Infinite-dimensional Stochastic Differential Equations with Symmetry

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Abstract We review recent progress in the study of infinite-dimensional stochastic differential equations with symmetry. This paper contains examples arising from random matrix theory.

1 Introduction

We consider $(\mathbb{R}^d)^{\mathbb{N}}$-valued infinite-dimensional stochastic differential equations (ISDEs) of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ of the form

$$dX_t^i = \sigma(X_t^i, \mathbf{X}_t^{i\diamondsuit})dB_t^i + b(X_t^i, \mathbf{X}_t^{i\diamondsuit})dt \quad (i \in \mathbb{N}). \tag{1}$$

Here $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is $(\mathbb{R}^d)^{\mathbb{N}}$-valued standard Brownian motion. For $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ we set $\mathbf{X}^{i\diamondsuit} = (X^j)_{j\in\mathbb{N}\setminus\{i\}}$. Coefficients $\sigma(x,\mathbf{y})$ and $b(x,\mathbf{y})$ are defined on a subset $\mathbf{S_0}$ of $\mathbb{R}^d \times (\mathbb{R}^d)^{\mathbb{N}}$ independent of $i \in \mathbb{N}$. By definition $\sigma(x,\mathbf{y})$ is $\mathbb{R}^{d^2}$-valued, and $b(x,\mathbf{y})$ is $\mathbb{R}^d$-valued. We assume that $\sigma(x,\mathbf{y})$ and $b(x,\mathbf{y})$ are symmetric in $\mathbf{y} = (y_i)_{i\in\mathbb{N}}$ for each $x \in \mathbb{R}^d$. Therefore, the set of (1) is referred to as "ISDEs with symmetry". In the present article, we review recent results in this regard. Using a Dirichlet form technique and an analysis on tail σ-fields of configuration spaces, we prove the existence and pathwise uniqueness of strong solutions of the ISDEs of (1). We emphasize that the coefficients are defined only on a *thin* subset in $(\mathbb{R}^d)^{\mathbb{N}}$ and the state space of the solution $\mathbf{X}$ is in this subset. Solving the ISDEs of (1) includes identifying such a subset.

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Let $\mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i};\ \mathsf{s}(K) < \infty \text{ for any compact } K \subset \mathbb{R}^d\}$ be the configuration space over $\mathbb{R}^d$. S is a Polish space equipped with the vague topology. With the symmetry of $\sigma(x,\mathbf{y})$ and $b(x,\mathbf{y})$ in $\mathbf{y}$, we regard σ and b as functions on $\mathbb{R}^d \times \mathsf{S}$. We denote these by the same symbol such that $\sigma(x,\mathbf{y}) = \sigma(x,\mathsf{y})$ and $b(x,\mathbf{y}) = b(x,\mathsf{y})$, where $\mathsf{y} = \sum_i \delta_{y_i}$ for $\mathbf{y} = (y_i)$. Then we rewrite the ISDEs of (1) for $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ as:

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i\diamondsuit})dB_t^i + b(X_t^i, \mathsf{X}_t^{i\diamondsuit})dt \quad (i \in \mathbb{N}). \tag{2}$$

Here $\mathsf{X}^{i\diamondsuit} = \sum_{j\neq i}^{\infty} \delta_{X^j}$, which is the S-valued process $\{\sum_{j\neq i}^{\infty} \delta_{X_t^j}\}$. $\mathbf{X}$ is called the *labeled* dynamics, and the associated *unlabeled* dynamics X is given by $\mathsf{X} = \sum_{i\in\mathbb{N}} \delta_{X^i}$.

If σ is a unit matrix and b is given by a pair interaction $\Psi(x-y)$, (1) becomes

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{\infty} \nabla\Psi(X_t^i - X_t^j)dt. \tag{3}$$

Here $\beta > 0$ is inverse temperature. Having Ψ of Ruelle class and $\Psi \in C_0^3(\mathbb{R}^d)$, Lang [10, 11] then solved (3), Fritz [2] constructed non-equilibrium solutions for $d \leq 4$, and Tanemura [25] provided solutions for hard core Brownian balls. The stochastic dynamics $\mathbf{X}$ given by the solution of (3) are called the interacting Brownian motions (IBM).

These solutions are strong solutions in the sense that $\mathbf{X}$ are functionals of the given Brownian motions $\mathbf{B}$ and initial starting points $\mathbf{s}$. The method used in these studies are based on the classic Itô scheme. Hence, if Ψ is of long range such as a polynomial decay, then it is difficult to apply this scheme. Tsai [26] solved (3) for the Dyson model. He used very cleverly a specific monotonicity of the logarithmic potential and its one-dimensional structure. As for the weak solution, we present a robust method based on the Dirichlet form technique from [15]. We present a general theory to give μ-pathwise unique strong solutions applicable to the logarithmic interaction from [22].

Thus, our demonstration is divided into two steps. In the first step, we obtain weak solutions of ISDEs (1). That is, we construct solutions $(\mathbf{X}, \mathbf{B})$ satisfying (1) (see Section 2–Section 4). In the second step, we prove the existence of strong solutions and the μ-pathwise uniqueness. For this, we perform a fine analysis of the tail σ-field of S (see Section 5 and Section 6). In Section 7, we give ISDEs arising from random matrix theory. In Section 8, we present the algebraic construction of the dynamics, and the coincidence of the algebraic dynamics with solutions of ISDEs.

