Zero Temperature Limits for Directed Polymers in Random Environment

Ryoki Fukushima (RIMS, Kyoto University)

The 12th Mathematical Society of Japan, Seasonal Institute
Stochastic Analysis, Random Fields and Integrable Probability
August 9, 2019

Joint works with F. Comets, S. Nakajima, N. Yoshida, S. Junk.

Disclaimer

The partition function of a directed polymer:

$$Z_n^\beta = \sum_{\gamma:\ \text{path}} \exp\left\{-\beta \sum_{j=1}^{n} \omega(j,\gamma_j)\right\} P(\gamma).$$

The free energy $\varphi(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^\beta$ is important.
(Existence by the subadditive ergodic theorem.)

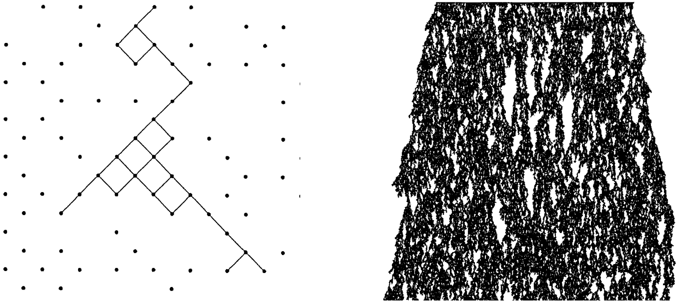
In the zero-temperature limit $\beta \to \infty$,

$$\lim_{\beta\to\infty} \lim_{n\to\infty} \frac{1}{\beta n} \log Z_n^\beta = -\lim_{n\to\infty} \frac{1}{n} \inf_{\gamma:\ \text{path}} \sum_{j=1}^{n} \omega(j,\gamma_j),$$

when the right-hand side is non-zero. This is the First Passage Percolation.

A problem on oriented percolation

Q How many open paths of length n in the oriented percolation cluster starting at $(0,0)$?



From *Durrett: Ten lectures on particle systems*

A problem on oriented percolation

Q How many open paths of length n in the oriented percolation cluster starting at $(0,0)$?

Let $N_n = \#\{\text{open paths from } (0,0) \text{ to level } n\}$.

- F.–Yoshida 2012: $N_n \geq e^{\delta n}$ when $\exists$ an infinite path.
- Garet–Gouéré–Marchand 2016: $\alpha(p) = \lim_{n\to\infty} \frac{1}{n} \log N_n$ exists when $\exists$ an infinite path.
- Duminil-Copin–Kesten–Nazarov–Peres–Sidoravicius 2019+: The number of maximizing paths grows exponentially.

A problem on oriented percolation

Q How many open paths of length n in the oriented percolation cluster starting at $(0,0)$?

Let $N_n = \#\{\text{open paths from } (0,0) \text{ to level } n\}$.

- F.–Yoshida 2012: $N_n \geq e^{\delta n}$ when $\exists$ an infinite path.
- Garet–Gouéré–Marchand 2016: $\alpha(p) = \lim_{n\to\infty} \frac{1}{n} \log N_n$ exists when $\exists$ an infinite path.
- Duminil-Copin–Kesten–Nazarov–Peres–Sidoravicius 2019+: The number of maximizing paths grows exponentially.

If ω is Ber(p), then $N_n = \lim_{\beta\to\infty} \sum_{\gamma:\text{ path}} \exp\left\{-\beta \sum_{j=1}^{n} \omega(j,\gamma_j)\right\}$.

Can we recover $\alpha(p)$ by taking zero-temperature limit?

We have corresponding results only for two toy models...

Model I: discrete time polymer with unbounded jumps

Toy model I

- $(\{\gamma_n\}_{n\in\mathbb{N}}, P)$: Random walk on $\mathbb{Z}^d$ with

$$P(\gamma_{n+1} = x | \gamma_n = y) = c_1 \exp\{-|x-y|_1^\alpha\};$$

- $(\{\omega(j,x)\}_{(j,x)\in\mathbb{N}\times\mathbb{Z}^d}, \mathbb{P})$: IID, Ber$(p)$.

