Cardy embedding of random planar maps

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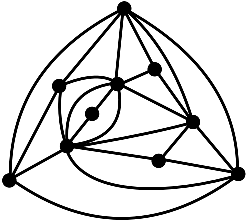
ETH Zürich, Institute for Theoretical Studies

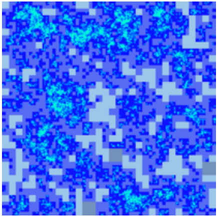
Collaboration with Xin Sun. Based on our joint works with Bernardi, Garban, Gwynne, Lawler, Li, and Sepúlveda.

August 1, 2019

Two random surfaces



random planar map (RPM)



Liouville quantum gravity (LQG)

Planar maps

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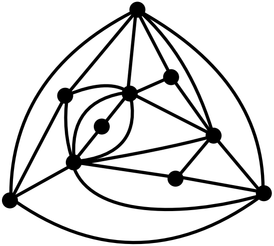

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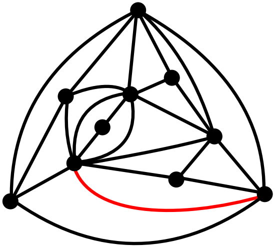

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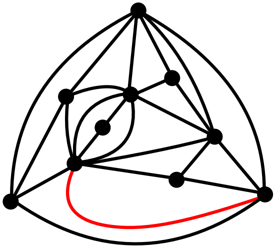

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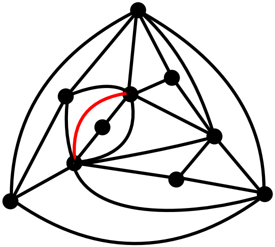

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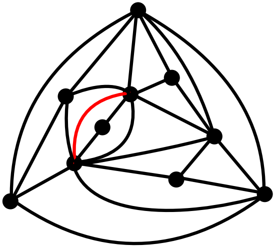

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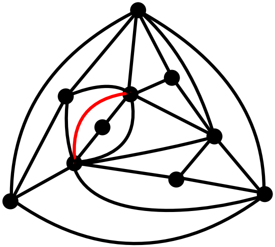

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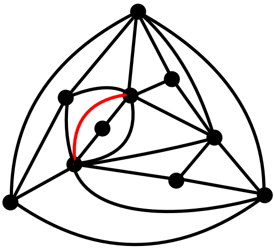
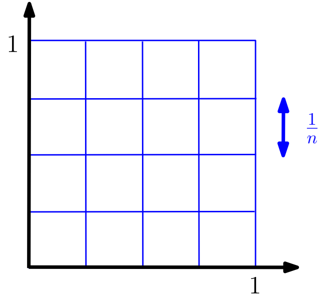
- Enumeration results by Tutte and Mullin in 60's.

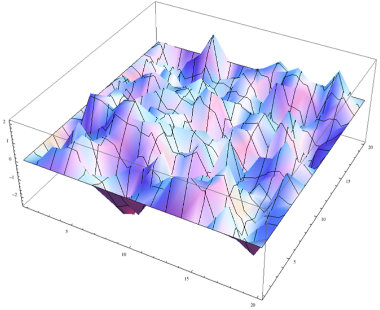


The Gaussian free field (GFF)

- Hamiltonian $H(f)$ quantifies how much f deviates from being harmonic

$$H(f) = \frac{1}{2}\sum_{x\sim y}(f(x) - f(y))^2, \qquad f : \frac{1}{n}\mathbb{Z}^2 \cap [0,1]^2 \to \mathbb{R}.$$





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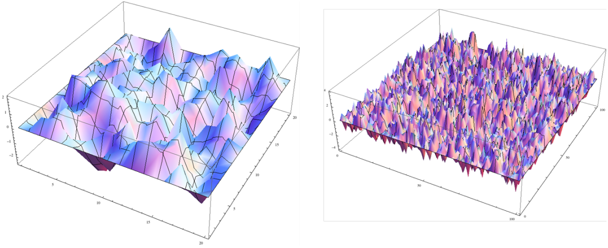

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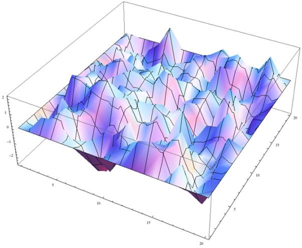

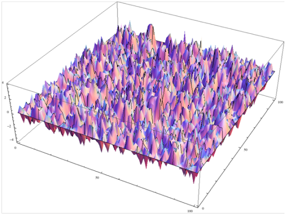

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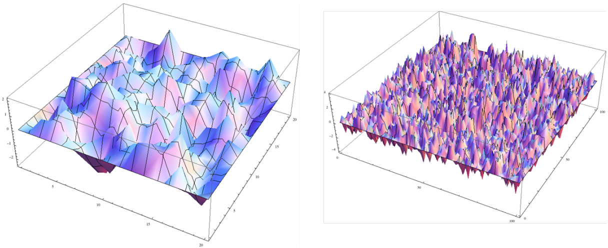

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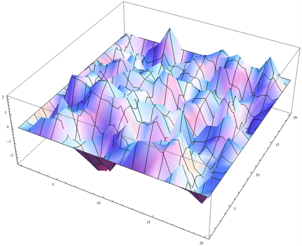
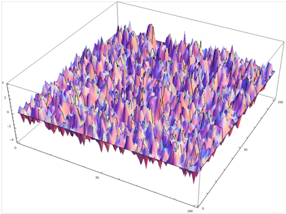
- The GFF is a **random distribution (i.e., random generalized function)**.





