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STOCHASTIC ANALYSIS OF INFINITE PARTICLE SYSTEMS — A NEW DEVELOPMENT IN CLASSICAL STOCHASTIC ANALYSIS AND DYNAMICAL UNIVERSALITY OF RANDOM MATRICES

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Dedicated to the memory of Hiroshi Kunita

Abstract. We examine the stochastic dynamics of infinite particle systems moving in $\mathbb{R}^d$. The classical stochastic analysis pursues the stochastic dynamics of a single particle. We explain how to extend the classical stochastic analysis when the object becomes an infinite particle system. The equilibrium states of a system of infinite number of particles resulting from random matrices have the logarithmic function as an interaction potential. Hence, we develop a theory that applies to interaction potentials having a dominating influence over long distances. As an application, we show that the universality of point processes related to random matrices holds for stochastic dynamics.

1. Beginnings of the general theory of interacting Brownian motion

Consider an infinite number of Brownian particles moving in Euclidean space $\mathbb{R}^d$ with interacting potential $\Psi:\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$. We denote the position of the particles at time t by $\mathbf{X}_t = (X^i_t)_{i\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$. Then $\mathbf{X}_t$ is given by the following stochastic differential (integral) equation:

$$X^i_t - X^i_0 = B^i_t - \frac{\beta}{2}\int_0^t \sum_{j\neq i}^{\infty} \nabla\Psi(X^i_u - X^j_u)dt \quad (i\in\mathbb{N}). \tag{1.1}$$

Here, β denotes a positive constant called the inverse temperature, $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ the $(\mathbb{R}^d)^{\mathbb{N}}$-valued, $\{\mathcal{F}_t\}$-Brownian motion, and $\mathbf{X}$ a continuous, stochastic process defined on a quadruple $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\})$. By definition, $\mathbf{X}$ is a function of time t and $\omega\in\Omega$. We write $\mathbf{X} = \mathbf{X}_t = \mathbf{X}(t) = \mathbf{X}_t(\omega) = \mathbf{X}(t,\omega)$, frequently omitting ω in $\mathbf{X}$.

We assume the probability space $(\Omega,\mathcal{F},P)$ is complete, and $\{\mathcal{F}_t\}$ is a right continuous, increasing family of σ-fields containing P-null sets. The $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion is an infinite number of independent copies of the $\mathbb{R}^d$-value Brownian motion starting at the origin. $\mathbf{B}$ is $\{\mathcal{F}_t\}$-Brownian motion, that is, $\mathbf{B}$ describes Brownian motion such that $\mathbf{B}_t$ is $\mathcal{F}_t$-measurable for each t, and $\mathbf{B}_t - \mathbf{B}_s$ is independent of $\{\mathcal{F}_t\}$ for each $t\geq s$.

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The meaning of (1.1) needs care because the coefficient on the right-hand side contains an infinite sum. We consider a set $\mathbf{S}$ such that the infinite sum

$$\sum_{j=1,\, j\neq i}^{\infty} \nabla\Psi(s_i - s_j)$$

makes sense for each $i \in \mathbb{N}$. Moreover, we have to solve equation (1.1) subject to constraints such that $\mathbf{X}$ does not exit from $\mathbf{S}$. How to capture and construct $\mathbf{S}$ is the critical problem. A possible method is first introducing a measure $\check{\mu}$ on $(\mathbb{R}^d)^{\mathbb{N}}$, and then taking $\mathbf{S}$ as a set such that $\check{\mu}(\mathbf{S}^c) = 0$. Indeed, if the number of particles is finite, the N-particle stochastic differential equation

$$X_t^i - X_0^i = B_t^i - \frac{\beta}{2}\int_0^t \sum_{j\neq i}^{N} \nabla\Psi(X_u^i - X_u^j)dt \quad (i = 1, \ldots, N) \tag{1.2}$$

can be solved (under natural assumptions such as Lipschitz continuity of the coefficients). The solution $\mathbf{X} = (X^1, \ldots, X^N)$ becomes symmetric with respect to the measure

$$\check{\mu}^N(ds_1 \cdots ds_N) = m_N(s_1 \cdots s_N)ds_1 \cdots ds_N, \tag{1.3}$$

where the density m_N is given by

$$m_N(s_1, \ldots, s_N) = \exp\{-\beta \sum_{i<j}^{N} \Psi(s_i - s_j)\}.$$

We construct our solution starting at $\check{\mu}^N$-a.s. $(s_1, s_2, \ldots, s_N) \in (\mathbb{R}^d)^N$. Hence, if we were to have a measure $\check{\mu}^\infty$ that has $\check{\mu}^N$ as a limit as N goes to infinity, then we could take the same strategy as the finite particle case. However, this is not the case. There exists no such measure (even if $\Psi = 0$).

The relationship between (1.2) and (1.3) in finite-dimensions corresponds to the fact that $m_N dx$ is an invariant measure of the semi-group with generator

$$\frac{1}{2}\Delta + \frac{1}{2}\nabla \log m_N \cdot \nabla.$$

Let us consider the same thing for infinite particle systems. Then invariant measures contain an infinite product of Lebesgue measures, which does not make sense. The classical theory of Gibbs measures resolved this difficulty in constructing the equilibrium state for the interaction potentials Ψ of Ruelle's class, which is a set of well-behaved interaction potentials enjoying super stability and integrability at infinity. We take the configuration space over $\mathbb{R}^d$ instead of $(\mathbb{R}^d)^{\mathbb{N}}$ and construct point process related to the potential μ via the Dobrushin–Lanford–Ruelle (DLR) equation. Here, the configuration space is a space of unlabeled particles in $\mathbb{R}^d$, and a point process is a probability measure on the configuration space. The primary examples considered in the present article are logarithmic potentials, or, more generally, the Coulomb potentials. The DLR equation does not make sense for these interaction potentials because of the non-integrability of the interaction at infinity. Therefore, we can not apply classical theory to construct the equilibrium states of these examples, and it thus poses a significant difficulty.

We expressed the infinite-dimensional stochastic differential equation (ISDE) in integral form in (1.1); however, usually, it is expressed in differential form,

$$dX_t^i = dB_t^i - \frac{\beta}{2}\sum_{j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt \quad (i \in \mathbb{N}). \tag{1.4}$$

We note that the meaning of (1.4) is contained in (1.1).

The study of (1.4) was initiated by Lang [32, 33] for $\mathbb{R}^d$ with $d \in \mathbb{N}$. He solved instances when $\Psi \in C_0^3(\mathbb{R}^d)$ (see also Shiga [56]). Fritz[10] constructed a non-equilibrium solution for $d \geq 4$. Tanemura solved the interacting Brownian motion of hard-core particles [60].

In finite dimensions, the standard method to solve the stochastic differential equations involves using the Itô scheme. That is, similar to ordinary differential equations, we take the Picard approximation based on Brownian motion. Hence, we need Lipschitz continuity of the coefficients at least locally. In infinite dimensions, we can not expect that Lipschitz continuity of the coefficients holds and, even if we could localize the coefficients, it would become very complicated. A feasible procedure would be to solve the equation for N-particle systems and take the limit of the solutions as $N \to \infty$. To undertake this procedure, we need to perform a robust estimate but the manipulations are tedious.

Lang succeeded in obtaining this for potentials $\Psi \in C_0^3(\mathbb{R}^d)$ combining estimates for the grand canonical Gibbs measures. However, for polynomial decay potentials Ψ, one can not solve the ISDE by Lang's method even if Ψ is in Ruelle's class.

Generally, we equip the ISDEs with an infinite number of coefficients σ_i and b_i $(i \in \mathbb{N})$ presenting them in the form

$$\begin{aligned} dX_t^1 &= \sigma_1(\mathbf{X}_t)dB_t^1 + b_1(\mathbf{X}_t)dt \\ dX_t^2 &= \sigma_2(\mathbf{X}_t)dB_t^2 + b_2(\mathbf{X}_t)dt \\ dX_t^3 &= \sigma_3(\mathbf{X}_t)dB_t^3 + b_3(\mathbf{X}_t)dt \\ &\cdots . \end{aligned} \tag{1.5}$$

If the (σ_i, b_i) converge sufficiently fast to $(0,0)$ as $i \to \infty$, then (according to the speed of convergence) we can solve ISDE (1.5) as usual for the (finite-dimensional) stochastic differential equations. In the present case, the problem is that the coefficients (σ_i, b_i) have symmetry and, as a result, they do not converge to $(0,0)$ as $i \to \infty$. That is, by a pair of functions $(\sigma, b) : \mathbb{R}^d \times \mathsf{S} \to (\mathbb{R}^{d^2} \times \mathbb{R}^d) \cup \{\infty\}$ the ISDE is given by

$$\begin{aligned} dX_t^1 &= \sigma(X_t^1, \mathsf{X}_t^{1\diamond})dB_t^1 + b(X_t^1, \mathsf{X}_t^{1\diamond})dt \\ dX_t^2 &= \sigma(X_t^2, \mathsf{X}_t^{2\diamond})dB_t^2 + b(X_t^2, \mathsf{X}_t^{2\diamond})dt \\ dX_t^3 &= \sigma(X_t^3, \mathsf{X}_t^{3\diamond})dB_t^3 + b(X_t^3, \mathsf{X}_t^{3\diamond})dt \\ &\cdots . \end{aligned}$$

Here, we set $\mathsf{X}_t^{i\diamond} = \sum_{j\neq i}^{\infty} \delta_{X_t^j}$. We emphasize again that the function $(\sigma, b)(x, \mathsf{s})$ does not depend on the label of the particle $i \in \mathbb{N}$. Such symmetries provide a natural framework for problems on statistical mechanics when treating a huge number of identical or finite species of particles. As we have seen above, symmetries are a barrier to using the conventional approach to solving ISDEs. However, they enable us to regard the object as a system in configuration space with the notion of unlabeled dynamics $\mathsf{X} = \sum_{i=1}^{\infty} \delta_{X^i}$ being available. The unlabeled dynamics X has an invariant probability measure (equilibrium state) μ, and thus a geometric stochastic analysis, the Dirichlet form theory, becomes effective.

In Section 5, we shall introduce the first theory, which solves the ISDEs for which the equilibrium states μ are a wide class of point processes. We construct solutions starting at $\mathbf{s} = (s_1, s_2, \ldots) \in (\mathbb{R}^d)^{\mathbb{N}}$ for μ-a.s. $\mathsf{s} = \sum_i \delta_{s_i}$. The solutions are of the form $(\mathbf{X}, \mathbf{B})$, that is, a pair of $\mathbf{X}$ and Brownian motion $\mathbf{B}$. This is a weak solution, and the first theory does not solve the uniqueness of the solution. Loosely speaking, a weak solution $(\mathbf{X}, \mathbf{B})$ is called a strong solution if $\mathbf{X}$ is a functional of $\mathbf{B}$. That is, if there exists a function F from the path space of Brownian motion to that of the solution and $\mathbf{X} = F(\mathbf{B})$, then $(\mathbf{X}, \mathbf{B})$ is called a strong solution. In Section 6, we shall introduce the second theory, which proves the pathwise uniqueness and the existence of strong solutions.

2. Set up and examples.

Let $S = \mathbb{R}^d$ and $S_r = \{|s| \le r\}$. The space of infinite particle systems on $\mathbb{R}^d$ is $(\mathbb{R}^d)^{\mathbb{N}}$, which is enormous. We introduce then the configuration space S over $\mathbb{R}^d$:

$$\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\,;\, \mathsf{s}(S_r) < \infty \quad \text{for all } r \in \mathbb{N}\}.$$

By definition, S denotes the set of Radon measures consisting of a finite or infinite sum of point measures. We equip S with the vague topology, under which, S is a Polish space. That is, S is homeomorphic to a complete and separable metric space. We call a probability measure μ on $(\mathsf{S}, \mathcal{B}(\mathsf{S}))$ a point process and also a random point field.

In the study of an ISDE (1.4), an equilibrium state μ plays an important role. By definition, μ becomes a point process. Below we present various examples of ISDEs. We give a point process μ first. Then we proceed with the explicit representation of the associated ISDE.

