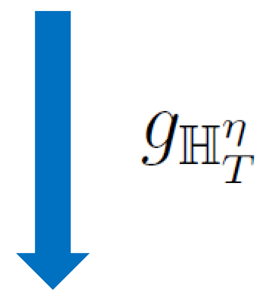
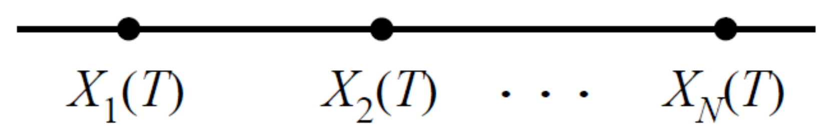


Sampling B



Sampling B





- **Note that at each time** $T \in [0, \infty)$**,**

$$(D_T^\eta, H_D^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}} \circ g_{D_T^\eta} - \chi \arg g'_{D_T^\eta}, \boldsymbol{X}(T)) \sim_\chi (D, H_D^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}}, \boldsymbol{X}(T)).$$

4. Main Theorems

Theorem 4.1 **The above two ways of sampling give the same result in probability law $\mathbb{P} \times \mathrm{P}$, that is,**

$$H_{\mathbb{H}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} \overset{\text{(law)}}{=} g_{\mathbb{H}_T^\eta} * H_{\mathbb{H}}^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}},$$

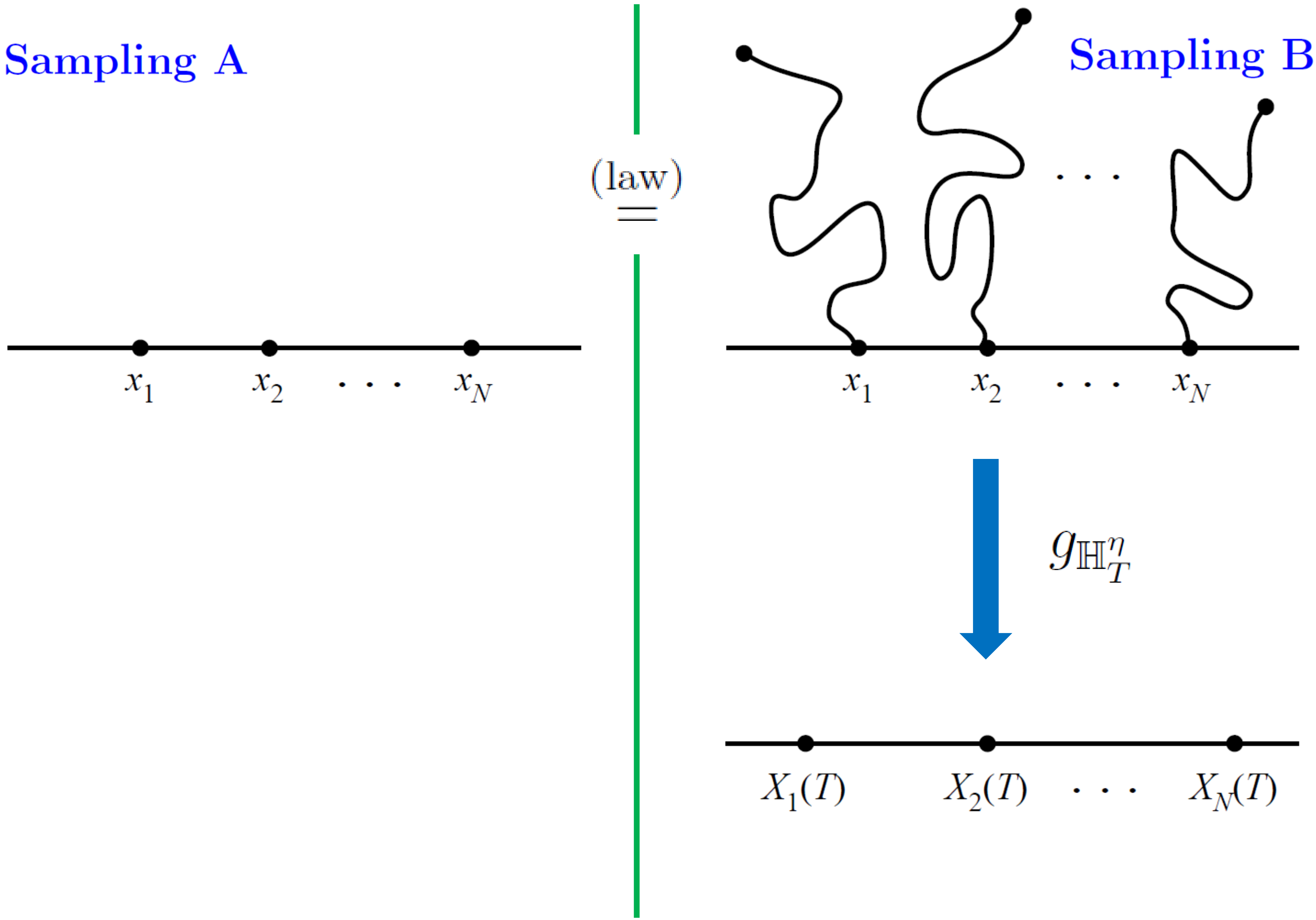
if the following three conditions are satisfied,

