Jump processes on the boundaries of random trees

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Consider an infinite tree T and a transient random walk $\{Z_n\}_{n\geq 0}$ on T. A transient random walk on a tree finally hits the Martin boundary, which is the collection of "infinities". Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form associated with $\{Z_n\}_{n\geq 0}$ and let ν be the hitting distribution (the harmonic measure) of M started from a certain point in T. By the theory of Martin boundaries, we have the map H which transforms functions on M into functions on $T \cup M$ with the following property.

Given a function f on M, Hf is harmonic on T and coincides with f on M.

Then the induced form $\mathcal{E}_M$ on the Martin boundary M is given by

$$\mathcal{E}_M(f, g) := \mathcal{E}(Hf, Hg)$$

for $f, g \in \mathcal{F}_M$, where $\mathcal{F}_M := \{f \in L^2(M, \nu) : Hf \in \mathcal{F}\}$. Since Hf is a solution of the Dirichlet problem at "infinity", $(\mathcal{E}_M, \mathcal{F}_M)$ can be regarded as the trace of $(\mathcal{E}, \mathcal{F})$ on M. In [1], the author constructs a Hunt process $\{X_t\}_{t>0}$ on M associated with $(\mathcal{E}_M, \mathcal{F}_M)$ and obtains estimates of the heat kernel $p_t(\cdot, \cdot)$ associated with $(\mathcal{E}_M, \mathcal{F}_M)$ for a deterministic tree. In particular, detailed two sided heat kernel estimates are obtained when ν has the volume doubling property.
In this paper, we are going to study properties of processes and heat kernel estimates associated with transient random walks on random trees instead of deterministic trees. In particular, we are interested in random trees which are generated by branching processes. Consider a Galton-Watson branching process with offspring distribution $\{p_k\}_{k\geq 0}$. Starting from a single individual called a root and denoted by ϕ, this process yields a random tree $\mathcal{T}$, which is called a Galton-Watson tree. In this paper, we assume that $p_0 = 0$ and $\mathcal{T}$ is supercritical (equivalently $m := \sum_{k\geq 0} kp_k > 1$) to make sure $\mathcal{T}$ is infinite for $\mathbb{P}_{GW}$-a.s. Furthermore, we assume $\sum_{k\geq 0} k^n p_k < \infty$ for all $n \geq 1$. Given a tree T, we consider *a λ-biased random walk* on T under the probability measure P^T as in [4]. Then it is shown in [2] that $\{Z_n^{(\lambda)}\}_{n\geq 0}$ on the supercritical Galton-Watson tree $\mathcal{T}$ is transient for $\mathbb{P}_{GW}$-a.s. if and only if $\lambda > 1/m$. Thus, for $\lambda > 1/m$, we have the harmonic measure $\nu^{(\lambda)}$, the induced Dirichlet form $(\mathcal{E}^{(\lambda)}, \mathcal{F}^{(\lambda)})$ and the heat kernel $p_t^{(\lambda)}(\cdot, \cdot)$ associated with $(\mathcal{E}^{(\lambda)}, \mathcal{F}^{(\lambda)})$ for $\mathbb{P}_{GW}$-a.s. As for $\nu^{(\lambda)}$, in [3] and [4], Lyons, Pemantle and Peres show that for $\lambda > 1/m$, $\beta_\lambda := \dim \nu^{(\lambda)}$ is a deterministic constant for $\mathbb{P}_{GW}$-a.s.

Our main results are the following short time log asymptotics of on-diagonal heat kernel estimates and estimates of mean displacements. Note that $d(\cdot,\cdot)$ is the natural metric on M.

Theorem 0.1. *For* $\lambda \geq 1, the\ following\ holds\ for\ \mathbb{P}_{GW}\ a.s..$

$$-\lim_{t\to 0}\frac{p_t^{(\lambda)}(\omega,\omega)}{\log t}=\frac{\beta_\lambda}{\beta_\lambda+\log\lambda}\qquad \nu^{(\lambda)}\ a.e.-\omega.$$

Theorem 0.2. *For* $\lambda \geq 1, the\ following\ holds\ for\ \mathbb{P}_{GW}\ a.s.$

$$\lim_{t\to 0}\frac{\log E_\omega[d(\omega,X_t)^\gamma]}{\log t}=\left(\frac{\gamma}{\beta_\lambda+\log\lambda}\right)\wedge 1\qquad \nu^{(\lambda)}\ a.e.-\omega.$$

The above results imply that the spectral dimension (resp, the walk dimension) is $2\beta_\lambda/(\beta_\lambda+\log\lambda)$ (resp, $(\beta_\lambda+\log\lambda)\vee 1$). In the above theorems, we assume $\lambda \geq 1$ because of the lack of the moment estimate of the effective resistance for $1/m < \lambda < 1$.

References

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