MULTIPLE ZETA VALUES

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In a paper published in 1776 ([11]), Euler studied the series

(1)
$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\cdots.$$

Born in 1707, Euler was already in his late sixties when he published this paper. More than forty years earlier, he had found the famous formula

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}$$

and its generalization for even m

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\cdots=|B_m|\frac{(2\pi)^m}{2m!}, \tag{2}$$

where B_m is the Bernoulli number. In [11] Euler examines in detail relations between the double series (1) and the series of the form (2), which is the value of Riemann's zeta function. The paper [11] is the origin of the study of what we call today *the multiple zeta values*. However, it was only in the twentieth century that this series regained the interest of many mathematicians. At first some sporadic research was done for its own sake, or motivated by analytic number theory. Since the 1990s the series (1) has drawn more and more attention, as it turned out that this series is related to many other fields of mathematics, including Galois representations, arithmetic geometry, quantum groups, invariants for knots, mathematical physics, etc.

This article gives a survey of the fundamentals of the multiple zeta values, and introduces an approach using regularization in order to understand the relation among various multiple zeta values. Due to the nature of this article, all the citations of references are grouped in the last section.

1. Definitions and problems

For an n-tuple of positive integers $(k_1,k_2,\dots,k_n)$, we define the multiple zeta value $\zeta(k_1,k_2,\dots,k_n)$ as the sum of the infinite series

$$\zeta(k_1,k_2,\dots,k_n)=\sum_{m_1>m_2>\cdots>m_n>0}\frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}}, \tag{3}$$

where the summation runs over all the ordered positive integers. We call $(k_1,k_2,\dots,k_n)$ the *index* of $\zeta(k_1,k_2,\dots,k_n)$. We assume $k_1\geq 2$, since the series does not converge if $k_1=1$. With this notation Euler's series (1) equals $\zeta(m,n)+\zeta(m+n)$.

The series (3) may be expressed in terms of a multiple integral:

$$\zeta(k_1,k_2,\cdots,k_n)=\int\cdots\int_{1>t_1>t_2>\cdots>t_k>0}\frac{dt_1}{A_1(t_1)}\frac{dt_2}{A_2(t_2)}\cdots\frac{dt_k}{A_k(t_k)}, \tag{4}$$

where $k = k_1 + k_2 + \cdots + k_n$ and

$$A_i(t) = \begin{cases} 1-t, & i = k_1, k_1+k_2, k_1+k_2+k_3, \ldots, k_1+k_2+\cdots+k_n, \\ t, & \text{otherwise.} \end{cases}$$

Since we assumed $k_1 \geq 2$, we have $A_1(t_1) = t_1$. We also have $A_k(t_k) = 1 - t_k$ by the definition of k. These facts assure the convergence of the integral. The multiple integral (4) can be written as an iterated integral (from right to left):

$$\int_0^1 \frac{dt_1}{A_1(t_1)} \int_0^{t_1} \cdots \int_0^{t_{k-2}} \frac{dt_{k-1}}{A_{k-1}(t_{k-1})} \int_0^{t_{k-1}} \frac{dt_k}{A_k(t_k)}. \tag{5}$$

Expanding $1/(1-t_i)$ and integrating it term by term, we see that the integral equals the sum of the series (3).

The sum $k = k_1 + k_2 + \cdots + k_n$ is called the weight of the multiple zeta value $\zeta(k_1, k_2, \ldots, k_n)$, and the number of k_i in the index, n, is called its depth. The condition $k_1 \geq 2$ implies that the weight is a positive integer at least 2. The depth n satisfies $1 \leq n \leq k - 1$. In the integral expression (4) the weight is nothing but the number of variables in the integral, and the depth is the number of $1/(1 - t_i)$ in the integrand.

Example. The only multiple zeta value of weight 2 is $\zeta(2)$. There are two multiple zata values of weight 3, namely $\zeta(3)$ and $\zeta(2,1)$. The depth of $\zeta(3)$ is 1, and the depth of $\zeta(2,1)$ is 2. There are four multiple zeta values of weitht 4: $\zeta(4)$, $\zeta(3,1)$, $\zeta(2,2)$ and $\zeta(2,1,1)$. Their depths are 1, 2, 2 and 3 respectively.

In (4) we have two possibilities for $A_i(t_i)$ for each i satisfying $2 \leq i \leq k-1$. This implies that there are 2^{k-2} different indices $(k_1, k_2, \ldots, k_n)$ of weight k. This, however, does not necessarily mean that there are 2^{k-2} different multiple zeta values of weight k. For example, Euler showed $\zeta(2,1) = \zeta(3)$. (It is a good exercise for a reader to prove this equality.) We also have $\zeta(2,1,1) = \zeta(4)$ for weight 4.

