On Poly-Bernoulli Numbers

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1. Main theorems

In our previous paper [1], we defined and studied "Poly-Bernoulli numbers" which generalize the classical Bernoulli numbers. As a continuation, we present here two results, one of which is a further investigation of Clausen-von Staudt type theorem that was treated only in "di-Bernoulli" case in [1], the other being a combinatorial closed formula for negative index poly-Bernoulli numbers.

Poly-Bernoulli numbers $B_n^{(k)}$ $(n = 0, 1, 2, \dots)$ are defined for each integer k by the generating series

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where $Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$. Table of values of $B_n^{(k)}$ for small k and n will be given at the end of the paper. In [1], we obtained an explicit formula for $B_n^{(k)}$:

$$B_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \left\{ {n \atop m} \right\}}{(m+1)^k}, \tag{1}$$

where $\left\{ {n \atop m} \right\}$ is an integer referred to as the Stirling number of the second kind ("Stirling subset number" in Knuth's terminology, we adopt his notation [2]).

Let p be a prime number. First of all, it is clear from the above formula (1) that the $B_n^{(k)}$ is p-integral when p is larger than $n+1$. Our first theorem gives an information on the p-part of $B_n^{(k)}$ for $p \le n+1$.

Theorem 1 (Clausen-von Staudt type theorem). *Assume $k \ge 2$. Let p be a prime number satisfying $k+2 \le p \le n+1$.*

(i) *If $n \equiv 0 \mod (p-1)$, then $p^k B_n^{(k)}$ is a p-adic integer and satisfies*

$$p^k B_n^{(k)} \equiv -1 \mod p\mathbb{Z}_p.$$

(ii) *If $n \not\equiv 0 \mod (p-1)$, then $p^{k-1} B_n^{(k)}$ is a p-adic integer. It satisfies*

$$p^{k-1} B_n^{(k)} \equiv \begin{cases} \dfrac{1}{p}\left\{ {n \atop p-1} \right\} - \dfrac{n}{2^k} \mod p\mathbb{Z}_p, & if\ n \equiv 1 \mod (p-1) \\ \dfrac{(-1)^{n-1}}{p}\left\{ {n \atop p-1} \right\} \mod p\mathbb{Z}_p & otherwise. \end{cases}$$

Remark. 1. That the $p^{k-1}B_n^{(k)}(p-1\not|n)$ and $p^kB_n^{(k)}(p-1|n)$ are p-integral $(k+2\le p)$ has also been obtained independently by Roberto Sánchez-Peregrino in [3].

2. If $n\not\equiv 0,1 \mod (p-1)$, the congruence in (ii) may be written as

$$p^{k-1}B_n^{(k)}\equiv (n-n')\frac{B_n^{(1)}}{n} \mod p\mathbb{Z}_p,$$

where n' is a unique integer with $n'\equiv n \mod p-1$ and $1<n'<p$. Actually, it was shown in [1] the congruence

$$\frac{(-1)^{n-1}}{p}\left\{ {n \atop p-1} \right\}\equiv (n-n')\frac{B_n^{(1)}}{n} \mod p\mathbb{Z}_p$$

if $n\not\equiv 0,1 \mod (p-1)$ (the assumption made there that n being even can be loosened to the present one).

3. When $p>n+1$, the formula (1) shows that the congruence $B_n^{(k)}\equiv B_n^{(k')} \mod p$ holds for any integers k and k' satisfying $k\equiv k' \mod (p-1)$.

The number $B_n^{(k)}$ is a positive integer when k is non-positive. Our second theorem is a closed formula (which is completely different from (1)) for this integer.

Theorem 2 (Closed formula). *For any $n,k\ge 0$,we have*

$$B_n^{(-k)}=\sum_{j=0}^{\min(n,k)}(j!)^2\left\{ {n+1 \atop j+1} \right\}\left\{ {k+1 \atop j+1} \right\}.$$

Remark. This formula gives another proof of the symmetry $B_n^{(-k)}=B_k^{(-n)}$ mentioned in [1].

The proofs of theorems 1 and 2 will be given in §2 and §3 respectively.