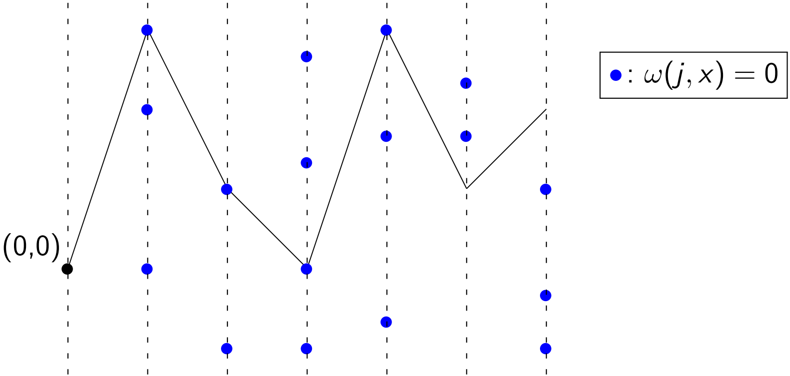
Directed polymer measure:

$$\mu_n^{\omega,\beta}(\gamma) = \frac{1}{Z_n^{\omega,\beta}} \exp\left\{-\beta \sum_{j=1}^{n} \omega(j,\gamma_j)\right\} P_0(\gamma),$$

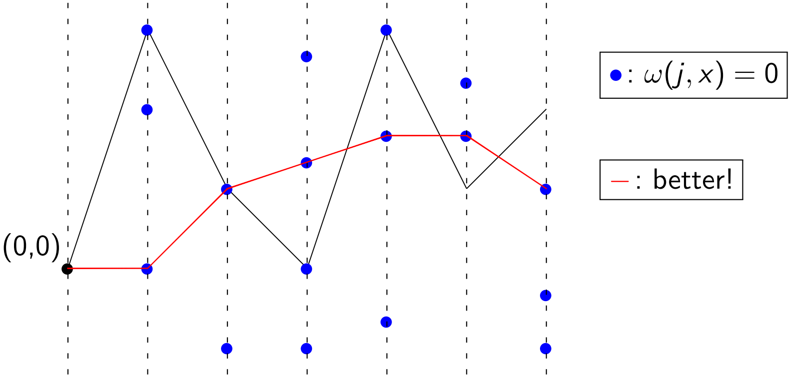
$$Z_n^{\omega,\beta} = E\left[\exp\left\{-\beta \sum_{j=1}^{n} \omega(j,\gamma_j)\right\}\right].$$

At $\beta = \infty$, we regard $\exp\{\cdots\} = 1_{\sum_{j=1}^{n} \omega(j,\gamma_j)=0}$.

$$Z_n^{\omega,\beta} = \sum_{\gamma:\ \text{path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[-\beta\omega(j,\gamma_j) - |\gamma_{j-1} - \gamma_j|_1^{\alpha} \right] \right\}.$$



$$Z_n^{\omega,\beta} = \sum_{\gamma:\ \text{path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[-\beta\omega(j,\gamma_j) - |\gamma_{j-1} - \gamma_j|_1^{\alpha} \right] \right\}.$$



Free energy I

It is standard to show the existence of the free energy:

$$\varphi(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\omega,\beta} = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\log Z_n^{\omega,\beta}].$$

If we naturally define $Z_n^{\omega,\infty} = P(\sum_{j=1}^n \omega(j,\gamma_j) = 0)$, this holds even at $\beta = \infty$.

The key ingredient is $\mathbb{E}[\log Z_n^{\omega,\infty}] < \infty$,

Free energy I

It is standard to show the existence of the free energy:

$$\varphi(\beta) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\omega,\beta} = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\log Z_n^{\omega,\beta}].$$

If we naturally define $Z_n^{\omega,\infty} = P(\sum_{j=1}^n \omega(j,\gamma_j) = 0)$, this holds even at $\beta = \infty$.

The key ingredient is $\mathbb{E}[\log Z_n^{\omega,\infty}] < \infty$, which fails to hold for some other models (2nd part).

Zero temperature limit I

In this model, we know $\varphi(\infty)$ exists.