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Liouville quantum gravity (LQG)

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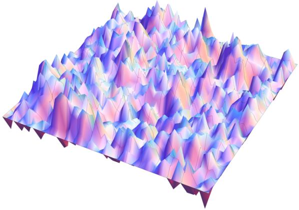

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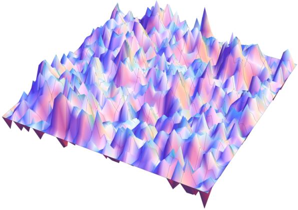

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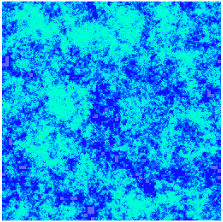

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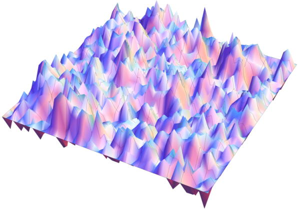

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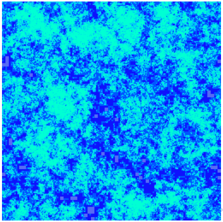

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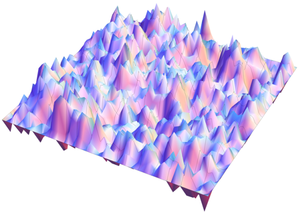
- The area measure is non-atomic and any open set has positive mass a.s., but the measure is a.s. singular with respect to Lebesgue measure.



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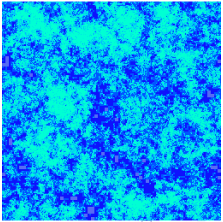

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Random planar maps converge to LQG

Two models for random surfaces:

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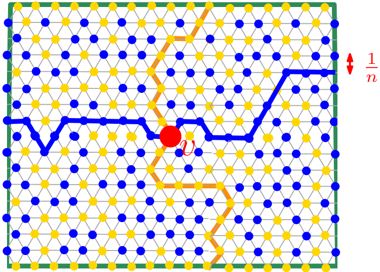

$\mathbb{P}[4 \text{ arms from } v] \approx n^{-5/4}$

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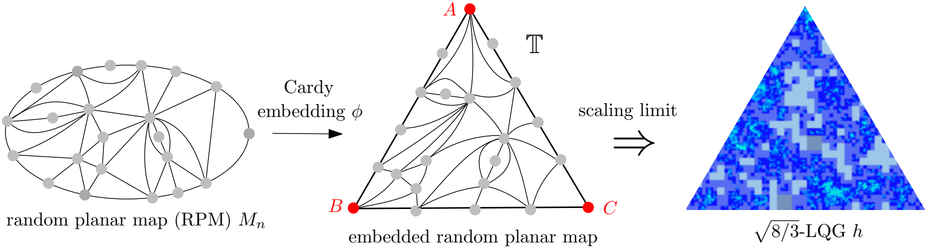
What does it mean for a RPM to converge?

- Metric structure (Le Gall'13, Miermont'13)
- Conformal structure (H.-Sun'19)
- Statistical physics observables (Duplantier-Miller-Sheffield'14, ...)

Conformally embedded RPM converge to $\sqrt{8/3}$-LQG



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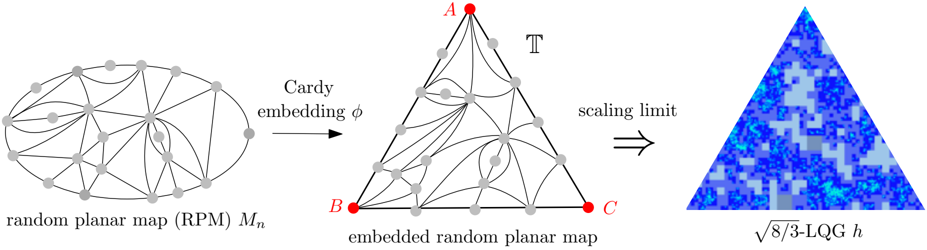

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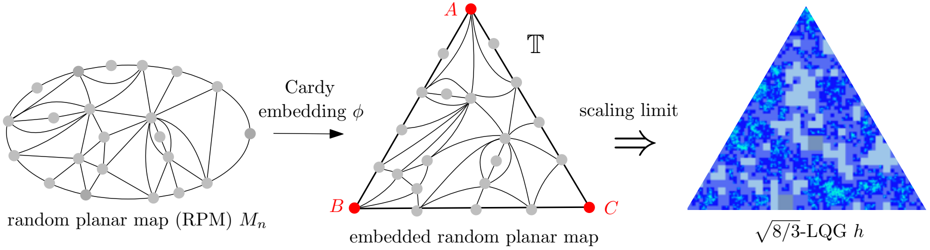