2. Proof of Clausen-von Staudt type theorem

Let $k\ge 2$ and p be a prime number satisfying $k+2\le p\le n+1$. To prove theorem 1, we estimate the p-order of each summand $\frac{(-1)^m m!\left\{ {n \atop m} \right\}}{(m+1)^k}$ in (1), which we denote hereafter by $b_n^{(k)}(m)$. We prove (i) and (ii) simultaneously. The p-order of an integer a is denoted by $ord_p(a)$ with the convention $ord_p(p^t)=t$. Write $m+1=ap^e, (a,p)=1, e\ge 0$. If $e=0$, then $b_n^{(k)}(m)$ is p-integral. We can ignore this term, because by the assumption $k\ge 2$ we have $p^{k-1}b_n^{(k)}(m)\equiv 0 \mod p\mathbb{Z}_p$. Since $\left\{ {n \atop m} \right\}$ is an integer, we have

$$ord_p(b_n^{(k)}(m))\ge ord_p\left(\frac{m!}{(m+1)^k}\right).$$

First, assume $e \geq 2$. We show that $p^k b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$ and moreover $p^{k-1} b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$ if $n \not\equiv 0 \mod (p-1)$. Using $ord_p(m!) = \sum_{j=1}^{\infty} \left[\frac{m}{p^j}\right]$, we have

$$\begin{aligned} ord_p\left(\frac{m!}{(m+1)^k}\right) &= \sum_{j=1}^{\infty}\left[\frac{m}{p^j}\right] - ek \\ &\geq \left[\frac{m}{p}\right] - ek = \left[\frac{ap^e - 1}{p}\right] - ek = ap^{e-1} - 1 - ek \\ &\geq p^{e-1} - 1 - ek = (1 + p - 1)^{e-1} - 1 - ek \\ &\geq 1 + (e-1)(p-1) - 1 - ek = (e-1)(p-1) - ek \\ &\geq (e-1)(k+1) - ek = -k + e - 1 \\ &\geq -k + 1. \end{aligned}$$

Thus we get $ord_p\left(\frac{m!}{(m+1)^k}\right) \geq -k+1$ and so $p^{k-1} b_n^{(k)}(m)$ is p-integral. Hence $p^k b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$. If any one of the above inequalities is strict (i.e. '$>$'), then we get $p^{k-1} b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$. The only case when the equalities hold everywhere is when $e = 2, m+1 = p^2$, and $p = k+2$. In this case, the following lemma ($a = p$) implies $p^{k-1} b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$ if $n \not\equiv 0 \mod (p-1)$.

Lemma. *Let n and a be natural numbers. We have the congruence*

$$\left\{ {n \atop ap-1} \right\} \equiv \begin{cases} \binom{c-1}{a-1} \mod p & \text{if } n = a - 1 + c(p-1)n \text{ for some } c \geq a \\ 0 \mod p & \text{otherwise.} \end{cases}$$

Proof. Use the following formula for a generating function of $\left\{ {n \atop m} \right\}$ ([4, (7.47)]):

$$\sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} x^n = \frac{x^m}{(1-x)(1-2x)\cdots(1-mx)}. \tag{2}$$

If $m = ap - 1$, the right-hand side of this formula is congruent modulo p to

$$\frac{x^{ap-1}}{(1-x^{p-1})^a} = x^{ap-1} \sum_{i=0}^{\infty} \binom{a+i-1}{i} x^{i(p-1)} = \sum_{i=0}^{\infty} \binom{a+i-1}{a-1} x^{a-1+(a+i)(p-1)}$$

(we have used $(1-x)(1-2x)\cdots(1-(p-1)x) \equiv 1 - x^{p-1} \mod p$). Putting $a + i = c$, we obtain the lemma.

Now suppose $e = 1$ ($m = ap-1$). If $a \geq 3$, then $p^2 | (ap-1)!$. Hence $ord_p\left(\frac{m!}{(m+1)^k}\right) > -k+1$, from which follows $p^{k-1} b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$. If $a = 2$, then $ord_p(m!) = 1$ and $ord_p\left(\frac{m!}{(m+1)^k}\right) = 1 - k$. Hence $ord_p(b_n^{(k)}(m)) = 1 - k + ord_p\left(\left\{ {n \atop m} \right\}\right)$ and so $p^k b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$. If $n \not\equiv 1 \mod (p-1)$, we see from the above lemma ($a = 2$) that $\left\{ {n \atop 2p-1} \right\} \equiv 0 \mod p$. From this we have $p^{k-1} b_n^{(k)}(m) \equiv 0 \mod p\mathbb{Z}_p$ if $n \not\equiv 1 \mod (p-1)$. If $n \equiv 1$

mod $(p-1)$ and $n = 1 + c(p-1)$ with $c \geq 2$ ($c = 1$ cannot occur because $n \geq m$), then we see by the lemma that $\left\{ {n \atop 2p-1} \right\} \equiv c - 1 \equiv -n \mod p$. From this, we obtain

$$\begin{aligned} p^{k-1} b_n^{(k)}(m) &= p^{k-1} \frac{(-1)^{2p-1}(2p-1)! \left\{ {n \atop 2p-1} \right\}}{(2p)^k} \\ &\equiv \frac{n}{2^k} \mod p\mathbb{Z}_p. \end{aligned}$$

