Extreme values for diffusion in random media

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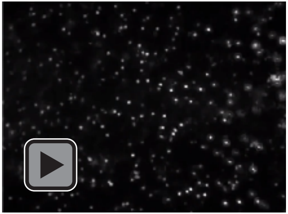
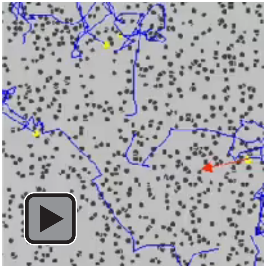
From pollen to Perrin

History: In 1827, Robert Brown observed that pollen suspended in water seemingly performed a random walk. Eighty years later, Einstein proposed a statistical description for this "Brownian motion" and an explanation: Water molecules jiggle and knock the pollen in small and seemingly random directions. This model was soon confirmed in experiments of Perrin.



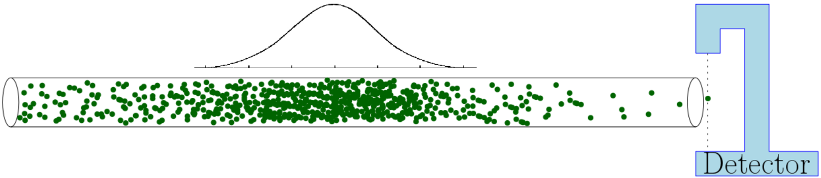
Questions for today:

- Are there senses in which Brownian motion fails to model such a physical system?
- Are there signatures of the underlying random media which can be recovered by studying the motion of particles?



I will argue that diffusion in random media has very different extreme value statistics / large deviations.

Diffusion in a random media



Many small particles moving in a viscous media:

- How does the bulk particle density evolve?
- What about the right-most particle?

Two models for such systems:

- Independent random walks.
- Independent random walks in a random environment (RWRE).

Punchline: Both models have same bulk behavior, but the RWRE drastically changes extreme value scalings / statistics to KPZ type.

Case 1: Independent (simple) random walk X_t on $\mathbb{Z}$

$$\mathsf{P}\big(X_{t+1} = X_t + 1\big) = \frac{\alpha}{\alpha+\beta}, \qquad \mathsf{P}\big(X_{t+1} = X_t - 1\big) = \frac{\beta}{\alpha+\beta}.$$

- **Law of Large Numbers (LLN)**:

$$\frac{X_t}{t} \longrightarrow \frac{\alpha-\beta}{\alpha+\beta}.$$

- **Central Limit Theorem (CLT)**: For $\sigma = \frac{2\sqrt{\alpha\beta}}{\alpha+\beta}$, $\mathcal{N}(0,1)$ Gaussian,

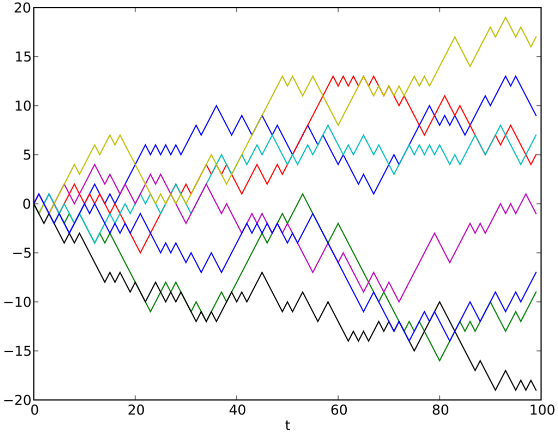
$$\frac{X_t - t\frac{\alpha-\beta}{\alpha+\beta}}{\sigma\sqrt{t}} \quad \Longrightarrow \quad \mathcal{N}(0,1).$$

- **Large Deviation Principle (LDP)**: For $\frac{\alpha-\beta}{\alpha+\beta} < x < 1$, with $I(x) = \sup_{z\in\mathbb{R}}\big(zx - \lambda(z)\big)$ and $\lambda(z) := \log\Big(\mathsf{E}\big[e^{zX_1}\big]\Big)$,

$$\frac{\log\Big(\mathsf{P}\big(X_t > xt\big)\Big)}{t} \longrightarrow -I(x),$$

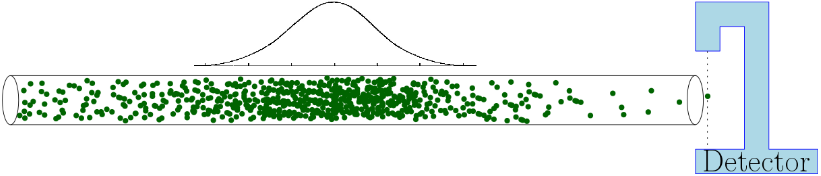
e.g. For $\alpha = \beta$, $I(x) = \frac{1}{2}\Big((1+x)\log(1+x) + (1-x)\log(1-x)\Big)$.