What type of questions do we ask ourselves about these values?

Euler's formula (2) gives the value of $\zeta(2k)$. It involves Bernoulli numbers, which are number-theoretically interesting objects. Other known examples include

$$\zeta(\underbrace{2, 2, \ldots, 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!} \tag{6}$$

and

$$\zeta(\underbrace{3, 1, 3, 1, \ldots, 3, 1}_{2n}) = \frac{2\pi^{4n}}{(4n+2)!}. \tag{7}$$

Both of them are of the form (rational number) $\cdot\, \pi^{(\text{weight})}$, but neither contains such a subtle quantity as Bernoulli numbers. Apart from a small number of such examples, there are very few known values. At present, the main interest of research is to find various relations among the multiple zeta values, rather than the values themselves. Zagier has conducted a large amount of numerical computations and found that there should be many

Q-linear relations among them. For example, there are $2^8 = 256$ indices of weight 10, but all the multiple zeta values of weight 10 seem to be expressed as a **Q**-linear combination of seven of those. He verified it up to 200 digits after the decimal. However, there seem to be no linear relations between the values of different weights. Based on the numerical experiments up to weight 12, Zagier advanced a conjecture.

Definition. For $k \geq 0$, define $\mathcal{Z}_k$ as the **Q**-vector space given by

$$\mathcal{Z}_0 = \mathbf{Q}, \quad \mathcal{Z}_1 = \{0\},$$

$$\mathcal{Z}_k = \sum_{\substack{1 \leq n \leq k-1 \\ k_1 + \cdots + k_n = k, k_1 \geq 2}} \mathbf{Q}\, \zeta(k_1, k_2, \ldots, k_n) \quad (k \geq 2).$$

In addition, we define $\mathcal{Z} = \sum_{k \geq 0} \mathcal{Z}_k$.

For $k \geq 2$, $\mathcal{Z}_k$ is the finite dimensional **Q**-vector subspace in **R** spanned by 2^{k-2} multiple zeta values of weight k. $\mathcal{Z}$ is the **Q**-vector subspace in **R** spanned by all the multiple zeta values.

Conjecture (Zagier)**.** *The dimension* $\dim_{\mathbf{Q}} \mathcal{Z}_k$ *should satisfy the recurrence relation*

$$\dim_{\mathbf{Q}} \mathcal{Z}_k = \dim_{\mathbf{Q}} \mathcal{Z}_{k-2} + \dim_{\mathbf{Q}} \mathcal{Z}_{k-3} \quad (k \geq 3).$$

This is the most fundamental conjecture in this subject. Let $\{d_k\}_{k \geq 0}$ be the sequence defined inductively by $d_0 = 1$, $d_1 = 0$, $d_2 = 1, d_k = d_{k-2} + d_{k-3}$ $(k \geq 3)$, then the conjecture states that $\dim_{\mathbf{Q}} \mathcal{Z}_k = d_k$. Recently a very important result was proved independently by Goncharov and Terasoma.

Theorem (Goncharov, Terasoma)**.** *For all* $k \geq 0$ *we have*

$$\dim_{\mathbf{Q}} \mathcal{Z}_k \leq d_k.$$

Both proofs use the fact that the integral (4) can be regarded as a period of certain cohomology, and use Borel's result on the rank of K-groups and a recent result on the mixed Tate motives in order to bound the dimension from above. As for a lower bound of $\dim_{\mathbf{Q}} \mathcal{Z}_k$, we have no known bound better than the trivial $1 \leq \dim_{\mathbf{Q}} \mathcal{Z}_k$.

We note that $\mathcal{Z}$ can be regarded not only as a **Q**-vector space but also as a **Q**-algebra. To see that the product of two multiple zeta values is again a linear combination of multiple zeta values, we can use either the series expression (3) or the integral expression, and prove it directly by reorganizing the range of the sum or the integral. We will introduce algebraic formulations describing these multiplication rules in §3.

Problem. Find the sturcture of $\mathcal{Z}$ as a **Q**-algebra.

Goncharov has made a specific conjecture concerning this problem, but very little is known. The theorem of Goncharov and Terasoma gives an upper bound for the number of algebraically independent elements of the **Q**-algebra $\mathcal{Z}$, but we have very little information as to which relations contribute to bound the dimension. In what follows, we will focus on the linear relations, as an algebraic relation can be reduced to a linear relation once the product is expanded.

