ON ORDINARY PRIMES FOR MODULAR FORMS AND THE THETA OPERATOR

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Abstract. We give a criterion for a prime being ordinary for a modular form, by using the theta operator of Ramanujan.

1. Introduction and statement of result

A normalized Hecke eigenform is said to be ordinary at a prime p if p does not divide its p-th Fourier coefficient. In the theory of p-adic modular forms and Galois representations attached to modular forms, this notion has a fundamental importance and there is an extensive literature on the subject.

In the present paper, we shall give a criterion for ordinariness in terms of certain polynomials attached to derivatives of given modular forms. Throughout the paper, the modular forms considered are those on the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

For any $f = f(z) = \sum_{n=0}^{\infty} a(n)q^n$ $(q = e^{2\pi iz})$, we define

$$\theta f := q\frac{d}{dq}f = \sum_{n=0}^{\infty} n\, a(n)q^n.$$

This is the derivative with respect to $2\pi iz$, and is often referred to as the "theta operator" of Ramanujan. The derivative of a modular form is no longer modular but "quasimodular", which means, in the case of $\mathrm{SL}_2(\mathbb{Z})$, that it is an isobaric element of the ring $\mathbb{C}[E_2, E_4, E_6]$. Here, $E_k = E_k(z)$ for even k is the standard Eisenstein series

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\Big(\sum_{d|n} d^{k-1}\Big)q^n,$$

B_k being the k-th Bernoulli number. For $k \geq 4$, the function $E_k(z)$ is modular of weight k, but $E_2(z)$ is not quite modular. The operator θ preserves the ring $\mathbb{C}[E_2, E_4, E_6]$ (as is seen by Ramanujan's formulae $\theta E_2 = (E_2^2 - E_4)/12$, $\theta E_4 = (E_2E_4 - E_6)/3$, $\theta E_6 = (E_2E_6 - E_4)/2$), and hence for any modular form f and non-negative integer n, $\theta^n f$ is an element in $\mathbb{C}[E_2, E_4, E_6]$.

To any $g \in \mathbb{C}[E_2, E_4, E_6]$, we attach a polynomial $F(g; X, Y, Z)$ in three variables so that

$$g(z) = F(g; E_2(z), E_4(z), E_6(z))$$

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holds. We also define its "modular part" $F^{(0)}(g; Y, Z)$ by

$$F^{(0)}(g; Y, Z) := F(g; 0, Y, Z).$$

If in particular g is modular (i.e., $g \in \mathbb{C}[E_4, E_6]$), then $F(g; X, Y, Z)$ is free from X and $F(g; X, Y, Z) = F^{(0)}(g; Y, Z)$. If g has p-integral Fourier coefficients, the polynomial F (and hence $F^{(0)}$) has also p-integral coefficients.

For a prime $p > 3$, set $H_p(Y, Z) = F^{(0)}(E_{p-1}; Y, Z)(= F(E_{p-1}; X, Y, Z))$. The polynomial $H_p(Y, Z)$ has p-integral coefficients and $H_p(Y, Z) \bmod p$ is known as the "Hasse invariant" ([3], [4]).

Now we can state our main theorem.

Theorem 1.1. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ be a normalized eigen cusp form of weight k and p a prime number greater than k. Then the following conditions are equivalent.*

(1) $a(p) \not\equiv 0 \mod p$.
(2) $H_p(Y, Z) \nmid F^{(0)}(\theta^{p-k+1} f; Y, Z) \mod p$.

2. Proof of Theorem and a corollary

In order to prove the theorem, we use the theory of filtration of modular forms modulo p developed by Swinnerton-Dyer [4], the theory of theta cycles by Tate [1], and a formula for the derivative $\theta^n f$. We first recall the definition of the filtration and then review theorems of Tate and Swinnerton-Dyer.