Theorem (Comets–F.–Nakajima–Yoshida 2015, N. 2018)

For any $\alpha > 0$,

$$\varphi(\beta) \xrightarrow{\beta \nearrow \infty} \varphi(\infty).$$

Remark

1. *The joint continuity in (p, β) is easy on $\beta < \infty$ region.*
2. *The proof shows that for any $\epsilon > 0$, we can choose $\beta \gg 1$ such that*

$$Z_n^{\omega,\infty} \leq Z_n^{\omega,\beta} \leq e^{\epsilon n} Z_n^{\omega,\infty}.$$

This gives an alternative proof of the existence of $\varphi(\infty)$.

Proof idea: $\alpha \leq 1$

The proof of $Z_n^{\omega,\infty} \leq Z_n^{\omega,\beta} \leq e^{\epsilon n} Z_n^{\omega,\infty}$ goes as follows:

$$
\begin{aligned}
Z_n^{\omega,\beta} &= \sum_{\gamma\text{: path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[-\beta\omega(j,\gamma_j) - |\gamma_{j-1} - \gamma_j|_1^\alpha \right] \right\} \\
&= \sum_{\text{no traps}} + \sum_{\text{few traps}} + \sum_{\text{many traps}} .
\end{aligned}
$$

$\sum_{\text{no traps}} = Z_n^{\omega,\infty}$ and $\sum_{\text{many traps}}$ is negligible when $\beta \sim \infty$.

Proof idea: $\alpha \le 1$

The proof of $Z_n^{\omega,\infty} \le Z_n^{\omega,\beta} \le e^{\epsilon n} Z_n^{\omega,\infty}$ goes as follows:

$$Z_n^{\omega,\beta} = \sum_{\gamma:\,\text{path}} c_1^n \exp\left\{ \sum_{j=1}^{n} \left[-\beta\omega(j,\gamma_j) - |\gamma_{j-1} - \gamma_j|_1^{\alpha} \right] \right\}$$
$$= \sum_{\text{no traps}} + \sum_{\text{few traps}} + \sum_{\text{many traps}} .$$

$\sum_{\text{no traps}} = Z_n^{\omega,\infty}$ and $\sum_{\text{many traps}}$ is negligible when $\beta \sim \infty$.

For $\sum_{\text{few traps}}$, we can deform paths to trap free paths without too much extra cost and multiplicity:

$$\Longrightarrow \sum_{\text{few traps}} \le e^{\epsilon n} \sum_{\text{no traps}} .$$

Proof idea: $\alpha > 1$

The "deformation cost" is too large in this case.
The proof is based on a control of the rate of convergence:

$$\log Z_n^\beta - n\varphi(\beta) = \underbrace{\log Z_n^\beta - \mathbb{E}[\log Z_n^\beta]}_{\text{random error}} + \underbrace{\mathbb{E}[\log Z_n^\beta] - n\varphi(\beta)}_{\text{non-random error}}$$

We need (uniformly in $\beta \in [0, \infty]$):

$$\mathbb{P}\left(\left|\log Z_n^\beta - \mathbb{E}[\log Z_n^\beta]\right| > n^{1-\delta}\right) \leq n^{-M},$$
$$\left|\mathbb{E}[\log Z_n^\beta] - n\varphi(\beta)\right| \leq n^{1-\delta}.$$

Proof idea: $\alpha > 1$

The "deformation cost" is too large in this case.
The proof is based on a control of the rate of convergence:

$$\log Z_n^\beta - n\varphi(\beta) = \underbrace{\log Z_n^\beta - \mathbb{E}[\log Z_n^\beta]}_{\text{random error}} + \underbrace{\mathbb{E}[\log Z_n^\beta] - n\varphi(\beta)}_{\text{non-random error}}$$

We need (uniformly in $\beta \in [0, \infty]$):

$$\mathbb{P}\left(\left|\log Z_n^\beta - \mathbb{E}[\log Z_n^\beta]\right| > n^{1-\delta}\right) \le n^{-M},$$
$$\left|\mathbb{E}[\log Z_n^\beta] - n\varphi(\beta)\right| \le n^{1-\delta}.$$

In fact, the first bound implies the second (Y. Zhang 2010).

Maximal jump

Proof of concentration requires a control on the *influence*, which is related to the jump size.

Lemma (Nakajima 2018)

For any $\alpha > 1$, "typical" polymers of length n jumps at most $n^{o(1)}$.