(i) $\kappa = \gamma^2$,

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right)$,

(iii) $F^{(i)}(\boldsymbol{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4}{x_i - x_j}, \quad i = 1, \dots, N$,

i.e., $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{W}_N(\mathbb{R}), t \geq 0$, **is a time change of the Dyson model with parameter** $\beta = \frac{8}{\kappa}$.


Sampling A
Sampling B
(law)
=
x_1
x_2
x_N
$g_{\mathbb{H}_T^\eta}$
$X_1(T)$
$X_2(T)$
$X_N(T)$

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$$H_{\mathbb{H}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} \stackrel{\text{(law)}}{=} g_{\mathbb{H}_T^\eta} * H_{\mathbb{H}}^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}},$$

if the following three conditions are satisfied,

(i) $\kappa = \gamma^2$, **Relation between SLE and IS**

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right)$, **Charges at BPs**

(iii) $F^{(i)}(\boldsymbol{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{4}{x_i - x_j}, \quad i = 1, \dots, N,$

System of Driving Process

i.e., $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{W}_N(\mathbb{R}), t \geq 0$, **is a time change of the Dyson model with parameter** $\beta = \frac{8}{\kappa}$.

Theorem 4.2 **The equivalence** $H_{\mathbb{O}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} \stackrel{\text{(law)}}{=} g_{\mathbb{O}_T^\eta} * H_{\mathbb{O}}^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}}$ **is established in the probability law** $\mathbb{P} \times \mathrm{P}$**, if the following three conditions are satisfied,**

(i) $\kappa = \gamma^2$,

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right), \quad \alpha_0 = \chi,$

(iii) $$F^{(i)}(\boldsymbol{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{4}{x_i - x_j} + \frac{4}{x_i + x_j} \right) + \frac{4(1+\delta) - \kappa/2}{x_i}, \quad i = 1, \dots, N,$$

i.e., $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{W}_N(\mathbb{R}_{\geq 0}), t \geq 0$**, is a time change of the Bru–Wishart process with parameters** $\beta = \frac{8}{\kappa}, \nu = \delta$**.**

Theorem 4.2 The equivalence $H_{\mathbb{O}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} \overset{\text{(law)}}{=} g_{\mathbb{O}_T^{\eta}} * H_{\mathbb{O}}^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}}$ is established in the probability law $\mathbb{P} \times \mathrm{P}$, if the following three conditions are satisfied,

(i) $\kappa = \gamma^2$,

Relation between SLE and IS

(ii) $(\alpha_1, \dots, \alpha_N) = \left(\frac{2}{\gamma}, \dots, \frac{2}{\gamma}\right), \quad \alpha_0 = \chi,$

Charges at BPs

(iii) $$F^{(i)}(\boldsymbol{x}) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{4}{x_i - x_j} + \frac{4}{x_i + x_j} \right) + \frac{4(1+\delta) - \kappa/2}{x_i}, \quad i = 1, \dots, N,$$

i.e., $\boldsymbol{X}(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{W}_N(\mathbb{R}_{\geq 0}), t \geq 0$, is a time change of the Bru–Wishart process with parameters $\beta = \frac{8}{\kappa}, \nu = \delta$.