Let $M_k(\mathbb{Z}_{(p)})$ be the set of modular forms of weight k (on $\mathrm{SL}_2(\mathbb{Z})$) whose Fourier coefficients belong to $\mathbb{Z}_{(p)}$, the local ring of $\mathbb{Q}$ at p. Following [4], let $\widetilde{M_k}$ be the $\mathbb{F}_p$-vector space (in $\mathbb{F}_p[[q]]$) obtained from $M_k(\mathbb{Z}_{(p)})$ by reducing Fourier coefficients modulo p. We note that, since we have $E_{p-1} \equiv 1 \bmod p$ and $E_2 \equiv E_{p+1} \bmod p$ by the Kummer congruences of Bernoulli numbers, any quasimodular forms having p-integral Fourier coefficients is congruent modulo p to a modular form of suitable weight.

Definition 2.1. For $f \in \widetilde{M_k}$, we define the filtration $w(f)$ of f to be the least ℓ such that f belongs to $\widetilde{M_\ell}$. For a modular or quasimodular form f whose Fourier coefficients are p-integral, we shall write $w(f)$ instead of $w(f \bmod p)$.

We call an element in $\widetilde{M_k}$ an eigenform if it is congruent modulo p to a Hecke-eigen cusp form. Tate's theory of theta cycles connects the ordinariness of an eigenform f to the filtration of the derivative of f.

Proposition 2.2 (Tate [1]). *Let $f = \sum_{n=1}^{\infty} a(n)q^n \in \widetilde{M_k}$ be an eigenform. We assume $k < p$ and $w(f) = k$. Then we have*

$$w(\theta^{p-k+1} f) = \begin{cases} 2p - k + 2 & \text{if } \ a(p) \not\equiv 0 \bmod p, \\ p - k + 3 & \text{if } \ a(p) \equiv 0 \bmod p. \end{cases}$$

(In [1] the assumption is weaker (that f is in the kernel of the "U-operator"), but for our purpose it is enough to restrict to the case of eigenform.)

On the other hand, the filtration of a modular form g is related to the divisibility of $F^{(0)}(g;Y,Z)$ mod p by the Hasse invariant.

Proposition 2.3. (Swinnerton-Dyer [4, Lemma 5]) *For $g \in \widetilde{M_{k'}}$, the following holds.*

(1) *If $w(g) = k'$ then $H_p(Y,Z) \nmid F^{(0)}(g;Y,Z) \mod p$.*
(2) *If $w(g) = k' - p + 1$ then $H_p(Y,Z) \mid F^{(0)}(g;Y,Z) \mod p$.*

Now assume that f is a normalized eigenform of weight k. The derivative $\theta^{p-k+1}f$ is quasimodular of weight $2p-k+2$. If $\theta^{p-k+1}f$ is congruent modulo p to a (true) modular form g of weight $2p-k+2$, then, combining Proposition 2.2 and Proposition 2.3 (with $k' = 2p-k+2$), the condition $a(p) \not\equiv 0 \mod p$ is equivalent to the polynomial $F^{(0)}(g;Y,Z) \mod p$ not being divisible by $H_p(Y,Z) \mod p$. Our theorem is therefore a consequence of the following observation that we can indeed take g to be the modular part of $\theta^{p-k+1}f$. Here, for a quasimodular form $g = \sum_{i=0}^{m} g_i E_2^i,\ g_i \in \mathbb{C}[E_4, E_6]$, we call g_0 its modular part.

Lemma 2.4. *Let $p > 3$ be a prime and f a modular form of weight $k < p$ with p-integral Fourier coefficients. Then we have*

$$\theta^{p-k+1}f \equiv (\theta^{p-k+1}f)_0 \mod p.$$

This is a consequence of a general formula for $\theta^n f$ given in [5]. Recall that, if f is modular of weight k, then

$$\partial f := \theta f - \frac{k}{12}E_2 f$$

is modular of weight $k+2$. For a modular form f of weight k, define a sequence of modular forms f_r of weight $k+2r$ recursively by

$$f_{r+1} = \partial f_r - \frac{r(r+k-1)}{144}E_4 f_{r-1} \quad (r \geq 0)$$

with initial condition $f_0 = f$. Then the formula (37) in [5] is equivalent to the following closed formula.