2.1. **Poisson point process.** For a Radon measure m on S the Poisson point process Poi^m with intensity m is the point process on S such that the distribution of the numbers of the particles on a Borel set A is given by the Poisson distribution

$$\mathrm{Poi}^m(\{\mathsf{s}(A) = k\}) = e^{-m(A)} \frac{m(A)^k}{k!}$$

and random variables π_A and π_B are independent under Poi^m. Here, $\pi_A : \mathsf{S} \to \mathsf{S}$ denotes the projection such that $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$.

Example 2.1 (Infinite-dimensional Brownian motion)**.** Let m be the Lebesgue measure. We write $\mathrm{Poi}^m = \Lambda$. Λ plays the role of a Lebesgue measure on the configuration space S and Λ is an invariant probability measure of the S-valued Brownian motion $\mathsf{B} = \{\mathsf{B}_t\}$ defined by $\mathsf{B}_t = \sum_{i\in\mathbb{N}} \delta_{B_t^i}$. The construction of a $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion $(B^i)_{i\in\mathbb{N}}$ is obvious because we can take B_i to be an independent Brownian motion $\{B^i\}_{i\in\mathbb{N}}$. Then B corresponds to unlabeled dynamics. For each initial starting point $\mathbf{s} = (s_i) \in (\mathbb{R}^d)^{\mathbb{N}}$ we can construct $\mathbf{B}^{\mathbf{s}}$ starting at $\mathbf{s}$ by taking $\mathbf{B}_t^{\mathbf{s}} = (B_t^i + s_i)_{i\in\mathbb{N}}$. Note that the associated unlabeled dynamics B is not necessary S-valued process. For example, if $\mathbf{s} = (0, 0, \ldots)$ is the origin in $(\mathbb{R}^d)^{\mathbb{N}}$, then $\mathsf{B}_t \notin \mathsf{S}$ for each t.

2.2. **Gibbs measures.** Let $\mu_{r,\xi}^n$ be the regular conditional probability of μ such that for $A \in \mathcal{B}(S_r^n)$

$$\mu_{r,\xi}^n(A) = \mu(\pi_{S_r}(\mathsf{x}) \in A | \pi_{S_r^c}(\mathsf{x}) = \pi_{S_r^c}(\xi),\, \mathsf{x}(S_r) = n). \tag{2.1}$$

We refer to [54, 55]. For $\mathsf{S}_r^n = \{\mathsf{s} \in \mathsf{S}; \mathsf{s}(S_r) = n\}$ we set $\Lambda_r^n = \Lambda(\cdot \cap \mathsf{S}_r^n)$. μ is called a (Φ, Ψ)-canonical Gibbs measure if it satisfies the relation (DLR equation).

$$\mu_{r,\xi}^n(d\mathsf{x}) = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}_{r,\xi}} \Lambda_r^n(d\mathsf{x}), \tag{DLR}$$

where $\mathcal{Z}$ denotes a normalizing constant and $\mathcal{H}_{r,\xi} = \mathcal{H}_r + \mathcal{I}_{r,\xi}$. Here, $\mathcal{H}_r$ is the Hamiltonian on S_r, and $\mathcal{I}_{r,\xi}$ is the term of the interaction between the inside and the outside. Moreover, $\beta > 0$ is a constant called the inverse temperature. Then, by definition,

$$\begin{aligned} \mathcal{H}_r(\mathsf{s}) &= \beta\{\sum_{i=1}^n \Phi(s_i) + \sum_{i<j,\, s_i, s_j \in S_r} \Psi(s_i - s_j)\}, \\ \mathcal{I}_{r,\xi} &= \beta \sum_{s_i \in S_r,\, \xi_k \in S_r^c} \Psi(s_i - \xi_k). \end{aligned} \tag{2.2}$$

Generally speaking, ISDEs with free and interaction potentials are of the form

$$dX_t^i = dB_t^i - \frac{\beta}{2} \nabla\Psi(X_t^i) dt - \frac{\beta}{2} \sum_{j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N}).$$

We present three examples of stochastic dynamics with equilibrium states given by Gibbs measures that are translation invariant.

Example 2.2 (Lennard-Jones 6-12 potential [45, 49])**.** Let $d = 3$ and $\Psi_{6,12}$ be the Lennard–Jones 6-12 potential given by $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$. The associated ISDE is

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j\neq i}^{\infty} \Big\{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \Big\} dt \quad (i \in \mathbb{N}).$$

Example 2.3 (Riesz potentials of Ruelle's class [45, 49])**.** $d < a$, $\Psi_a(x) = (\beta/a)|x|^{-a}$,

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j\neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}} dt \quad (i \in \mathbb{N}). \tag{2.3}$$

At first glance, ISDE (2.3) resembles ISDEs (2.6) and (2.10). Indeed, (2.3) corresponds to (2.6) and (2.10) with $a = 0$. The drift terms in (2.3) converge absolutely unlike (2.6) and (2.10).

Example 2.4 (Hard core Brownian balls. [60])**.** Let $d \in \mathbb{N}$ and Ψ_R be the hard core potential such that $\Psi_R(x) = \infty \cdot 1_{S_R}(x)$. Then the associated ISDE is

$$dX_t^i = dB_t^i + \sum_{j\neq i}^{\infty} 1_{\{R\}}(|X_t^i - X_t^j|) dL_t^{ij}. \tag{2.4}$$

Here L_t^{ij} are non-decreasing continuous processes with $L_0^{ij} = 0$, $L_t^{ij} = L_t^{ji}$, and

$$L_t^{ij} = \int_0^t 1_{\{R\}}(|X_s^i - X_s^j|) dL_s^{ij}.$$

ISDE (2.4) describes the stochastic dynamics consisting of infinitely many, Brownian particles, all with a hard core of radius R. The stochastic dynamics corresponding to the lattice gas are simple exclusion processes on $\mathbb{Z}^d$, which is the most simple and natural model of a lattice gas obeying Kawasaki dynamics.

2.3. **Determinantal point process.** For a point process μ on S we call a symmetric function $\rho^n : S^n \to [0,\infty)$ the n-point correlation function of μ with respect to Radon measure m if ρ^n satisfies (2.5),

$$\int_{A_1^{k_1}\times\cdots\times A_m^{k_m}} \rho^n(x_1,\dots,x_n)m(dx_1)\cdots m(dx_n) = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i)-k_i)!}d\mu. \tag{2.5}$$

Here, $A_1,\dots,A_m \in \mathcal{B}(S)$, $k_1,\dots,k_m \in \mathbb{N}$, and $k_1+\cdots+k_m = n$. If $\mathsf{s}(A_i)-k_i < 0$, then we interpret $\mathsf{s}(A_i)!/(\mathsf{s}(A_i)-k_i)! = 0$. A determinantal point process μ with kernel $K : S\times S \to \mathbb{C}$ and m is a point process for which the n-point correlation function ρ^n with respect to m is given by

$$\rho^n(x_1,\dots,x_n) = \det[K(x_i,x_j)]_{i,j=1}^n.$$

It is known that for given (K,m) the associated determinantal point process exists uniquely if the operator $Kf(x) = \int K(x,y)f(y)m(dy)$ on $L^2(S,m)$ is Hermitian symmetric with spectrum $\mathrm{Spec}(K) \subset [0,1]$ [34, 57, 58]. Determinantal point processes emerge in unexpected places. For example, we shall introduce the planner Gaussian analytic function (GAF) in Section 10. Although the planner GAF is not a determinantal point process, its counterpart on the Poincaré disk $\{|z|<1\}$ is and is called the hyperbolic GAF. This point process consists of the zero points of the analytic function F_{disk} with random coefficients.

$$F_{\mathrm{disk}}(z) = \sum_{k=0}^{\infty} \xi_k z^k.$$

Here $\{\xi_k\}$ is i.i.d. with distribution $\frac{1}{\pi}e^{-|z|^2}dz$. The formula gives the kernel function of the hyperbolic GAF given in [53],

$$K_{\mathrm{disk}}(x,y) = \frac{1}{\pi}\frac{1}{(x-\bar{y})^2}.$$

In the rest of this section, we shall show three examples arising from random matrix theory. In the following, if $\beta = 2$, then the point processes are determinantal. For $d = 1$, we have a class of point processes referred to as Pfaffian with $\beta = 1,4$, having a structure similar to determinantal point processes. In the following examples, it is not easy to obtain explicit presentations of the formula associated with ISDEs with point processes.

In the Gibbsian case, the potentials are *a priori* given, and the coefficients of the ISDEs are easily determined. It is not apparent the coefficients for these ISDEs exist because of the infinite sums of derivatives of pair interactions in the coefficients. The classical theory has developed the Gibbs measures under the constraints such that such sums converge.

We give the kernel functions first in the following examples. Hence, identifying the associated potentials is required. In the first theory, we have introduced the notion of a logarithmic derivative of the point processes, and we have solved the ISDEs determined by these logarithmic derivatives. In doing so, we encounter two issues: the existence of the logarithmic derivatives and their explicit representations.

Example 2.5 (sine$_\beta$ interacting Brownian motion [59, 45, 63])**.** Let $d = 1$. We take $\Psi(x, y) = -\log|x - y|$. Consider the ISDE,

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}). \tag{2.6}$$

For $\beta = 2$, ISDE (2.6) is called **Dyson's model in infinite-dimensions**. The equilibrium state $\mu_{\sin,\beta}$ of the unlabeled dynamics $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ has translation invariance and the limit in the coefficient does not converge absolutely. Here, $\mu_{\sin,\beta}$ is called the sine$_\beta$ point process. Suppose $\beta = 2$. Then the sine$_2$ point process is determinantal and the kernel function $K_{\sin,2}$ is given by

$$K_{\sin,2}(x - y) = \frac{\sin 2(x - y)}{\pi(x - y)}. \tag{2.7}$$

ISDE (2.6) was solved for $\beta = 1, 2, 4$ in [45, 49] and for $\beta \geq 1$ by Tsai [63].

Example 2.6 (Airy$_\beta$ interaction Brownian motion. [50])**.** Let $d = 1$ and $\beta = 1, 2, 4$. We take the logarithmic potential $\Psi(x, y) = -\log|x - y|$ and consider the ISDE,

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \Big\{ \Big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} \Big) - \frac{2\sqrt{r}}{\pi} \Big\} dt \quad (i \in \mathbb{N}). \tag{2.8}$$

The equilibrium state of the ISDE is $\mu_{\mathrm{Ai},\beta}$ for $\beta = 1, 2, 4$. If $\beta = 2$, then $\mu_{\mathrm{Ai},2}$ is a determinantal point process with continuous kernel given by

$$K_{\mathrm{Ai},2}(x, y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y} \quad (x \neq y).$$

Here $\mathrm{Ai}'(x) = d\mathrm{Ai}(x)/dx$ and $\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{i(zk+k^3/3)}$ $(z \in \mathbb{R})$ is the Airy function. The representation of ISDE (2.8) was given in [50], and (2.8) was solved for $\beta = 1, 2, 4$.

Example 2.7 (Bessel$_{\alpha,\beta}$ interacting Brownian motion. [18])**.** Let $d = 1$ and $S = [0, \infty)$. We take $1 \leq \alpha < \infty$ and $\beta > 0$. We consider the ISDE,

$$dX_t^i = dB_t^i + \{\frac{\alpha}{2X_t^i} + \frac{\beta}{2} \sum_{j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j}\} dt \quad (i \in \mathbb{N}). \tag{2.9}$$

The equilibrium state $\mu_{\mathrm{Be},\alpha,\beta}$ are called the Bessel$_{\alpha,\beta}$ point process. Here, $\beta = 1, 2, 4$. If $\beta = 2$, then $\mu_{\mathrm{Be},\alpha,2}$ is the determinantal point process with continuous kernel such that

$$\mathsf{K}_{\mathrm{Be},\alpha,2}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - \sqrt{x}J_\alpha'(\sqrt{x})J_\alpha(\sqrt{y})}{2(x - y)} \quad (x \neq y).$$

The n-point correlation function $\rho_{\mathrm{Be},\alpha,2}^n$ with respect to the Lebesgue measure on $[0, \infty)$ is

$$\rho_{\mathrm{Be},\alpha,2}^n(\mathbf{x}^n) = \det[\mathsf{K}_{\mathrm{Be},\alpha,2}(x_i, x_j)]_{i,j=1}^n.$$

The ISDE (2.9) is given in [18] for $\beta = 2$.