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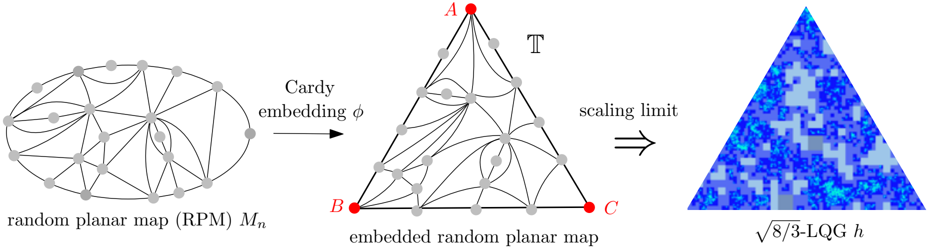

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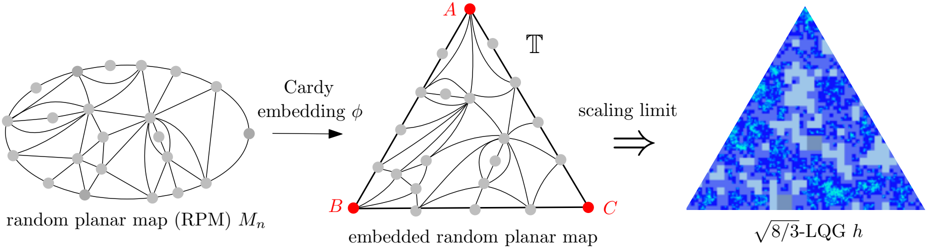

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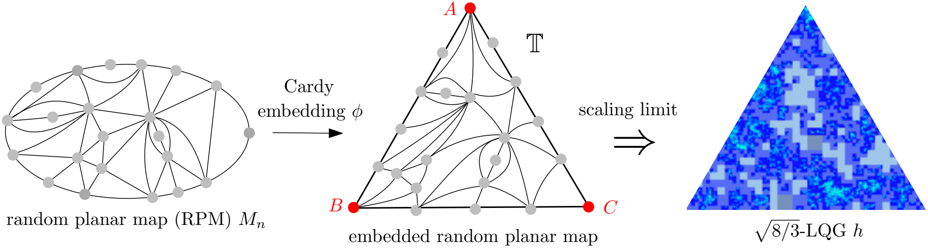

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Theorem (H.-Sun'19)

In the above setting, $(\mu_n, d_n) \Rightarrow (\mu, d)$.

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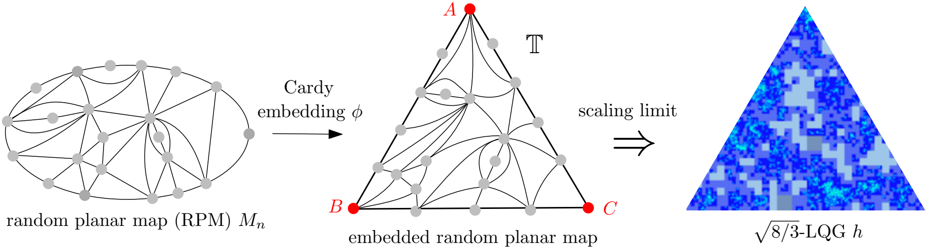

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More precisely, $\exists$ coupling of M_n and h s.t. with probability 1, as $n \to \infty$,

- $\int f d\mu_n \to \int f d\mu$ $\forall$ continuous $f : \mathbb{T} \to [0, 1]$ (measure convergence)
- $d_n(z, w) \to d(z, w)$, uniformly in $z, w \in \mathbb{T}$ (metric convergence)

The Schramm-Loewner evolution (SLE)

- One-parameter family of random fractal curves indexed by $\kappa \geq 0$, which describe the scaling limit of statistical physics models
  - loop-erased random walk, $\kappa = 2$
  - Ising, $\kappa = 3$, and FK-Ising, $\kappa = 16/3$
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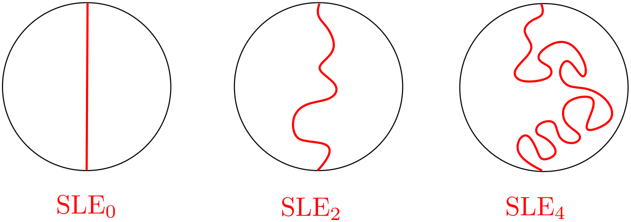

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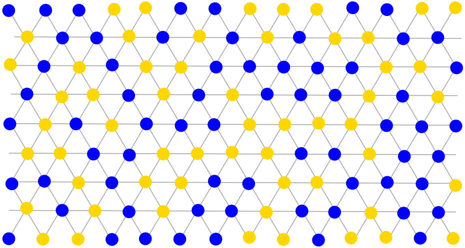

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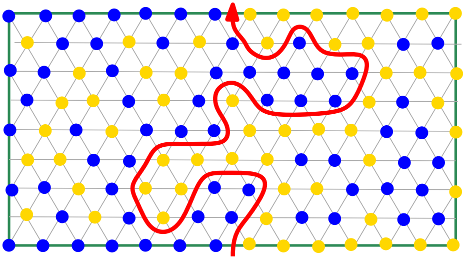