Proposition 2.5. *Let f be a modular form of weight k. Then for any $n \geq 0$ we have*

$$\frac{\theta^n f}{n!} = \sum_{i=0}^{n} \binom{k+n-1}{i} \frac{f_{n-i}}{(n-i)!}\left(\frac{E_2}{12}\right)^i.$$

When $n = p-k+1$, the binomial coefficients $\binom{k+n-1}{i}$ are divisible by p for all $i > 0$, and hence Lemma 2.4 follows ($f_n = (\theta^n f)_0$). This completes the proof of the theorem. □

We give here a corollary to the theorem. As in the theorem, assume that $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ is a normalized eigenform of weight k and p is a prime number greater than k. We denote by $b(l,m,n)$ the coefficient of $X^l Y^m Z^n$ in $F(\theta^{p-k+1}f;X,Y,Z)$:

$$F(\theta^{p-k+1}f;X,Y,Z) = \sum_{2l+4m+6n=2p-k+2} b(l,m,n)X^l Y^m Z^n.$$

Corollary 2.6. (1) *Assume that* $k \equiv 0 \bmod 6$ *and* $p \equiv 2 \bmod 3$.

$$\textit{If } b(0,0,\frac{2p-k+2}{6}) \not\equiv 0 \bmod p, \textit{ then } a(p) \not\equiv 0 \bmod p.$$

(2) *Assume that* $k \equiv 0 \bmod 4$ *and* $p \equiv 3 \bmod 4$.

$$\textit{If } b(0,\frac{2p-k+2}{4},0) \not\equiv 0 \bmod p, \textit{ then } a(p) \not\equiv 0 \bmod p.$$

Proof. We only prove (1), the proof of (2) being similar. Write

$$H_p(Y,Z) = \sum_{4m+6n=p-1} c(m,n) Y^m Z^n.$$

By the assumption, $p-1$ is not divisible by 6 and hence the term with $m=0$ does not occur on the right. Therefore, if $b(0,0,\frac{2p-k+2}{6}) \not\equiv 0 \bmod p$, the polynomial $F(\theta^{p-k+1}f;X,Y,Z) \bmod p$ never be a multiple of $H_p(Y,Z) \bmod p$ and thus $a(p) \not\equiv 0 \bmod p$ by Theorem 1.1. □

3. Relation to supersingular j-invariants of elliptic curves

We may rephrase the theorem in terms of the supersingular j-polynomial.

Let f be a modular form of weight k. Write $k = 12m + 4\delta + 6\varepsilon$ with $m \geq 0$, $\delta \in \{0,1,2\}$, $\varepsilon \in \{0,1\}$. Then there exists a unique polynomial $G(f;x)$ such that

$$f(z) = \Delta(z)^m E_4(z)^\delta E_6(z)^\varepsilon G(f;j(z)),$$

where $\Delta(z) = (E_4(z)^3 - E_6(z)^2)/1728$ is the discriminant function and $j(z) = E_4(z)^3/\Delta(z)$ is the modular invariant. Moreover we put

$$\widetilde{G}(f;x) := x^\delta (x-1728)^\varepsilon G(f;x).$$

For a prime number p, we define the supersingular j-polynomial $S_p(x)$ by

$$S_p(x) := \prod_{E/\overline{\mathbb{F}}_p:\,\text{supersingular}} (x - j(E)) \in \mathbb{F}_p[x],$$

where the product runs over the isomorphism classes of supersingular elliptic curves in characteristic p and $j(E)$ is the j-invariant of E. Assume $p > 3$. A theorem of Deligne (cf. [3], [2]) then asserts that

$$\widetilde{G}(E_{p-1};x) \equiv S_p(x) \bmod p.$$

By this and Theorem 1.1, we have the following.

Theorem 3.1. *Assumption being the same as in Theorem 1.1, the following conditions are equivalent.*

(1) $a(p) \not\equiv 0 \bmod p$.

(2) $S_p(x) \nmid \widetilde{G}((\theta^{p-k+1}f)_0;x) \bmod p$.

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