2. Various linear relations

In this section we first state some known linear relations. We begin with the so-called duality. Any index with $k_1 \geq 2$ can be written uniquely in the form

$$\mathbf{k} = (a_1+1, \underbrace{1,\ldots,1}_{b_1-1}, a_2+1, \underbrace{1,\ldots,1}_{b_2-1}, \ldots, a_s+1, \underbrace{1,\ldots,1}_{b_s-1}),$$

where $a_1, b_1, a_2, b_2, \ldots, a_s, b_s \geq 1$, and $s \geq 1$. Then

$$\mathbf{k}' = (b_s+1, \underbrace{1,\ldots,1}_{a_s-1}, b_{s-1}+1, \underbrace{1,\ldots,1}_{a_{s-1}-1}, \ldots, b_1+1, \underbrace{1,\ldots,1}_{a_1-1})$$

is an index whose first element is greater than 1. We call $\mathbf{k}'$ the dual index of $\mathbf{k}$. Note that the dual index of $\mathbf{k}'$ is $\mathbf{k}$.

Theorem (Duality)**.** *For mutually dual indices* $\mathbf{k}$ *and* $\mathbf{k}'$ *we have*

$$\zeta(\mathbf{k}) = \zeta(\mathbf{k}').$$

This equality can be seen easily from the integral expression (4) by the change of coordinate $(t_1, t_2, \ldots, t_k) \to (1-t_k, \ldots, 1-t_2, 1-t_1)$.

The first example of the duality is Euler's relation $\zeta(2,1) = \zeta(3)$. For instance, by letting $s = 1$, $a_1 = 1$, $b_1 = k-1$, we have $\zeta(2, \underbrace{1,1,\ldots,1}_{k-2}) = \zeta(k)$. We note that the indices of known values (6) and (7) are self-dual. Is there any reason to believe that the self-dual case is easier?

We list some examples of linear relations.

Theorem (Hoffman's relation)**.** *Fix an index* $(k_1, k_2, \ldots, k_n)$*. Then*

$$\sum_{l=1}^{n} \zeta(k_1, \ldots, k_l+1, \ldots, k_n)$$
$$= \sum_{\substack{1\leq l\leq n\\ k_l\geq 2}} \sum_{j=0}^{k_l-2} \zeta(k_1, \ldots, k_{l-1}, k_l-j, j+1, k_{l+1}, \ldots, k_n).$$

Theorem (Sum formula)**.** *Fix* n *and* k *satisfying* $1 \leq n \leq k-1$*. Then*

$$\sum_{\substack{k_1\geq 2, k_2,\ldots,k_n\geq 1,\\ k_1+\cdots+k_n=k}} \zeta(k_1, \ldots, k_n) = \zeta(k).$$

In other words, the sum of all the multiple zeta values of a fixed weight and depth equals Riemann's zeta value of that weight.

Theorem (Ohno's relation)**.** *Let* $(k_1, k_2, \ldots, k_n)$ *and* $(k'_1, k'_2, \ldots, k'_{n'})$ *be mutually dual indices, and let* $l \geq 0$*. Then*

$$\sum_{e_1+e_2+\cdots+e_n=l} \zeta(k_1+e_1, k_2+e_2, \ldots, k_n+e_n)$$
$$= \sum_{e'_1+e'_2+\cdots+e'_{n'}=l} \zeta(k'_1+e'_1, k'_2+e'_2, \ldots, k'_{n'}+e'_{n'}).$$

As a matter of fact, all the theorems in this section can be deduced from this last one. We leave to the reader how to deduce them.

3. Double shuffle relations

As we mentioned earlier, the product of two multiple zeta value is a linear combination of multiple zeta values with rational (actually integer) coefficients. In order to describe the conversion rule from a product to sums, we proceed as follows.

Let $\mathfrak{H} = \mathbf{Q}\langle x, y\rangle$ be the nonocommutative polynomial ring over $\mathbf{Q}$ with two variables. Define $\mathfrak{H}^1 = \mathbf{Q} + \mathfrak{H}y$ and $\mathfrak{H}^0 = \mathbf{Q} + x\mathfrak{H}y$. In other words, let $\mathfrak{H}^1$ be the subalgebra of $\mathfrak{H}$ generated by 1 and monomials ending y, and $\mathfrak{H}^0$ the subalgebra generated by monomials starting with x and ending with y, together with 1.

For a monomial $w = u_1u_2\cdots u_k \in \mathfrak{H}^0$ with $u_i = x$ or y, define

$$A_i(t) = \begin{cases} t & \text{if } u_i = x, \\ 1-t & \text{if } u_i = y, \end{cases}$$

and let

$$Z(w) = \int\cdots\int_{1>t_1>t_2>\cdots>t_k>0} \frac{dt_1}{A_1(t_1)}\frac{dt_2}{A_2(t_2)}\cdots\frac{dt_k}{A_k(t_k)}.$$

Since $w \in \mathfrak{H}^0$, we have $A_1(t_1) = t_1$ and $A_k(t_k) = 1 - t_k$; thus the above integral converges. Extending this $\mathbf{Q}$-linearly, we obtain a map

$$Z : \mathfrak{H}^0 \longrightarrow \mathbf{R}.$$

(Here, we set $Z(1) = 1$.) It is clear from the integral expression (4) that

$$Z(x^{k_1-1}\,y\,x^{k_2-1}\,y\cdots x^{k_n-1}\,y) = \zeta(k_1, k_2, \ldots, k_n).$$

The total degree of w is the weight, and the degree with respect to y is the depth of $\zeta(k_1, \ldots, k_n)$.

