$L^{\alpha-1}$ distance between two one-dimensional stochastic differential equations driven by a symmetric α-stable process

Takuya Nakagawa

Department of Mathematical Sciences Ritsumeikan University

— @Kyushyu University
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Main result

Let X and $\widetilde{X}$ be solutions of the one-dimensional SDEs. For $t \geq 0$,

$$X_t = x_0 + \int_0^t \sigma(X_{s-})\mathrm{d}Z_s \quad \widetilde{X}_t = \widetilde{x}_0 + \int_0^t \widetilde{\sigma}(\widetilde{X}_{s-})\mathrm{d}Z_s \quad x_0, \widetilde{x}_0 \in \mathbb{R},$$

- $0 < \inf_{x\in\mathbb{R}} \sigma(x)$, $\qquad (Z_t)_{t\in[0,T]}$ is SαS process, $\alpha \in (1,2)$.
- $\sigma \in C_b^{\eta}$, $\eta \in (0,1]$
- $\widetilde{\sigma} \in C_b^{\gamma}$, $\gamma \in [1/\alpha, 1]$
- $\|\sigma - \widetilde{\sigma}\| := \left(\int_{\mathbb{R}} |\sigma(x) - \widetilde{\sigma}(x)|^{\alpha}(|x - x_0|^{-1-\alpha} \wedge 1)\mathrm{d}x\right)^{1/\alpha} \leq 1$

Lemma (Kulik (2014))

X_t has a probability density function $y \mapsto p_t(x_0, y)$ such that

$$p_t(x_0, y) \leq Ct^{-\frac{1}{\alpha}}\left(t^{1+\frac{1}{\alpha}} \vee 1\right)\left(|y - x_0|^{-1-\alpha} \wedge 1\right).$$

Theorem

$$\sup_{0\le t\le T}\mathbb{E}\left[|X_t-\widetilde{X}_t|^{\alpha-1}\right]\le|x_0-\widetilde{x}_0|^{\alpha-1}+\begin{cases}C\|\sigma-\widetilde{\sigma}\|^{\frac{\alpha\gamma-1}{\gamma}} & \gamma\in(1/\alpha,1],\\ C\left(\log\frac{1}{\|\sigma-\widetilde{\sigma}\|}\right)^{-1} & \gamma=1/\alpha.\end{cases}$$

$$h\mathbb{P}\left(\sup_{0\le t\le T}|X_t-\widetilde{X}_t|^{\alpha-1}>h\right)\le|x_0-\widetilde{x}_0|^{\alpha-1}$$
$$+\begin{cases}C\|\sigma-\widetilde{\sigma}\|^{\frac{\alpha\gamma-1}{\gamma}} & \gamma\in(1/\alpha,1],\\ C\left(\log\frac{1}{\|\sigma-\widetilde{\sigma}\|}\right)^{-1} & \gamma=1/\alpha.\end{cases}$$

Remark

The solution of SDE $(X_t)_{t\in[0,T]}$ fails the pathwise uniqueness property if $\eta<1/\alpha$. Still, the theorem holds for such solutions.