2 Unlabeled dynamics: quasi-Gibbs property

We next construct a natural μ-reversible unlabeled diffusion, where μ is a point process. The key point is the quasi-Gibbs property of μ, which we proceed to describe.

Let $S_r = \{x \in \mathbb{R}^d; |x| < r\}$. Let $\pi_r, \pi_r^c : \mathsf{S} \to \mathsf{S}$ be projections such that $\pi_r(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r)$, $\pi_r^c(\mathsf{s}) = \mathsf{s}(\cdot \cap S_r^c)$. For a point process μ, we set

$$\mu_{r,\mathsf{t}}^{m}(\cdot) = \mu(\pi_r(\mathsf{s}) \in \cdot | \mathsf{s}(S_r) = m, \pi_r^c(\mathsf{s}) = \pi_r^c(\mathsf{t}))$$

Let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\Psi : (\mathbb{R}^d)^2 \to \mathbb{R} \cup \{\infty\}$ be potentials. We set

$$\mathscr{H}_r = \sum_{s_i \in S_r} \Phi(s_i) + \sum_{s_i, s_j \in S_r, i<j} \Psi(s_i, s_j).$$

A point process μ is called a canonical Gibbs measure if μ satisfies Dobrushin-Lanford-Ruelle (DLR) equation, that is, for μ-a.s. $\mathsf{t} = \sum_j \delta_{t_j}$

$$\mu_{r,\mathsf{t}}^{m} = c_{r,\mathsf{t}}^{m} e^{-\mathscr{H}_r - \sum_{x_i \in S_r, t_j \in S_r^c} \Psi(x_i, t_j)} d\Lambda_r^m. \tag{4}$$

Here $\Lambda_r^m = \Lambda(\cdot | \mathsf{s}(S_r) = m)$ and Λ_r is the Poisson PP with intensity $1_{S_r} dx$.

Point processes appear in random matrix theory in the form sine, Airy, Bessel, and Ginibre point processes having logarithmic potentials

$$\Psi(x,y) = -\beta \log |x-y|.$$

However, the DLR equation (4) does not make sense for a logarithmic potential. Hence we introduce the notion of quasi-Gibbs measures:

Definition 1. μ is (Φ, Ψ)-quasi-Gibbs measure if $\exists\, c_{r,\mathsf{t}}^{m}$ such that

$$c_{r,\mathsf{t}}^{m\ -1} e^{-\mathscr{H}_r} d\Lambda_r^m \le \mu_{r,\mathsf{t}}^{m} \le c_{r,\mathsf{t}}^{m} e^{-\mathscr{H}_r} d\Lambda_r^m$$

By definition a canonical Gibbs measure is a quasi-Gibbs measure. We refer to [16, 17] for a sufficient condition for quasi-Gibbs property. We assume:
(A1) μ is a quasi-Gibbs measure with upper semi-continuous (Φ, Ψ). Furthermore, $a(x,\mathsf{s}) = \sigma^t(x,\mathsf{s})\sigma(x,\mathsf{s})$ is bounded and uniformly elliptic.
(A2) There exists a $1 < p \le \infty$ such that the k-point correlation function ρ^k of μ is in $L^p_{\mathrm{loc}}((\mathbb{R}^d)^m)$ for each $k \in \mathbb{N}$.

For a given point process μ we introduce a Dirichlet form such that

$$\mathscr{E}^{\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}[f,g] d\mu, \quad \mathbb{D}[f,g] = \frac{1}{2} \sum_i a(s_i, \mathsf{s}^{i\diamondsuit}) \frac{\partial \check{f}}{\partial s_i} \cdot \frac{\partial \check{g}}{\partial s_i}. \tag{5}$$

Here we set $\mathsf{s}^{i\diamondsuit} = \sum_{j \ne i} \delta_{s_j}$ for $\mathsf{s} = \sum_i \delta_{s_i}$, and $f(\mathsf{s}) = \check{f}(s_1, s_2, \ldots)$, where $\check{f}$ is symmetric in $(s_1, s_2, \ldots)$. Note that $\mathbb{D}[f,g]$ is a function of s by construction.

A function f defined on the configuration space S is called local if f is $\sigma[\pi_r]$-measurable for some $r \in \mathbb{N}$. f is called smooth if $\check{f}$ is smooth. Let $\mathscr{D}_\circ$ be the set of local, smooth functions on S. We set $\mathscr{D}_\circ^{\mu} = \{f \in \mathscr{D}_\circ \cap L^2(\mu) ; \mathscr{E}^{\mu}(f,f) < \infty\}$.

Theorem 1 ([13, 16, 22]). (i) *Assume* (A1). *Then* $(\mathscr{E}^{\mu}, \mathscr{D}_\circ^{\mu})$ *is closable on* $L^2(\mu)$.
(ii) *Assume* (A1) *and* (A2). *Then there exists a diffusion* $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ *associated with the closure* $(\mathscr{E}^{\mu}, \mathscr{D}^{\mu})$ *of* $(\mathscr{E}^{\mu}, \mathscr{D}_\circ^{\mu})$ *on* $L^2(\mu)$.

The local boundedness of the correlation functions is used for the quasi-regularity of the Dirichlet form. Once quasi-regularity is established, the existence of μ-reversible diffusion is immediate from the general theory [12, 3].