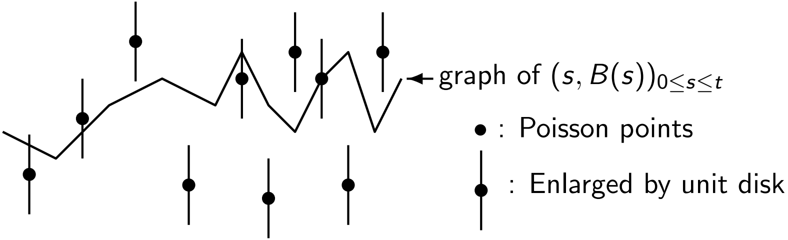
Remark

Numerical experiment shows that there is a big jump when $\alpha < 1$. I have a proof that the maximal jump is larger than $(\log n)^c$ but for all $\alpha \in (0, \infty)$.

Model II: Brownian polymer in Poissonian environment

Toy model II

- $((B(t))_{t\geq 0}, P_x)$: standard Brownian motion on $\mathbb{R}^d$, $B(0) = x$.
- $(\omega = \sum_i \delta_{(t_i,x_i)}, \mathbb{P})$: Poisson point process on $(0,\infty) \times \mathbb{R}^d$ with unit intensity.



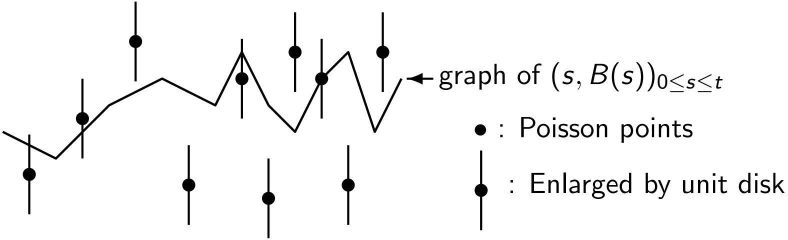
Directed polymer measure:

$$\mu_t^{\omega,\beta}(\mathrm{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta \#\{\text{hitting to } \phi \text{ up to } t\}} P_0(\mathrm{d}B).$$

See a survey article by Comets–Cosco for known results.

Toy model II

- $((B(t))_{t\geq 0}, P_x)$: standard Brownian motion on $\mathbb{R}^d$, $B(0) = x$.
- $(\omega = \sum_i \delta_{(t_i,x_i)}, \mathbb{P})$: Poisson point process on $(0,\infty) \times \mathbb{R}^d$ with unit intensity.



Directed polymer measure:

$$\mu_t^{\omega,\beta}(\mathrm{d}B) = \frac{1}{Z_t^{\omega,\beta}} e^{-\beta\#\{\text{hitting to } \phi \text{ up to } t\} - \int_0^t |\dot{B}(s)|^2 \mathrm{d}s}.$$

See a survey article by Comets–Cosco for known results.

Free energy II

Existence of the free energy $\varphi(\beta)$ for $\beta \in \mathbb{R}$ is standard:

$$\varphi(\beta) = \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\beta} = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\Big[\log Z_t^{\omega,\beta}\Big].$$

At $\beta = \infty$, the model makes sense by setting $\tau(\omega)$ to be the hitting time to ϕ and $Z_t^{\omega,\infty} = P_0(\tau(\omega) > t)$.

Free energy II

Existence of the free energy $\varphi(\beta)$ for $\beta \in \mathbb{R}$ is standard:

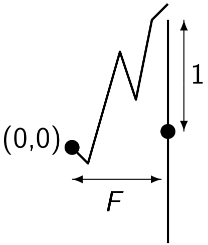
$$\varphi(\beta) = \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\beta} = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\Big[\log Z_t^{\omega,\beta}\Big].$$

At $\beta = \infty$, the model makes sense by setting $\tau(\omega)$ to be the hitting time to $\bullet$ and $Z_t^{\omega,\infty} = P_0(\tau(\omega) > t)$. But $\mathbb{E}\Big[\log Z_t^{\omega,\infty}\Big] = -\infty$.

Proof.

