Analogues of the Aoki-Ohno and Le-Murakami relations for finite multiple zeta values

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Abstract
We establish finite analogues of the identities known as the Aoki-Ohno relation and the Le-Murakami relation in the theory of multiple zeta values. We use an explicit form of a generating series given by Aoki and Ohno.

1 Introduction and statement of the results
For an index set of positive integers \(k = (k_1, \ldots, k_r)\) with \(k_1 > 1\), the multiple zeta value \(\zeta(k)\) and the multiple zeta-star value \(\star \zeta(k)\) are defined respectively by the nested series

\[
\zeta(k) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}
\]

and

\[
\star \zeta(k) = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

We refer to the sum \(k_1 + \cdots + k_r\), the length \(r\) and the number of components \(k_i\) with \(k_i > 1\) as the weight, depth, and height of the index \(k\) respectively.

For given \(k\) and \(s\), let \(I_0(k, s)\) be the set of indices \(k = (k_1, \ldots, k_r)\) with \(k_1 > 1\) of weight \(k\) and height \(s\). We naturally have \(k \geq 2s\) and \(s \geq 1\); otherwise \(I_0(k, s)\) is empty. Aoki and Ohno proved in [1] the identity

\[
\sum_{k \in I_0(k, s)} \star \zeta(k) = 2 \left( \frac{k-1}{2s-1} \right) (1 - 2^{1-k}) \zeta(k).
\] (1.1)

On the other hand, for \(\zeta(k)\), the following identity is known as the Le-Murakami relation ([6]): for even \(k\),

\[
\sum_{k \in I_0(k, s)} (-1)^{\text{dep}(k)} \zeta(k) = \frac{(-1)^{k/2}}{(k+1)!} \sum_{r=0}^{k/2-s} \binom{k+1}{2r} (2 - 2^{2r}) B_{2r} \pi^k,
\]

where \(B_n\) denotes the Bernoulli number. As Euler discovered, the right-hand side is a rational multiple of the Riemann zeta value \(\zeta(k)\).
In this short article, we establish the analogous identities for finite multiple zeta values. For an index set of positive integers \( k = (k_1, \ldots, k_r) \), the finite multiple zeta value \( \zeta_A(k) \) and the finite multiple zeta-star value \( \zeta^*_A(k) \) are elements in the quotient ring \( A := \left( \prod_p \mathbb{Z}/p\mathbb{Z} \right) / \left( \bigoplus_p \mathbb{Z}/p\mathbb{Z} \right) \) \((p \text{ runs over all primes})\) represented respectively by

\[
\left( \sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} \mod p} \right)_p \text{ and } \left( \sum_{p > m_1 \geq \cdots \geq m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} \mod p} \right)_p.
\]

Studies of finite multiple zeta(-star) values go back at least to Hoffman [2] (the preprint was available around 2004) and Zhao [10]. But it was only recently that Zagier proposed (in 2012 to the first-named author) considering them in the (characteristic 0) ring \( A \) \((\text{see also } [3, 4])\). In \( A \), the naive analogue \( \zeta_A(k) \) of the Riemann zeta value \( \zeta(k) \) is zero because \( \sum_{n=1}^{p-1} 1/n^k \) is congruent to 0 modulo \( p \) for all sufficiently large primes \( p \). However, the “true” analogue of \( \zeta(k) \) in \( A \) is considered to be

\[
Z(k) := \left( \frac{B_{p-k}}{k} \right)_p.
\]

We note that this value is zero when \( k \) is even because the odd-indexed Bernoulli numbers are 0 except \( B_1 \). It is still an open problem whether \( Z(k) \neq 0 \) for any odd \( k \geq 3 \).

We now state our main theorem, where the role of \( Z(k) \) as a finite analogue of \( \zeta(k) \) is evident.

**Theorem 1.1.** The following identities hold in \( A \):

\[
\sum_{k \in I_0(k,s)} \zeta_A(k) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) Z(k),
\]

\[
\sum_{k \in I_0(k,s)} (-1)^{\text{dep}(k)} \zeta_A(k) = 2 \binom{k-1}{2s-1} (1 - 2^{1-k}) Z(k).
\]

We should note that the right-hand sides are exactly the same. In the next section, we give a proof of the theorem.