Unlabeled dynamics are also obtained in [1, 27] with a different frame work. It is now proved these are the same dynamics as in [21, 20]. We remark that ergodicity of unlabeled dynamics with grand canonical Gibbs measures with small enough activity constant is obtained in [1].

3 Labeled dynamics: A scheme of Dirichlet spaces

We next lift the unlabeled dynamics X in Theorem 1 to a labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ solving (1). For this we present a natural scheme of Dirichlet spaces describing the labeled dynamics $\mathbf{X}$. We assume a pair of mild assumptions:
(A3) $\{X^i\}$ do not collide with each other (non-collision)
(A4) each tagged particle X^i never explode (non-explosion)

Let $\mathsf{S}_{s,i} = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(\{x\}) = 0 \text{ for all } x \in S,\ \mathsf{s}(S) = \infty\}$. Then (A3) is equivalent to $\mathrm{Cap}^{\mu}(\mathsf{S}_{s,i}^{c}) = 0$. (A4) follows from $\rho^1(x) = O(e^{|x|^{\alpha}})$, $\alpha < 2$.

We call $\mathfrak{u}$ the unlabeling map if $\mathfrak{u}((s_i)) = \sum_i \delta_{s_i}$. We call $\mathfrak{l}$ a label if $\mathfrak{l}$ is defined for μ-a.s. s, and $\mathfrak{u}\circ\mathfrak{l}(\mathsf{s}) = \mathsf{s}$. For a unlabeled dynamics satisfying (A3) and (A4), the particles can keep the initial label $\mathfrak{l}(\mathsf{s})$. Thus we can construct a map $\mathfrak{l}_{\mathrm{path}}$ to $C([0,\infty);(\mathbb{R}^d)^{\mathbb{N}})$ such that $\{\mathfrak{u}(\mathfrak{l}_{\mathrm{path}}(\mathsf{X}_t))\}_{t\in[0,\infty)} = \mathsf{X}$. Hence we obtain:

Theorem 2 ([14]). *Assume* (A1)–(A4)*. Then there exists a labeled dynamics* $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ *such that* $\mathsf{X} = \sum_{i\in\mathbb{N}} \delta_{X^i}$ *and that* $\mathbf{X}_0 = \mathfrak{l}(\mathsf{X}_0)$.

Remark that $(\mathbb{R}^d)^{\mathbb{N}}$ has no good measures. Then no Dirichlet forms on $(\mathbb{R}^d)^{\mathbb{N}}$ associated with the labeled dynamics $\mathbf{X}$ exist. We hence introduce the scheme of spaces $(\mathbb{R}^d)^m \times \mathsf{S}$ with Campbell measures $\mu^{[m]}$ such that $d\mu^{[m]} = \rho^m(\mathbf{x}_m)\mu_{\mathbf{x}_m}(d\mathsf{s})d\mathbf{x}_m$, where ρ^m is the m-point correlation function of μ and $\mu_{\mathbf{x}_m}$ is the reduced Palm measure conditioned at $\mathbf{x}_m$. For $a = \sigma^t\sigma$, let $\mathbb{D}^{[m]}$ be the square field on $(\mathbb{R}^d)^m \times \mathsf{S}$ defined similarly as $\mathbb{D}$ on S given by (5) in Section 2. Let

$$\mathscr{E}^{[m]}(f,g) = \int_{(\mathbb{R}^d)^m\times\mathsf{S}} \mathbb{D}^{[m]}[f,g]d\mu^{[m]}.$$

Let $\mathscr{D}_{\circ}^{[1],\mu} = \{f \in C_0^{\infty}(\mathbb{R}^d)\otimes\mathscr{D}_{\circ}; \mathscr{E}^{[m]}(f,f) < \infty, f \in L^2(\mu^{[m]})\}$.

Theorem 3 ([14]). *Assume* (A1) *and* (A2)*. Then* $(\mathscr{E}^{[m]}, \mathscr{D}_{\circ}^{[1],\mu})$ *is closable on* $L^2(\mu^{[m]})$*, and its closures is quasi-regular. Hence the associated diffusion* $(\mathbf{X}^{[m]}, \mathsf{X}^{[m]*})$ *exists. Here we write* $(\mathbf{X}^{[m]}, \mathsf{X}^{[m]*}) = (X^{[m],1}, \ldots, X^{[m],m}, \sum_{i=m+1}^{\infty} \delta_{X^{[m],i}})$.

Let $(\mathscr{E}^{\mu}, \mathscr{D}^{\mu}, L^2(\mu))$ be the original Dirichlet form. Let $\mathsf{X} = \sum_{i=1}^{\infty}\delta_{X^i}$ be the associated unlabeled diffusion. We fix a label $\mathfrak{l}$. Let $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be the labeled dynamics given by $\mathfrak{l}$. We set $(\mathbf{X}^m, \mathsf{X}^{m*}) = (X^1, \ldots, X^m, \sum_{i=m+1}^{\infty}\delta_{X^i})$.

Theorem 4 ([14]). *Assume* (A1)–(A4). *Assume* $(\mathbf{X}^{[m]}, \mathsf{X}^{[m]*})$ *and* $(\mathbf{X}^{m}, \mathsf{X}^{m*})$ *start at the same initial point. Then* $(\mathbf{X}^{[m]}, \mathsf{X}^{[m]*}) = (\mathbf{X}^{m}, \mathsf{X}^{m*})$ *in distribution for each* $m \in \mathbb{N}$.