Example 2.8 (Ginibre interacting Brownian motion [45])**.** Let $d=2$ and $\Psi(x,y) = -\log|x-y|$. We consider two ISDEs, (2.10) and (2.11):

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i\in\mathbb{N}), \tag{2.10}$$

$$dX_t^i = dB_t^i - X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i\in\mathbb{N}). \tag{2.11}$$

The equilibrium states of these ISDEs coincide and are the Ginibre point process μ_{Gin}, which is a determinantal point process with kernel

$$K_{\mathrm{Gin}}(x,y) = \frac{1}{\pi}\exp\{-\frac{1}{2}|x|^2 + x\bar{y} - \frac{1}{2}|y|^2\}.$$

Here we identify $\mathbb{R}^2$ as $\mathbb{C}$ in an obvious manner. Clearly, (2.10) and (2.11) are different ISDEs. Nevertheless, they have the same unique, strong solution on the support of the Ginibre point process μ_{Gin}. For μ_{Gin}-a.s. s, the solution of (2.10) and (2.11) starting at $\mathbf{s} = \mathfrak{l}(\mathsf{s})$ is pathwise unique. Here $\mathfrak{l}\colon\mathsf{S}\to(\mathbb{R}^2D)^{\mathbb{N}}$ is a label.

3. Random matrices and interacting Brownian motions

All the examples in Section 2.3 are related to random matrix theory in such a way that the equilibrium states are limits of the distributions of the eigenvalues of random matrices as $N\to\infty$. There are many random matrices of various kinds. We describe the Gaussian random matrices that have become prototypical examples. A Gaussian random matrix of order N is a square matrix $M^N = [m_{ij}]_{i,j=1}^N$, the elements of which are independent Gaussian random variables subject to symmetry constraints –real/Hermitian/quaternion–in accordance with the orthogonal/unitary/symplectic invariance of the distribution. These random matrices are called GOE, GUE, GSE [3, 9, 37]. Choosing a suitable variance and mean, the distribution of the eigenvalues of M^N is

$$m_\beta^N(d\mathbf{x}_N) = \frac{1}{Z}\{\prod_{i<j}^N |x_i - x_j|^\beta\}\exp\left\{-\frac{\beta}{4}\sum_{k=1}^N |x_k|^2\right\} d\mathbf{x}_N, \tag{3.1}$$

Here GOE, GUE, and GSE correspond to $\beta = 1,2,4$, respectively.

Let $\mathcal{P}$ denote the set consisting of all the probability measures on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. We consider a $\mathcal{P}$-valued random variable

$$\mathbb{X}_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i/\sqrt{N}}$$

under $m_\beta^N(d\mathbf{x}_N)$ and denote the distribution of $\mathbb{X}_N$ by μ_β^N. Let

$$\sigma_{\mathrm{semi}}(x) = \frac{1}{\pi}\sqrt{4-x^2}\,\mathbf{1}_{(-2,2)}(x). \tag{3.2}$$

Then, by definition, $\sigma_{\mathrm{semi}}(x)dx \in \mathcal{P}$. $\sigma_{\mathrm{semi}}(x)dx$ is called the semi-circle distribution. The celebrated Wigner semi-circle law asserts that the distribution $\{\mu_\beta^N\}$ converges weakly to the non-random measure $\delta_{\sigma_{\mathrm{semi}}(x)dx}$.

From the view point of a semi-circle distribution $\sigma_{\mathrm{semi}}(x)dx$, we call each point $\theta\in\mathbb{R}$ a macro position. For each macro position $\theta\in\mathbb{R}$ of (3.2) we may take an effective scaling of (3.1). This is possible for $|\theta| < 2$ and $\theta = \pm 2$; the former is called the bulk, and the latter a soft edge.

3.1. **A bulk limit and small universality.** For a macro position θ such that $\{|\theta| < 2\}$, we rescale (3.1) with

$$x_i \mapsto \frac{s_i + \theta N}{\sqrt{N}}.$$

Then the distribution of $m_\beta^N(d\mathbf{s}_N) = m_\beta^N(\mathbf{s}_N)d\mathbf{s}_N$ is

$$m_\beta^N(d\mathbf{s}_N) = \frac{1}{Z}\{\prod_{i<j}^{N}|s_i - s_j|^\beta\}\exp\left\{-\frac{\beta}{4}\sum_{k=1}^{N}|\frac{s_i+\theta N}{\sqrt{N}}|^2\right\}d\mathbf{s}_N. \tag{3.3}$$

We denote by $\mu_{\beta,\theta}^N$ the corresponding distribution of the configuration space S. The limit $\mu_{\beta,\theta}$ is called the sine$_{\beta,\theta}$ point process. If $\beta = 2$, then $\mu_{2,\theta}$ is a determinantal point process with kernel

$$K_\theta(x,y) = \frac{\sin\{\sqrt{4-\theta^2}(x-y)\}}{\pi(x-y)}.$$

Here, the instance $\theta = 0$ was treated in (2.7). Each sine$_{\beta,\theta}$ point process $\mu_{\beta,\theta}$ is a constant scaling in the space of $\mu_{2,\theta}$. Therefore, the bulk limit has a (very small) universality.

We next focus on the dynamical counterpart of universality given above. We first introduce the Dirichlet form describing the reversible stochastic dynamics of N particles with equilibrium state $m_\beta^N(d\mathbf{s}_N) = m_\beta^N(\mathbf{s}_N)d\mathbf{s}_N$. Let

$$\mathcal{E}^{m_\beta^N}(f,g) = \int_{\mathbb{R}^N}\frac{1}{2}\sum_{i=1}^{N}\frac{\partial f}{\partial s_i}\frac{\partial g}{\partial s_i}m_\beta^N d\mathbf{s}_N$$

on the L^2 space

$$L^2(\mathbb{R}^N, m_\beta^N).$$

Because the measures of the energy form and the L^2 space are common, the Dirichlet form is called a distorted Brownian motion. We calculate the generator of the Dirichlet space using integration by parts,

$$\mathcal{E}^{m_\beta^N}(f,g) = -\int_{\mathbb{R}^N}\Big\{\frac{1}{2}\Delta f + \frac{1}{2}\sum_{i=1}^{N}\frac{\partial \log m_\beta^N}{\partial s_i}\frac{\partial f}{\partial s_i}\Big\}g m_\beta^N d\mathbf{s}_N. \tag{3.4}$$

Then, by a straightforward calculation, we have from (3.3)

$$\frac{\partial \log m_\beta^N(\mathbf{s}_N)}{\partial s_i} = \beta\sum_{j\neq i}^{N}\frac{1}{s_i - s_j} - \frac{\beta}{N}s_i - \beta\theta. \tag{3.5}$$

Hence, we deduce from (3.5) the stochastic differential equation of $\mathbf{X}^N = (X^{N,i})_{i=1}^N$,

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{N}\frac{1}{X_t^{N,i} - X_t^{N,j}}dt - \frac{\beta}{2N}X_t^{N,i}dt - \frac{\beta}{2}\theta dt. \tag{3.6}$$

Taking $N \to \infty$ in (3.6), we arrive at the ISDE in the limit

$$dX_t^{\infty,i} = dB_t^i + \frac{\beta}{2}\sum_{j\neq i}^{\infty} \frac{1}{X_t^{\infty,i} - X_t^{\infty,j}}dt - \frac{\beta}{2}\theta dt. \tag{3.7}$$

The ISDE however does not provide a correct answer except for $\theta = 0$. Indeed, we prove,

Theorem 3.1 (Kawamoto-O.[27])**.** *Let $\beta = 2$. Take the initial distribution of the unlabeled particles as $\mu_{2,\theta}^N$ and choose the label suitably. Then for each $m \in \mathbb{N}$ the distributions of the first m-particles of $\mathbf{X}^N = (X^{N,i})_{i=1}^N$ converge weakly in $C([0,\infty);\mathbb{R}^m)$ to the distribution of the first m-particles of the solution of ISDE (2.6).*

Theorem 3.1 is an application of the result given in [28], where we constructed a general result on the convergence and transition of a stochastic differential equation for N particles to that for infinite particle systems. The assumption required to apply this result is not simple and therefore is skipped here. Intuitively, the assumption requires the uniform control of the tail of the coefficients of the stochastic differential equations for finite particle systems.

We apply the result in [28] to non-symmetric infinite particle systems. In addition, if the interaction potential Ψ is of Ruelle's class, then we can apply the result given in [28] easily. However, we need a fine calculation concerning the logarithmic potential, specifically Theorem 3.1. Later, we shall introduce another convergence theorem (Theorem 9.1). Although Theorem 9.1 is restricted to particle systems given by the symmetric Dirichlet form, the necessary assumption is simple and easily checked. Theorem 9.1 has various applications and provides a second and simple proof of Theorem 3.1. If $\theta = 0$, then Theorem 3.1 can be proved by a calculation of the space-time correlation functions [52]. We therefore see that, for the one-dimensional infinite particle systems with logarithmic potential and $\beta = 2$, there exist three completely different methods proving the convergence of Theorem 3.1. We expect Theorem 3.1 holds also for general β.

3.2. **Soft edge limit and the Airy interacting Brownian motion.** We consider a scaling at $\theta = \pm 2$, called soft edge scaling, such that

$$x \longmapsto 2\sqrt{N} + \frac{s}{N^{1/6}}. \tag{3.8}$$

Then, the distribution of the N particle systems is

$$m_{\mathrm{Ai},\beta}^N(d\mathbf{s}_N) = \frac{1}{Z}\{\prod_{i<j}^N |s_i - s_j|^\beta\} \exp\Big\{ -\frac{\beta}{4}\sum_{k=1}^N \Big|2\sqrt{N} + \frac{s_k}{N^{1/6}}\Big|^2\Big\} d\mathbf{s}_N.$$

Taking $N \to \infty$, we obtain $\mu_{\mathrm{Ai},\beta}$. The logarithmic derivative of $m_{\mathrm{Ai},\beta}^N$ is

$$\frac{\partial \log m_{\mathrm{Ai},\beta}^N}{\partial s_i}(\mathbf{s}_N) = \beta\sum_{j\neq i}^N \frac{1}{s_i - s_j} - \beta\Big\{N^{1/3} + \frac{s_i}{2N^{1/3}}\Big\}.$$

Hence, the stochastic differential equation of the N-particle systems $\mathbf{X}^N = (X_t^{N,i})_{i=1}^N$ is

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1,\, j\neq i}^{N} \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2}\Big\{ N^{1/3} + \frac{X_t^{N,i}}{2N^{1/3}} \Big\} dt. \tag{3.9}$$

The difficulty of taking the limit $N \to \infty$ in (3.9) is that the coefficient of (3.9) includes the divergent term $-\frac{\beta}{2}N^{1/3}dt$. In [50], we solved the ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \Big\{ \Big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} \Big) - \frac{2\sqrt{r}}{\pi} \Big\} dt \tag{3.10}$$

and we proved the convergence of the solutions of the N-particle systems (3.9) to that of ISDE (3.10) in [28].

We derive the representation (3.10) from the semi-circle distribution (3.2) and the scaling (3.8) intuitively. Indeed, rescaling the semi-circle function by (3.8), we obtain $\varsigma(x) = \frac{1}{\pi} 1_{(-\infty,0)}(x)\sqrt{-x}$. Hence

$$\frac{2\sqrt{r}}{\pi} = \int_{-r}^{0} \frac{\varsigma(x)}{-x} dx.$$

We expect such a relation holds for a general limit distribution other than the semi-circle distribution, and the corrector function according to the limit distribution exists. Such a derivation seems to be universal.