Extreme value statistics for random walks



$$\mathsf{P}\Big(\max(X_t^{(1)},\ldots,X_t^{(N)}) \leq x\Big) = \mathsf{P}(X_t \leq x)^N = \big(1 - \mathsf{P}(X_t > x)\big)^N$$

Extreme value statistics for random walks



- **How does the bulk particle density evolve?**
- **What about the right-most particle?**

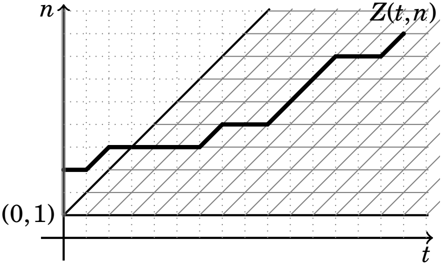
Let $X_t^{(1)}, \ldots X_t^{(N)}$ be N-independent copies of X_t. Then we have:

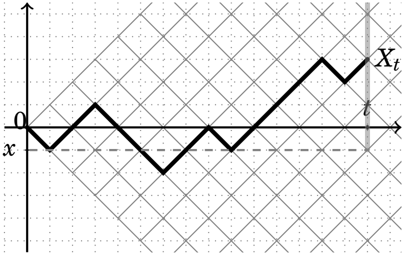
- Centered bulk density solves heat equation and is Gaussian.
- If $N = e^{ct}$ and $c < c_{\text{saturated}}$, then for $c_1 = I^{-1}(c)$ (and similarly explicit constants c_2, c_3)

$$\max_{i=1,\ldots,N} \{X_t^{(i)}\} \approx c_1 \cdot t + c_2 \cdot \log(t) + c_3 \cdot \text{Gumbel}$$

where Gumbel has distribution function $e^{-e^{-x}}$.

Deriving exact formulas via a recurrence





Recurrence formula

Define a function $Z(t,n)$ via the recursion (with $Z(0,n) = \mathbf{1}_{n\geq 1}$)

$$Z(t,n) = \frac{\alpha}{\alpha+\beta} \cdot Z(t-1,n) + \frac{\beta}{\alpha+\beta} \cdot Z(t-1,n-1).$$

We have equality of

$$Z(t,n) = \mathsf{P}\big(X_t \geqslant t-2n+2\big).$$

This recursion is easily solved in terms of Binomial coefficients.

Asymptotics via contour integrals

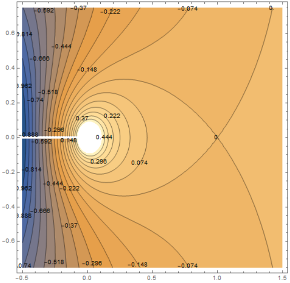
Binomial coefficients can be written in terms of contour integrals:

$$\binom{n}{k} = \frac{1}{2\pi i} \oint_{|z|<1} (1+z)^n z^{-k} \frac{dz}{z}.$$

Can study various asymptotic regimes for n and k.

$$\binom{n}{n/2} = \frac{1}{2\pi i} \oint_{|z|<1} e^{nf(z)} \frac{dz}{z}, \qquad \text{with} \quad f(z) = \log(1+z) - \tfrac{1}{2}\log z.$$

Steepest descent analysis expands around $f(z)$'s critical point $z = 1$.



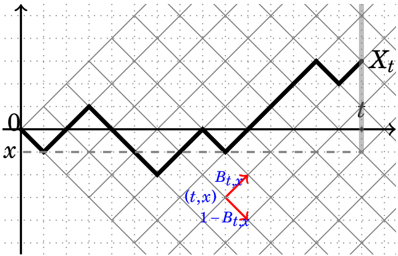
Case 2: Random walks in random environment (RWRE)

Let $B = (B_{t,x})_{t,x}$ be independent random variables with a common fixed distribution on $[0,1]$. Call $\mathbb{P}$ the probability measure on B.