System of Driving Process

5. Proof of Theorem 4.1

- Define

$$\mathcal{M}_t(z) := \phi_{\mathbb{H}}^{\{\boldsymbol{X}(t),\infty\},\alpha}(g_{\mathbb{H}_t^\eta}(z)) - \chi \arg g'_{\mathbb{H}_t^\eta}(z)$$
$$= -\sum_{i=1}^N \alpha_i \arg(g_{\mathbb{H}_t^\eta}(z) - X_i(t)) - \chi \arg g'_{\mathbb{H}_t^\eta}(z), \quad t \in [0,T],$$

  and put

$$\mathcal{I}_t := H_{\mathbb{H}} \circ g_{\mathbb{H}_t^\eta} + \mathcal{M}_t, \quad t \in [0,T].$$

- By definition, the equivalence

$$H_{\mathbb{H}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} \overset{\text{(law)}}{=} g_{\mathbb{H}_T^\eta} * H_{\mathbb{H}}^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}}$$

  is equal to

$$\boxed{\mathcal{I}_0 \overset{\text{(law)}}{=} \mathcal{I}_T \quad \textbf{in } \mathbb{P} \otimes \mathrm{P}.}$$

Lemma 5.1 **The stochastic process $\mathcal{M}_t(z)$, $z \in \mathbb{H}$, $t \in [0, T]$ is a local martingale with increment**

$$d\mathcal{M}_t(z) = \sum_{i=1}^{N} \operatorname{Im} \frac{2}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} dB_i(t), \qquad z \in \mathbb{H}, \qquad t \in [0, T],$$

if the three conditions of Theorem 4.1 are satisfied.

<u>**Proof**</u> **Note that $\mathcal{M}_t(z)$ is the imaginary part of**

$$\mathcal{M}_t^*(z) = -\sum_{i=1}^{N} \alpha_i \log(g_{\mathbb{H}_t^\eta}(z) - X_i(t)) - \chi \log g'_{\mathbb{H}_t^\eta}(z), \qquad \chi = \frac{2}{\gamma} - \frac{\gamma}{2}.$$

Then Itô's formula gives

$$\begin{aligned} d\mathcal{M}_t^*(z) &= \sum_{i=1}^{N} \frac{\alpha_i \sqrt{\kappa}}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} dB_i(t) \\ &+ \sum_{i=1}^{N} \frac{1}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} \left(\alpha_i F^{(i)}(\boldsymbol{X}(t)) - \sum_{\substack{1 \le j \le N, \\ j \neq i}} \frac{2(\alpha_i + \alpha_j)}{X_i(t) - X_j(t)} \right) dt \\ &+ \sum_{i=1}^{N} \frac{1}{(g_{\mathbb{H}_t^\eta}(z) - X_i(t))^2} \left[\frac{1}{2}(\kappa - 4) \left(\alpha_i - \frac{2}{\gamma} \right) + \frac{1}{\gamma}(\kappa - \gamma^2) \right] dt, \qquad t \in [0, T]. \end{aligned}$$

This proves the statement. ▮

Lemma 5.1 **The stochastic process $\mathcal{M}_t(z)$, $z \in \mathbb{H}$, $t \in [0,T]$ is a local martingale with increment**

$$d\mathcal{M}_t(z) = \sum_{i=1}^{N} \operatorname{Im} \frac{2}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} dB_i(t), \qquad z \in \mathbb{H}, \qquad t \in [0,T],$$