Instead of the huge space $(\mathbb{R}^d)^{\mathbb{N}}$, we use a scheme of countably infinite *good* infinite-dimensional spaces $\{(\mathbb{R}^d)^m \times \mathsf{S}\}_{m\in\{0\}\cup\mathbb{N}}$. Using the diffusion X on the original unlabeled space S, we construct a scheme of the *coupled* diffusions $(\mathbf{X}^{m}, \mathsf{X}^{m*})$ on $(\mathbb{R}^d)^m \times \mathsf{S}$ associated with the scheme of Dirichlet spaces $(\mathscr{E}^{[m]}, L^2(\mu^{[m]}))$ on $(\mathbb{R}^d)^m \times \mathsf{S}$. This construction is key for the ISDE-representation below.

4 ISDE-representation: Logarithmic derivative

We pursue the ISDE describing the labeled dynamics $\mathbf{X}$ obtained in Theorem 2. The key notion for this is the logarithmic derivative of μ introduced below.

Definition 2 ([15]). Let ∇_x be the nabla on $\mathbb{R}^d$. $\mathsf{d}^{\mu} \in L^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathsf{S}, \mu^{[1]})^d$ is called the *logarithmic derivative* of μ if, for all $f \in C_0^{\infty}(\mathbb{R}^d) \otimes L^{\infty}(\mathsf{S})$,

$$\int_{\mathbb{R}^d\times\mathsf{S}} \nabla_x f d\mu^{[1]} = -\int_{\mathbb{R}^d\times\mathsf{S}} f \mathsf{d}^{\mu} d\mu^{[1]}. \tag{6}$$

Let $a(x,\mathsf{s}) = \sigma^t(x,\mathsf{s})\sigma(x,\mathsf{s})$ as before. We set $\nabla_x a$ such that $\nabla_x a = \sum_{j=1}^d \frac{\partial}{\partial x_j} a_{ij}$. We introduce a "geometric" differential equation on $\mathsf{d}^{\mu}(x,\mathsf{s}) =: \nabla_x \log \mu^{[1]}(x,\mathsf{s})$:

$$\nabla_x a(x,\mathsf{s}) + a(x,\mathsf{s})\nabla_x \log \mu^{[1]}(x,\mathsf{s}) = 2b(x,\mathsf{s}). \tag{7}$$

(A5) μ has a logarithmic derivative d^{μ}.
(A6) The logarithmic derivative d^{μ} satisfies (7).

Theorem 5 ([15]). *Assume* (A1)–(A6). *Then there exists an* $\mathsf{S}_0 \subset \mathsf{S}$ *such that* $\mu(\mathsf{S}_0) = 1$ *and that, for each* $\mathbf{s} \in \mathfrak{u}^{-1}(\mathsf{S}_0)$*, ISDE* (1) *has a solution* $(\mathbf{X}, \mathbf{B})$ *satisfying* $\mathbf{X}_0 = \mathbf{s}$ *and* $\mathbf{X}_t \in \mathfrak{u}^{-1}(\mathsf{S}_0)$ *for all* t.

From the coupling in Theorem 4 and Fukushima decomposition (Itô formula), we prove that $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ satisfies the ISDEs of (1). We use the m-labeled process $(\mathbf{X}^{m}, \mathsf{X}^{m*})$, to apply Itô formula to coordinate functions $x_1, \ldots, x_m$.

5 Strong solutions of ISDEs and pathwise uniqueness

We lift the weak solutions $(\mathbf{X}, \mathbf{B})$ to pathwise unique strong solutions. For this purpose, we introduce a scheme consisting of *an infinite system of finite-dimensional SDEs with consistency* (IFC), and perform an analysis of the tail σ-field of the path space $W((\mathbb{R}^d)^{\mathbb{N}}) = C([0,T]; (\mathbb{R}^d)^{\mathbb{N}})$. The key idea is the following interpretations:

- a single ISDE $\Longleftrightarrow$ a scheme of IFC.

• the tail σ-field of $W((\mathbb{R}^d)^{\mathbb{N}}) \iff$ the boundary condition of the ISDEs.

The method is robust and may be applied to many other models. We consider *non-Markovian* ISDEs because the argument is general. Let $(W_{\text{sol}}, \mathbf{S_0}, \{\sigma^i\}, \{b^i\})$ be

$$\begin{aligned}
&W_{\text{sol}}: \text{ a Borel subset of } W((\mathbb{R}^d)^{\mathbb{N}}) && \text{(space of solutions of the ISDEs)},\\
&\mathbf{S_0}: \text{a Borel subset of } (\mathbb{R}^d)^{\mathbb{N}} && \text{(initial starting points of the ISDEs)},\\
&\sigma^i, b^i: W_{\text{sol}} \to W((\mathbb{R}^d)^{\mathbb{N}}) && \text{(coefficients of the ISDEs)}.
\end{aligned}$$

We consider the ISDEs on $(\mathbb{R}^d)^{\mathbb{N}}$ of the form: $\mathbf{X} = \{(X_t^i)_{i\in\mathbb{N}}\}_{t\in[0,T]} \in W((\mathbb{R}^d)^{\mathbb{N}})$

$$\begin{aligned}
&dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}),\\
&\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S_0}, \quad \mathbf{X} \in W_{\text{sol}}.
\end{aligned} \tag{8}$$

Note that (1) is a special case of (8). We assume:

(P1) The ISDEs (8) has a weak solution $(\mathbf{X}, \mathbf{B})$. (not a strong solution!)