Define $z_k = x^{k-1}y$. Then $Z(z_k)$ is the Riemann zeta value $\zeta(k)$. The algebra $\mathfrak{H}^1$ is a (noncommutative) free $\mathbf{Q}$-algebra generated by z_k ($k = 1, 2, 3, \ldots$), and we have $Z(z_{k_1}z_{k_2}\cdots z_{k_n}) = \zeta(k_1, k_2, \ldots, k_n)$ ($k_1 \geq 2$).

The "*harmonic product* $*$" on $\mathfrak{H}^1$ is defined as follows. First, for k, $l \geq 1$ and for w, w_1, $w_2 \in \mathfrak{H}^1$, we define

$$\begin{aligned} 1 * w \;&=\; w * 1 \;=\; w, \\ z_kw_1 * z_lw_2 \;&=\; z_k(w_1 * z_lw_2) + z_l(z_kw_1 * w_2) + z_{k+l}(w_1 * w_2), \end{aligned}$$

and then we extend it by $\mathbf{Q}$-bilinearlity.

The harmonic product $*$ gives the $\mathbf{Q}$-vector space $\mathfrak{H}^1$ a structure of commutative $\mathbf{Q}$-algebra, and $\mathfrak{H}^0$ becomes its $\mathbf{Q}$-subalgebra. We denote these algebras by $\mathfrak{H}^1_*$ and $\mathfrak{H}^0_*$, respectively. Then, the multiplication law of multiple zeta values using series expression can be stated as follows: the evaluation map $Z : \mathfrak{H}^0 \to \mathbf{R}$ is an algebra homomorphism with respect to the multiplication $*$; that is,

$$Z(w_1 * w_2) = Z(w_1)Z(w_2) \tag{8}$$

for all w_1, $w_2 \in \mathfrak{H}^0$. For example, the harmonic product $z_k * z_l = z_k z_l + z_l z_k + z_{k+l}$ corresponds to the formula $\zeta(k)\zeta(l) = \zeta(k,l) + \zeta(l,k) + \zeta(k+l)$. As a product of the series we see that

$$\begin{aligned}
\zeta(k)\zeta(l) &= \sum_{m>0} \frac{1}{m^k} \sum_{n>0} \frac{1}{n^l} = \sum_{\substack{m>0\\n>0}} \frac{1}{m^k n^l} \\
&= \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0} \right) \frac{1}{m^k n^l} \\
&= \zeta(k,l) + \zeta(l,k) + \zeta(k+l).
\end{aligned}$$

The definition of the harmonic product $*$ describes the way of developing series in such a way inductively.

Another product ш is called "*shuffle product*" and is defined on all of $\mathfrak{H}$. It is defined by

$$\begin{aligned}
1 \,ш\, w \;&=\; w \,ш\, 1 \;=\; w, \\
uw_1 \,ш\, vw_2 \;&=\; u(w_1 \,ш\, vw_2) + v(uw_1 \,ш\, w_2)
\end{aligned}$$

(w, w_1, $w_2 \in \mathfrak{H}$ and u, $v \in \{x, y\}$), and is extended by $\mathbf{Q}$-bilinearlity. With respect to this multiplication, a $\mathbf{Q}$-vector space $\mathfrak{H}$ becomes a commutative $\mathbf{Q}$-algebra. We denote it by $\mathfrak{H}_{ш}$. It is easy to see that $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are subalgebras of $\mathfrak{H}_{ш}$, and we denote them by $\mathfrak{H}^1_{ш}$ and $\mathfrak{H}^0_{ш}$ respectively. Then, the classically known shuffle product for iterated integrals translates to the fact that the map Z is a homomorphism with respect to ш:

$$Z(w_1 \,ш\, w_2) = Z(w_1)Z(w_2). \tag{9}$$

For example, corresponding to the shuffle product $xy \,ш\, xy = 4x^2y^2 + 2xyxy$, we have the relation