4. Space-time correlation functional method.

In one-dimensional particle systems with the logarithmic interaction potential and $\beta = 2$, one constructs the stochastic dynamics by the explicit formula given by the space–time correlation functions via the extended kernels. Following Katori–Tanemura [23, 24, 25, 26], we present the extended kernels for sine, Airy, and Bessel point processes with $\beta = 2$.

We set the multi-time, moment generating functions of the S-valued process X_t by

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \mathbb{E}\left[\exp\left\{ \sum_{m=1}^{M} \int_{\mathbb{R}} f_m d\mathsf{X}_{t_m} \right\} \right].$$

Let $\mathbb{K}(s,x;t,y)$ be an extended kernel. Using the Fredholm determinant of $\mathbb{K}(s,x;t,y)$, we represent $\Psi^{\mathbf{t}}[\mathbf{f}]$ as

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \underset{\substack{(s,t)\in\{t_1,t_2,\dots,t_M\}^2, \\ (x,y)\in\mathbb{R}^2}}{\mathrm{Det}} \Big[\delta_{st}\delta(x-y) + \mathbb{K}(s,x;t,y)\chi_t(y) \Big], .$$

Here $M \in \mathbb{N}$, $\mathbf{f} = (f_i)_{i=1}^M \in C_0(\mathbb{R})^M$, $\mathbf{t} = (t_i)_{i=1}^M$ $(0 < t_1 < \cdots < t_M)$, and $\chi_{t_i} = e^{f_i} - 1$ $(1 \le i \le M)$ [23, 26].

(i)**Extended sine kernel** $\mathbb{K}_{\sin}$**:** For $s,t \in \mathbb{R}^+$ and $x,y \in \mathbb{R}$, we set

$$\mathbb{K}_{\sin}(s,x;t,y) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^1 du\, e^{u^2(t-s)/2} \cos\{u(y-x)\} & \text{if } s<t, \\ K_{\sin}(x,y) & \text{if } s=t, \\ -\dfrac{1}{\pi}\displaystyle\int_1^{\infty} du\, e^{u^2(t-s)/2} \cos\{u(y-x)\} & \text{if } s>t. \end{cases}$$

(ii) **Extended Airy kernel** $\mathbb{K}_{\mathrm{Ai}}$**:** For $s,t \in \mathbb{R}^+$ and $x,y \in \mathbb{R}$, we set

$$\mathbb{K}_{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \displaystyle\int_0^\infty du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } s<t, \\ K_{\mathrm{Ai}}(x,y) & \text{if } s=t, \\ \displaystyle -\int_{-\infty}^0 du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y) & \text{if } s>t. \end{cases}$$

(iii) **Extended Bessel kernel** $\mathbb{K}_{J_\nu}$**:** For $s,t \in \mathbb{R}^+$ and $x,y \in \mathbb{R}^+$, we set

$$\mathbb{K}_{J_\nu}(s,x;t,y) = \begin{cases} \displaystyle\int_0^1 du\, e^{-2u(s-t)}J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) & \text{if } \quad s<t, \\ K_{J_\nu}(x,y) & \text{if } \quad s=t, \\ \displaystyle -\int_1^\infty du\, e^{-2u(s-t)}J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) & \text{if } \quad s>t. \end{cases}$$

Using these kernels, we construct the S-valued stochastic dynamics. The corresponding representation for finite particle systems also exists, and the infinite particle systems are the limit. Combining the construction above with the results in [49, 50, 51, 52, 30], we see that the stochastic dynamics derived from the space-time correlation functions and that arising out of the stochastic analysis are the same. We also remark that other constructions are known [17, 7] for one-dimensional infinite particle systems with logarithmic interaction potentials and $\beta = 2$.

5. Dirichlet form approach and weak solutions: The first theory.

The aim of this section is to develop a general theory solving the ISDE,

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i\diamondsuit})dB_t^i + b(X_t^i, \mathsf{X}_t^{i\diamondsuit})dt \quad (i \in \mathbb{N}), \quad \mathbf{X} \in \mathbf{W}^{\mathrm{sol}}. \tag{5.1}$$

We shall solve the equation in the time interval $[0,\infty)$. $\mathbf{W}^{\mathrm{sol}}$ is a symmetric subset of $W((\mathbb{R}^d)^{\mathbb{N}}) = C([0,\infty);(\mathbb{R}^d)^{\mathbb{N}})$ and is the space of solutions of the ISDE. We regard the coefficients as functions defined on subsets of $W((\mathbb{R}^d)^{\mathbb{N}})$ and suppose $\mathbf{W}^{\mathrm{sol}}$ are contained by the domains of the functions.

5.1. Construction of the unlabeled diffusion processes.

In the present section, we introduce a general theorem constructing the unlabeled diffusions from the Dirichlet form associated with point process μ. We begin by introducing the notion of **quasi-Gibbs measure**.

Definition 5.1 (**quasi-Gibbs measures**)**.** Let Φ and Ψ be the free and the interaction potentials, respectively. We say μ is a (Φ,Ψ)-quasi-Gibbs measure if the regular conditional probability $\mu_{r,\xi}^n$ defined in (2.1) satisfies, for μ-a.s. ξ and $r,n \in \mathbb{N}$ with positive constant $C = C(r,\xi,n)$ depending on (r,ξ,n),

$$C(r,\xi,n)^{-1}e^{-\mathcal{H}_r(\mathsf{s})}d\Lambda_r^n \le \mu_{r,\xi}^n(d\mathsf{s}) \le C(r,\xi,n)e^{-\mathcal{H}_r(\mathsf{s})}d\Lambda_r^n. \tag{5.2}$$

Here, for the two measures μ and ν we set $\mu \le \nu$ if $\mu(A) \le \nu(A)$ for any A. $\mathcal{H}_r(\mathsf{s})$ is a Hamiltonian defined on $\mathbf{S}_r$ defined by (2.2).

Remark 5.2. We note that $C(r,\xi,n)$ in (5.2) depends on ξ. The notion of quasi-Gibbs is robust under the perturbation of free potentials. Indeed, if μ is a (Φ,Ψ)-quasi-Gibbs measure, then for any locally bounded Φ_0 μ is a $(\Phi{+}\Phi_0,\Psi)$-quasi-Gibbs measure. In particular, the sine, Airy, Bessel, and Ginibre point processes are all $(0, -\beta \log |x-y|)$-quasi-Gibbs measures.

Let σ be as in (5.1) and take $a(x,\mathsf{s}) = \sigma(x,\mathsf{s})\, {}^t\sigma(x,\mathsf{s})$. We assume:
(**A1**) a is uniformly elliptic and bounded. $a(\cdot,\mathsf{s})$ is smooth for each s. μ is a (Φ,Ψ)-quasi-Gibbs measure, and there exist upper semi-continuous potentials (Φ_0,Ψ_0) and positive constants c_1 and c_2 such that

$$c_1(\Phi_0,\Psi_0) \le (\Phi,\Psi) \le c_2(\Phi_0,\Psi_0).$$

(**A2**) $\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty$ for each $r \in \mathbb{N}$, where $\mathsf{S}_r^k = \{\mathsf{s}(S_r) = k\}$. For some $p > 1$, the n-point correlation function of μ is locally L^p-bounded for each $n \in \mathbb{N}$.

With μ and a, we define the Dirichlet form as

$$\mathcal{E}^{a,\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}^a[f,g]d\mu, \quad \mathbb{D}^a[f,g] = \frac{1}{2}\sum_i a(s_i,\mathsf{s}^{i\diamondsuit})\frac{\partial \check{f}}{\partial s_i}\cdot\frac{\partial \check{g}}{\partial s_i}.$$

Here for $\mathsf{s} = \sum_i \delta_{s_i}$ we set $\mathsf{s}^{i\diamondsuit} = \sum_{j\ne i}\delta_{s_j}$. Let $\mathcal{D}_\circ$ be the set consisting of local and smooth functions on S. Here we say f is local if f is $\sigma[\pi_{S_r}]$-measurable for some $r \in \mathbb{N}$. We also say f is smooth if $\check{f}(s_1,\ldots,)$ is smooth by regarding $f(\mathsf{s})$ as a symmetric function $\check{f}(s_1,\ldots,)$ on a symmetric subset of $S^{\mathbb{N}} \cup \sum_i S^i$. We refer the reader to [41, 5] for a more rigorous definition. We set

$$\mathcal{D}_\circ^{a,\mu} = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S},\mu); \mathcal{E}^{a,\mu}(f,f) < \infty\} \tag{5.3}$$

Theorem 5.3 ([41, 42, 46, 49])**.** *Assume* (**A1**) *and* (**A2**)*. Then,* $(\mathcal{E}^{a,\mu},\mathcal{D}_\circ^{a,\mu})$ *is closable on* $L^2(\mathsf{S},\mu)$*. There exists a diffusion* $(\mathsf{X},\{\mathsf{P_s}\}_{\mathsf{s}\in\mathsf{S}})$ *associated with the closure* $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu})$*.*

In [2, 64], a space of polynomials on S was taken as a core of the domain of the Dirichlet form. These two Dirichlet forms coincide with each other [51].

5.2. **Label dynamics and coupling of Dirichlet spaces.** We set

$$\mathsf{S}_{\mathrm{s,i}} = \{\mathsf{s}; \mathsf{s}(\{x\}) = 0 \text{ or } 1\ \forall x \in \mathbb{R}^d,\ \mathsf{s}(\mathbb{R}^d) = \infty\}.$$

We assume the following:
(**A3**) Non-collision and infinite: $\mathsf{P}_\mu(\mathsf{X}_t \in \mathsf{S}_{\mathrm{s,i}}\ \forall t \in [0,\infty)) = 1$.
(**A4**) Non-explosion: $\mathsf{P}_\mu(\bigcap_{i=1}^\infty\{\sup_{0\le u\le t}|X_u^i| < \infty\ \forall t \in [0,\infty)\}) = 1$.

See [43] for the non-collision property in (**A3**) and [49] for other conditions. It is known that non-explosion holds if the one-point correlation function grows at most $\exp(|x|^c)$ $(c < 2)$ as $|x| \to \infty$.

Under these assumptions, we initially label the path of unlabeled particles $\{\mathsf{X_t}\}$ at time $t = 0$ using label $\mathfrak{l}$ such that $\mathfrak{l}(\mathsf{X}_0) = \mathbf{X}_0$. It then is retained forever under (**A3**) and (**A4**), and defines the map from the unlabeled path to the labeled path. We denote this correspondence by $\mathfrak{l}_{\mathrm{path}}$.

By definition $\mathfrak{l}_{\mathrm{path}}$ is a map from $C([0,\infty);\mathsf{S}_{\mathrm{s,i}})$ to $C([0,\infty);(\mathbb{R}^d)^{\mathbb{N}})$. Generally, $\mathfrak{l}_{\mathrm{path}}(\mathsf{X})_t \ne \mathfrak{l}(\mathsf{X}_t)$.

Theorem 5.4 ([44])**.** *We impose assumptions* (**A1**)*–*(**A4**)*. For the unlabeled dynamics* X *in Theorem 5.3 we define the labeled dynamics* $\mathbf{X}$ *by* $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathsf{X})$*.* $\mathbf{X}$ *is an* $(\mathbb{R}^d)^{\mathbb{N}}$*-valued diffusion.*

Once the labeled dynamics $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathsf{X})$ has been constructed, we define the m-labeled dynamics by labeling only the first m-particles,

$$(X^1, \dots, X^m, \sum_{i=m+1}^{\infty} \delta_{X^i}).$$

We next introduce the Dirichlet space for the m-labeled dynamics. Let $\mu^{[m]}$ be the m-reduced Campbell measure of μ:

$$\mu^{[m]}(d\mathbf{x}d\mathsf{s}) = \rho^m(\mathbf{x})d\mathbf{x}\mu_{\mathbf{x}}(d\mathsf{s}).$$

Here, ρ^m is the m-point correlation function of μ and $\mu_{\mathbf{x}}$ is the reduced Palm measure conditioned at $\mathbf{x} = (x_1, \dots, x_m) \in (\mathbb{R}^d)^m$. Loosely, $\mu_{\mathbf{x}}$ is given by

$$\mu_{\mathbf{x}}(d\mathsf{s}) = \mu(d\mathsf{s} - \sum_{i=1}^{m} \delta_{x_i} \,|\, \mathsf{s}(x_i) \geq 1 \text{ for all } i).$$

For $\mu^{[m]}$ we have the analogy of Theorem 5.3 on the m-labeled dynamics. Hence, we set by $\Xi^{[m]}(\mu)$ the Dirichlet space on $(\mathbb{R}^d)^m \times \mathsf{S}$ for $\mu^{[m]}$, where $m \in \mathbb{N}$. Let $\Xi^{[0]}(\mu)$ be the Dirichlet space given by Theorem 5.3.