Finally, the case $a = 1$ ($m = p - 1$) gives us $b_n^{(k)}(m) = \frac{(p-1)! \left\{ {n \atop p-1} \right\}}{p^k}$. From the lemma, we have $\left\{ {n \atop p-1} \right\} \equiv 0 \mod p$ if $n \not\equiv 0 \mod (p-1)$ and thus $p^{k-1} b_n^{(k)}(m) \equiv -\frac{1}{p} \left\{ {n \atop p-1} \right\} \mod p\mathbb{Z}_p$. If $n \equiv 0 \mod (p-1)$, then $\left\{ {n \atop p-1} \right\} \equiv 1 \mod p$ and $p^k b_n^{(k)}(m) \equiv -1 \mod p\mathbb{Z}_p$. Summing up, and noting the factor $(-1)^n$ before the summation in (1) (also note n is odd if $n \equiv 1 \mod (p-1)$), we obtain the theorem.

3. Proof of the closed formula for negative index poly-Bernoulli numbers

In this section we prove Theorem 2. In the course of our proof, we obtain

Proposition. *For all* $n > 0$,

$$\sum_{l=0}^{n} (-1)^l B_{n-l}^{(-l)} = 0.$$

Example. $B_2^{(0)} - B_1^{(-1)} + B_0^{(-2)} = 1 - 2 + 1 = 0$, $B_4^{(0)} - B_3^{(-2)} + B_2^{(-2)} - B_1^{(-3)} + B_0^{(-4)} = 1 - 8 + 14 - 8 + 1 = 0$, etc.

This is trivial when n is odd because of the symmetry mentioned in the remark after the theorem.

In order to prove the theorem, we calculate the generating function $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} x^n y^k$ of $B_n^{(-k)}$ in the following form:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} x^n y^k = \sum_{j=0}^{\infty} p_j(x) p_j(y), \tag{3}$$

where

$$p_j(x) = \frac{j! x^j}{(1-x)(1-2x)\cdots(1-(j+1)x)}.$$

Once we establish this, the theorem follows by equating the coefficients of both sides, because we have by the formula (2) in §2

$$p_j(x) = j! \sum_{n=j}^{\infty} \left\{ {n+1 \atop j+1} \right\} x^n. \tag{4}$$

Put the left-hand side of (3) $= B(x,y)$. Using (1) we have

$$\begin{aligned} B(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left((-1)^n\sum_{m=0}^{n}(-1)^m m!\left\{ n \atop m \right\}(m+1)^k\right)x^n y^k \\ &= \sum_{n=0}^{\infty}(-1)^n\sum_{m=0}^{n}(-1)^m m!\left\{ n \atop m \right\}x^n\sum_{k=0}^{\infty}(m+1)^k y^k \\ &= \sum_{m=0}^{\infty}(-1)^m m!\sum_{n=m}^{\infty}(-1)^n\left\{ n \atop m \right\}x^n\frac{1}{1-(m+1)y}. \end{aligned} \tag{5}$$

Here we use (2) to get

$$B(x,y) = \sum_{m=0}^{\infty}\frac{m!x^m}{(1+x)(1+2x)\cdots(1+mx)(1-(m+1)y)}.$$

The proposition follows from this. Namely, putting $y=-x$ gives

$$\begin{aligned} B(x,-x) &= \sum_{m=0}^{\infty}\frac{m!x^m}{(1+x)(1+2x)\cdots(1+mx)(1+(m+1)x)} \\ &= \sum_{m=1}^{\infty}\frac{(m-1)!x^{m-1}}{(1+x)\cdots(1+mx)} \\ &= \sum_{m=1}^{\infty}(-1)^m(m-1)!\sum_{n=m}^{\infty}\left\{ n \atop m \right\}(-1)^n x^{n-1} \qquad \text{(by (2))} \\ &= \sum_{n=1}^{\infty}(-1)^n\left(\sum_{m=1}^{n}(-1)^m(m-1)!\left\{ n \atop m \right\}\right)x^{n-1} \\ &= 1 \qquad \text{(by [4, (6.16)])}, \end{aligned}$$

while by definition

$$\begin{aligned} B(x,-x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k B_n^{(-k)}x^{n+k} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^l B_{n-l}^{(-l)}\right)x^n \qquad (n+k\to n, k\to l), \end{aligned}$$

and hence the proposition.

Let us return to the proof of (3). We need the following lemma.