$$\begin{aligned}
\zeta(2)^2 &= \iint_{1>t_1>t_2>0} \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \cdot \iint_{1>s_1>s_2>0} \frac{ds_1}{s_1} \frac{ds_2}{1-s_2} \\
&= \int_{\substack{1>t_1>t_2>0\\1>s_1>s_2>0}} \cdots \int \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{ds_1}{s_1} \frac{ds_2}{1-s_2} \\
&= \Bigg(\int_{1>t_1>t_2>s_1>s_2>0} \cdots \int + \int_{1>t_1>s_1>t_2>s_2>0} \cdots \int \\
&\qquad + \int_{1>s_1>t_1>t_2>s_2>0} \cdots \int + \int_{1>t_1>s_1>s_2>t_2>0} \cdots \int \\
&\qquad + \int_{1>s_1>t_1>s_2>t_2>0} \cdots \int + \int_{1>s_1>s_2>t_1>t_2>0} \cdots \int \Bigg) \frac{dt_1}{t_1} \frac{dt_2}{1-t_2} \frac{ds_1}{s_1} \frac{ds_2}{1-s_2}
\end{aligned}$$

$$= 4 \int\cdots\int_{1>u_1>u_2>u_3>u_4>0} \frac{du_1}{u_1}\frac{du_2}{u_2}\frac{du_3}{1-u_3}\frac{du_4}{1-u_4}$$

$$+ 2 \int\cdots\int_{1>v_1>v_2>v_3>v_4>0} \frac{dv_1}{v_1}\frac{dv_2}{1-v_2}\frac{dv_3}{v_3}\frac{dv_4}{1-v_4}$$

$$= 4\zeta(3,1) + 2\zeta(2,2).$$

Combining (8) and (9), we obtain

$$Z(w_1 ш w_2) = Z(w_1 * w_2) \qquad (w_1, w_2 \in \mathfrak{H}^0). \tag{10}$$

The relations among multiple zeta values obtained in this way are called *double shuffle relations*. The expression "double shuffle" comes from the fact that we consider the definition of the harmonic product as a shuffle of the range of the sum. The first example of such relations is

$$4\zeta(3,1) + 2\zeta(2,2) = 2\zeta(2,2) + \zeta(4) \ (= \zeta(2)^2), \tag{11}$$

which gives $4\zeta(3,1) = \zeta(4)$.

Although the double shuffle relations arise very naturally, they do not account for all the linear relations. For example, the relation $\zeta(2,1) = \zeta(3)$ for weight 3 does not come from double shuffle relations. In weight 4 we should have three independent linear relations, but the double shuffle relations account for only one of them. We would like to obtain more linear relations systematically, and in a natural way if possible. We will introduce the idea of "regularization" in the next section. We will see that most (conjecturally all) of the theorems quoted in the previous section can be obtained using this idea.

4. Regularization and generalized double shuffle relations

For an index $(k_1, k_2, \ldots, k_n)$ with $k_1 = 1$, the series (3) and the integral (4) do not converge, but we can use the technique called *regularization* to associate a finite value to such an index. We first explain it algebraically. First, note that we have the isomorphisms of commutative algebras $\mathfrak{H}^1_* \simeq \mathfrak{H}^0_*[y]$ and $\mathfrak{H}^1_ш \simeq \mathfrak{H}^0_ш[y]$; the former was shown by Hoffman, while the latter has been known classically. It claims that they are isomorphic to the polynomial rings in the variable y with coefficients in $\mathfrak{H}^0_*$ and $\mathfrak{H}^0_ш$, respectively. Together with the fact that Z is a homomorphism with respect to each multiplication, we have the following.

Proposition. *There are two homomorphisms*

$$Z^* : \mathfrak{H}^1_* \longrightarrow \mathbf{R}[T] \quad \textit{and} \quad Z^ш : \mathfrak{H}^1_ш \longrightarrow \mathbf{R}[T]$$

such that the restriction of each of these homomorphisms to $\mathfrak{H}^0$ coincides with $Z : \mathfrak{H}^0 \to \mathbf{R}$, and that the image of y is T.

From now on, we cosider indices $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_1 \geq 1$ (allowing $k_1 = 1$). For such an index $\mathbf{k} = (k_1, \ldots, k_n)$, the images of the monomial $x^{k_1-1}y\cdots x^{k_n-1}y \in \mathfrak{H}^1$ by the maps Z^* and $Z^ш$ are denoted by $Z^*_{\mathbf{k}}(T)$ and $Z^ш_{\mathbf{k}}(T)$ respectively. If $k_1 \geq 2$, we have $Z^*_{\mathbf{k}}(T) = Z^ш_{\mathbf{k}}(T) = \zeta(\mathbf{k})$.

Table 1.