Theorem 5.5 ([44]). *Assume* (**A1**)–(**A4**). *Let $\mathbf{X}^{[m]}$ be the diffusion given by $\Xi^{[m]}(\mu)$. Then $\Xi^{[m]}(\mu)$ satisfies*

$$\mathbf{X}^{[m]} = (X^1, \dots, X^m, \sum_{i=m+1}^{\infty} \delta_{X^i}) \quad \textit{(in law)}. \tag{5.4}$$

Here, X^i denotes the component of the labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ given by Theorem 5.4.

The right–hand side of (5.4) is a functional of the diffusion given by the Dirichlet space $\Xi^{[0]}(\mu)$. Note that $\Xi^{[0]}(\mu)$ is independent of m. In this sense, Theorem 5.5 indicates the coupling of the sequence of the Dirichlet spaces $\{\Xi^{[m]}(\mu)\}_m$.

5.3. **ISDE.** We nexst solve ISDE (5.1). We want to construct a solution $\mathbf{X}$ of (5.1) such that the equilibrium state of the associated unlabeled dynamics X is μ. For this purpose, we introduce the notion of the logarithmic derivative d^{μ} of μ.

Definition 5.6. d^{μ} is called the logarithmic derivative of μ if

$$\int_{\mathbb{R}^d\times\mathsf{S}} \mathsf{d}^{\mu} f d\mu^{[1]} = -\int_{\mathbb{R}^d\times\mathsf{S}} \nabla_x f d\mu^{[1]} \quad \text{for any } f \in C_0^{\infty}(\mathbb{R}^d) \otimes \mathcal{D}_{\circ}.$$

Taking Definition 5.6 into account, we set

$$\mathsf{d}^{\mu}(x,\mathsf{s}) = \nabla_x \log \mu^{[1]}(x,\mathsf{s}).$$

With this notation, we introduce the geometric, differential equation (5.5) for μ.

(**A5**) μ has a logarithmic derivative d^{μ} such that

$$2b(x,\mathsf{s}) = \nabla_x a(x,\mathsf{s}) + a(x,\mathsf{s})\nabla_x \log \mu^{[1]}(x,\mathsf{s}). \tag{5.5}$$

Suppose that a is the unit matrix. For a given interaction potential Ψ we set

$$b(x,\mathsf{s}) = -\frac{1}{2}\sum_i \nabla_x \Psi(x, s_i).$$

If Ψ is an interacting potential of Ruelle's class, then there exists a μ satisfying the DLR equation. Then μ becomes a solution of (5.5).

For determinantal point processes, a point process μ is *a priori* given by a kernel. Solving (5.5) becomes an inverse problem. In [45, 47], we prepared a general theory for the calculation of the logarithmic derivative of μ. Using this theory, we can calculate the logarithmic derivative of all examples in the present article. The key idea for the calculation is proving geometric rigidity according to point processes μ as well as the proof of the quasi-Gibbs property. This part of the calculation requires fine estimates depending on the free potential of μ [45, 50, 18, 5].

Once the logarithmic derivative d^{μ} of μ is obtained and shown to satisfy (5.5), we can solve the ISDE (5.1). The next theorem has high versatility and applies to all examples given in this article.

Theorem 5.7 ([45])**.** *Assume* (**A1**)–(**A5**). *Then the ISDE* (5.1) *has a solution* $(\mathbf{X}, \mathbf{B})$ *starting at* $\mu \circ \mathfrak{l}^{-1}$*-a.s.* $\mathbf{s}$ *for a given label* $\mathfrak{l}$. $\mathbf{X}$ *is a diffusion on* $(\mathbb{R}^d)^{\mathbb{N}}$ *and the associated unlabeled dynamics* X *is* μ*-reversible.*

The key point of the proof of Theorem 5.7 is Theorem 5.5. Indeed, for $\mathbf{x} = (x_i)_{i=1}^{\infty} \in (\mathbb{R}^d)^{\mathbb{N}}$, each coordinate function x_i belongs locally to the domain of the Dirichlet form $\Xi^{[m]}$ $(i \leq m)$. Then, applying the Itô formula to each x_i, we derive the set of Dirichlet processes given by the coordinate function and the m-labeled process satisfying the stochastic differential equation (5.1). Here, we use the Fukushima decomposition and the Revuz correspondence as a substitution of the classical Itô formula. At this stage, we describe the motion of the first m-particles by the stochastic differential equation. This holds for each $m \in \mathbb{N}$. Accordingly, we have solved the stochastic differential equation (5.1) for each $i \in \mathbb{N}$ using the coupling obtained in Theorem 5.5. Therefore, we have solved ISDE (5.1).

6. Existence of strong solutions and pathwise uniqueness: the second theory

In this section, we introduce the second theory due to the author and Tanemura, and using this, we show the existence and pathwise uniqueness of strong, solutions of the ISDE [49].

The solution obtained by the first theory is a weak solution in the sense that the solution is a pair $(\mathbf{X}, \mathbf{B})$ of the $(\mathbb{R}^d)^{\mathbb{N}}$-valued continuous process $\mathbf{X}$ and the $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion $\mathbf{B}$. If $\mathbf{X}$ is a function of the Brownian motion $\mathbf{B}$ in addition, $(\mathbf{X}, \mathbf{B})$ is called a strong solution. This is not the case for the solution in the first theory. We see the solution associated with the given Dirichlet form is unique in law. Because the uniqueness of Dirichlet forms is not established by the first theory, we have not yet proved the uniqueness of solutions by the first theory.

In this section, we consider a sequence of *tiny* infinite-dimensional spaces $(\mathbb{R}^d)^m \times \mathsf{S}$ instead of the *huge* infinite-dimensional space $(\mathbb{R}^d)^{\mathbb{N}}$. On $(\mathbb{R}^d)^m \times \mathsf{S}$, we regard S as a random environment and interpret $(\mathbb{R}^d)^m$ $(m \in \mathbb{N})$ as a sequence of spaces where the existence and uniqueness theorems of the solution of stochastic differential equations hold. We therefore introduce a sequence of finite-dimensional stochastic differential equations of random-environment type. We construct couplings between these stochastic differential equations (6.1) on $(\mathbb{R}^d)^m$, where $m \in \mathbb{N}$.

The critical point is the equivalence of the ISDE as a sequence of finite-dimensional stochastic differential equations with consistency (IFC). For this equivalence, we use the weak solution $(\mathbf{X}, \mathbf{B})$ obtained in the first theory.

6.1. **Existence and uniqueness of solutions of the ISDE..** Let $(\mathbf{X}, \mathbf{B})$ be a weak solution of (5.1) defined on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$. Let $\mathfrak{l}$ be a label and μ be a point process. We suppose the distribution of $\mathbf{X}_0$ is given by $\mu \circ \mathfrak{l}^{-1}$. The solution starting $\mathbf{s}$ is given by $(\mathbf{X}, \mathbf{B})$ under the conditional distribution $P_{\mathbf{s}} = P(\cdot|\mathbf{X}_0 = \mathbf{s})$.

We consider a family of $(\mathbb{R}^d)^m$-valued stochastic differential equations by $(\mathbf{X}, \mathbf{B})$. For each $m \in \mathbb{N}$ we consider the stochastic differential equation of $\mathbf{Y}^m = (Y^{m,i})_{i=1}^m$:

$$dY_t^{m,i} = \sigma(Y_t^{m,i}, \mathsf{Y}_t^{m,i\diamondsuit} + \mathsf{X}_t^{m*})dB_t^i + b(Y_t^{m,i}, \mathsf{Y}_t^{m,i\diamondsuit} + \mathsf{X}_t^{m*})dt, \quad \mathbf{Y}_0^m = \mathbf{s}^m. \tag{6.1}$$

Here for $\mathbf{s} = (s_i)_{i\in\mathbb{N}}$ we set $\mathbf{s}^m = (s_1, \ldots, s_m)$ and

$$\mathsf{Y}^{m,i\diamondsuit} = \sum_{j\neq i}^{m} \delta_{Y^{m,j}}, \quad \mathsf{X}^{m*} = \sum_{k=m+1}^{\infty} \delta_{X^k}, \quad \mathbf{X}^{m*} = (X^k)_{k=m+1}^{\infty}.$$

Fix $\mathbf{X}$. Then (6.1) is a time-inhomogeneous dm-dimensional stochastic differential equation. Hence for well-behaved $\mathbf{X}$, the finite-dimensional stochastic differential equation (6.1) has a pathwise unique, strong solution for each m. Therefore, we assume

(**IFC**) For each $m \in \mathbb{N}$, (6.1) has a pathwise unique strong solution.

The solution $\mathbf{Y}^m$ of (6.1) is a function of $(\mathbf{B}^m, \mathbf{X}^{m*})$ and $\mathbf{s}^m$. Hence, we write

$$\mathbf{Y}^m = \mathbf{Y}^m(\mathbf{s}^m, \mathbf{B}^m, \mathbf{X}^{m*}) = \mathbf{Y}^m(\mathbf{s}, \mathbf{B}, \mathbf{X}^{m*}).$$

We see $\mathbf{Y}^m$ is $\sigma[\mathbf{s}, \mathbf{B}, \mathbf{X}^{m*}]$-measurable. Using the assumption that $(\mathbf{X}, \mathbf{B})$ is a weak solution of the ISDE, as well as the pathwise uniqueness of solutions of (6.1), we obtain

$$\mathbf{X}^m = \mathbf{Y}^m. \tag{6.2}$$

Hence, the limit $\lim_{m\to\infty} \mathbf{Y}^m$ clearly exists. That is, the following holds;

$$\mathbf{X} = \lim_{m\to\infty} \mathbf{Y}^m(\mathbf{s}, \mathbf{B}, \mathbf{X}^{m*}). \tag{6.3}$$

The solution $\mathbf{X}$ is a *fix point* by (6.2). The next section considers the general case.

Let $\mathcal{T}(\mathsf{S})$ be the tail σ-field of the configuration space S of $\mathbb{R}^d$;

$$\mathcal{T}(\mathsf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}].$$

Here. $S_r^c = \{s \in \mathbb{R}^d; \, |s| > r\}$ and $\pi_{S_r^c}(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$.

The critical point for the passage from the property of the unlabeled dynamics X to that of the labeled dynamics $\mathbf{X}$ is the control of the random variable

$$\mathsf{m}_{T,r}(\mathbf{X}) = \inf\{m \in \mathbb{N}; X^i \in C([0,T]; S_r^c) \text{ for } m < \forall i \in \mathbb{N}\}.$$

Here, $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ and $T, r \in \mathbb{N}$.

We set $\mathsf{X} = \sum_{i=1}^{\infty} \delta_{X^i}$ and recall that the distribution of $\mathbf{X}_0$ is $\mu \circ \mathfrak{l}^{-1}$. Then, the distribution of X_0 is μ. We introduce a condition for $(\mathbf{X}, \mathbf{B})$,

(**TT**) $\mu(A) \in \{0, 1\}$ for all $A \in \mathcal{T}(\mathsf{S})$ (tail triviality)
(**AC**) $P \circ \mathsf{X}_t^{-1} \prec \mu$ for all $0 < t < \infty$ (absolute continuity)
(**NBJ**) $P(\mathsf{m}_{T,r}(\mathbf{X}) < \infty) = 1$ for all $T, r \in \mathbb{N}$ (no big jump)

A weak solution $(\mathbf{X}, \mathbf{B})$ satisfying (**IFC**) is called an IFC solution.