Lemma. (i) $\dfrac{1}{1-(m+1)y} = \displaystyle\sum_{j=0}^{m}\binom{m}{j}p_j(y).$

(ii) $\displaystyle\sum_{m=j}^{n}(-1)^m m!\binom{m}{j}\left\{ n \atop m \right\} = (-1)^n j!\left\{ n+1 \atop j+1 \right\} \quad (n \geq j \geq 0).$

Proof will be given later. From (5),

$$\begin{aligned}
B(x,y) &= \sum_{m=0}^{\infty}(-1)^m m!\sum_{n=m}^{\infty}(-1)^n \left\{ {n \atop m} \right\} x^n \frac{1}{1-(m+1)y} \\
&= \sum_{m=0}^{\infty}\left((-1)^m m!\sum_{n=m}^{\infty}(-1)^n \left\{ {n \atop m} \right\} x^n\right)\sum_{j=0}^{m}\binom{m}{j}p_j(y) \qquad \text{(by Lemma (i))} \\
&= \sum_{j=0}^{\infty}p_j(y)\left(\sum_{m=j}^{\infty}(-1)^m m!\binom{m}{j}\sum_{n=m}^{\infty}(-1)^n \left\{ {n \atop m} \right\} x^n\right) \\
&= \sum_{j=0}^{\infty}p_j(y)\sum_{n=j}^{\infty}(-1)^n x^n\left(\sum_{m=j}^{n}(-1)^m m!\binom{m}{j}\left\{ {n \atop m} \right\}\right) \\
&= \sum_{j=0}^{\infty}p_j(y)\sum_{n=j}^{\infty}j!\left\{ {n+1 \atop j+1} \right\} x^n \qquad \text{(by Lemma (ii))} \\
&= \sum_{j=0}^{\infty}p_j(x)p_j(y) \qquad \text{(by (4))}.
\end{aligned}$$

This is (3) and thus the theorem is proved.

Proof of Lemma. (i) The following partial fraction expansion is easily established by residue calculation:

$$\frac{1}{z(z-1)\cdots(z-m)} = \frac{(-1)^m}{m!}\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{z-l}.$$

From this we have

$$\begin{aligned}
yp_j(y) &= \frac{j!y^{j+1}}{(1-y)(1-2y)\cdots(1-(j+1)y)} = \frac{j!}{\left(\frac{1}{y}-1\right)\left(\frac{1}{y}-2\right)\cdots\left(\frac{1}{y}-(j+1)\right)} \\
&= \frac{j!(-1)^{j+1}}{y(j+1)!}\sum_{l=0}^{j+1}\frac{(-1)^l\binom{j+1}{l}}{\frac{1}{y}-l} = \frac{(-1)^{j+1}}{j+1}\sum_{l=0}^{j+1}\frac{(-1)^l\binom{j+1}{l}}{1-ly},
\end{aligned}$$

and therefore,

$$\begin{aligned}
y\sum_{j=0}^{m}\binom{m}{j}p_j(y) &= \sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{j+1}\sum_{l=0}^{j+1}\frac{(-1)^l\binom{j+1}{l}}{1-ly} \\
&= \sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{j+1} + \sum_{l=1}^{m+1}\frac{(-1)^l}{1-ly}\sum_{j=l-1}^{m}\frac{(-1)^{j+1}}{j+1}\binom{m}{j}\binom{j+1}{l}.
\end{aligned}$$

Now, since

$$\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{j+1}}{j+1} = -\frac{1}{m+1}$$

(take $\int_0^1 dx$ of $(1-x)^m = \sum_{j=0}^{m}(-1)^j \binom{m}{j} x^j$), and

$$\begin{aligned}
\sum_{j=l-1}^{m} \frac{(-1)^{j+1}}{j+1}\binom{m}{j}\binom{j+1}{l} &= \frac{1}{l}\sum_{j=l-1}^{m}(-1)^{j+1}\binom{m}{j}\binom{j}{l-1} \\
&= \frac{1}{l}\sum_{j=l-1}^{m}(-1)^{j+1}\binom{m}{l-1}\binom{m-l+1}{j-l+1} \\
&= \frac{1}{l}\binom{m}{l-1}\sum_{j=l-1}^{m}(-1)^{j+1}\binom{m-l+1}{j-l+1} \\
&= \begin{cases} \dfrac{(-1)^{m+1}}{m+1} & l = m+1 \\ 0 & l < m+1 \end{cases}
\end{aligned}$$

(the second equality is by $\binom{p}{q}\binom{q}{r} = \binom{p}{r}\binom{p-r}{q-r}$), we have

$$\begin{aligned}
y\sum_{j=0}^{m}\binom{m}{j}p_j(y) &= -\frac{1}{m+1} + \frac{(-1)^{m+1}}{1-(m+1)y}\cdot\frac{(-1)^{m+1}}{m+1} \\
&= \frac{y}{1-(m+1)y}.
\end{aligned}$$

This gives Lemma (i).