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Proof Note that $\mathcal{M}_t(z)$ is the imaginary part of

$$\mathcal{M}_t^*(z) = -\sum_{i=1}^{N} \alpha_i \log(g_{\mathbb{H}_t^\eta}(z) - X_i(t)) - \chi \log g'_{\mathbb{H}_t^\eta}(z), \qquad \chi = \frac{2}{\gamma} - \frac{\gamma}{2}.$$

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three conditions

$$\begin{aligned} d\mathcal{M}_t^*(z) &= \sum_{i=1}^{N} \frac{\alpha_i \sqrt{\kappa}}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} dB_i(t) \\ &+ \sum_{i=1}^{N} \frac{1}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} \cancel{\left(\alpha_i F^{(i)}(\boldsymbol{X}(t)) - \sum_{\substack{1 \le j \le N, \\ j \ne i}} \frac{2(\alpha_i + \alpha_j)}{X_i(t) - X_j(t)} \right)} dt \\ &+ \sum_{i=1}^{N} \frac{1}{(g_{\mathbb{H}_t^\eta}(z) - X_i(t))^2} \left[\frac{1}{2}(\kappa - 4) \cancel{\left(\alpha_i - \frac{2}{\gamma} \right)} + \frac{1}{\gamma} \cancel{(\kappa - \gamma^2)} \right] dt, \qquad t \in [0,T]. \end{aligned}$$

This proves the statement. ∎

- **In the following, we assume the three conditions of Theorem 4.1.**
- **The above lemma implies that, at each point $z \in \mathbb{H}$, the stochastic process $\{\mathcal{M}_t(z) : t \in [0,T]\}$ can be regarded as a Brownian motion modulo time change.**
- **Moreover, the above lemma gives the cross variation between two points $z, w \in \mathbb{H}$ as**

$$d\langle \mathcal{M}(z), \mathcal{M}(w)\rangle_t = \sum_{i=1}^{N} \left(\mathrm{Im}\, \frac{2}{g_{\mathbb{H}_t^\eta}(z) - X_i(t)} \right) \left(\mathrm{Im}\, \frac{2}{g_{\mathbb{H}_t^\eta}(w) - X_i(t)} \right) dt, \quad t \in [0,T].$$

Lemma 5.2 Let $G_{\mathbb{H}_t^\eta}(z,w)$ be the Green function of GFF with the Dirichlet boundary condition in $\mathbb{H}_t^\eta$, $t\in[0,T]$. Then

$$d\langle \mathcal{M}(z),\mathcal{M}(w)\rangle_t = -dG_{\mathbb{H}_t^\eta}(z,w),\quad t\in[0,T],\quad z,w\in\mathbb{H}.$$

<u>Proof</u> This can be verified by direct computation. Due to the conformal invariance of the Green's function of GFF with the Dirichlet boundary condition,

$$G_{\mathbb{H}_t^\eta}(z,w) = G_{\mathbb{H}}(g_{\mathbb{H}_t^\eta}(z), g_{\mathbb{H}_t^\eta}(w)) = \log\left|\frac{g_{\mathbb{H}_t^\eta}(z)-\overline{g_{\mathbb{H}_t^\eta}(w)}}{g_{\mathbb{H}_t^\eta}(z)-g_{\mathbb{H}_t^\eta}(w)}\right| \qquad t\in[0,T],\qquad z,w\in\mathbb{H}.$$