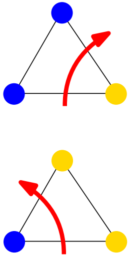

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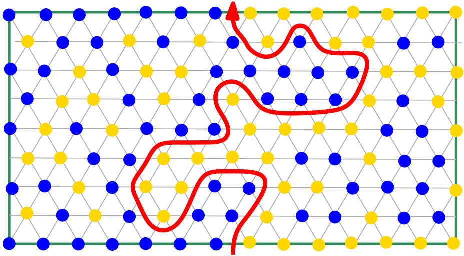
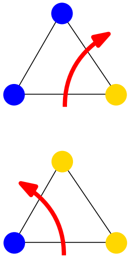
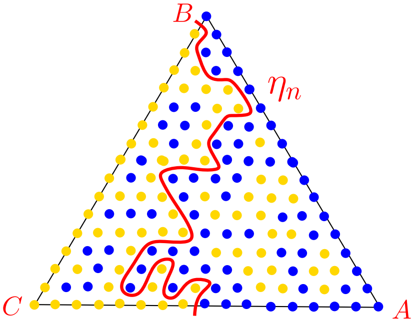
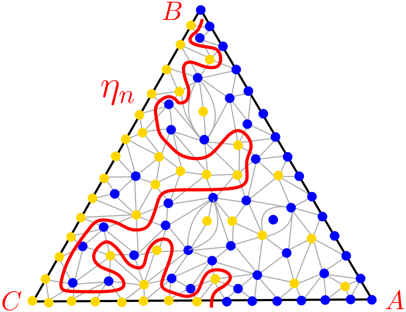
- Introduced by Schramm'99: SLE uniquely characterized by **conformal invariance** and **domain Markov property**.





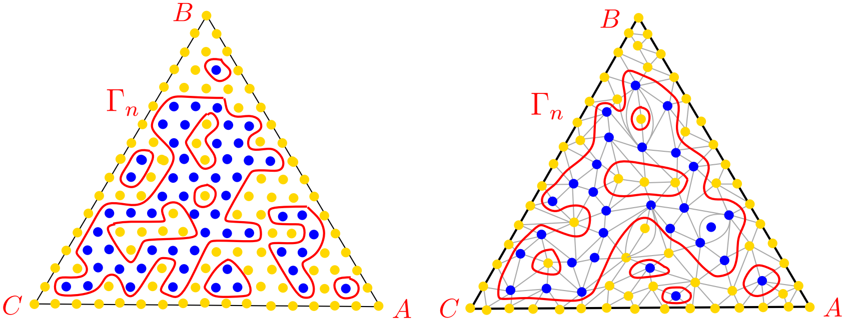
Percolation on uniform triangulations $\Rightarrow$ SLE$_6$





- Smirnov'01, Camia-Newman'06: $\eta_n \Rightarrow$ SLE$_6$ on triangular lattice.
- H.-Sun'19: $\eta_n \Rightarrow$ SLE$_6$ on Cardy embedded triangulation in a *quenched* sense.

Percolation on uniform triangulations ⇒ CLE_6



- The conformal loop ensemble (CLE_6) is the loop version of SLE_6.
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Convergence of percolation crossing probability

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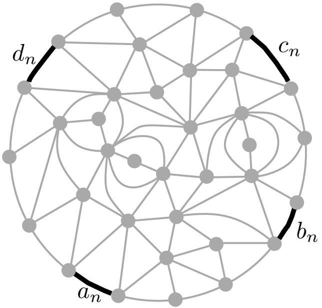

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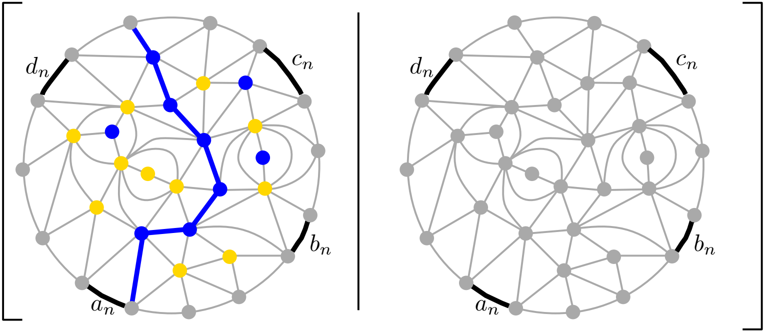

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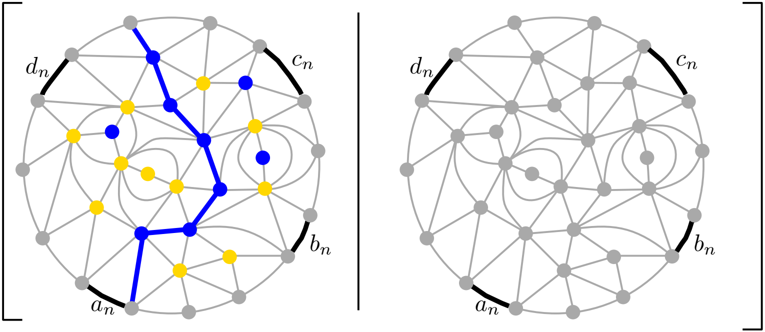

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- P_n gives some notion of **extremal distance** between $a_n b_n$ and $c_n d_n$.

