Martingales and Determinantal Processes

Makoto Katori (Chuo University)

Let $Y(t), t \in \mathcal{T}(\ni 0)$ be a diffusion process in the one-dimensional space $\mathcal{S}$ with $(\Omega, \mathcal{F}, \mathrm{P})$ and the natural filtration $\mathcal{F}_t = \sigma(Y(s) : 0 \leq s \leq t), t \in \mathcal{T}$. For a fixed integer $N \in \mathbb{N}$, we choose an N distinct points in $\mathcal{S}$, $u_1 < u_2 < \cdots < u_N$. Then we assume a set of analytic functions $M_j(t,x), t \in \mathcal{T}, x \in \mathcal{S}$, $j = 1, 2, \dots, N$ depending on $\mathbf{u} = (u_1, \dots, u_N)$ and satisfying the following.
(i) For any $t \in \mathcal{T}$, $\{M_j(t,x)\}_{j=1}^N$ are linearly independent functions of $x \in \mathcal{S}$.
(ii) $M_j(0, u_k) = \delta_{jk}, j, k = 1, \dots, N$.
(iii) $\mathrm{E}[M_j(t, Y(t)) | \mathcal{F}_s] = M_j(s, Y(s))$ a.s. for all $0 \leq s \leq t \in \mathcal{T}$.
We define an N-dimensional vector $\mathbf{M}(t,x) = (M_1(t,x), \dots, M_N(t,x))$. Let $Y_k(t), k = 1, 2, \dots, N$ be independent copies of $Y(t), t \in \mathcal{T}$ and set $Y_k^{u_k}(t) = u_k + Y_k(t), k = 1, 2, \dots, N$. Thus we have N independent vector-valued martingales $\{\mathbf{M}(t, Y_k^{u_k}(t))\}_{k=1}^N, t \in \mathcal{T}$, and consider time-evolution of a random polyhedron in $\mathcal{S}^N$ spanned by them;

$$\Lambda(t) = \left\{ \sum_{k=1}^N a^{(k)} \mathbf{M}(t, Y_k^{u_k}(t)) : 0 \leq a^{(k)} \leq 1, k = 1, 2 \dots, N \right\}, \quad t \in \mathcal{T}.$$

Its signed volume is given by $\mathrm{vol}^*(\Lambda(t)) = \det_{1 \leq j,k \leq N} \Big[M_j(t, Y_k^{(u_k)}(t)) \Big], t \in \mathcal{T}$, which is a martingale; $\mathrm{E}[\mathrm{vol}^*(\Lambda(t))] = \mathrm{vol}^*(\Lambda(0)) = 1, 0 \leq t \in \mathcal{T}$.

We consider the interacting particle systems $\mathbf{X}(t) = (X_1(t), \dots, X_N(t)), t \in \mathcal{T}$ starting from $\mathbf{u}$, whose probability laws have the following absolutely continuity relations to $\mathrm{P}^{\mathbf{u}}$,

$$\mathbb{P}^{\mathbf{u}} \Big|_{\mathcal{F}_t} = \mathrm{vol}^*(\Lambda(t \wedge \tau)) \mathrm{P}^{\mathbf{u}} \Big|_{\mathcal{F}_t}, \quad t \in \mathcal{T},$$

where $\tau = \inf\{t > 0 : \mathrm{vol}^*(\Lambda(t)) = 0\}$. We regard the above as N-dimensional extensions of the relation between the three-dimensional Bessel process and the Brownian motion $B \sim \mathrm{W}$,

$$\mathbb{P}^u_{\mathrm{BES}^{(3)}} \Big|_{\mathcal{F}_t} = M(B(t \wedge \tau)) \mathrm{W}^u \Big|_{\mathcal{F}_t}, \quad u > 0, \quad t \geq 0$$

with $M(x) = x/u, \tau = \inf\{t > 0 : B(t) = 0\}$. In the present talk, we will show that such processes have been realized as determinantal processes related to random matrix theory and the KPZ universality class.