For a given instance of B let P_B denote the probability measure on simple random walks on $\mathbb{Z}$ with left / right jump probabilities

$$\mathsf{P}_B\big(X_{t+1} = x+1 \,\big|\, X_t = x\big) = B_{t,x}, \qquad \mathsf{P}_B\big(X_{t+1} = x-1 \,\big|\, X_t = x\big) = 1 - B_{t,x}.$$

Consider independent P_B-distributed copies $X_t^{(1)}, \ldots, X_t^{(N)}$ of X_t.



CLT and LDP

Theorem (Rassoul-Agha and Seppäläinen, 2004)

Assume $\mathbb{P}\big(0 < B_{t,x} < 1\big) > 0$ *and let* $v = 2\mathbb{E}\big[B_{t,x}\big] - 1$ *and* $\sigma = \sqrt{1 - v^2}$.
Then for $\mathbb{P}$*-almost every choice of jump rates,*

$$\frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{\sigma \sqrt{n}} \xrightarrow[n\to\infty]{\text{as a process in } t} BM(t).$$

Theorem (Rassoul-Agha, Seppäläinen and Yilmaz, 2013)

Assume $\mathbb{E}\Big[\big(\log(B_{t,x})\big)^3\Big] < \infty$. *Then* $\lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log\Big(\mathsf{E}_B\big[e^{zX_t}\big]\Big)$ *exists and is constant* $\mathbb{P}$*-almost surely. For* $I(x)$ *the Legendre transform of* $\lambda(z)$

$$\frac{\log\Big(\mathsf{P}_B\big(X_t > xt\big)\Big)}{t} \xrightarrow[t\to\infty]{\mathbb{P}-\text{almost surely}} -I(x).$$

- Finding an explicit formula for $\lambda(z)$ or $I(x)$ is generally not possible.
- Random rate $I(x) \geq$ deterministic rate $I(x)$ (by Jensen's inequality).
- Lower order fluctuations of $\mathsf{P}_B\big(X_t > xt\big)$ are lost in this result.

Integrable probability to the rescue

In a lab, how could we distinguish deterministic or random media?

- × Extreme value **speed** depends non-universally on the underlying random walk model or media.
- ✓ Extreme value **fluctuations** have different behaviors than in the deterministic and random cases. (See below!)

Definition

The **Beta RWRE** has *Beta*(α, β)-distributed jump probabilities $B_{t,x}$:

$$\mathbb{P}\big(B_{t,x} \in [y, y + \mathrm{d}y]\big) = y^{\alpha-1}(1-y)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{d}y.$$

If $\alpha = \beta = 1$, we recover the uniform distribution on $[0, 1]$.

Aim

We will show how to compute the distribution of $\mathsf{P}_B(X_t \geq x)$ *exactly.*

Large deviations and cube-root fluctuations

For simplicity lets take $\alpha = \beta = 1$ (i.e. $B_{t,x}$ uniform on $[0,1]$).

Theorem (Barraquand-C '15)

For $B_{t,x}$ uniform on $[0,1]$, the large deviation principle rate function is

$$\lim_{t\to\infty} -\frac{\log\Big(\mathsf{P}_B\big(X_t > xt\big)\Big)}{t} = I(x) = 1 - \sqrt{1-x^2}.$$

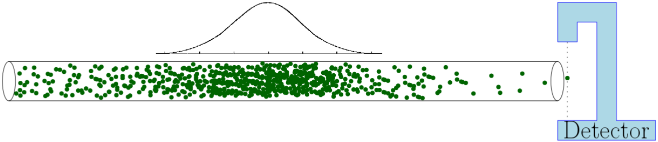
Moreover, as $t \to \infty$, we have convergence in distribution of

$$\frac{\log\Big(\mathsf{P}_B\big(X_t > xt\big)\Big) + I(x)t}{\sigma(x)\cdot t^{1/3}} \quad \Longrightarrow \quad \mathcal{L}_{GUE},$$

where $\mathcal{L}_{GUE}$ is the GUE Tracy-Widom distribution, and $\sigma(x)^3 = \frac{2I(x)^2}{1-I(x)}$.