## 2 Proof of Theorem 1.1

Let \( \text{Li}_k^*(t) \) be the ‘nonstrict’ version of the multiple-polylogarithm:

\[
\text{Li}_k^*(t) = \sum_{m_1 \geq \cdots \geq m_r \geq 1} \frac{t^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}}.
\]

Aoki and Ohno [1] computed the generating function

\[
\Phi_0 := \sum_{k,s \geq 1} \left( \sum_{k \in I_0(k,s)} \text{Li}_k^*(t) \right) x^{k-2s} z^{2s-2},
\]

We denote the right-hand side by \( \mathcal{P}_0 \). It is shown that \( \mathcal{P}_0 \) satisfies the following recursion relation:

\[
\mathcal{P}_0(t) = \left( \frac{1}{1-t} - \frac{1}{1-t^2} \right) \mathcal{P}_0(t) + \left( \frac{1}{1-t} - \frac{1}{1-t^2} \right) \mathcal{P}_0(t).
\]

Since \( \mathcal{P}_0(t) \) is the generating function of the sequence of finite multiple zeta values, we have

\[
\sum_{k \in I_0(k,s)} \zeta_A(k) = \mathcal{P}_0(t) - \Phi_0(t).
\]
and, in view of $\text{Li}_k^*(1) = \zeta^*(k)$ (if $k_1 > 1$), evaluated it at $t = 1$ to obtain the identity (1.1). For our purpose, the function $\text{Li}_k^*(t)$ is useful because the truncated sum

$$
\sum_{p > m_1 \geq \cdots \geq m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}
$$

used to define $\zeta^*(k)$ is the sum of the coefficients of $t^i$ in $\text{Li}_k^*(t)$ for $i = 1, \ldots, p - 1$. In [1, Section 3], Aoki and Ohno showed that

$$
\Phi_0 = \sum_{n=1}^{\infty} a_n t^n,
$$

where

$$
a_n = \sum_{l=1}^{n} \left( \frac{A_{n,l}(z)}{x + z - l} + \frac{A_{n,l}(-z)}{x - z - l} \right)
$$

and

$$
A_{n,l}(z) = (-1)^l \binom{n-1}{l} \frac{(z - l + 1) \cdots (z - 1)(z + 1) \cdots (z + n - l - 1)}{(2z - l + 1) \cdots (2z - 1)2z(2z + 1) \cdots (2z + n - l)}.\]

The problem is then to compute the coefficient of $x^{k-2s}z^{2s-2}$ in $\sum_{n=1}^{p-1} a_n$ modulo $p$.

We proceed as follows:

$$
\sum_{n=1}^{p-1} a_n = \sum_{n=1}^{p-1} \sum_{l=1}^{n} \left( \frac{A_{n,l}(z)}{x + z - l} + \frac{A_{n,l}(-z)}{x - z - l} \right) = \sum_{l=1}^{p-1} \sum_{n=1}^{p-1} \left( \frac{A_{n,l}(z)}{x + z - l} + \frac{A_{n,l}(-z)}{x - z - l} \right) = \sum_{l=1}^{p-1} \sum_{n=0}^{p-l-1} \left( \frac{A_{n+l,l}(z)}{x + z - l} + \frac{A_{n+l,l}(-z)}{x - z - l} \right).
$$

Writing $A_{n+l,l}(z)$ as

$$
A_{n+l,l}(z) = (-1)^l \binom{n-l+1}{l-1} (l)_n \frac{z}{2z} \frac{(2z - l + 1)(2z + 1)n!}{(2z - l + 1)(2z + 1)n!},
$$

where $(a)_n = a(a + 1) \cdots (a + n - 1)$, we have

$$
\sum_{n=0}^{p-l-1} A_{n+l,l}(z) = \frac{(-1)^l (z - l + 1)(l-1)}{2z (2z - l + 1)} \sum_{n=0}^{p-l-1} \frac{(l)_n z^n}{(2z + 1)n!}.
$$

We view the sum on the right as

$$
\sum_{n=0}^{p-l-1} \frac{(l)_n z^n}{(2z + 1)n!} = F(-p + l, z; 2z + 1; 1) - \frac{(l)_{p-l}(z)_{p-l}}{(2z + 1)_{p-l}(p - l)!} \mod p.
$$
By the binomial expansion, 

\[ F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \]

where \((a)_n\) for \(n \geq 1\) is as before and \((a)_0 = 1\). Note that if \(a\) (or \(b\)) is a nonpositive integer \(-m\), then \(F(a, b; c; z)\) is a polynomial in \(z\) of degree at most \(m\), and the renowned formula of Gauss

\[ F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \]

becomes

\[ F(-m, b; c; 1) = \frac{(c-b)_m}{(c)_m}. \]

Hence

\[ F(-p + l, z; 2z + 1; 1) = \frac{(z + 1)_{p-l}}{(2z + 1)_{p-l}} \equiv \frac{z^{p-1} - 1}{(2z)^{p-1} - 1} \frac{(2z - l + 1)_{l-1}}{(z - l + 1)_{l-1}} \mod p. \]

We also compute

\[ \frac{(l)_{p-l}(z)_{p-l}}{(2z + 1)_{p-l}(p - l)!} \equiv (-1)^{l-1} z^{(p-1) - 1} (2z - l + 1)_{l-1} \mod p. \]