$\mathbf{k}$	(1)	(1,1)	(1,2)
$Z^*_{\mathbf{k}}(T)$	T	$\frac{1}{2}T^2 - \frac{1}{2}\zeta(2)$	$\zeta(2)\,T - \zeta(2,1) - \zeta(3)$
$Z^{ш}_{\mathbf{k}}(T)$	T	$\frac{1}{2}T^2$	$\zeta(2)\,T - 2\zeta(2,1)$

Analytically, these polynomials may be interpreted as follows. For $Z^*_{\mathbf{k}}(T)$, the finite sum

$$\zeta_N(k_1,\ldots,k_n) := \sum_{N>m_1>m_2>\cdots>m_n>0} \frac{1}{m_1^{k_1}m_2^{k_2}\cdots m_n^{k_n}} \tag{12}$$

can be shown to diverge with the order of $Z^*_{\mathbf{k}}(\log N)$ when we let $N \to \infty$. To be precise, there is a nonnegative integer j such that

$$(12) = Z^*_{\mathbf{k}}(\log N + \gamma) + O(N^{-1}\log^j N) \quad (N \to \infty),$$

where γ is Euler's constant. For the harmonic series we have the formula

$$1 + \frac{1}{2} + \cdots + \frac{1}{N-1} = \log N + \gamma + O(N^{-1}).$$

The left-hand side is $\zeta_N(1)$, which corresponds to y in $\mathfrak{H}^1$. On the other hand, for the index $\mathbf{k} = (1)$, we have $Z^*_{\mathbf{k}}(T) = T$. Thus the right-hand side is equal to $Z^*_{\mathbf{k}}(\log N+\gamma)+O(N^{-1})$ with $\mathbf{k} = (1)$. As for $Z^{ш}_{\mathbf{k}}(T)$, we consider the integration obtained by replacing the limit of integral 1 by $1-\epsilon$ ($\epsilon > 0$) in (4):

$$\int\cdots\int_{1-\epsilon>t_1>t_2>\cdots>t_k>0} \frac{dt_1}{A_1(t_1)}\frac{dt_2}{A_2(t_2)}\cdots\frac{dt_k}{A_k(t_k)}. \tag{13}$$

This has a finite value even when $k_1 = 1$. The behavior as we let $\epsilon \to 0$ is such that there is a positive integer j satisfying

$$(13) = Z^{ш}_{\mathbf{k}}(\log(1/\epsilon)) + O(\epsilon\log(1/\epsilon)^j).$$

Note that $Z^{ш}_{\mathbf{k}}(T) = T$ when $\mathbf{k} = (1)$. Since the series (12) and the integral (13) are compatible with the harmoing product and the shuffle product respectively, we see that the asymptotic behavior of these series and integrals can be described by $Z^*_{\mathbf{k}}(T)$ and $Z^{ш}_{\mathbf{k}}(T)$ inductively from the case $\mathbf{k} = (1)$. As for the concrete expressions of $Z^*_{\mathbf{k}}(T)$ and $Z^{ш}_{\mathbf{k}}(T)$, we see, again by induction, that for an index $\mathbf{k}$ of the form $\mathbf{k} = (\underbrace{1,1,\ldots,1}_{s},\mathbf{k}')$ ($s \geq 0$, $k'_1 \geq 2$), both $Z^*_{\mathbf{k}}(T)$ and $Z^{ш}_{\mathbf{k}}(T)$ are of the form

$$\zeta(\mathbf{k}')\frac{T^s}{s!} + (\text{lower degree terms}).$$

For each i, the coefficient of T^i is a $\mathbf{Q}$-linear combination of multiple zeta values with weight $k-i$, where k is the weight of $\mathbf{k}$. We list a few examples in Table 1.

In particular, the constant terms $Z^*_{\mathbf{k}}(0)$ and $Z^{ш}_{\mathbf{k}}(0)$ are called the regularized (renormalized) values of $\zeta(k_1,k_2,\ldots,k_n)$. Thus, for an index $\mathbf{k}$ we have two different regularized values, one obtained using the series and the other using the integral. For $w \in \mathfrak{H}^1$, the constant term of $Z^*(w)$ and of $Z^{ш}(w)$ are denoted by $Z_0^*(w)$ and $Z_0^{ш}(w)$, respectively.

Let us describe the polynomials $Z_{\mathbf{k}}^*(T)$ and $Z_{\mathbf{k}}^{ш}(T)$ further, especially the relation between them. It gives us an important clue when we try to generalize the double shuffle relations. First, define a power series $G(u)$ with $\mathbf{R}$-coefficents by

$$G(u) \;=\; \exp\Bigl(\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\,\zeta(n)\,u^n\Bigr).$$

