Two manifestations of rigidity phenomena in random point sets: forbidden regions and maximal degeneracy

Subhro Ghosh
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Point processes and rigidity

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- Rigidity of particle numbers was also established for the zeros of the planar Gaussian analytic function [G. - Peres]

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f(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}. \]
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Projection kernel in the above is necessary! [G.-Krishnapur]
In general, for a point process $\Pi$ and a bounded domain $D$, let us denote by $\Pi_{\text{in}}$ the point configuration inside $D$, and by $\Pi_{\text{out}}$ the point configuration outside $D$. The observable $\chi(\Pi_{\text{in}})$ is said to be rigid if $\chi(\Pi_{\text{in}})$ is a deterministic function of $\Pi_{\text{out}}$.

An important class of examples are linear statistics:

$$\chi(\Pi_{\text{in}}) = \sum_{\lambda \in \Pi_{\text{in}}} \phi(\lambda)$$

for some function $\phi$. Setting $\phi = 1_D$ gives the number of points in $D$. Natural to ask about rigidity of more general functionals of a point process (other than the particle count), particularly higher moments of the points in $D$.
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\( \alpha = 1 \) recovers the standard planar case.

For \( \alpha \in \left(\frac{1}{m}, \frac{1}{m-1}\right] \), the first \( m \) moments of the zero process are rigid. [G.-Krishnapur]
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General picture

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(Moment-matching) [G.] Consider a point process $\Pi$ having precisely the first $m$ moments rigid, and two collections of points $\zeta = (\zeta_1, \cdots, \zeta_k)$ and $\eta = (\eta_1, \cdots, \eta_l)$. Then Palm measures $[\Pi]_\zeta$ and $[\Pi]_\eta$ are mutually absolutely continuous iff the first $m$ moments of $\zeta$ and $\eta$ match,
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However, very few rigorous theorems establishing general implications like the above between these concepts.
We say that the disk $D$ is a hole if there are no particles inside $D$. When $\text{radius}(D) \to \infty$, how does the outside configuration behave? In other words, what causes a large hole (a rare event) to occur?
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![Diagram of GUE distribution]

\[ f_1(x) \]
We consider this problem for the zeros of the standard planar Gaussian analytic function.
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We show that:
Theorem (G.- Nishry)

The conditional intensity for zeroes of Gaussian random polynomials has the following behaviour:

- There is a singular component at the edge of the hole.
- There is a subsequent "forbidden region", namely, in the annulus $R < r < \sqrt{e}R$, the conditional intensity $\to 0$ as $R \to \infty$.
- Beyond $\sqrt{e}R$, the conditional intensity behaves, in the limit $R \to \infty$, like the equilibrium intensity.
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$I$ is the LDP rate function. No zeros in the hole $D$ is the same as $Z(D) = 0$. To find the "most likely configuration" given that there is hole is roughly the same as minimizing the rate functional $I$ over the space of probability measures (on $\mathbb{C}$) under the constraint that there is zero mass on $D$. 

Subhro Ghosh National University of Singapore Rigidity Phenomena
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- The functional to be optimized is highly non-smooth:

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I(\mu) = 2 \sup_{z \in \mathbb{C}} \left\{ U_\mu(z) - \frac{|z|^2}{2} \right\} - \Sigma(\mu) - C,
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where \( U_\mu \) is the logarithmic potential and \( \Sigma(\mu) \) is the logarithmic energy of the measure \( \mu \) and \( C \) is a constant.

No clear variational method available. Tackled by "guessing" the solution and then establishing that it is indeed the minimizer using potential theoretic methods. Heuristics made rigorous by obtaining "effective" versions of large deviation estimates and approximating the analytic function zeros by those of the polynomials.
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Stealthy particle systems conjectured to have deterministically bounded holes [Zhang-Stillinger-Torquato].
Stealthy random fields

Theorem (G.-Lebowitz)

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Special case: Gaussian process with a gap (or fast decay) in the spectrum.
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- (Anti-concentration) The particle number in a domain is bounded deterministically.
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- **(Anti-concentration)** The particle number in a domain is bounded deterministically by (a constant multiple of) the expected number of points in the domain.
The existence of a gap / fast decay in the spectrum can be exploited to construct linear functionals of the process which have low variance.

A linear functional with a low variance is approximately constant, so this gives an approximate linear constraint.

Sufficiently rich class of such constraints can be exploited to deduce degenerate behaviour.
Thank you !!