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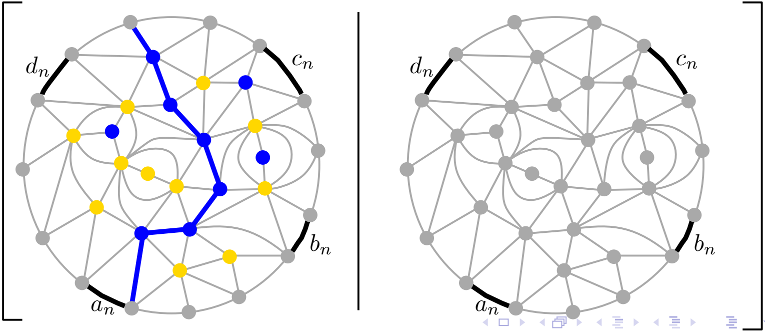
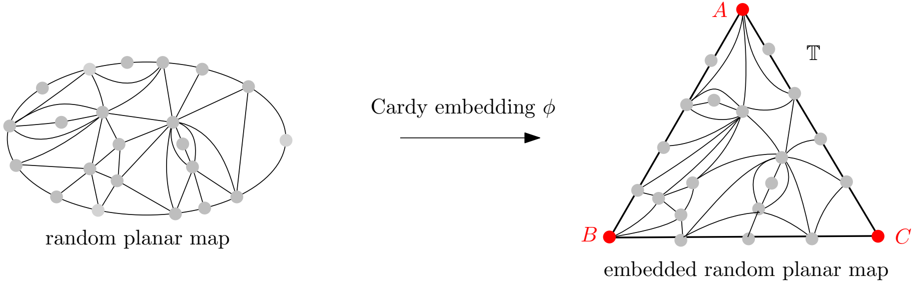


Cardy embedding: percolation-based embedding



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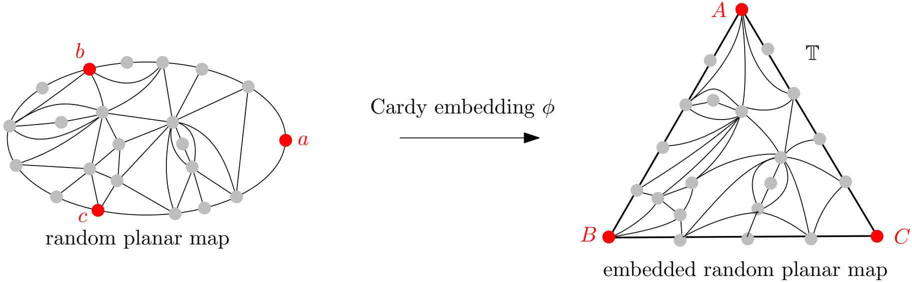

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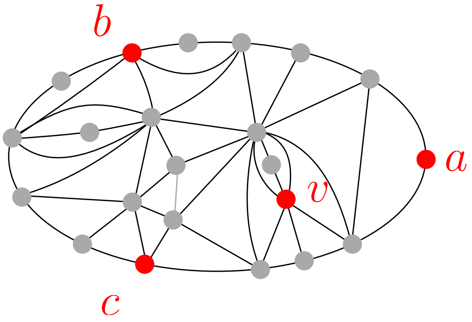
- What is the "correct" position of v in $\mathbb{T}$?



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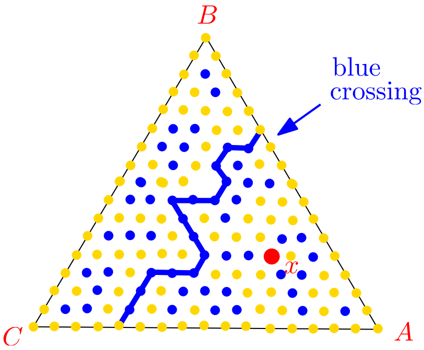

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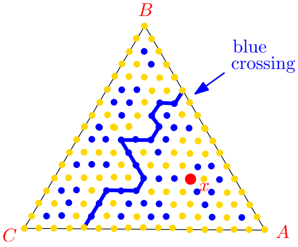

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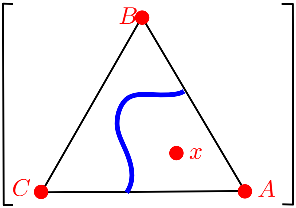

Cardy embedding: percolation-based embedding

- What is the "correct" position of v in $\mathbb{T}$?
- Map $v \in V(M)$ to $x \in \mathbb{T}$ such that

$$(p_A(x), p_B(x), p_C(x)) = (\widehat{p}_a(v), \widehat{p}_b(v), \widehat{p}_c(v)).$$

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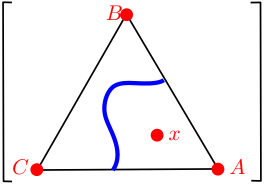

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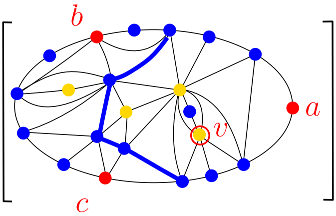
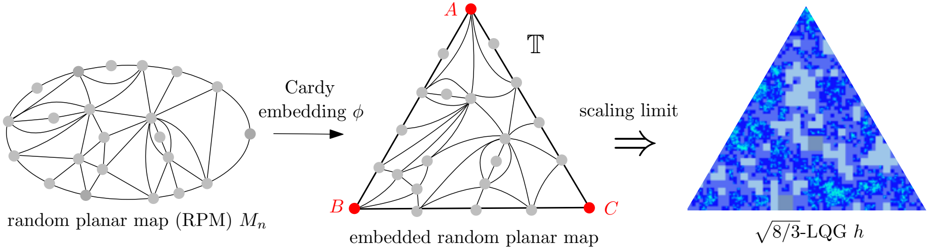