Theorem 6.1 ([49]). *Assume that μ satisfies* (**TT**). *Let* $(\mathbf{X},\mathbf{B})$ *be an IFC solution satisfying* (**AC**) *and* (**NBJ**). *Then* (5.1) *has a unique strong solution* $F_{\mathbf{s}}$ *starting at* $\mu\circ\mathfrak{l}^{-1}$*-a.s.* $\mathbf{s}$ *under the constraints* (**AC**), (**NBJ**), *and* (**IFC**).

Here, the claim that $F_{\mathbf{s}}$ is a unique strong solution under the constraints (**AC**), (**NBJ**), and (**IFC**) means that arbitrary weak solutions $(\mathbf{X}',\mathbf{B}')$ satisfying (**AC**), (**NBJ**), (**IFC**) satisfy $\mathbf{X}'=F_{\mathbf{s}}(\mathbf{B}')$, and that for any $\{\mathcal{F}_t\}$-Brownian motion $\hat{\mathbf{B}}$, $F_{\mathbf{s}}(\hat{\mathbf{B}})$ becomes a strong solution satisfying (**AC**), (**NBJ**), and (**IFC**).

The weak solution obtained in the first theory satisfies (**AC**) and (**NBJ**). A general theorem verifying (**IFC**) is given by [49]. This result requires that the solution is associated with the quasi-regular Dirichlet form. In on-going work, we are preparing a result without quasi-regularity of the Dirichlet form.

We assume (**IFC**) and generalize the consistency in (6.2) in an asymptotic form. Let $\mathbf{W}^{\mathrm{sol}}$ be the space of solutions of ISDE (5.1); then

$$\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}=\{\mathbf{X}\in\mathbf{W}^{\mathrm{sol}};\mathbf{X}_0=\mathbf{s}\},\quad \mathbf{W}_{\mathbf{0}}=\{\mathbf{X}\in W((\mathbb{R}^d)^{\mathbb{N}})\,;\,\mathbf{X}_0=\mathbf{0}\}\ .$$

We set $F_{\mathbf{s}}^m:\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}\times\mathbf{W}_{\mathbf{0}}\to\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}$ by

$$F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B})=\{(Y_t^{m,1},\ldots,Y_t^{m,m},X_t^{m+1},X_t^{m+2},\ldots)\}_{0\le t\le T}.$$

Here $\mathbf{Y}^m=(Y^{m,i})_{i=1}^m$ is a unique strong solution of (6.1) given by (**IFC**).

6.2. **IFC solution and pathwise unique strong solution.** For each $(\mathbf{s},\mathbf{B})$, $F_{\mathbf{s}}^m(\,\cdot\,,\mathbf{B})$ defines the map $F_{\mathbf{s}}^m=(F_{\mathbf{s}}^{m,i})_{i=1}^m$ from $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}$ to $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}$. Let $\bar{P}_{\mathbf{s}}$ be a probability measure on $\mathbf{W}_{\mathbf{s}}^{\mathrm{sol}}\times\mathbf{W}_{\mathbf{0}}$. Then we write

$$F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B})=\lim_{m\to\infty}F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B})\text{ in }\mathbf{W}^{\mathrm{sol}}\text{ under }\bar{P}_{\mathbf{s}}\tag{6.4}$$

if, for each $i\in\mathbb{N}$ the limits (6.5)–(6.7) in $W(\mathbf{S})$ exist for $\bar{P}_{\mathbf{s}}$-a.s. $(\mathbf{X},\mathbf{B})$

$$\lim_{m\to\infty}F_{\mathbf{s}}^{m,i}(\mathbf{X},\mathbf{B})=F_{\mathbf{s}}^{\infty,i}(\mathbf{X},\mathbf{B}),\tag{6.5}$$

$$\lim_{m\to\infty}\int_0^{\cdot}\sigma^i(F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B})_u)dB_u^i=\int_0^{\cdot}\sigma^i(F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B})_u)dB_u^i,\tag{6.6}$$

$$\lim_{m\to\infty}\int_0^{\cdot}b^i(F_{\mathbf{s}}^m(\mathbf{X},\mathbf{B})_u)du=\int_0^{\cdot}b^i(F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B})_u)du,\tag{6.7}$$

and $F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B})\in\mathbf{W}^{\mathrm{sol}}$. Here $\sigma^i(\mathbf{Z}_t)=\sigma(Z_t^i,\mathsf{Z}_t^{i,\diamondsuit})$, and we set b^i similarly.

Definition 6.2. $\bar{P}_{\mathbf{s}}$ is called an AIFC solution of (5.1) if it satisfies (6.4) and $\bar{P}_{\mathbf{s}}(\mathbf{B}\in\cdot)=P_{\mathrm{Br}}^{\infty}$.

We now construct a weak solution of (5.1) from an AIFC solution.

Lemma 6.3 (O.-Tanemura [49]). *Let* $\mathbf{s}$ *be fixed and assume* (**IFC**). *Let* $\bar{P}_{\mathbf{s}}$ *be an AIFC solution of* (5.1). *Then* $(F_{\mathbf{s}}^{\infty}(\mathbf{X},\mathbf{B}),\mathbf{B})$ *under* $\bar{P}_{\mathbf{s}}$ *is a weak solution of* (5.1).

We next define the tail σ-field $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ of the labeled path space with respect to the label

$$\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})=\bigcap_{m=1}^{\infty}\sigma[\mathbf{X}^{m*}].$$

We denote by $\bar{P}_{\mathbf{s},\mathbf{b}}(\cdot)=P((\mathbf{X},\mathbf{B})\in\cdot\,|(\mathbf{X}_0,\mathbf{B}_0)=(\mathbf{s},\mathbf{b}))$ the regular conditional distribution conditioned at the initial stating point $\mathbf{X}_0$ and Brownian motion $\mathbf{B}$. Let P_{Br}^{∞} be the distribution of $\mathbf{B}$ and fix $\mathbf{s}$.

(**PT1**) For P_{Br}^{∞}-a.s. $\mathbf{b}$, $\bar{P}_{\mathbf{s},\mathbf{b}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}$ is trivial.
(**PT2**) For P_{Br}^{∞}-a.s. $\mathbf{b}$, $\bar{P}_{\mathbf{s},\mathbf{b}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}$ is unique.
The uniqueness means for any IFC solutions $(\mathbf{X},\mathbf{B})$ and $(\mathbf{X}',\mathbf{B}')$ starting at $\mathbf{s}$

$$\bar{P}_{\mathbf{s},\mathbf{b}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})} = \bar{P}'_{\mathbf{s},\mathbf{b}}|_{\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})}.$$

Theorem 6.4 (O.-Tanemura [49]). *Fix* $\mathbf{s}$ *and let* $(\mathbf{X},\mathbf{B})$ *be an IFC solution of* (5.1) *staring at* $\mathbf{s}$*. Then*
(1) $(\mathbf{X},\mathbf{B})$ *is a strong solution if and only if* (**PT1**) *holds.*
(2) $(\mathbf{X},\mathbf{B})$ *is a unique strong solution if and only if* (**PT1**)–(**PT2**) *hold.*

In the second theory, we regard the tail σ-field $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ of the labeled path space for the label as the boundary condition of the ISDE. If $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ is trivial and the restriction of the distributions of solutions $\mathcal{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$, then the solutions of the original ISDE is unique. We remark the claim in Theorem 6.4 is necessary and sufficient.

The claim of Theorem 6.4 holds beyond ISDEs of infinite particle systems, even if the ISDEs do not have symmetry. What we need is a criterion such that we can interpret the infinite-dimensional equation as a scheme of finite-dimensional equations.

(**PT1**)–(**PT2**) involve the tail σ-field of the enormous infinite-dimensional space. Hence, one may think it is challenging to prove triviality. As we shall explain in the next section, we deduce this from μ-triviality of the configuration space.

6.3. **A sufficient condition of (PT1)–(PT2).** We present a sufficient condition of triviality of the path space in the previous section.

Theorem 6.5 (O.-Tanemura [49]). *Assume* (**TT**), (**IFC**), (**AC**), *and* (**NBJ**). *Then* (**PT1**)–(**PT2**) *hold for* $\mu \circ \mathfrak{l}^{-1}$*-a.s.* $\mathbf{s}$.

Therefore, tail triviality of the labeled path space with respect to the label follows from tail triviality of the configuration space. We have obtained triviality of the huge infinite-dimensional space from that of the tiny infinite-dimensional space.

Determinantal point processes satisfy (**TT**) [48, 35, 6]. Any quasi-Gibbs measures can be decomposed to tail trivial components, and we can apply Theorem 6.1 to each tail trivial component. Indeed, for quasi-Gibbs measure μ, we consider the regular conditional probability $\mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{s})$ with respect to the tail σ-field $\mathcal{T}(\mathsf{S})$ and decompose μ as

$$\mu(\cdot) = \int_{\mathsf{S}} \mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{s})\mu(d\mathsf{s}). \tag{6.8}$$

Then, s μ $\mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{s})$ becomes tail trivial for μ-a.e. [49]. Because $\mathcal{T}(\mathsf{S})$ is not countably determined, this fact is non-trivial.

For (**A1**)–(**A3**) and (**A5**), we see these properties of μ are inherited by $\mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{s})$ through (6.8). A feasible sufficient condition of (**A4**) is a growth condition of the one-point correlation function at infinity. This condition is also inherited by $\mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{s})$ through the Fubini theorem. Therefore, if (**TT**) does not hold, then we have an unlabeled dynamics that keeps the tail σ-field invariant, and such a solution is unique under the constraint that keeps the tail σ-field. This result does not deny the possibility of the existence of solutions with a varying tail σ-fields. We conjecture that under a mild assumption, the unlabeled dynamics preserve the tail σ-field. We also expect that there exist unlabeled dynamics that vary the tail σ-field.

We consider the solution of (5.1) such that the associated unlabeled dynamics X is μ-reversible. Hence (**AC**) is automatically satisfied. Using the μ-reversibility, we apply the Lyons–Zheng decomposition to the solution to obtain (**NBJ**). We do not necessarily assume that the solution is associated with a quasi-regular Dirichlet form. We perform the Lyons–Zheng decomposition because of the representation of the solution given by (5.1) and the fact that the unlabeled dynamics is a μ-reversible Markovian semi-group.

7. Classical stochastic analysis and infinite particle systems

We review the results in the previous two sections from the viewpoint of the development of the classical stochastic analysis to the stochastic analysis of infinite particle systems. The stochastic analysis was initiated by Kiyosi Itô in 1942, and prototypes of the theory such as the stochastic integral based on Brownian motion, the stochastic differential equation, and the Itô formula, which are indispensable now, were born.

After a period of stagnation, Kunita and Watanabe revived the theory in 1967 by extending the stochastic analysis to the framework of martingales from that of Brownian motion. This tide was followed by the martingale problem by Stroock and Varadhan, the Yamada–Watanabe theory on stochastic differential equations, the Dirichlet form theory by Fukushima, the Malliavin calculus, stochastic flow, and other advancements. The field has developed significantly from the 1960s to the 1980s.

Stochastic analysis was disseminated in Japan through the book "Stochastic Differential Equations" (*Kakuritsu Bibun Houteisiki* in Japanese) written by Shinzo Watanabe [65]. Moreover, the new developments of the Malliavin calculus, stochastic flow, and stochastic analysis on manifolds, were also added in the book "Stochastic differential equations", Ikeda and Watanabe also published the seminal book "Stochastic differential equations and diffusion processes" in 1981, which was reprinted in 1989 [20, 21]. This book brought the stochastic analysis of the Japanese school to the world.

