Sharp interface limit for stochastically perturbed mass conserving Allen-Cahn equation

Satoshi Yokoyama
(The University of Tokyo)

We consider the solution $u = u^\varepsilon(t,x)$ of the following stochastic partial differential equation (1) in a bounded domain D in $\mathbb{R}^n$ having a smooth boundary ∂D:

$$
\text{(1)} \qquad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \Delta u^\varepsilon + \varepsilon^{-2}\left(f(u^\varepsilon) - \fint_D f(u^\varepsilon)\right) + \alpha \dot{w}^\varepsilon(t), & \text{in } D \times \mathbb{R}_+, \\ \dfrac{\partial u^\varepsilon}{\partial \nu} = 0, & \text{on } \partial D \times \mathbb{R}_+, \\ u^\varepsilon(0,\cdot) = g^\varepsilon(\cdot), & \text{in } D, \end{cases}
$$

where $\varepsilon > 0$ is a small parameter, $\alpha > 0$, ν is the inward normal vector on ∂D, $\mathbb{R}_+ = [0,\infty)$,

$$
\fint_D f(u^\varepsilon) = \frac{1}{|D|}\int_D f(u^\varepsilon(t,x))dx,
$$

g^ε are continuous functions having the property

$$
\text{(2)} \qquad \lim_{\varepsilon \downarrow 0} g^\varepsilon(x) = \chi_{\gamma_0},
$$

where γ_0 is a smooth hypersurface in D without boundary with finitely many connected components and it has the form $\gamma_0 = \partial D_0$ with a smooth domain D_0 such that $\overline{D}_0 \subset D$ and $\chi_\gamma(x) = +1$ or -1 according to the outside or inside of the hypersurface γ. The noise $\dot{w}^\varepsilon(t)$ is the derivative of $w^\varepsilon(t) \equiv w^\varepsilon(t,\omega) \in C^\infty(\mathbb{R}_+)$ in t defined on a certain probability space $(\Omega, \mathcal{F}, P)$ such that $w^\varepsilon(t)$ converges to a 1D standard Brownian motion $w(t)$ as $\varepsilon \downarrow 0$ in a suitable sense. We assume that the reaction term $f \in C^\infty(\mathbb{R})$ is bistable and satisfies the following three conditions:

(i) $f(\pm 1) = 0$, $f'(\pm 1) < 0$, $\displaystyle\int_{-1}^{1} f(u)du = 0$,

(ii) f has only three zeros ± 1 and one another between ± 1,

(iii) there exists $\bar{c}_1 > 0$ such that $f'(u) \le \bar{c}_1$ for every $u \in \mathbb{R}$.

The equation (1) with $\alpha = 0$ and without the averaged reaction term is called the Allen-Cahn equation. When $\alpha = 0$, the mass of the solution u^ε of (1) is conserved, namely,

$$
\text{(3)} \qquad \frac{1}{|D|}\int_D u^\varepsilon(t,x)dx = C,
$$

holds for some constant $C \in \mathbb{R}$. For a mass conserving Allen-Cahn equation without noise ((1) with $\alpha = 0$), its sharp interface limit as $\varepsilon \downarrow 0$ is studied by Chen et al. [1].

Our goal is to show that the solution $u^\varepsilon(t,x)$ of (1) converges as $\varepsilon \downarrow 0$ to $\chi_{\gamma_t}(x)$ with certain hypersurface γ_t in D, if this holds for the initial data g^ε with a certain γ_0, and the time evolution of γ_t is governed by

$$V = \kappa - \fint_{\gamma_t} \kappa + \frac{\alpha |D|}{2|\gamma_t|} \circ \dot{w}(t), \quad t \in [0, \sigma], \tag{4}$$

up to a certain stopping time $\sigma > 0$ (a.s.), where V is the inward normal velocity of γ_t, κ represents the mean curvature of γ_t multiplied by $n-1$, $\fint_{\gamma_t} \kappa = \frac{1}{|\gamma_t|}\int_{\gamma_t} \kappa d\bar{s}$, $\dot{w}(t)$ is the white noise process and $\circ$ means the Stratonovich stochastic integral. When $\alpha = 0$, the equation (4) coincides with the limit of the mass conserving Allen-Cahn equation studied in [1]. On the other hand, in the case where the fluctuation caused by $\alpha w^\varepsilon(t)$ is added, the rigid mass conservation law is destroyed and in place of (3), we have the conservation law in a stochastic sense

$$\frac{1}{|D|}\int_D u^\varepsilon(t,x)dx = C + \alpha w^\varepsilon(t), \quad t \in \mathbb{R}_+, \tag{5}$$

which implies that the total mass per volume behaves like a Brownian motion multiplied by α as ε tends to 0. For our equation, the comparison argument does not work, so that to study the limit we adopt the asymptotic expansion method, which extends that for deterministic equations used in Chen et al. [1]. Differently from the deterministic case, each term except the leading term appearing in the expansion of the solution in a small parameter ε diverges as ε tends to 0, since our equation contains the noise which converges to a white noise and the products or the powers of the white noise diverge. To derive the error estimate for our asymptotic expansion, we need to establish the Schauder estimate for a diffusion operator with coefficients determined from higher order derivatives of the noise and their powers. We show that one can choose the noise sufficiently mild in such a manner that it converges to the white noise and at the same time its diverging speed is slow enough for establishing a necessary error estimate.

This is a joint work with Tadahisa Funaki.

[1] X. Chen, D. Hilhorst, E. Logak, *Mass conserving Allen-Cahn equation and volume preserving mean curvature flow*, Interfaces Free Bound., **12** (2010), 527–549.

[2] T. Funaki, S. Yokoyama, *Sharp interface limit for stochastically perturbed mass conserving Allen-Cahn equation*, arXiv:1610.01263.