RPM $\Rightarrow$ LQG under conformal embedding

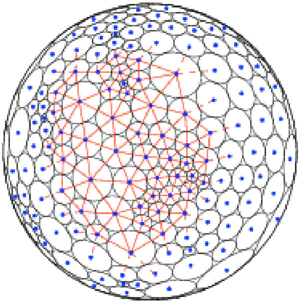


Our result is for **uniform triangulations** and the **Cardy embedding**, but is also believed to hold for other

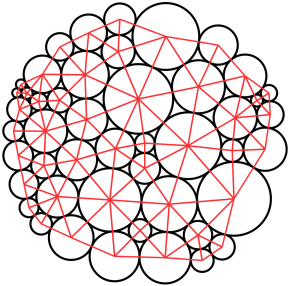
1. conformal embeddings,
2. local map constraints, and
3. universality classes of random planar maps.

Discrete conformal embeddings

- Circle packing
- Riemann uniformization
- Tutte embedding
- Cardy embedding



circle packing (sphere topology)



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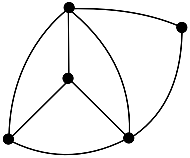
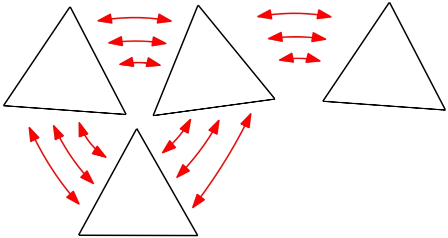


Random planar map



Riemannian manifold

Discrete conformal embeddings

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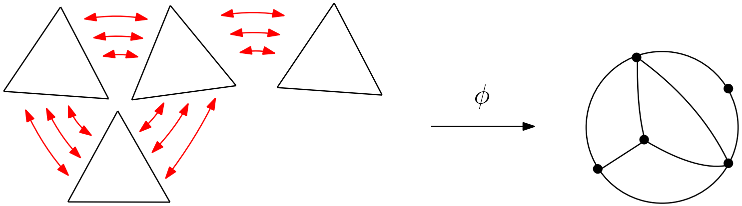
Uniformization theorem: For any simply connected Riemann surface M there is a conformal map ϕ from M to either $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{S}^2$.



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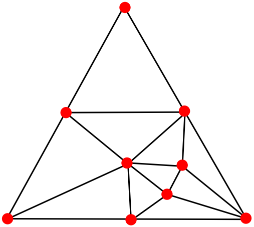
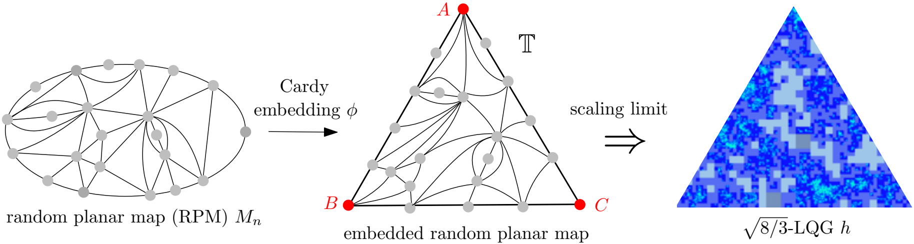


Tutte embedding

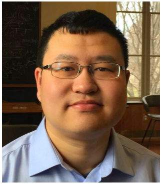
Conformally embedded RPM converge to $\sqrt{8/3}$-LQG



Conformally embedded RPM converge to $\sqrt{8/3}$-LQG

The proof is based on multiple works, including:

- Percolation on triangulations: a bijective path to Liouville quantum gravity (Bernardi-H.-Sun)
- Minkowski content of Brownian cut points (Lawler-Li-H.-Sun)
- Natural parametrization of percolation interface and pivotal points (Li-H.-Sun)
- Uniform triangulations with simple boundary converge to the Brownian disk (Albenque-H.-Sun)
- Joint scaling limit of site percolation on random triangulations in the metric and peanosphere sense (Gwynne-H.-Sun)
- Liouville dynamical percolation (Garban-H.-Sepúlveda-Sun)
- Convergence of uniform triangulations under the Cardy embedding (H.-Sun)









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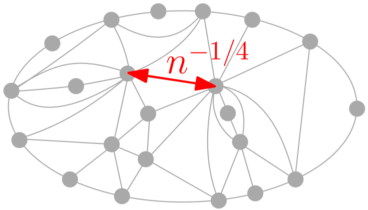

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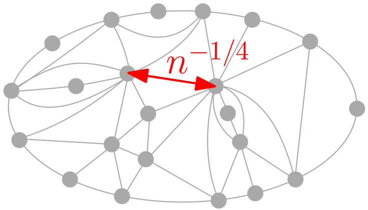

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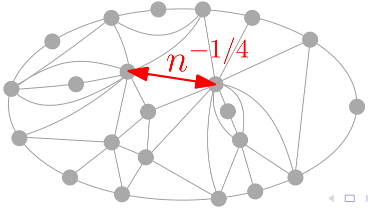
Theorem (Albenque-H.-Sun'19)

$M_n \Rightarrow M$ in the GHP topology, where M is $\sqrt{8/3}$-LQG (equivalently, the Brownian disk).