From a weak solution $(\mathbf{X}, \mathbf{B})$, we define a new SDE of $\mathbf{Y}^m = (Y_t^{m,i})_{i=1}^m$ such that

$$\begin{aligned}
&dY_t^{m,i} = \sigma^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dt \quad (i = 1, \dots, m),\\
&\mathbf{Y}_0^m = (s_1, \dots, s_m), \quad (\mathbf{Y}^m, \mathbf{X}^{m*}) \in W_{\text{sol}}
\end{aligned} \tag{9}$$

for each $\mathbf{X} \in W_{\text{sol}}^{\mathbf{s}}$, $\mathbf{s} = (s_i)_{i=1}^{\infty} \in \mathbf{S_0}$, and $m \in \mathbb{N}$. Here $\mathbf{X}^{m*} := (X^n)_{n>m}$ is interpreted as a part of the coefficients of the SDE (9) and $W_{\text{sol}}^{\mathbf{s}} = \{\mathbf{X} \in W_{\text{sol}}; \mathbf{X}_0 = \mathbf{s}\}$. Indeed, we regard (9) as *finite-dimensional* SDEs of $\mathbf{Y}^m$. (9) become automatically time-inhomogeneous SDEs. We have therefore obtained a scheme of finite-dimensional SDEs of $\{\mathbf{Y}^m\}_{m\in\mathbb{N}}$. We assume:

(P2) The SDE (9) has a unique, strong solution for each $\mathbf{s} \in \mathbf{S_0}$, $\mathbf{X} \in W_{\text{sol}}^{\mathbf{s}}$, and $m \in \mathbb{N}$.

Let $\bar{P}_{\mathbf{s}}$ be the distribution of solution $(\mathbf{X}, \mathbf{B})$ on $W_{\text{sol}}^{\mathbf{s}} \times W((\mathbb{R}^d)^{\mathbb{N}})$. Let

$$\begin{aligned}
&\bar{P}_{\mathbf{s},\mathbf{B}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot | \mathbf{B}) \text{ and } P_{\text{Br}}^{\infty} = \bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot),\\
&\mathscr{T}_{\text{path}}((\mathbb{R}^d)^{\mathbb{N}}) = \cap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}],\\
&\mathscr{T}_{\text{path}}^{(1)}[\bar{P}_{\mathbf{s},\mathbf{B}}] = \{\mathbf{A} \in \mathscr{T}_{\text{path}}((\mathbb{R}^d)^{\mathbb{N}}); \bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) = 1\}.
\end{aligned}$$

(P3) $\mathscr{T}_{\text{path}}((\mathbb{R}^d)^{\mathbb{N}})$ is $\bar{P}_{\mathbf{s},\mathbf{B}}$-trivial for each $\mathbf{s} \in \mathbf{S_0}$ and P_{Br}^{∞}-a.s. $\mathbf{B}$.

Theorem 6 ([22]). *Assume* (P1)–(P3). *Then*
(i) $\mathbf{X}$ *is a strong solution of the ISDEs* (8) *for each* $\mathbf{s} \in \mathbf{S}_0$.
(ii) *Let* $\mathbf{X}_{\mathbf{s}}$ *and* $\mathbf{X}'_{\mathbf{s}}$ *be strong solutions of the ISDEs* (8) *starting at* $\mathbf{s} \in \mathbf{S}_0$ *defined on the same space of Brownian motions* $\mathbf{B}$. *Then* $\mathbf{X}_{\mathbf{s}} = \mathbf{X}'_{\mathbf{s}}$ *for* P_{Br}^{∞}*-a.s.* $\mathbf{B}$ *if and only if* $\mathscr{T}_{\text{path}}^{(1)}[\bar{P}_{\mathbf{s},\mathbf{B}}] = \mathscr{T}_{\text{path}}^{(1)}[\bar{P}'_{\mathbf{s},\mathbf{B}}]$ *for* P_{Br}^{∞}*-a.s.* $\mathbf{B}$.

Idea of the proof (i): Let $(\mathbf{X}, \mathbf{B})$ be a weak solution of ISDE given by (P1), and fix it. Let $\mathbf{Y}^m$ be the unique strong solution of (9) given by (P2). By construction $\mathbf{Y}^m$ is $\sigma[\mathbf{B}] \bigvee \sigma[\mathbf{X}^{m*}]$-measurable. Because the solution (9) is unique, we see that

$\mathbf{Y}^m = \mathbf{X}^m := (X^1, \dots, X^m)$. Let $\mathbf{Y}$ be the limit $\mathbf{Y} = \lim_{m\to\infty} \mathbf{Y}^m$. Then $\mathbf{Y} = \mathbf{X}$ and $\mathbf{Y}$ is $\sigma[\mathbf{B}] \bigvee \mathscr{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$-measurable. Because $\mathscr{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ is $\bar{P}_{\mathbf{s},\mathbf{B}}$-trivial by ($\mathsf{P3}$), $\mathbf{Y}$ depends only on $\mathbf{s}$ and $\mathbf{B}$. This means $\mathbf{Y} = \mathbf{X}$ is a strong solution. □

In [22] we intoduce a notion of IFC solution, with which we generalize Theorem 6.