Cube-root $\mathcal{L}_{GUE}$ fluctuations are a hallmark of random matrix theory and the Kardar-Parisi-Zhang universality class.

Extreme value fluctuations



Corollary (Barraquand-C '15)

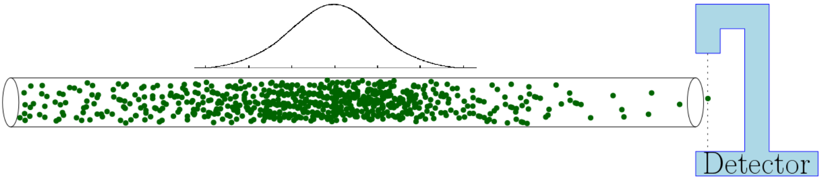
For $B_{t,x}$ uniform on $[0,1]$, let $X_t^{(1)},\dots,X_t^{(N)}$ be random walks drawn independently according to P_B. For $N=e^{ct}$ with $c\in(0,1)$,

$$\frac{\max_{i=1}^{N}\left\{X_t^{(i)}\right\}-t\sqrt{1-(1-c)^2}}{d(c)\cdot t^{1/3}} \quad\Longrightarrow\quad \mathscr{L}_{GUE}.$$

Compare $\max_{\mathrm{r(andom\ probabilities)}}$ to $\max_{\mathrm{d(eterministic\ probabilities)}}$:

- $\max_{\mathrm{r}}$ has a slower speed than $\max_{\mathrm{d}}$ (the random $B_{t,x}$ routes many walkers along the same path and hence decreases entropy).
- $\max_{\mathrm{r}}$ fluctuates $O(t^{1/3})$ versus $O(1)$ for $\max_{\mathrm{d}}$.

Diffusion in a (random) media



Many small particles moving in a viscous media:

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Two models for such systems:

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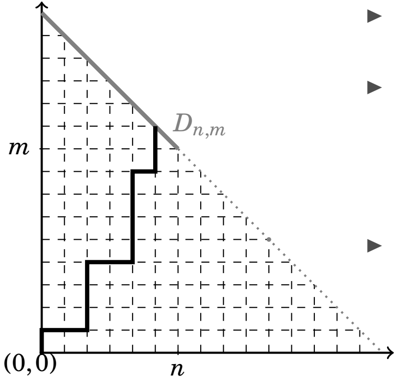
Walking across a city (the $\alpha, \beta \to 0$ limit)



- For every edge, let E_e be i.i.d. exp(1) and for each vertex $\xi_{i,j}$ i.i.d. Bernoulli(1/2).
- Define the passage time of an edge

$$t_e = \begin{cases} \xi_{i,j} E_e \text{ if vertical } (i,j) \to (i,j+1), \\ (1-\xi_{i,j}) E_e \text{ if horizontal } (i,j) \to (i+1,j). \end{cases}$$

- Define the first passage-time $T(n,m)$ from $(0,0)$ to the half-line $D_{n,m}$ by

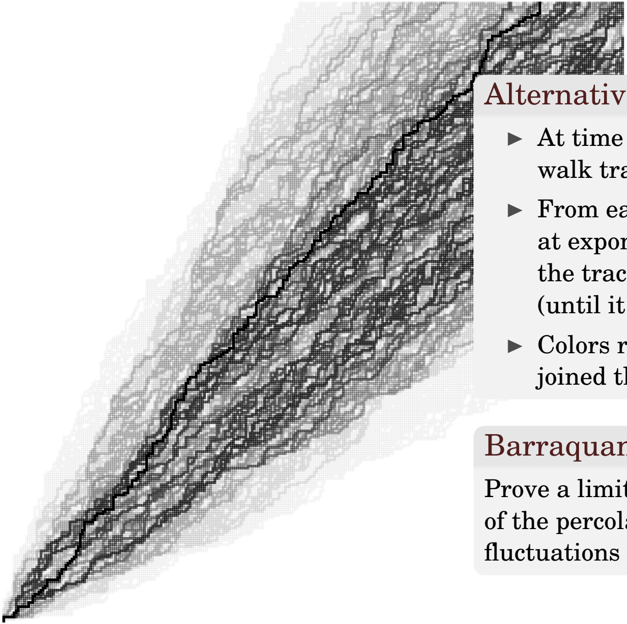
$$T(n,m) = \min_{\pi:(0,0)\to D_{n,m}} \sum_{e\in\pi} t_e.$$