Brownian motion has to avoid the first disaster in $[0,\infty] \times [-\frac{1}{2}, \frac{1}{2}]$. If it occurs at time F, then

$$\log P_0(\tau(\omega) > t) \lesssim \log \exp\left(-(\tfrac{1}{2})^2/F\right) = -\frac{1}{4F}.$$



Since $F \overset{\mathrm{d}}{=} \mathrm{Exp}(1)$, $1/F$ is not integrable. □

Free energy II

Existence of the free energy $\varphi(\beta)$ for $\beta \in \mathbb{R}$ is standard:

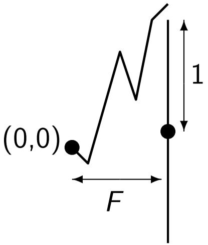
$$\varphi(\beta) = \lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\beta} = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\Big[\log Z_t^{\omega,\beta}\Big].$$

At $\beta = \infty$, the model makes sense by setting $\tau(\omega)$ to be the hitting time to ⸸ and $Z_t^{\omega,\infty} = P_0(\tau(\omega) > t)$. But $\mathbb{E}\Big[\log Z_t^{\omega,\infty}\Big] = -\infty$.

Proof.

Brownian motion has to avoid the first disaster in $[0,\infty] \times [-\frac{1}{2}, \frac{1}{2}]$. If it occurs at time F, then

$$\log P_0(\tau(\omega) > t) \lesssim \log \exp\left(-(\tfrac{1}{2})^2/F\right) = -\frac{1}{4F}.$$



Since $F \overset{\mathrm{d}}{=} \mathrm{Exp}(1)$, $1/F$ is not integrable. □

$\longrightarrow$ Direct sub-additivity argument fails.

Zero temperature limit II

Theorem

There exists $p(\infty) \in (-\infty, 0)$ *such that the following hold:*

(i) $\mathbb{P}$*-almost surely,* $\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\infty} = p(\infty)$*;*

(ii) $\lim_{\beta\to\infty} p(\beta) = p(\infty)$.

Zero temperature limit II

Theorem

There exists $p(\infty) \in (-\infty, 0)$ *such that the following hold:*

(i) $\mathbb{P}$*-almost surely,* $\lim_{t\to\infty} \frac{1}{t} \log Z_t^{\omega,\infty} = p(\infty)$*;*

(ii) $\lim_{\beta\to\infty} p(\beta) = p(\infty)$.

The proof follows the same line as $\alpha > 1$ case of Model I.

Modified death time

Lemma (non-integrability is due to the first disaster)

Let F_t *be the first disaster in* $[0,t] \times [-\frac{7}{2}, \frac{7}{2}]^d$*. Then there exists* $c > 0$ *such that*

$$\mathbb{E}\left[\log P_0(\tau(\omega) > t) \,\middle|\, F_t\right] \geq -c(t + F_t^{-1}).$$

Thus the following modification ensures the integrability:

$$\tau^1(\omega) := \inf\{s \geq 1 \colon (s, B_s) \text{ hits a disaster}\}.$$

Modified death time

Lemma (non-integrability is due to the first disaster)

Let F_t be the first disaster in $[0, t] \times [-\frac{7}{2}, \frac{7}{2}]^d$. Then there exists $c > 0$ such that

$$\mathbb{E}\left[\log P_0(\tau(\omega) > t) \,\middle|\, F_t\right] \geq -c(t + F_t^{-1}).$$

Thus the following modification ensures the integrability:

$$\tau^1(\omega) := \inf\{s \geq 1 \colon (s, B_s) \text{ hits a disaster}\}.$$

Problem: Standard argument for super-additivity yields

$$\begin{aligned}
&\mathbb{E}\left[\log P(\tau^1(\omega) \geq s + t)\right] \\
&\quad \geq \mathbb{E}\left[\log P(\tau^1(\omega) \geq s)\right] + \mathbb{E}\left[\log P(\tau(\omega) \geq t)\right].
\end{aligned}$$

Effect of changing disasters in a slab

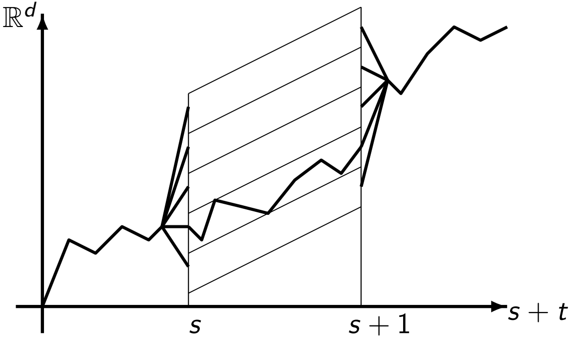
We show an almost super-additivity by estimating

$$\log P\big(\tau^1(\omega) \geq s+t\big) - \log P\big(\tau^1(\omega_{[s,s+1]^c}) \geq s+t\big) \\ = \log P\left(\tau^1(\omega) \geq s+t \;\middle|\; \tau^1(\omega_{[s,s+1]^c}) \geq s+t\right).$$