6 Tail triviality: Application to interacting Brownian motions.

We return to the Markovian-type ISDEs of (1). We assume ($\mathsf{A1}$)–($\mathsf{A6}$). We apply Theorem 6 to ISDEs of (1) by checking ($\mathsf{P1}$)–($\mathsf{P3}$). ($\mathsf{P1}$) follows from Theorem 5. Controlling the capacity of $(\mathscr{E}^{\mu}, \mathscr{D}^{\mu}, L^2(\mu))$ we obtain ($\mathsf{P2}$). Because $\mathbf{X} \in W_{\mathrm{sol}}$ and W_{sol} is a nice subset of $W((\mathbb{R}^d)^{\mathbb{N}})$, we can assume ($\mathsf{P2}$) for the solution of (1). Dirichlet form theory proves that $\mathbf{X}$ stays in W_{sol}. Indeed, such a condition is reduced to a calculation of capacity related to the unlabeled Dirichlet space [3, 12]. Roughly speaking, ($\mathsf{P2}$) is satisfied if $\nabla_x^m \mathsf{d}^{\mu} \in \mathscr{D}_{\mathrm{loc}}^{\mu^{[1]}}$ for a suitable m, see [22, Sections 8,9].

Theorem 7 ([22]). *Assume* ($\mathsf{Q1}$)–($\mathsf{Q3}$) *below. Then* ($\mathsf{P3}$) *holds.*
($\mathsf{Q1}$) μ *is tail trivial. That is,* $\mu(A) \in \{0,1\}$ *for all* $A \in \mathscr{T}(\mathsf{S}) := \cap_{r=1}^{\infty} \sigma[\pi_r^c]$.
($\mathsf{Q2}$) $P_{\mu} \circ \mathsf{X}_t^{-1} \prec \mu$ *for all* t. *(absolute continuity condition).*
($\mathsf{Q3}$) $P_{\mu}(\cap_{r=1}^{\infty} \{\mathsf{m}_r(\mathsf{X}) < \infty\}) = 1$,
where $\mathsf{m}_r = \inf\{m \in \mathbb{N}; X^i \in C([0,T]; S_r^c)$ *for* $m < \forall i \in \mathbb{N}\}$ *for* $\mathsf{X} = \sum_{i\in\mathbb{N}} \delta_{X^i}$.

Remark 1. (i) Determinantal point processes satisfy ($\mathsf{Q1}$) (see [18]).
(ii) ($\mathsf{Q2}$) is obvious because the unlabeled dynamics is μ-reversible.
(iii) ($\mathsf{Q3}$) is satisfied if the one-point correlation function ρ^1 satisfies$\rho^1(x) = O(e^{|x|^{\alpha}})$ $(|x| \to \infty)$ for some $\alpha < 2$.

Let $\mathbf{T} = \{\mathbf{t} = (t_1, \dots, t_m)\,;\, t_i \in [0,T], m \in \mathbb{N}\}$ and $\mathbf{X}_{\mathbf{t}}^{n*} = (\mathbf{X}_{t_1}^{n*}, \dots, \mathbf{X}_{t_m}^{n*})$. Let

$$\tilde{\mathscr{T}}_{\mathrm{path}}(\mathsf{S}) = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{r=1}^{\infty} \sigma[\pi_r^c(\mathsf{X}_{\mathbf{t}})], \quad \tilde{\mathscr{T}}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}].$$

Hence by definition, $\tilde{\mathscr{T}}_{\mathrm{path}}(\mathsf{S})$ is the cylindrical tail σ-field of the unlabeled path space and $\tilde{\mathscr{T}}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ is the cylindrical tail σ-field of the labeled path space $W((\mathbb{R}^d)^{\mathbb{N}})$. We deduce the triviality of $\mathscr{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}})$ from that of S. We do this step-by-step following the scheme:

$$\begin{array}{cccc} \mathscr{T}(\mathsf{S}) \xrightarrow[(\mathsf{Q1}),(\mathsf{Q2})]{\text{(Step I)}} & \tilde{\mathscr{T}}_{\mathrm{path}}(\mathsf{S}) \xrightarrow[(\mathsf{Q3})]{\text{(Step II)}} & \tilde{\mathscr{T}}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) \xrightarrow[\text{(IFC)}]{\text{(Step III)}} & \mathscr{T}_{\mathrm{path}}((\mathbb{R}^d)^{\mathbb{N}}) \\ \mu & \mathsf{P}_{\mu} & \mathbf{P}_{\mu^{\mathfrak{l}}} = \int \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot) d\mu^{\mathfrak{l}} & \bar{P}_{\mathbf{s},\mathbf{B}} \text{ a.s.}(\mathbf{s},\mathbf{B}). \end{array}$$

We denote by $\mathscr{B}(\mathsf{S})^{\mu}$ the completion of the σ-field $\mathscr{B}(\mathsf{S})$ with respect to μ.