Using the well-known properties of the gamma function, we see that

$$G(u) = e^{\gamma u}\,\Gamma(1+u) \quad (|u|<1),$$

where γ is once again Euler's constant. Then, define an $\mathbf{R}$-linear map $\rho : \mathbf{R}[T] \to \mathbf{R}[T]$ by the formula

$$\rho(e^{Tu}) = G(u)\,e^{Tu}.$$

In other words, if we write $G(u) \;=\; \sum_{k=0}^{\infty} \gamma_k u^k$ ($\gamma_0 = 1$, $\gamma_1 = 0$, $\gamma_2 = \zeta(2)/2, \dots$), ρ is defined by the formula

$$\rho\Bigl(\frac{T^l}{l!}\Bigr) = \sum_{k=0}^{l} \gamma_k \frac{T^{l-k}}{(l-k)!} \quad (l = 0, 1, 2, \dots),$$

and extended by $\mathbf{R}$-linearlity. By the definition of $G(u)$, its coefficient γ_k is an element of weight k of the $\mathbf{Q}$-subalgebra of $\mathcal{Z}$ generated by the Riemann zeta values.

Theorem (Zagier, Boutet de Monvel)**.** *For any index* $\mathbf{k}$ *we have*

$$Z_{\mathbf{k}}^{ш}(T) = \rho(Z_{\mathbf{k}}^*(T)). \tag{14}$$

As an example, consider $\mathbf{k} = (1, 2)$. From Table 1 we see

$$\zeta(2)\,T - 2\zeta(2,1) \;=\; \zeta(2)\,T - \zeta(2,1) - \zeta(3).$$

(Note that $\rho(T) = T$.) From this we obtain Euler's relation $\zeta(2,1) \;=\; \zeta(3)$, and we see that the space $\mathcal{Z}_3$ of weight 3 is one dimensional. Similarly, comparing both sides of (14) for $\mathbf{k} = (1,3)$ and $\mathbf{k} = (1,2,1)$ respectively, we have

$$\zeta(3)\,T - 2\zeta(3,1) - \zeta(2,2) = \zeta(3)\,T - \zeta(4) - \zeta(3,1),$$
$$\zeta(2,1)\,T - 3\zeta(2,1,1) = \zeta(2,1)\,T - \zeta(3,1) - \zeta(2,2) - 2\zeta(2,1,1).$$

Thus we have

$$\zeta(4) \;=\; \zeta(3,1) + \zeta(2,2) \;=\; \zeta(2,1,1)$$

These relations together with $4\zeta(3,1) = \zeta(4)$ obtained from (11) show that the dimension of $\mathcal{Z}_4$ is 1. The proof of the theorem is based on the comparison between (12) and (13). See the references for detail. Now, here are some generalizations of the double shuffle relations.

Theorem. *The following relations hold.*

(i) *For any* $w_1 \in \mathfrak{H}^1$ *and* $w_0 \in \mathfrak{H}^0$, *we have* $Z_0^{ш}(w_1 \,ш\, w_0 - w_1 * w_0) = 0$.
(i′) *For any* $w_1 \in \mathfrak{H}^1$ *and* $w_0 \in \mathfrak{H}^0$, *we have* $Z_0^{*}(w_1 \,ш\, w_0 - w_1 * w_0) = 0$.
(ii) *For any* $m \geq 1$ *and* $w_0 \in \mathfrak{H}^0$, *we have* $Z_0^{ш}(y^m * w_0)) = 0$.

(ii′) *For any* $m \geq 1$ *and* $w_0 \in \mathfrak{H}^0$, *we have* $Z_0^*(y^m \sqcup\!\sqcup w_0 - y^m * w_0)) = 0$.

If we take $w_1 \in \mathfrak{H}^0$ in (i) or (i′), we obtain the double shuffle relation (10). (ii) and (ii′) are specializations of (i) and (i′), respectivley. (Set $w_1 = y^m$.) Since $Z_0^{\sqcup\!\sqcup}(y^m) = 0$, we do not have the term $y^m \sqcup\!\sqcup w_0$ in (ii). All of (i), (i′), (ii) and (ii′) can be obtained from (14). Conversely, we can derive (14) from one of these. The totality of the relations obtained from (8), (9) and (14) coincides with the relations obtained form (i). Also, it coincides with the relations (ii) or (ii′) together with (10). Thus, all of these relations are called *generalized double shuffle relations.*

Conjecture. *Any linear relation among multiple zeta values is a consequence of generalized double shuffle relations. As for algebraic relations, any relation can be obtained from* (8), (9) *and the relations in the theorem above.*

It will be a very important task to understand these relations from a different point of view.

5. Supplements and references

§1. To see that $k_1 \geq 2$ is a sufficient condition for the convergence, regard (3) as a Dirichlet series with respect to m_1 and estimate the coefficients. The analytic continuation and the values at negative integers of the function obtained by regarding the indices of the multiple zeta values as the variable are studied, for example, in Akiyama-Egami-Tanigawa [1], Akiyama-Tanigawa [2], Arakawa-Kaneko [4] and Matsumoto [26]. [26] contains a history of analytic number theory and detailed references.