(ii) First we show

$$\sum_{n=j}^{\infty}(-1)^n j! \left\{ {n+1 \atop j+1} \right\} \frac{t^n}{n!} = (e^{-t}-1)^j \cdot e^{-t}.$$

For this, we start with

$$\frac{(e^t-1)^j}{j!} = \sum_{n=j}^{\infty} \left\{ {n \atop j} \right\} \frac{t^n}{n!} \tag{6}$$

(this is [4, (7.49)]). Replacing j by $j+1$,

$$\begin{aligned}
\frac{(e^t-1)^{j+1}}{(j+1)!} &= \sum_{n=j+1}^{\infty} \left\{ {n \atop j+1} \right\} \frac{t^n}{n!} \\
&= \sum_{n=j}^{\infty} \left\{ {n+1 \atop j+1} \right\} \frac{t^{n+1}}{(n+1)!} \quad (n \to n+1).
\end{aligned}$$

Differentiation by t gives

$$\frac{(e^t-1)^j \cdot e^t}{j!} = \sum_{n=j}^{\infty} \left\{ {n+1 \atop j+1} \right\} \frac{t^n}{n!}.$$

From this we have

$$(e^{-t}-1)^j \cdot e^{-t} = \sum_{n=j}^{\infty} (-1)^n j! \left\{ {n+1 \atop j+1} \right\} \frac{t^n}{n!}.$$

The proof will be finished if we show

$$\sum_{n=j}^{\infty} \left(\sum_{m=j}^{n} (-1)^m m! \binom{m}{j} \left\{ {n \atop m} \right\} \right) \frac{t^n}{n!} = (e^{-t}-1)^j \cdot e^{-t}.$$

The left-hand side is equal to

$$\begin{aligned} & \sum_{m=j}^{\infty} (-1)^m m! \binom{m}{j} \sum_{n=m}^{\infty} \left\{ {n \atop m} \right\} \frac{t^n}{n!} \\ = & \sum_{m=j}^{\infty} (-1)^m m! \binom{m}{j} \frac{(e^t-1)^m}{m!} \qquad (6) \\ = & \sum_{m=j}^{\infty} (-1)^m \binom{m}{j} (e^t-1)^m. \end{aligned}$$

Since $\sum_{m=j}^{\infty} \binom{m}{j} X^m = \frac{X^j}{(1-X)^{j+1}}$, (replace m by $m-j$ in $\sum_{m=0}^{\infty} \binom{m+j}{j} X^m = (1-X)^{-(j+1)}$) we obtain

$$\sum_{m=j}^{\infty} (-1)^m \binom{m}{j} (e^t-1)^m = \sum_{m=j}^{\infty} \binom{m}{j} (1-e^t)^m \frac{(1-e^t)^j}{(1-(1-e^t))^{j+1}} = (e^{-t}-1)^j \cdot e^{-t}.$$

This completes the proof of the lemma and Theorem 2 is thus established.

Table 1: $B_n^{(k)}$ $(-5 \le k \le 5,\ 0 \le n \le 7)$

k \ n	0	1	2	3	4	5	6	7
-5	1	32	454	4718	41506	329462	2441314	17234438
-4	1	16	146	1066	6902	41506	237686	1315666
-3	1	8	46	230	1066	4718	20266	85310
-2	1	4	14	46	146	454	1394	4246
-1	1	2	4	8	16	32	64	128
0	1	1	1	1	1	1	1	1
1	1	$\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0
2	1	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{24}$	$\frac{7}{450}$	$\frac{1}{40}$	$-\frac{38}{2205}$	$-\frac{5}{168}$
3	1	$\frac{1}{8}$	$-\frac{11}{216}$	$-\frac{1}{288}$	$\frac{1243}{54000}$	$-\frac{49}{7200}$	$-\frac{75613}{3704400}$	$\frac{599}{35280}$
4	1	$\frac{1}{16}$	$-\frac{49}{1296}$	$\frac{41}{3456}$	$\frac{26291}{3240000}$	$-\frac{1921}{144000}$	$\frac{845233}{1555848000}$	$\frac{1048349}{59270400}$
5	1	$\frac{1}{32}$	$-\frac{179}{7776}$	$\frac{515}{41472}$	$-\frac{216383}{194400000}$	$-\frac{183781}{25920000}$	$\frac{4644828197}{653456160000}$	$\frac{153375307}{49787136000}$

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