COUPLED KPZ EQUATIONS

MASATO HOSHINO (THE UNIVERSITY OF TOKYO)
(JOINT WORK WITH TADAHISA FUNAKI (THE UNIVERSITY OF TOKYO))

We consider the $\mathbb{R}^d$-valued coupled KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ given by

(1)
$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \xi^\alpha, \quad t>0,\ x\in\mathbb{T}=\mathbb{R}/\mathbb{Z},\ \alpha=1,\dots,d,$$

where $(\sigma^\alpha_\beta)_{1\le\alpha,\beta\le d}$ and $(\Gamma^\alpha_{\beta\gamma})_{1\le\alpha,\beta,\gamma\le d}$ are given constants, and $\xi(t,x) = (\xi^\alpha(t,x))$ is an $\mathbb{R}^d$-valued space-time white noises. In (1), we omit the summation over β and γ (Einstein's convention).

The difficulty in making sense of the equation (1) is that the nonlinear term is not well-defined because we expect that $h^\alpha(t,\cdot)\in\mathcal{C}^{\frac{1}{2}-\kappa}$ for any $\kappa>0$. A natural approach is replacing ξ by a smeared noise $\xi^\epsilon(t,x) = (\xi(t)*\eta^\epsilon)(x)$ with an even mollifier $\eta^\epsilon(x)=\epsilon^{-1}\eta(\epsilon^{-1}x)$, and introducing a suitable renormalization for the nonlinear term. We study the following two types of approximations.

(2) $$\partial_t h^{\epsilon,\alpha} = \frac{1}{2}\partial_x^2 h^{\epsilon,\alpha} + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^{\epsilon,\beta}\partial_x h^{\epsilon,\gamma} - c^\epsilon A^{\beta\gamma} - B^{\epsilon,\beta\gamma}) + \sigma^\alpha_\beta\xi^{\epsilon,\alpha},$$

(3) $$\partial_t \tilde{h}^{\epsilon,\alpha} = \frac{1}{2}\partial_x^2 \tilde{h}^{\epsilon,\alpha} + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}^{\epsilon,\beta}\partial_x \tilde{h}^{\epsilon,\gamma} - c^\epsilon A^{\beta\gamma} - \tilde{B}^{\epsilon,\beta\gamma}) * \eta_2^\epsilon + \sigma^\alpha_\beta\xi^{\epsilon,\alpha},$$

where $c^\epsilon = \epsilon^{-1}\|\eta\|^2_{L^2(\mathbb{R})}$, $A^{\beta\gamma} = \sum_{\delta=1}^d \sigma^\beta_\delta\sigma^\gamma_\delta$, and B^ϵ and $\tilde{B}^\epsilon$ are suitably chosen constant matrixes which behave as $O(|\log\epsilon|)$ in $\epsilon\downarrow 0$. In (3), $\eta_2^\epsilon = \eta^\epsilon * \eta^\epsilon$.

Theorem 1. *For every initial condition $h_0\in\mathcal{C}^\kappa(\mathbb{T},\mathbb{R}^d)$, the solutions of* (2) *and* (3) *converge to a common limit h in probability, in a short time. This limit is independent to the choice of mollifier η.*

Unlike a scalar-valued case, global existence of the solution h is nontrivial because Cole-Hopf transform does not work for the coupled case in general. We show global existence under the equilibrium setting. Note that $\hat{h}^\alpha = (\sigma^{-1})^\alpha_\beta h^\beta$ solves (1) with (σ,Γ) replaced by $(I,\hat{\Gamma})$, where

$$\hat{\Gamma}^\alpha_{\beta\gamma} = (\sigma^{-1})^\alpha_{\alpha'}\Gamma^{\alpha'}_{\beta'\gamma'}\sigma^{\beta'}_\beta\sigma^{\gamma'}_\gamma.$$

Theorem 2. *We assume*

$$\hat{\Gamma}^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\gamma\beta} = \hat{\Gamma}^\beta_{\alpha\gamma}$$

for all α,β,γ. There exists a μ-full set H in $\mathcal{C}_0^{-\frac{1}{2}-\kappa} = \{u\in\mathcal{C}^{-\frac{1}{2}-\kappa}; \int_\mathbb{T} u = 0\}$ such that the limit h starting at h_0 with $\partial_x h_0\in H$ exists on whole $[0,\infty)$ almost surely, where μ is a distribution of $\mathbb{R}^d$-valued spatial white noise.