Zagier [38] gives a credit to Kontsevich for the integral expression (4). Although the direction is different, K. T. Chen has studied far more general iterated integrals. See the special issue of the *Illinois Journal of Math.* **34** (1990).

The value $\zeta(2, 2, \ldots, 2)$ can be obtained easily by comparing the infinite product expansion and the Taylor expansion of $\sin x$. For the evaluation of $\zeta(3, 1, \ldots, 3, 1)$ and its generalization, see Borwein-Bradley-Broadhurst-Lisonek [5] and Bowman-Bradley [7]. See [38] for Zagier's conjecture. See Broadhurst-Kreimer[8] for its generalization that takes the depth into account, and the generalization to the case of so-called Euler sums. Goncharov has also done some broad research on the series generalizing multiple zeta values and Euler sums. In particular, he studied them in relation to the Galois representation of the fundamental group of $\mathbf{P}^1 \setminus \{0, \infty, \text{roots of unity}\}$. A serious reader can learn the conjecture together with its background from Goncharov [13, 14, 15, 16, 17], etc. For Terasoma's work, see [36]. As for the independence of multiples zeta values, we see at least that $\mathcal{Z}$ is infinite dimensional, using Euler's formular for Riemann zeta values and the transcendence of π. Recently, Rivoal [35] showed that there are infinitely many $\mathbf{Q}$-independent numbers among the Riemann zeta values at odd integers.

§2. See Zagier [38], Hoffman [19] and Ohno [28] for theorems in this section. For a recent attempt to understand Ohno's relation from the connection formula for the multi-polylogarithm function, see Okuda-Ueno [30]. Also, Hoffman-Ohno [21] gives a direction to its generalization.

We did not mention the relations obtained from the theory of knots in the text. See Le-Murakami [25] and Ihara-Takamuki [23] for this direction. Ohno-Zagier [29] gave a direct proof for some of the relations in [25]. They obtained beautiful results on combination of multiple zeta values expressible in terms of Riemann zeta values, a generalization of Euler's attempt for the case of depth 2.

§3. Description of the products and relations of multiple zeta values using the **Q**-algebra $\mathfrak{H}$ was initiated by Hoffman [20]. For shuffle products, see Reutenauer's textbook [34]. For shuffle products of iterated integrals, see Ree [33]. See also Minh-Petitot [27] for algebras of multiple zeta values and polylogarithms.

At the end of §3 we mentioned that almost all relations can be deduced from the generalized double shuffle relations. Curiously, it is still unknown whether the simplest relation, the duality, is subject to the generalized double shuffle relations.

§4. The main reference for this section is [22]. While the fact that the two regularizations are related by the map ρ has long been known by Zagier, Boutet de Monvel [6] also obtained it independently.

We explain here briefly the relation between the regularization and the so-called Drinfel'd associator [10].

Let us use the same notation $Z_0^{ш}$ for the extension of the regularization $Z_0^{ш}$ to $\mathfrak{H}$ through the isomorphism $\mathfrak{H}_{ш} \simeq \mathfrak{H}^0_{ш}[x,y]$ (namely, take an element of $\mathfrak{H}_{ш}$, regard it as a polynomial in two variables x and y with $\mathfrak{H}^0_{ш}$ coefficients, take its constant term, and then apply Z). Define a formal power series $\Phi(X,Y)$ in two variables X and Y with real coefficients by

$$\Phi(X,Y) = \sum_W Z_0^{ш}(w)W,$$

where W runs over all the monomials in X and Y, and w is defined as the element of $\mathfrak{H}$ obtained by replacing uppercase letters in W by lowercase letters. This is the generating function of the regularized values of the integrals (4) allowing both $A(t_1) = 1 - t_1$ and $A(t_k) = t_k$.It is not difficult to see that this generating function coincides with the KZ-associator of Drinfel'd up to some conventional sign and constant multiple. It is an important problem to understand the relation between the functional equation of the Drinfel'd associator and the relations among the multiple zeta values. Racinet has done some interesting work ([31] [32]) in this direction recently. Recent work of Furusho [12] also points in that diredction, and links it to the "stable derivation algebra" ([24]). The last theorem in §4 has yet another equivalent description in terms of the derivations on $\mathfrak{H}$. The Lie algebra generated by the derivations arising there has a suggestive structure, but its importance is yet to be clarified.

The references given here are not at all complete. For more extensive references on the multiple zeta values, see the bibliographies in the survey articles by Cartier [9] and Waldschmidt [37], as well as the useful Web site maintained by Hoffman:

`http://www.usna.edu/Users/math/meh/biblio.html`

It has links to the electronically available preprints.

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