Indeed, the spread of stochastic analysis was triggered socially by its application to mathematical finance in the 1990s. A wide variety of textbooks appeared but the Ikeda–Watanabe book became prototypical of many books on stochastic analysis that appeared around the world. The Dirichlet form theory initiated by Masatoshi Fukushima was spread throughout Japan by the book "Dirichlet forms and Markov process (in Japanese, *Dirichlet keishiki to Markov katei*)" (1975) [11] and to the world by the book "Dirichlet forms and Markov processes" (1980) [12], which was an extension of the earlier book. Further developments were presented in the book by Fukushima, Oshima, and Takeda [13, 14]. The classical stochastic analysis in the present review means the theory was developed over 50 years since 1942.

How can the classical stochastic analysis be developed in the context of infinite particle systems? We explain this by taking up the Yamada–Watanabe theory as an example, which is the method used to prove (**IFC**) in the previous section.

Roughly speaking, (a part of) the Yamada–Watanabe theory of stochastic differential equations derives the existence and uniqueness of strong solutions from the existence of weak solutions and pathwise uniqueness. When demonstrating the existence of strong solutions to stochastic differential equations, Lipschitz continuity and boundedness of the coefficients are required at least locally. If pathwise uniqueness of the solution is demonstrated, the Yamada–Watanabe theory requires only Lipschitz continuity of the coefficients. Hence, this theory provides a unique, strong solution under milder conditions than usual.

The details of the Yamada-Watanabe theory are complicated, but the point of the idea for deriving a strong solution from the existence of weak solutions and pathwise uniqueness is simple and presented below.

Let X and Y be two random variables defined on a Polish space that are (1) independent and (2) $X = Y$ a.s. Then X and Y are constant a.s. Taking this into account, let P and Q be the distributions of two given weak solutions (X, B) and (Y, B). Let $P_{x,B}$ and $Q_{x,B}$ be the conditional probabilities conditioned at the initial starting point as x and Brownian motion B. Note that B is common here. Under the product probability measure of $P_{x,B}$ and $Q_{x,B}$, we consider (X, B) and (Y, B). Then, by pathwise uniqueness, we deduce that $X = Y$ a.s. under the product probability measure. Hence, by the result above, X and Y are constant depending only on x and B. They are then functions of x and B; that is, X and Y are strong solutions. Here, measurability on (x, B) and others are non-trivial and require a delicate discussion, but the essence is exhausted by this idea.

The coefficients of the finite-dimensional stochastic differential equations in Section 6.1 contains $\mathbf{X}^{m*}$. Regarding $\mathbf{X}^{m*}$ as a random environment, the finite-dimensional stochastic differential equations are seen as those with random environments. These are different from the conventional one.

Even if we generalize the notion of a strong solution from the function of $(\mathbf{s}, \mathbf{B})$ to that of $(\mathbf{x}, \mathbf{B}, \mathbf{X}^{m*})$ and accordingly modify the relevant statements, some parts of the Yamada–Watanabe theory are still valid. Meanwhile, we can control the behavior of $\mathbf{X}^{m*}$ by the Dirichlet form of the unlabeled dynamics. We see that the coefficients of (6.1) are locally Lipschitz continuous if the interaction potentials are of Ruelle's class or are logarithmic functions. Hence, we obtain (**IFC**) in this case.

A single infinite-dimensional system with symmetry is
infinitely many finite-dimensional systems with consistency

is the grand design of the theory, and

Classic stochastic analysis can be carried in finite-dimensions
even if the random environment is equipped

is what we believe. We introduce a scheme of finite-dimensional objects depending on the problem and apply classical stochastic analysis to it via each finite-dimensional stochastic differential equation of random environment type. Then, we arrive with an object of the ISDE of infinite particle systems by consistency. This is the essence of the theory.

We introduced the scheme of the Dirichlet spaces describing the m-labeled dynamics for the construction of weak solutions in the first theory (Theorem 5.7). We considered the scheme of the finite-dimensional stochastic differential equations of random environment type for the existence and uniqueness of strong solutions of the ISDE in the second theory (Theorem 6.4). Therefore, schemes of finite-dimensions depend on the problems and differ from each other.

We expect that with this grand design above, we can perform an analogous classical stochastic analysis on a single particle in the world of infinite particle systems. Other than the two examples above, we expect that we can apply the strategy to many problems. Examples include the construction of stochastic flow of infinite particle systems, non-equilibrium solutions, a kind of smoothness of probability density of stochastic dynamics for the initial starting points, ergodicity of unlabeled dynamics, and ergodic decomposition of state space of unlabeled dynamics.

8. Two approximation schemes and uniqueness of Dirichlet forms.

Let $S_R = \{s \in S\,;\,|s| \le R\}$, and consider square fields capturing energy only inside S_R.

$$\mathbb{D}^a_R[f,g] = \frac{1}{2}\sum_{s_i \in S_R} a(s_i, \mathsf{s}^{i\diamondsuit})\frac{\partial \check{f}}{\partial s_i}\cdot\frac{\partial \check{g}}{\partial s_i}.$$

From this, we define the bilinear form

$$\underline{\mathcal{E}}^{a,\mu}_R(f,g) = \int_{\mathsf{S}} \mathbb{D}^a_R[f,g]d\mu.$$

From (**A1**), we see $(\underline{\mathcal{E}}^{a,\mu}_R, \mathcal{D}^{a,\mu}_\circ)$ is closable on $L^2(\mathsf{S},\mu)$. We then denote its closure by $(\underline{\mathcal{E}}^{a,\mu}_R, \underline{\mathcal{D}}^{a,\mu}_R)$.

Clearly, $(\underline{\mathcal{E}}^{a,\mu}_R, \underline{\mathcal{D}}^{a,\mu}_R)$ is increasing in $R \in \mathbb{N}$. Then, we denote by $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu})$, the closed form in the limit. It is known that $(\underline{\mathcal{E}}^{a,\mu}_R, \underline{\mathcal{D}}^{a,\mu}_R)$ converges to $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu})$ in the strong resolvent sense.

Let $\mathcal{D}^{a,\mu}_\circ$ be as in (5.3) and let $\pi^c_R(\mathsf{s}) = \mathsf{s}(\cdot \cap S^c_R)$. We set

$$\mathcal{D}^{a,\mu}_{\circ,R} = \{f \in \mathcal{D}^{a,\mu}_\circ; f \text{ is } \sigma[\pi^c_R]\text{-measurable}\}.$$

Suppose $f \in \mathcal{D}^{a,\mu}_{\circ,R}$. Then f is continuous on S and $\sigma[\pi^c_R]$-measurable by definition. Hence, $f(\mathsf{s})$ is constant in $\pi_R(\mathsf{s})$ and depends only on $\pi^c_R(\mathsf{s})$. In particular, when a single particle s_j in the configuration $\pi_R(\mathsf{s}) = \sum_{s_i \in S_R}\delta_{s_i}$ approaches the boundary ∂S, the limit points of $f(\mathsf{s})$ depends only on $\mathsf{s}' = \sum_{s_i \notin S_R}\delta_{s_i}$, which is the particles in s outside S_R.

By construction, $\mathcal{D}^{a,\mu}_\circ \supset \mathcal{D}^{a,\mu}_{\circ,R}$ and $\underline{\mathcal{E}}^{a,\mu}_R(f,f) = \mathcal{E}^{a,\mu}(f,f)$ for any $f \in \mathcal{D}^{a,\mu}_{\circ,R}$. Then

$$(\underline{\mathcal{E}}^{a,\mu}_R, \mathcal{D}^{a,\mu}_\circ) \le (\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_{\circ,R}).$$

Hence, from closability of $(\underline{\mathcal{E}}^{a,\mu}_R, \mathcal{D}^{a,\mu}_\circ)$ on $L^2(\mathsf{S},\mu)$ we see that of $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}_{\circ,R})$. We denote its closure by $(\mathcal{E}^{a,\mu}_R, \mathcal{D}^{a,\mu}_R)$. By construction

$$(\underline{\mathcal{E}}^{a,\mu}_R, \underline{\mathcal{D}}^{a,\mu}_R) \le (\mathcal{E}^{a,\mu}_R, \mathcal{D}^{a,\mu}_R). \tag{8.1}$$

Clearly, $(\underline{\mathcal{E}}^{a,\mu}_R, \underline{\mathcal{D}}^{a,\mu}_R)$ $(R \in \mathbb{R})$ are non-decreasing in R and converge in a strong resolvent sense to $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu})$. Similarly, $(\mathcal{E}^{a,\mu}_R, \mathcal{D}^{a,\mu}_R)$ $(R \in \mathbb{R})$ are non-increasing in R and converge in a strong resolvent sense to $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$. Therefore, with (8.1), we see

$$(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu}) \le (\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}). \tag{8.2}$$

We pursue the problem when the equality holds in (8.2), that is,

$$(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu}) = (\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu}). \tag{8.3}$$

This equality plays a critical role in Section 9.

A sufficient condition of (8.3) was given by [30]. We verify this condition in all examples but GAF in the present article. Following [30], we outline the idea of the proof of (8.3).

A Dirichlet form on $L^2(\mathsf{S},\mu)$ is a closed bilinear form with Markov property. If its domain is regular or, more generally, quasi-regular, then there exists an associated Markov process [14, 36]. This Markov process is a family of probability measures starting at each point of the state space S. The construction of the stationary Markovian process is immediate from the existence of the Markovian semi-group given by the bilinear form, whereas the construction of the Markov process starting at each point is not. The regularity of Dirichlet forms guarantees this. Once (quasi) regularity of Dirichlet forms is established, one uses the tools of stochastic analysis such as Fukushima decomposition. We have already proved quasi-regularity of $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu})$ in the proof of Theorem 5.3, and hence we can apply stochastic analysis. This is a crucial point of solving the ISDE in Theorem 5.7.

Although $(\underline{\mathcal{E}}^{a,\mu},\underline{\mathcal{D}}^{a,\mu})$ is a Dirichlet form, we do not know whether it is quasi-regular. Therefore, it is not clear whether $(\underline{\mathcal{E}}^{a,\mu},\underline{\mathcal{D}}^{a,\mu})$ gives a solution of ISDE (5.1) by the method valid for $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu})$.

However, $(\underline{\mathcal{E}}^{a,\mu},\underline{\mathcal{D}}^{a,\mu})$ has an associated Markovian semi-group on $L^2(\mathsf{S},\mu)$. Then, the stationary Markov process can be constructed. In [30], under a mild assumption on the coefficients of the ISDE, we proved the convergence of the solutions of the stochastic differential equations associated with $(\underline{\mathcal{E}}_R^{a,\mu},\underline{\mathcal{D}}_R^{a,\mu})$ to that associated with $(\underline{\mathcal{E}}^{a,\mu},\underline{\mathcal{D}}^{a,\mu})$ using the strong resolvent convergence of the Dirichlet forms. From this, we find that the stationary Markov process associated with $(\underline{\mathcal{E}}^{a,\mu},\underline{\mathcal{D}}^{a,\mu})$ is a solution of ISDE (5.1).

Here, $(\underline{\mathcal{E}}_R^{a,\mu},\underline{\mathcal{D}}_R^{a,\mu})$ is a regular Dirichlet form, and the particles are reflected on the boundary ∂S_R. From this, we deduce that finite particle systems satisfy finite-dimensional stochastic differential equations. Taking the limit with the strong resolvent convergence as above, we see the limit points are solutions of ISDE (5.1).

For the Markov semi-groups associated with Dirichlet forms $(\underline{\mathcal{E}}^{a,\mu},\underline{\mathcal{D}}^{a,\mu})$ and $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu})$, there exist stochastic processes that are solutions of (5.1) for almost-surely starting points. Hence, from the uniqueness in law of weak solutions starting for almost-surely starting points we obtain the coincidence of these two Markovian semi-groups, which implies the coincidence of the Dirichlet forms (8.3).

The critical idea is that the uniqueness of the Dirichlet forms follows from the uniqueness of the solutions of the associated ISDE. This result is due to Tanemura [61]. Originally, (8.3) is not a problem on stochastic differential equations, but on Dirichlet forms. We do not know a direct proof based on Dirichlet form theory.

We explain more about the behavior of particles in the bounded domain S_R.