Theorem

For any $\kappa > 1$, there are explicit functions $\rho(\kappa)$ and $\tau(\kappa)$ such that

$$\frac{T(n,\kappa n) - \tau(\kappa) n}{\rho(\kappa) n^{1/3}} \implies \mathscr{L}_{GUE}.$$

Dynamical construction of percolation cluster



Alternative description

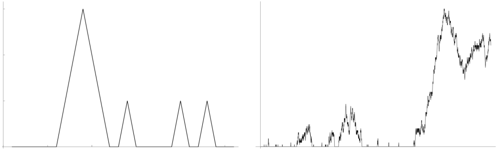
- At time 0, only one random walk trajectory (in black).
- From each point in the cluster, at exponential rate one we add the trace of a new random walk (until it rejoins the cluster).
- Colors represent when a point joined the cluster.

Barraquand-Rychnovsky ’18

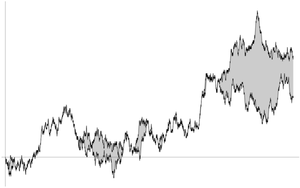
Prove a limit theorem for the shape of the percolation cone and that its fluctuations have a 4/9 exponent!

Sticky Brownian motion (another $\alpha, \beta \to 0$ limit)

Brownian motion **sticky at the origin** (Feller '52): Random walk away from origin; at origin, escape with probability $n^{-1/2}$



A pair of sticky Brownian motions has difference sticky at the origin.



***N*-particle sticky Brownian motion**: Diffusive limit of N particles in the same random environment, when the $B_{t,x}$ are close to 0 or 1.

Need to specify rate for clusters of $k+\ell$ particles to "split" into separate clusters of size k and ℓ. Rate for limit of Beta RWRE is $\frac{k+\ell}{k\ell}$. Barraquand-Rychnovsky '19 prove KPZ extreme value results for this model.

KPZ equation limit

Theorem (C-Gu '16)

Consider the RWRE with $B_{t,x} = \frac{1}{2}(1+\epsilon^{1/2} w_{t,x})$ *for i.i.d. bounded, mean zero* $w_{t,x}$*. Fix any velocity* $v \in (0,1)$*, and any* $t > 0$ *and* $x \in R$*. Then*

$$\frac{\epsilon^{-1}}{2} e^{\epsilon^{-2} t I(v) + \epsilon^{-1} x J(v)} \mathsf{P}_B\left(X_{\epsilon^{-2}t} = \epsilon^{-2} vt + \epsilon^{-1} x\right) \Longrightarrow \mathbf{U}(\mathbf{t},\mathbf{x}),$$

where U *solves the multiplicative stochastic equation equation*

$$\partial_{\mathbf{t}} \mathbf{U}(\mathbf{t},\mathbf{x}) = \frac{1-v^2}{4} \cdot \partial_{\mathbf{xx}} \mathbf{U}(\mathbf{t},\mathbf{x}) + v^2 \mathbb{E}[w^2] \cdot \mathbf{U}(\mathbf{t},\mathbf{x}) \xi(\mathbf{t},\mathbf{x})$$

with space time white noise ξ *and initial data* $\mathbf{U}(\mathbf{0},\mathbf{x}) = \delta_{x=0}$*. Here*

$$I(v) = \frac{1-v}{2} \log\left(\frac{1-v}{1+v}\right) + \log(1+v), \quad \textit{and} \qquad J(v) = \frac{1}{2} \log\left(\frac{1+v}{1-v}\right).$$

The logarithm of the SHE solves the KPZ equation!