Definition 3. For $\mathbf{s} \in (\mathbb{R}^d)^{\mathbb{N}}$, we set $\mathbf{X_s} = \{\mathbf{X}_{\mathbf{s},t}\}_{t\in[0,\infty)}$ such that $\mathbf{X}_{\mathbf{s},0} = \mathbf{s}$. We set

$$\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})} = \{\{\mathbf{X}_{\mathbf{s},t}\}_{t\in[0,\infty)}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})}.$$

(i) We call $(\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})}, \mathbf{B})$ a μ-solution of (1) if $\mathsf{H} \in \mathscr{B}(\mathsf{S})^{\mu}$ satisfying $\mu(\mathsf{H}) = 1$ and $\mathfrak{u}^{-1}(\mathsf{H}) \subset \mathbf{S_0}$, and if $(\mathbf{X_s}, \mathbf{B})$ is a solution of ISDE (1) for each $\mathbf{s} \in \mathfrak{l}(\mathsf{H})$.
(ii) We call $(\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})}, \mathbf{B})$ a μ-strong solution if it is a μ-solution such that $(\mathbf{X_s}, \mathbf{B})$ is a strong solution for each $\mathbf{s} \in \mathfrak{l}(\mathsf{H})$.

Definition 4. We say that the μ-strong uniqueness holds if the following holds.
(i) The μ-uniqueness in law holds. That is, $\mathbf{X_s} = \mathbf{X}'_\mathbf{s}$ in law for each $\mathbf{s} \in \mathfrak{l}(\mathsf{H}\cap\mathsf{H}')$ for any pair of μ-solutions $(\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})}, \mathbf{B})$ and $(\{\mathbf{X}'_\mathbf{s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H}')}, \mathbf{B}')$ satisfying (Q2).
(ii) A μ-solution $(\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})}, \mathbf{B})$ satisfying (Q2) is a μ-strong solution $(\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H}')}, \mathbf{B})$ for some $\mathsf{H}' \subset \mathsf{H}$.
(iii) The μ-pathwise uniqueness holds. That is, $P^{\mathbf{B}}(\mathbf{X_s} = \mathbf{X}'_\mathbf{s}) = 1$ for each $\mathbf{s} \in \mathfrak{l}(\mathsf{H}\cap\mathsf{H}')$, where $\{\mathbf{X_s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H})}$ and $\{\mathbf{X}'_\mathbf{s}\}_{\mathbf{s}\in\mathfrak{l}(\mathsf{H}')}$ are any pair of μ-strong solutions defined for the same Brownian motion $\mathbf{B}$ satisfying (Q2).

Theorem 8 ([22]). *Make the same assumptions as for Theorem 5. Assume* (P2), (Q1)*, and* (Q3)*. Then* (1) *has a* μ*-strong solution* $\mathbf{X}$ *such that the associated unlabeled dynamics* X *is* μ*-reversibile, and the* μ*-strong uniqueness holds.*

7 Examples arising from random matrix theory.

The first three examples are particle systems in $\mathbb{R}$ ($[0,\infty)$ for Bessel), whereas the last example is in $\mathbb{R}^2$. All examples have logarithmic interaction potential.
sine, Airy, and Bessel IBM [15, 26, 23, 4]: Let $d = 1$.

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^i - X_t^j|<r,\ j\neq i}\frac{1}{X_t^i - X_t^j}dt, \qquad \text{(Dyson model, sine)}$$

$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\{(\sum_{j\neq i,\ |X_t^j|<r}\frac{1}{X_t^i - X_t^j}) - \frac{1}{\pi}\int_{-r}^{0}\frac{\sqrt{-x}}{-x}dx\}dt, \qquad \text{(Airy)}$$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i}dt + \frac{\beta}{2}\sum_{j\neq i}^{\infty}\frac{1}{X_t^i - X_t^j}dt \qquad a \geq 1. \qquad \text{(Bessel)}$$

The equilibrium states of these dynamics are sine, Airy, and Bessel point processes ($\beta = 1, 2, 4$ (sine, Airy), $\beta = 2$ (Bessel)). These point processes correspond to bulk, soft edge, and hard edge scaling limits respectively. The relationships to inverse temperature are: $\beta = 1 \Rightarrow$GOE, $\beta = 2 \Rightarrow$GUE, and $\beta = 4 \Rightarrow$GSE, respectively.
Ginibre IBM [15]: Let $d = 2$ and $\beta = 2$. We consider two ISDEs.

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt, \tag{10}$$

$$dX_t^i = dB_t^i - X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt. \tag{11}$$

The equilibrium state μ_{gin} is the Ginibre point process, which has various rigidities such as small variance [24], number rigidity [5], and dichotomy in its reduced Palm measures [19]. The drift coefficients are equal on and tangential to the support of μ_{gin}, yielding the coincidence of the solutions of (10) and (11). This dynamical rigidity reflects rigidity of μ_{gin}.

8 Algebraic construction and finite particle approxiations

An algebraic construction is known for stochastic dynamics related to point processes appearing in random matrix theory in $\mathbb{R}$ with $\beta = 2$, which is given by space-time correlation functions *e.g.* [6, 7, 8]. For example, as for the Airy$_2$ point process, the multi-time, moment generating function is

$$\mathbf{E}\big[\exp\big\{\sum_{m=1}^{M} \langle f_{t_m}, \mathsf{X}_{t_m}\rangle\big\}\big] = \underset{(s,t)\in\mathbf{t}^2,\,(x,y)\in\mathbb{R}^2}{\mathrm{Det}} \Big[\delta_{st}\delta(x-y) + \mathbf{K}_{\mathrm{Airy}}(s,x;t,y)\chi_t(y)\Big].$$