Thus its increment is computed as

$$\begin{aligned}
dG_{\mathbb{H}_t^\eta}(z,w) &= \operatorname{Re}\frac{dg_{\mathbb{H}_t^\eta}(z)-d\overline{g_{\mathbb{H}_t^\eta}(w)}}{g_{\mathbb{H}_t^\eta}(z)-g_{\mathbb{H}_t^\eta}(w)} - \operatorname{Re}\frac{dg_{\mathbb{H}_t^\eta}(z)-dg_{\mathbb{H}_t^\eta}(w)}{g_{\mathbb{H}_t^\eta}(z)-g_{\mathbb{H}_t^\eta}(w)}\\
&= -\sum_{i=1}^N \operatorname{Re}\frac{2dt}{(g_{\mathbb{H}_t^\eta}(z)-X_i(t))(\overline{g_{\mathbb{H}_t^\eta}(w)}-X_i(t))} + \sum_{i=1}^N \operatorname{Re}\frac{2dt}{(g_{\mathbb{H}_t^\eta}(z)-X_i(t))(g_{\mathbb{H}_t^\eta}(w)-X_i(t))}\\
&= \sum_{i=1}^N\left(\operatorname{Re}\frac{2\sqrt{-1}}{g_{\mathbb{H}_t^\eta}(z)-X_i(t)}\right)\left(\operatorname{Im}\frac{2}{g_{\mathbb{H}_t^\eta}(w)-X_i(t)}\right)dt\\
&= -\sum_{i=1}^N\left(\operatorname{Im}\frac{2}{g_{\mathbb{H}_t^\eta}(z)-X_i(t)}\right)\left(\operatorname{Im}\frac{2}{g_{\mathbb{H}_t^\eta}(w)-X_i(t)}\right)dt
\end{aligned}$$

which is the same as $-d\langle\mathcal{M}(z),\mathcal{M}(w)\rangle_t$, $z,w\in\mathbb{H}$. ∎

Proof of Theorem 4.1

- For any function $f \in \mathcal{C}_0^\infty \subset \mathsf{D}((-\Delta)^{-1})$, we have
$$d\langle(\mathcal{M}, f), (\mathcal{M}, f)\rangle_t = -dE_t(f),$$
where
$$E_t(f) = \int_{\mathbb{H}^2} f(z) G_{\mathbb{H}_t^\eta}(z, w) f(w) d\mu(z) d\mu(w).$$

- Since the process $\mathbb{H}_t^\eta := \mathbb{H} \setminus \bigcup_{i=1}^{N} \eta_i(0, t], t \geq 0$ is decreasing, the Dirichlet energy $E_t(f)$ is non-increasing in the time variable $t \in [0, T]$.

- This implies that $(\mathcal{M}_t, f)$, $t \in [0, T]$, is a Brownian motion such that we can regard $-E_t(f)$ as time variable.

- Thus $(\mathcal{M}_T, f)$ is normally distributed with mean $(\mathcal{M}_0, f)$ and variance $-E_T(f) - (-E_0(f)) = -E_T(f) + E_0(f)$.

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Dirichlet energy

where

$$E_t(f) = \int_{\mathbb{H}^2} f(z) G_{\mathbb{H}_t^\eta}(z, w) f(w) d\mu(z) d\mu(w).$$

- Since the process $\mathbb{H}_t^\eta := \mathbb{H} \setminus \bigcup_{i=1}^N \eta_i(0, t], t \geq 0$ is decreasing, the Dirichlet energy $E_t(f)$ is non-increasing in the time variable $t \in [0, T]$.
- This implies that $(\mathcal{M}_t, f)$, $t \in [0, T]$, is a Brownian motion such that we can regard $-E_t(f)$ as time variable.
- Thus $(\mathcal{M}_T, f)$ is normally distributed with mean $(\mathcal{M}_0, f)$ and variance $-E_T(f) - (-E_0(f)) = -E_T(f) + E_0(f)$.

- The random variable $(H_{\mathbb{H}} \circ g_{\mathbb{H}^\eta_T}, f)$ is also normally distributed with mean zero and variance $E_T(f)$ by the conformal invariance of the GFF.