Building on Le Gall'13, Miermont'13, Bettinelli–Miermont'17, Poulalhon–Schaeffer'06, Addario-Berry–Albenque'17, Addario-Berry–Wen'17



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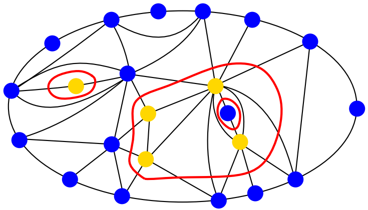

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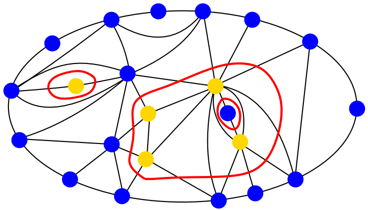

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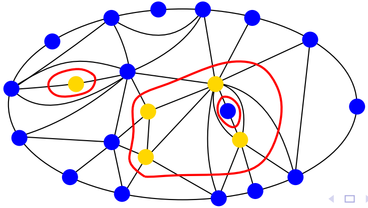
$(M_n, P_n) \Rightarrow (M, \Gamma)$ in the GHPU topology, where Γ is the conformal loop ensemble CLE_6.

Building on Gwynne-Miller'17, Bernardi-H.-Sun'18



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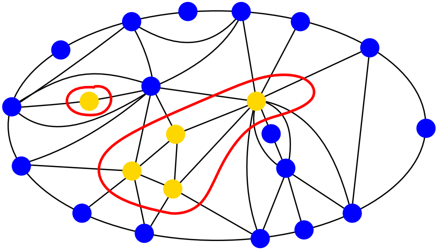

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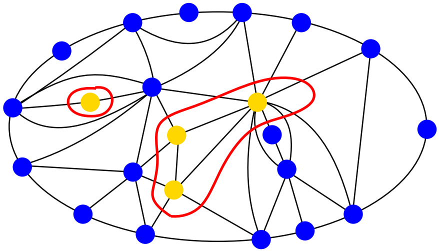

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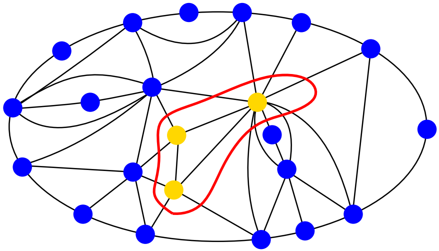

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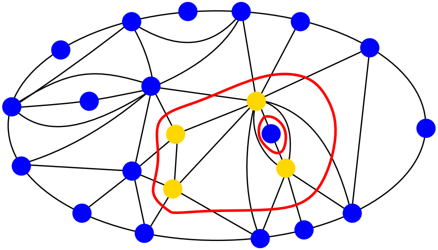

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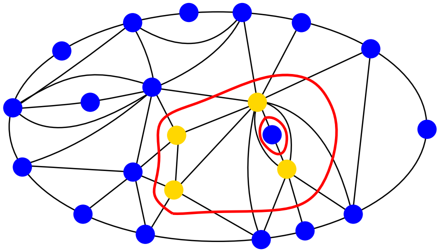

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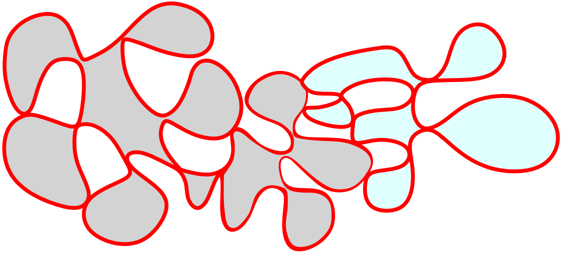

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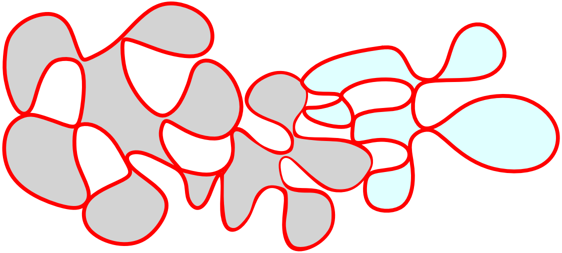

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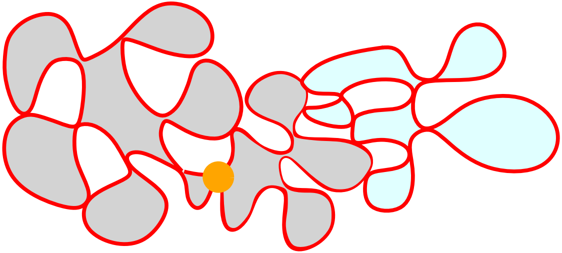

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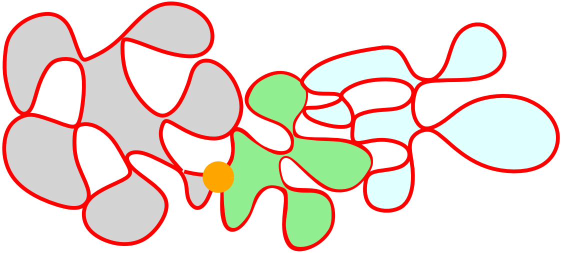