A first step into integrable probability

The following result shows that this model is exactly solvable:

Proposition (Barraquand-C '15)

For $t, n, k \geqslant 1$,

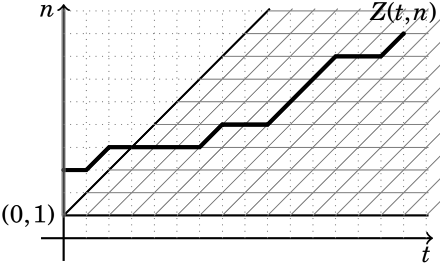
$$\mathbb{E}\left[\mathsf{P}_B\big(X_t \geqslant t-2n+2\big)^k\right] =$$

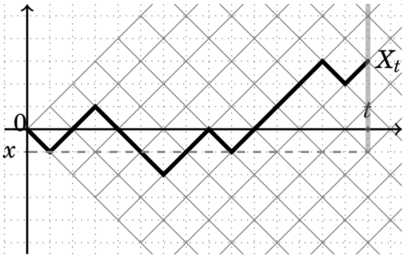
$$\frac{1}{(2i\pi)^k}\int\cdots\int \prod_{1\leqslant A<B\leqslant k}\frac{z_A-z_B}{z_A-z_B-1}\prod_{j=1}^{k}\left(\frac{\alpha+\beta+z_j}{z_j}\right)^n\left(\frac{\alpha+z_j}{\alpha+\beta+z_j}\right)^t\frac{\mathrm{d}z_j}{\alpha+\beta+z_j}$$

where the contour for z_k is a small circle around the origin, and the contour for z_j contains the contour for $z_{j+1}+1$ for all $j = 1,\dots,k-1$, as well as the origin, but all contours exclude $-\alpha-\beta$.

Since $\mathsf{P}_B \in [0,1]$, its moments uniquely identify its distribution. Combining these into a formula for $\mathbb{E}\left[e^{u\mathsf{P}_B(X_t\geq x)}\right]$ we may extract asymptotics.

Random recurrence





Recurrence formula

Define a function $Z(t,n)$ via the recursion (with $Z(0,n) = \mathbf{1}_{n\geq 1}$)

$$Z(t,n) = B_{t,n} \cdot Z(t-1,n) + (1-B_{t,n}) \cdot Z(t-1,n-1).$$

For fixed t, n, we have equality in law of

$$Z(t,n) = \mathsf{P}_B\big(X_t \geqslant t - 2n + 2\big).$$

Recursion for moments

$$Z(t,n) = B_{t,n} \cdot Z(t-1,n) + (1-B_{t,n}) \cdot Z(t-1,n-1).$$

We wish to compute formulas for moments of $Z(t,n)$, and more generally

$$u(t,\vec{n}) := \mathbb{E}\big[Z(t,n_1)Z(t,n_2)\cdots Z(t,n_k)\big].$$

When $k=1$, u satisfies $u(t+1,n) = \frac{\alpha}{\alpha+\beta} \cdot u(t,n) + \frac{\beta}{\alpha+\beta} \cdot u(t,n-1)$.

True evolution equation for general k

For $\vec{n} = (n,\ldots,n)$

$$\begin{aligned} u(t+1,\vec{n}) &= \sum_{j=0}^{k} \binom{k}{j} \mathbb{E}\Big[B^j (1-B)^{k-j} Z(t,n)^j Z(t,n-1)^{k-j}\Big] \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{(\alpha)_j (\beta)_{k-j}}{(\alpha+\beta)_k}\, u\big(t,(n,\ldots,n,n-1,\ldots,n-1)\big). \end{aligned}$$

where B is $Beta(\alpha,\beta)$ distributed and $(a)_k = a(a+1)\ldots(a+k-1)$.

Non-commutative binomial identity

For general $\vec{n} \in \mathbb{W}^k = \{\vec{n} \in \mathbb{Z}^k : n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k\}$, we find that

$$u(t+1,\vec{n}) = \mathscr{L} u(t,\vec{n}),$$

where $\mathscr{L}$ acts on functions from $\mathbb{W}^k \to \mathbb{C}$ as the direct sum of the previous action on each cluster of equal coordinates in $\vec{n}$.