Effect of changing disasters in a slab

We show an almost super-additivity by estimating

$$\log P\big(\tau^1(\omega) \geq s+t\big) - \log P\big(\tau^1(\omega_{[s,s+1]^c}) \geq s+t\big)$$
$$= \log P\left(\tau^1(\omega) \geq s+t \;\middle|\; \tau^1(\omega_{[s,s+1]^c}) \geq s+t\right).$$



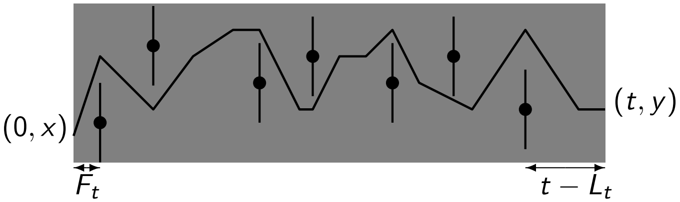
We need a control on the survival in tubes and that the polymer is "spread out" under $P(\cdot \mid \tau^1(\omega_{[s,s+1]^c}) \geq s+t)$.

Survival in tube

Lemma

Let F_t and L_t be the first and last disaster in $[0,t] \times [-\frac{7}{2}, \frac{7}{2}]$ respectively. Then

$$\inf_{x,y \in [-5/2, 5/2]^d} \mathbb{E}\left[\log P_{0,x}^{t,y}(\tau(\omega) \wedge \tau_{[-3,3]} > t) \,\middle|\, F_t, L_t\right] \geq -c(t + F_t^{-1} + (t - L_t)^{-1}).$$



Concentration bound

Previous Lemma and "spread-out" estimate for polymer measure (skipped) yield almost super-additivity

$\Rightarrow$ Existence of $\displaystyle\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log P(\tau^1(\omega) > t)]$.

Control on the effect of changing disasters in a slab

$\Rightarrow$ Concentration around the mean

$\Rightarrow$ Existence of $\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1(\omega) > t)$, $\mathbb{P}$-a.s.

Concentration bound

Previous Lemma and "spread-out" estimate for polymer measure (skipped) yield almost super-additivity

$\Rightarrow$ Existence of $\displaystyle\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log P(\tau^1(\omega) > t)]$.

Control on the effect of changing disasters in a slab

$\Rightarrow$ Concentration around the mean

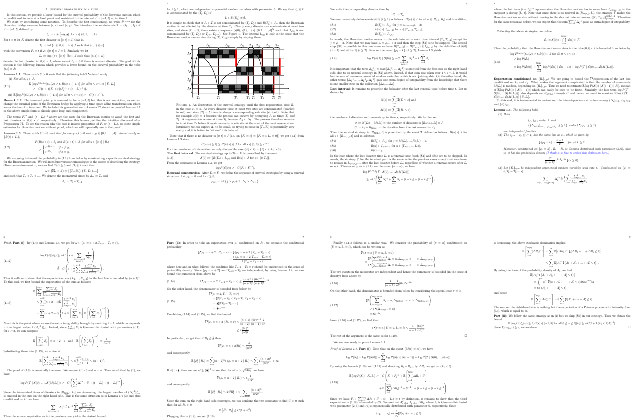
$\Rightarrow$ Existence of $\lim_{t\to\infty} \frac{1}{t} \log P(\tau^1(\omega) > t)$, $\mathbb{P}$-a.s.

Once we get a concentration around the mean, as before,

$$\left| \frac{1}{t} \log P(\tau^1(\omega) > t) - p(\infty) \right| \le t^{-\delta},$$

which extends to the positive temperature uniformly in $\beta \in \mathbb{R}$.
This yields the continuity of $p(\beta)$.

Proof of survival in tube Lemma



Thank you for your attention!