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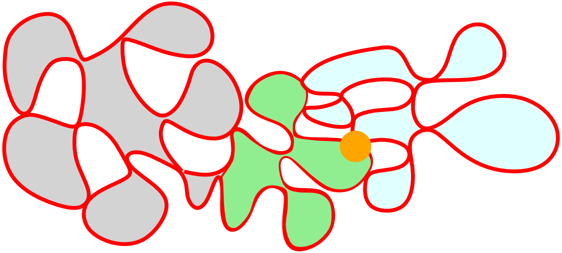

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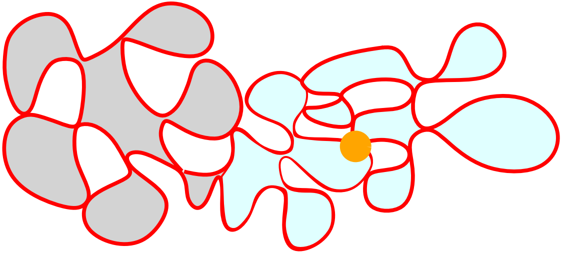

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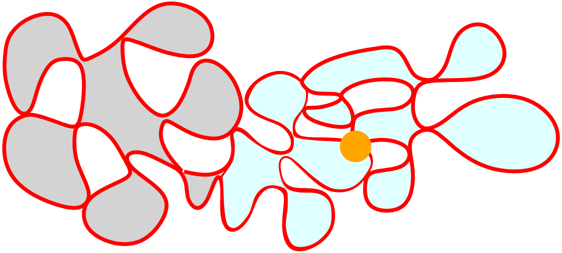

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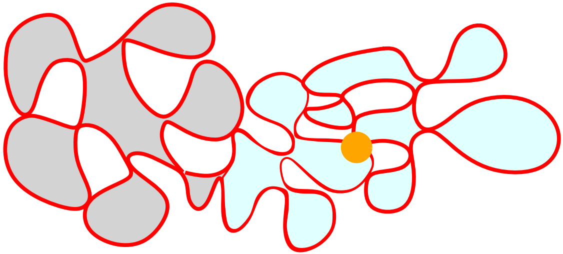

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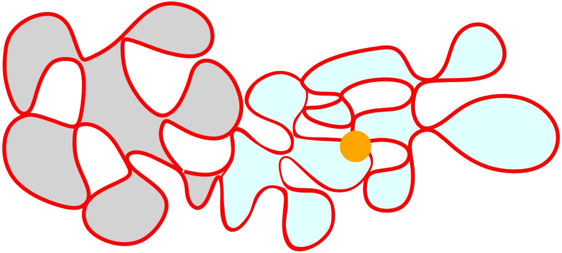
- **Dynamical percolation** $(P_t)_{t\geq 0}$ **on** M: Each vertex has an exponential clock and its color is resampled when its clock rings.
- $(P_{n^{-1/4}t})_{t\geq 0} \Rightarrow (\Gamma_t)_{t\geq 0}$, for $(\Gamma_t)_{t\geq 0}$ Liouville dynamical percolation.
  - Γ_t is a CLE_6 for each $t \geq 0$.
- $(\Gamma_t)_{t\geq 0}$ is **mixing** (in particular, ergodic): Γ_t is asymptotically indep. of Γ_0.
  - $\lim_{t\to\infty} \text{Cov}(E_1(\Gamma_0), E_2(\Gamma_t)) = 0$ for all events E_1, E_2.



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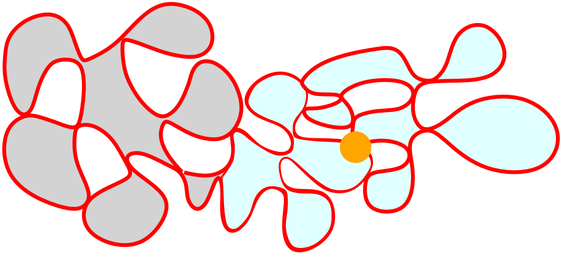

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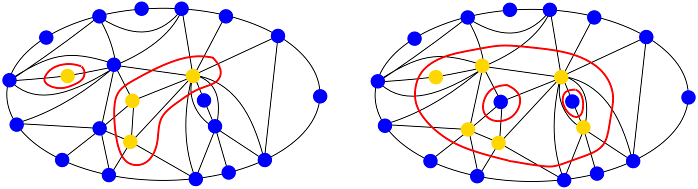

(M, P) $(M, \widetilde{P})$

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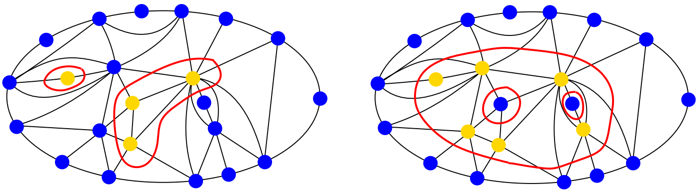

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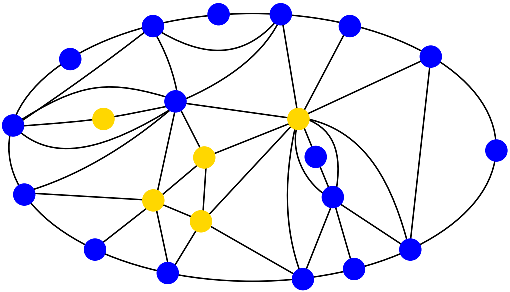

Thanks!