Lemma (Rosengren '00, Povolotsky '13)

Let X, Y generate an associative algebra such that

$$XX + (\alpha+\beta-1)XY + YY - (\alpha+\beta+1)YX = 0.$$

Then we have the following non-commutative binomial identity:

$$\left(\frac{\alpha}{\alpha+\beta}X + \frac{\beta}{\alpha+\beta}Y\right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{(\alpha)_j(\beta)_{k-j}}{(\alpha+\beta)_k} X^j Y^{k-j}.$$

Factorizing $\mathscr{L}$

Let $\tau^{(i)}$ act on a function $f(\vec{n})$ by changing n_i to $n_i - 1$.
Define the operator L on functions $f : Z^k \to \mathbb{C}$ by $(X \mapsto 1, Y \mapsto \tau)$

$$\mathsf{L} = \prod_{i=1}^{k} \left(\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} \tau^{(i)} \right)$$

This equals $\mathscr{L}$ for $\vec{n}$ strictly in $\mathbb{W}^k$.

Define the boundary condition

$$B^{(i,i+1)} = 1 + (\alpha+\beta-1)\tau^{(i+1)} + \tau^{(i)}\tau^{(i+1)} - (1+\alpha+\beta)\tau^{(i)}.$$

Corollary

Any function $u : Z^k \to \mathbb{C}$ which satisfies (for all $1 \le i \le k-1$)

$$B^{(i,i+1)} u(\vec{n}) \Big|_{n_i = n_{i+1}} = 0$$

has, for all $\vec{n} \in \mathbb{W}^k$,

$$\mathsf{L} u(\vec{n}) = \mathscr{L} u(\vec{n}).$$

Moment formula

It is now easy to check the following formula.

Proposition (Barraquand-C '15)

For $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$,

$$\mathbb{E}\Big[Z(t,n_1)\cdots Z(t,n_k)\Big] =$$

$$\frac{1}{(2i\pi)^k}\int\cdots\int \underbrace{\prod_{1\leqslant A<B\leqslant k}\frac{z_A-z_B}{z_A-z_B-1}}_{\text{boundary condition}}\underbrace{\prod_{j=1}^{k}\left(\frac{\alpha+\beta+z_j}{z_j}\right)^{n_j}\left(\frac{\alpha+z_j}{\alpha+\beta+z_j}\right)^{t}}_{\text{solution of } u(t+1)=\mathsf{L}u(t)}\underbrace{\frac{\mathrm{d}z_j}{\alpha+\beta+z_j}}_{\text{initial condition}}$$

where the contour for z_k *is a small circle around the origin, and the contour for* z_j *contains the contour for* $z_{j+1}+1$ *for all* $j=1,\dots,k-1$*, as well as the origin, but all contours exclude* $-\alpha-\beta$*.*

Stochastic quantum integrable systems

Beta RWRE: moments solved a closed evolution equation which could be “factorized” and solved explicitly via contour integrals.

KPZ equation / SHE: moments solve the δ-Bose gas which is explicitly diagonalizable via Bethe ansatz (see, e.g. Kardar ’87).

These are special cases of a general theory of *stochastic vertex models* which come from the theory of quantum integrable systems.

- Model $\rightsquigarrow$ transfer matrix for representations of $U_q(\widehat{\mathfrak{sl}_2})$ R matrix.
- Moment evolution equation $\rightsquigarrow$ Markov self duality.
- Moment formulas $\rightsquigarrow$ Bethe ansatz eigenfunctions

$$\psi_{\vec{n}}(\vec{z};t,\nu) := \sum_{\sigma\in S_k} \prod_{1\leq a<b\leq k} \frac{z_{\sigma(b)} - t z_{\sigma(a)}}{z_{\sigma(b)} - z_{\sigma(a)}} \prod_{j=1}^{k} \left(\frac{1-\nu z_{\sigma(j)}}{1-z_{\sigma(j)}}\right)^{n_j}$$

and Plancherel theory (i.e., completeness and orthogonality).

Summary

Physics goal: Study the effect of space-time random jump probabilities on the behavior of random walks in one dimension.

- Bulk behaviors are unchanged from deterministic case.
- Extreme value statistics show different scaling and statistics (connected to Kardar-Parisi-Zhang universality class).
- This is only demonstrated for special Beta distribution case.

Math goal: Use quantum integrable system tools in probability.

- Relate to a random recurrence relation whose moments solve a Bethe ansatz diagonalizable evolution equation.
- Utilize moment formulas to compute the distribution (and subsequently perform asymptotics).
- Connect to theory of stochastic vertex models.

Tomorrow we will further study stochastic vertex models.