As seen above, $(\underline{\mathcal{E}}_R^{a,\mu},\underline{\mathcal{D}}_R^{a,\mu})$ is a Dirichlet form posited in the reflecting boundary condition on S_R. The particles out of S_R are frozen and affect the behavior of particles inside S_R via interaction potentials. For particles inside, this effect is interpreted as an effect of free potentials. The particles are reflected from the boundary. Hence, local-time-type terms appear in the stochastic differential equations.

In contrast, $(\mathcal{E}_R^{a,\mu},\mathcal{D}_R^{a,\mu})$ is not a regular Dirichlet form. Hence, we can not construct the associated Markov (diffusion) process as is. We impose an equivalence relation on the configuration space, which is the state space of the Dirichlet form. Then, we construct the diffusion process associated with the Dirichlet form (with respect to the new domain). Note that S is equipped with a new topology. We emphasize here that the original Dirichlet forms are not regular.

Nevertheless, all is well. The limit Dirichlet form $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ of $(\mathcal{E}_R^{a,\mu}, \mathcal{D}_R^{a,\mu})$ is proved directly to be quasi-regular, and the associated diffusion process exists. Moreover, quasi-regularity enables us to apply stochastic analysis. That is, using the Fukushima decomposition and the schemes of the m-labeled Dirichlet forms, we can *directly* prove that the diffusion generates the solution of ISDE (5.1). We emphasize again; we do not show that the stochastic dynamics of the finite particles given by the Dirichlet form $(\mathcal{E}_R^{a,\mu}, \mathcal{D}_R^{a,\mu})$ is a solution of stochastic differential equations.

In summary, we have two natural Dirichlet forms $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu})$ and $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$. Both are the limits of different finite particle systems. When Lang [32] solved ISDE (5.1), he took the finite particle systems associated with $(\underline{\mathcal{E}}_R^{a,\mu}, \underline{\mathcal{D}}_R^{a,\mu})$. Therefore, his solutions are associated with $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu})$. Although the Dirichlet form $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ are natural, it was introduced in [41] and the difference between $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu})$ and $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ were distinguished from the view point of ISDE (5.1).

9. Dynamical universality of random matrices

In Section 3, we stated limits of the point processes consisting of eigenvalues of Gaussian random matrices as $N \to \infty$. In the last two decades, the universality of random matrices has been actively studied. There exist two types of frameworks for this problem. One is a generalization of Gaussian random variables to general random variables. Another is a generalization of free potentials x^2 in the logarithmic gas to general functions. They have been solved under quite general assumptions [62, 4]. The former is the extension of the classical central limit theorem to dependent variables originating with the eigenvalues of random matrices with independent variables. The latter also presents new results arising from the strong and long-range effects of the logarithmic potential.

The non-random probability measure appearing in Wigner's semi-circle law is a counterpart of the mean of the classical law of large numbers and varies depending on the models. In contrast, the rescaled point processes are much more universal and depend only on the macro-positions such as the bulk, the soft edge, and the hard edge in one dimension. They correspond to sine, Airy, and Bessel point process in one dimension. In two dimensions, the bulk corresponds to the Ginibre point process. These point fields have universality in the sense that they do not depend on the details of the models.

The results in [62, 4] demonstrated the weak convergence of the correlation functions, being on the same level as the classical central limit theorem. Several works are at a level of the local central limit theorem in the sense that they give a local uniform convergence of the correlation functions [1, 8].

We next state a dynamical counterpart. Recall that the equality $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu}) = (\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ in (8.3) holds under mild assumptions.

Theorem 9.1 (Kawamoto-O.[29]). *Assume $(\underline{\mathcal{E}}^{a,\mu}, \underline{\mathcal{D}}^{a,\mu}) = (\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$. Assume that the m-point correlation function of μ^N converges uniformly on each compact set to that of μ and the capacity of the zero points of the limit correlation functions vanish. Then the first m-component of the stochastic dynamics of the N-particle systems converge weakly in $C([0,\infty); S^m)$ to that of the limit stochastic dynamics for each m.*

The idea of the proof of Theorem 9.1 is an application of the generalized Mosco convergence due to Kuwae and Shioya [31] to two sequences of approximations of the Dirichlet forms.

Theorem 9.1 is valid not only for random matrices but also for a wide range of finite particle approximations μ^N. In particular, the two convergence theorems in Section 3 follow immediately from Theorem 9.1.

We present two applications; the necessary uniform convergence of correlation functions for Theorem 9.1 is already known for these examples. The assumption concerning the capacity of the zero points of the limit correlation functions follows from [43, 22].

9.1. **Universality of the Airy interaction Brownian motion.** Let $l \in \mathbb{N}$, $\kappa_{2l} > 0$ and $V(x) = \sum_{i=0}^{2l} \kappa_i x^i$. We set $\mu^N_{\mathrm{Ai},V}$ by

$$\mu^N_{\mathrm{Ai},V}(d\mathbf{s}^N) = \frac{1}{Z}\prod_{i<j}^{N}|s_i - s_j|^2 \prod_{k=1}^{N}\exp(-NV(N^{-\frac{1}{2l}}(c_N\Big(1+\frac{s_k}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N)))\, d\mathbf{s}^N.$$

Here, α_N, c_N, and d_N are constants depending on V and N [8]. The stochastic differential equation of the N-particle systems is

$$dX_t^{N,i} = dB_t^i + \sum_{j\neq i}^{N}\frac{1}{X_t^{N,i}-X_t^{N,j}}dt - \frac{N^{\frac{1}{3}-\frac{1}{2l}}c_N}{2\alpha_N}V'\Big(\frac{1}{N^{\frac{1}{2l}}}\{c_N\Big(1+\frac{X_t^{N,i}}{\alpha_N N^{\frac{2}{3}}}\Big)+d_N\}\Big)dt.$$

The solutions converge to the Airy interacting Brownian motion (2.8) as $N \to \infty$.

9.2. **Universality of the Ginibre interacting Brownian motion (Non-Hermitian model).** Let $\gamma \geq 0$, $K_p \in \mathbb{R}$, $\tau \in [0,1)$ be constants. We define the probability measure on the space $\mathcal{J}(N)$ of normal matrices of order N by

$$\sigma(J) = \frac{1}{\mathcal{Z}}\exp\Big\{-\frac{N}{1-\tau^2}\mathrm{Tr}(JJ^* - \frac{\tau}{2}(J^2+J^{*2})) - \gamma(\mathrm{Tr}JJ^* - NK_p)^2\Big\}.$$

Then the density of the eigenvalues is given by a constant multiple,

$$\prod_{i<j}^{N}|z_i - z_j|^2 \times \exp\Big\{-\frac{N}{1-\tau^2}\Big(\sum_{i=1}^{N}|z_i|^2 - \frac{\tau}{2}\sum_{i=1}^{N}(z_i^2+\bar{z}_i{}^2)\Big) - \gamma\Big(\sum_{i=1}^{N}|z_i|^2 - NK_p\Big)^2\Big\}.$$

Here c_1, c_2, $c_3 > 0$ are constants depending on K_p and γ, τ, and we set

$$E = \{z \in \mathbb{C}; c_1(\Re z)^2 + c_2(\Im z)^2 < 1\}.$$

Lemma 9.2 ([1, Theorem 1])**.** *For $\zeta \in E$, $k \in \mathbb{N}$, we have*

$$\lim_{N\to\infty}\frac{1}{N}\rho_N^1(\zeta) = \frac{c_3}{\pi}1_E(\zeta),$$
$$\frac{1}{(c_3N)^k}\rho_N^k\Big(\zeta+\frac{z_1}{\sqrt{c_3N}},\dots,\zeta+\frac{z_k}{\sqrt{c_3N}}\Big) = \rho^k_{\mathrm{gin}}(z_1,\dots,z_k) + o\Big(\frac{1}{\sqrt{N}}\Big).$$

The N particle systems $\mathbf{X}^N = (X^{N,i})$ satisfy for $i = 1,\dots,N$

$$dX_t^{N,i} = dB_t^i + \frac{1}{2}\Big\{\Big(\sum_{j\neq i}^{N}\frac{2(X_t^{N,i}-X_t^{N,j})}{|X_t^{N,i}-X_t^{N,j}|^2}\Big) - \frac{\tau N}{1-\tau^2}\Big(\zeta+\frac{X_t^{N,i}}{\sqrt{c_3N}}\Big)\frac{1}{\sqrt{c_3N}}$$
$$+\frac{\tau N}{1-\tau^2}\Big(\zeta+\frac{X_t^{N,i}}{\sqrt{c_3N}}\Big)^{\dagger}\frac{1}{\sqrt{c_3N}} - \Big(\zeta+\frac{X_t^{N,i}}{\sqrt{c_3N}}\Big)\frac{2\gamma}{\sqrt{c_3N}}\Big\{\sum_{k=1}^{N}\Big|\zeta+\frac{X_t^{N,k}}{\sqrt{c_3N}}\Big|^2 - NK_p\Big\}dt.$$

Here, $(x, y)^\dagger = (x, -y) \in \mathbb{R}^2$. Then $\mathbf{X}^{N,m} = (X^{N,1}, \dots, X^{N,m})$ converge weakly to the first m-components of the Ginibre interaction Brownian motion (2.10) as $N \to \infty$.

10. Ginibre, GAF, and vortices

In the final section, we introduce two more stochastic dynamics that look similar to the Ginibre interacting Brownian motion but have essentially different properties.

10.1. **Planner Gaussian analytic function.** Let F_{plane} be the entire function with random Gaussian coefficients such that [19, 16, 15].

$$F_{\text{plane}}(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\sqrt{k!}} z^k.$$

Here, $\{\xi_k\}$ is a sequence of i.i.d. with distribution $(1/\pi)e^{-|z|^2}dz$. F is called the planner Gaussian analytic function (GAF).

Let μ_{GAF} be the point process on $\mathbb{C}$ consisting of zero points of F_{plane}. It is known that μ_{GAF} is invariant under translations and rotations. It resembles the Ginibre point process but has a stronger rigidity than the Ginibre point process. Indeed, if we impose a condition outside of $S_r = \{|x| \le r\}$, then the mean of the inside particles is deterministic. This implies μ_{GAF} is not a quasi-Gibbs measure but has a density with respect to the low-dimensional Lebesgue measure [15]. By generalizing the first theory, we prove closability and quasi-regularity of the associated Dirichlet form and construct the diffusion. The existence of the logarithmic derivative of μ_{GAF} is known but its explicit representation is not known. It does not come from the sum of the derivatives of two-body potentials, and hence may be difficult to describe. [1]

Random analytic functionals appear in various situations. Therefore, point processes consisting of zero points or poles of random analytic functions by the method described in the present article would be interesting to investigate.

10.2. **Vortex equation.** We conclude this article with an ISDE not yet solved. Consider infinite-many vortices in a viscous fluid in the whole plane. Taking the Ginibre interacting Brownian motion into account, we suppose all the vortices have the same sign and the same strength. Let $\mathbf{X}_t = (X_t^i)_{i\in\mathbb{N}} \in (\mathbb{R}^2)^{\mathbb{N}}$ be the position of the i vortices. Choosing suitable viscosity and vorticity, we have

$$dX_t^i = dB_t^i + \sum_{j\ne i}^{\infty} \frac{(X_t^i - X_t^j)^\dagger}{|X_t^i - X_t^j|^2} dt \qquad (i \in \mathbb{N}).$$

The ISDE is the same as the Ginibre ISDE except for the presence of $\dagger$ in the sum. The dynamics behind the ISDE is very different from the Ginibre interacting Brownian motion. Indeed, this describes a skew motion of the particles.

With the number of vortices finite, the equation was solved for arbitrary vorticities of the vortices. Indeed, the associated heat equation is of generalized divergence form in the sense of [38] and was solved by using the global Gaussian both sides esitimates [38]. Using this estimate, an unique strong solution of the stochastic differential equation was obtained [38, 39, 40].

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[1] We obtained an explicit formula of the logarithmic derivative of the planner GAF after the original review was published in Japanese.

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