Since we only need the coefficient of \(z^{2s-2}\), we may work modulo higher powers of \(z\) and, in particular, we may replace \((z^{p-1} - 1)/(2z)^{p-1} - 1\) by 1, assuming \(p\) is large enough. (We may assume this because an identity in \(A\) holds true if the \(p\)-components on both sides agree in \(\mathbb{Z}/p\mathbb{Z}\) for all large enough \(p\).) Hence,

\[
\sum_{n=1}^{p-1} a_n \equiv \sum_{l=1}^{p-1} \left\{ \frac{(-1)^l}{2z} \left( \frac{1}{x + z - l} - \frac{1}{x - z - l} \right) + \frac{1}{2} \left( \frac{1}{(x + z - l)(z - l)} - \frac{1}{(x - z - l)(z + l)} \right) \right\} \mod p.
\]

By the binomial expansion,

\[
\sum_{l=1}^{p-1} \frac{(-1)^l}{x + z - l} = \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l} \sum_{m=0}^{\infty} \frac{x + z}{l}^m = \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l} \sum_{m=0}^{\infty} \frac{1}{l^m} \sum_{i=0}^{m} \binom{m}{i} x^{m-i} z^i = \sum_{m \geq i \geq 0} \binom{m}{i} \left( \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^{m+1}} \right) x^{m-i} z^i.
\]

From this we obtain

\[
\sum_{l=1}^{p-1} \frac{(-1)^l}{2z} \left( \frac{1}{x + z - l} - \frac{1}{x - z - l} \right) = \sum_{m \geq 2i+1 \geq 0} \binom{m}{2i+1} \left( \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^{m+1}} \right) x^{m-2i-1} z^{2i}
\]
and, by letting $i \rightarrow s - 1$ and $m \rightarrow k - 1$, the coefficient of $x^{k-2s}z^{2s-2}$ in this is
\[
\binom{k-1}{2s-1} \sum_{l=1}^{p-1} \frac{(-1)^{l-1}}{l^k}.
\]
This is known to be congruent modulo $p$ to
\[
2\binom{k-1}{2s-1}(1-2^{1-k})B_{p-k}
\]
(see for example, [11, Theorem 8.2.7]). Concerning the other term,
\[
\sum_{l=1}^{p-1} \left\{ \frac{1}{x} \left( \frac{1}{z-l} - \frac{1}{z+l} \right) \right\}.
\]
every quantity that appears as a coefficient in the expansion into power series in $x$ and $z$ is a multiple of the sum of the form $\sum_{l=1}^{p-1} 1/l^m$, and all are congruent to 0 modulo $p$.
This concludes the proof of (1.2).
We may prove (1.3) in a similar manner by using the generating series of Ohno-Zagier [7], but we deduce (1.3) from (1.2) by showing that the left-hand sides of both formulas are equal up to sign.
Set $S_{k,s} := \sum_{k \in I_0(k,s)} (-1)^{\text{dep}(k)} \zeta_A(k)$ and $S_{k,s}^\star := \sum_{k \in I_0(k,s)} \zeta_A^\star(k)$.

**Lemma 2.1.** $S_{k,s}^\star = (-1)^{k-1}S_{k,s}$.

**Proof.** We use the well-known identity (see, for instance, [8, Corollary 3.16])
\[
\sum_{i=0}^{r} (-1)^i \zeta_A(k_i, \ldots, k_1) \zeta_A(k_{i+1}, \ldots, k_r) = 0. 
\] (2.1)
Taking the sum of this over all $k \in I_0(k,s)$ and separating the terms corresponding to $i = 0$ and $i = r$, we obtain
\[
S_{k,s}^\star + \sum_{k' \in I_0(k')} (-1)^{\text{dep}(k')} \zeta_A(\overline{k'}) \left( \sum_{k'' \in I(k'', s''')} \zeta_A(k'') \right) + (-1)^k S_{k,s} = 0.
\]
Here, $\overline{k'}$ denotes the reversal of $k'$, and the set $I(k'', s''')$ consists of all indices (no restriction on the first component) of weight $k''$ and height $s'''$. We have used $\zeta_A(\overline{k'}) = (-1)^k \zeta_A(k)$ in computing the last term ($i = r$). Since the second sum in the middle is symmetric and hence 0 (by Hoffman [2, Theorem 4.4] and $\zeta_A(k) = 0$ for all $k \geq 1$), the lemma follows.

Since $Z(k) = 0$ if $k$ is even, we see from Lemma 2.1 that the formula for $S_{k,s}$ is the same as that for $S_{k,s}^\star$. This concludes the proof of our theorem.

**Remark 2.2.** K. Yao [9] proved the lemma in the case $s = 1$ and T. Murakami (unpublished) in general for all odd $k$.5
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References