Here $\mathbf{t} = \{t_1, \ldots, t_M\}$, $\chi_t = e^{f_t} - 1$, and $\mathbf{K}_{\mathrm{Ai}}$ is the extended Airy kernel

$$\mathbf{K}_{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \displaystyle\int_0^\infty du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y), & t \geq s \\ \displaystyle -\int_{-\infty}^0 du\, e^{-u(t-s)/2}\mathrm{Ai}(u+x)\mathrm{Ai}(u+y), & t < s. \end{cases}$$

Theorem 9 ([21, 20]). *The algebraic construction and the ISDEs define the same stochastic dynamics for sine$_2$, Airy$_2$, and Bessel$_2$.*

By algebraic method, the finite particle approximation for sine$_2$, Airy$_2$, and Bessel$_2$ is proved [21]. By analytic method, the same is proved for these point processes with $\beta = 1, 2, 4$ and also the Ginibre point process [9]. The latter approach is robust and valid for many other examples.

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References

1. Albeverio, S., Kondratiev, Yu. G., Röckner M., *Analysis and geometry on configuration spaces: the Gibbsian Case*, J. Funct. Anal. 157 (1998), 242–291.
2. Fritz, J., *Gradient dynamics of Infinite Point Systems*, Ann. Probab. **15** (1987), 478–514.
3. Fukushima, M., Oshima, Y., Takeda M., *Dirichlet forms and symmetric Markov processes*, 2nd ed., De Gruyter Studies in Mathematics, **19**. Walter de Gruyter & Co., Berlin, 2011.
4. Honda, R., Osada, H, *Infinite-dimensional stochastic differential equations related to Bessel random point fields*, Stochastic Process. Appl. **125** (2015), 3801–3822.
5. Ghosh, S., Peres, Y., *Rigidity and tolerance in point processes: Gaussian zeroes and Ginibre eigenvalues*, Duke Math. **166** (2017), 1789-1858.
6. Johansson, K., *Non-intersecting paths, random tilings and random matrices*, Probab. Theory Relat. Fields **123** (2002), 225–280.
7. Katori, M., Tanemura, H.: *Noncolliding Brownian motion and determinantal processes*, J. Stat. Phys. **129** (2007), 1233–1277.
8. Katori, M., Tanemura, H., *Markov property of determinantal processes with extended sine, Airy, and Bessel kernels*, Markov Process. Related fields **17** (2011), 541–580.
9. Kawamoto, Y., Osada, H., *Finite-particle approximations for interacting Brownian particles with logarithmic potentials*, (to appear in J. Math. Soc. Japan), arXiv:1607.06922
10. Lang, R., *Unendlich-dimensionale Wienerprocesse mit Wechselwirkung I*, Z. Wahrschverw. Gebiete **38** (1977), 55–72.
11. Lang, R., *Unendlich-dimensionale Wienerprocesse mit Wechselwirkung II*, Z. Wahrschverw. Gebiete **39** (1978), 277–299.
12. Ma, Z.-M. and Röckner, M., *Introduction to the theory of (non-symmetric) Dirichlet forms*, Universitext. Springer-Verlag, Berlin, 1992.
13. Osada, H., *Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions*, Commun. Math. Phys. **176** (1996), 117–131.
14. Osada, H., *Tagged particle processes and their non-explosion criteria*, J. Math. Soc. Japan, **62** (2010), 867–894.
15. Osada, H., *Infinite-dimensional stochastic differential equations related to random matrices*, Probab. Theory Relat. Fields, **153** (2012), 471–509.
16. Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials*, Ann. Probab. **41** (2013), 1–49.
17. Osada, H., *Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II : Airy random point field*, Stochastic Process. Appl. **123** (2013), 813–838.
18. Osada, H., Osada, S., *Discrete approximations of determinantal point processes on continuous spaces: tree representations and tail triviality*, arXive:1603.07478-v3.
19. Osada, H., Shirai, T., *Absolute continuity and singularity of Palm measures of the Ginibre point process*, Probab. Theory Relat. Fields, **165** (2016), 725–770.
20. Osada, H., Tanemura, H., *Cores of Dirichlet forms related to Random Matrix Theory*, Proc. Japan Acad. Ser. A Math. Sci., **90** (2014), 145–150.
21. Osada, H., Tanemura, H., *Strong Markov property of determinantal processes with extended kernels*, Stochastic Process. Appl. **126** (2016), 186–208.
22. Osada, H., Tanemura, H., *Infinite-dimensional stochastic differential equations and tail σ-fields*, arXiv:1412.8674.
23. Osada, H., Tanemura, H., *Infinite-dimensional stochastic differential equations related to Airy random point fields*, arXiv:1408.0632.
24. Shirai, T., *Large deviations for the Fermion point process associated with the exponential kernel*, J. Stat. Phys. **123** (2006), 615–629.
25. Tanemura, H., *A system of infinitely many mutually reflecting Brownian balls in* $\mathbb{R}^d$, Probab. Theory Relat. Fields **104** (1996), 399–426.
26. Tsai, Li-Cheng, *Infinite dimensional stochastic differential equations for Dyson's model*, Probab. Theory Relat. Fields **166** (2015), 801–850.
27. Yoshida, M.,W., *Construction of infinite-dimensional interacting diffusion processes through Dirichlet forms*, Probab. Theory Relat. Fields **106** (1996), 265–297.