- Since the random variable $(H_{\mathbb{H}} \circ g_{\mathbb{H}^\eta_T}, f)$ is conditionally independent of $(\mathcal{M}_T, f)$, their sum
$$(\mathcal{I}_T, f) := (H_{\mathbb{H}} \circ g_{\mathbb{H}^\eta_T} + \mathcal{M}_T, f)$$
is a normal random variable with mean $(\mathcal{M}_0, f)$ and variance
$$\{-E_T(f) + E_0(f)\} + E_T(f) = E_0(f)$$
coinciding with $(\mathcal{M}_0 + H_{\mathbb{H}}, f) = (\mathcal{I}_0, f)$ in probability.

- This implies $\mathcal{I}_T \overset{\text{(law)}}{=} \mathcal{I}_0$ as distribution-valued random fields. The proof of Theorem 4.1 is complete. ∎

6. Concluding Remarks

- Theorem 4.1 is a multi-slit extension of the result by Sheffield [She16], in which the GFF is coupled with a single SLE curve (i.e., $N = 1$).
- In the case $N = 1$, the location of single BP is irrelevant, since a shift does not change conformal equivalence. For general IS with N BPs, time evolution of BPs is essential;

$$H_{\mathbb{H}}^{\{\boldsymbol{x},\infty\},\boldsymbol{\alpha}} \stackrel{\text{(law)}}{=} g_{\mathbb{H}_T^\eta} * H_{\mathbb{H}}^{\{\boldsymbol{X}(T),\infty\},\boldsymbol{\alpha}}.$$

[She16] S. Sheffield : Conformal weldings of random surfaces: SLE and the quantum gravity zipper, Ann. Probab. 44, 3474–3545 (2016).

- Sheffield [She16] addressed very interesting geometrical problems, which we call the conformal welding problems and the flow line problems, and solved these problems by coupling the GFFs with a single SLE curve.
- The present result for the imaginary surface (IS) with $(N+1)$-boundary points (BPs) and the counterpart result for the quantum surface (QS) with $(N+1)$-boundary points (BPs) also solve the N-slit extensions of these geometrical problems. See [KK19+].

[She16] S. Sheffield : Conformal weldings of random surfaces: SLE and the quantum gravity zipper, Ann. Probab. 44, 3474–3545 (2016).

[KK19+] M. K., S. Koshida : Conformal welding problem, flow line problem, and multiple Schramm–Loewner evolution, arXiv:math/PR:1903.09925

- As I mentioned at the beginning of the present talk, the Dyson model and the Bru–Wishart process are dynamical extensions of different ensembles in random matrix theory.

- We have constructed two systems of distribution-valued random fields with $(N+1)$ BPs in $\mathbb{H}$ and $\mathbb{O}$ coupled with multiple SLEs driven by these two processes, respectively.

- The obtained two systems are in the same equivalence class induced by the conformal equivalence, that is, in the same χ-IS with $(N+1)$-BPs,

$$(\mathbb{H}, H_{\mathbb{H}}^{\{\boldsymbol{X},\infty\},\boldsymbol{\alpha}}) \sim_{\chi} (\mathbb{O}, H_{\mathbb{O}}^{\{\widehat{\boldsymbol{X}},\infty\},\boldsymbol{\alpha}}).$$

- We hope that the notion of χ-IS (and γ-QS) will provide us a new and universal view point for the variety of (stochastic) log-gas systems and random matrix theory as well as for other variants of SLEs.

- **It has been reported that random planar maps converge to SLE-decorated Liouville quantum gravity (LQG) in several topology (see [GMS17] and references therein).**
- **As the chordal SLE describes the scaling limit of a single interface in various critical lattice models, the multiple SLEs describe scaling limits of collections of interfaces in critical lattice models with alternating boundary conditions (see [BPW18] and references therein).**
- **Here we have discussed χ-IS (and γ-QS). Discrete counterparts of these random systems will be studied.**

[GMS17] E. Gwynne, J. Miller, S. Sheffield . The Tutte embedding of the mated-CRT map converges to Liouville quantum gravity, arXiv:1705.11161.

[BPW18] V. Beffara, E. Peltola, H. Wu: On the uniqueness of global multiple SLE, arXiv:1801.07699.

Thank you very much
for your attention.